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On the Rise and Fall of Networked Societies

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Abstract. We review recent results on the dynamics of social networks which suggest that the interplay between the network formation process and volatility may lead to the occurrence of discontinuous phase transitions and phase coexistence in a large class of models. We then investigate the effects of negative links – links inhibiting local growth of the network – and of a geographical distribution of the agents in such models. We show, by extensive numerical simulations, that both effects enhance this phenomenology, i.e. it increases the size of the coexistence region.

INTRODUCTION

Recent phenomenological studies on complex networks in the social sciences have uncovered ubiquitous nontrivial statistical properties, such as scale free distribution of connectivity or small world phenomena [1, 2, 3]. These properties have striking consequences on the processes which take place on such networks, such as percolation [4], diffusion [5, 6], phase transitions [7, 8] and epidemic spreading [9]. The research on complex networks raises questions of a new type as it addresses phenomena where the topology of interactions is part of the dynamic process. This contrasts with traditional statistical mechanics, where the topology of the interaction is fixed *a priori* by the topology of the embedding space.

Phenomena of this type are quite common in social sciences where agents purposefully establish cooperative links [10]. Links between individuals in a social network support not only the socioeconomic interactions that determine their payoffs, but also carry information about the state of the network. This aspect has important consequences in the long run if the underlying environment is volatile. In this case, former choices tend to become obsolete and individuals must swiftly search for new opportunities to offset negative events. The role of the network for information diffusion is particularly apparent, for example, pertaining to the way in which individuals find new job opportunities. For example, it has been consistently shown by sociologists and economists alike [11, 12] that personal acquaintances play a prominent role in job search. This, in turn, leads to a significant correlation in employment across friends, relatives, or neighbours. The common thesis proposed to explain this evidence is that, in the presence of environmental volatility, the quantity and quality of one's social links – sometimes referred to as her *social capital* [13]– is a key basis for search and adaptability to change.

A recent statistical mechanics approach to simple models of social networks has recently shown that the interplay between volatility and the quest for efficiency leads, in

a broad class of models, to a positive feedback loop between the network's structure and its dynamics [14, 15]. As a result, social networks may exhibit sharp phase transitions – i.e. a dense network may emerge or disappear [16] suddenly – coexistence of different network phases for the same parameters and resilience – i.e. robustness of a dense social network even when external conditions deteriorate beyond the point where a dense network first came into existence. This generic conclusion was derived in two qualitatively different setups: Ref. [14] addressed the interplay between volatility and search in a model where agents use their links to look for new fruitful collaborations. Ref. [15] found instead the same phenomenology in generic models where proximity or similarity favours the formation of links among agents and, conversely, the presence of links between two agents enhances similarity. As discussed in Ref. [15], there are several socio-economic phenomena, ranging from job contact networks and research collaborations to the spread of crime and other social pathologies, for which anecdotal evidence has been reported.

Such dynamic effects (e.g. sharp transitions) are much more difficult to detect in empirical studies than static properties (e.g. scale-freeness or small-worldness). Hence, the empirical verification of the scenarios proposed in Refs. [14, 15] requires very accurate data, which is rarely available in socio-economic sciences. A different way to check the validity of the scenario in generic cases is to challenge its robustness on theoretical grounds, including effects which have been neglected so far. Here, in particular, we address two simplifying assumptions of the models of Ref. [14, 15]: *i)* that network formation is long ranged, i.e. independent of a geographical distribution of the agents and *ii)* that links always have a positive effect on the link formation process.

We discuss these effects in the framework of the model of Ref. [14] where they enter in an important way into the dynamics of the network (see later). In both cases, we find by extensive numerical simulations, that inclusion of these effects enhances the character of our conclusions (i.e. it increases the co-existence region in parameter space). This supports the conclusion that sharp transitions, co-existence and resilience are generic dynamic properties of social networks.

In what follows we shall first review the model of Ref. [14], then turn to the study of negative links and finally discuss the inclusion of geographical effects.

SEARCHING PARTNERS IN A VOLATILE WORLD

Ref. [14] proposes a stylized model of a society that embodies the following three features: (i) agent interaction, (ii) search and (iii) volatility (i.e. random link removal). Individuals are involved in bilateral interactions, as reflected by the prevailing network. Through occasional update, some of the existing links have their value deteriorate and are therefore lost. In contrast, the individuals also receive opportunities to search that, when successful, allow the establishment of fresh new links.

Formally, the network is given by a set of nodes N and the corresponding adjacency matrix $A(t)$ with elements $a_{ij}(t) = 1$ if there is a link connecting nodes i and j at time t, and $a_{ij} = 0$ otherwise (we assume no on-site loops, $a_{ii} = 0$ and un-oriented links $a_{ij} = a_{ji}$). Denote by $F_i = \{j | a_{ij} = 1\}$ the set of neighbours ("friends") of the node i.

The matrix $A(t)$ follows a stochastic process governed by the following three processes:

Long distance search: At rate η, each node i gets the opportunity to make a link to a node j randomly selected (if the link is already there nothing happens).

Short distance search: At rate ξ, each node i picks at random one of its neighbours $j \in F_i$ and j then randomly selects (i.e. "refers to") one of its other neighbours $k \in F_j \backslash \{i\}$. If $k \notin F_i$ then the link between i and k is formed. If $F_i = \emptyset$ or $F_j = \{i\}$ or $k \in F_i$ nothing happens.

*Decay*: At rate λ, each existing link decays and it is randomly deleted.

Over time, this process leads to an evolving social network that is always adapting to changing conditions. For $\xi = 0$, the dynamics is very simple and the stationary network is a random graph with average degree $c = 2\eta/\lambda$. For $\eta \ll \lambda$ the network is composed of many disconnected parts. Fig. 1 reports what happens when the local search rate ξ is turned on. For small ξ, network growth is limited by the global search process that proceeds at rate η. Clusters of more than 2 nodes are rare and, when they form, local search quickly saturates the possibilities of forming new links. Suddenly, at a critical value ξ_2, a giant component connecting a finite fraction of the nodes emerges. The average degree c indeed jumps abruptly at ξ_2. The distribution $p(c)$ of c_i is peaked with an exponential decrease for large c and a power law $p(c) \sim c^\mu$ for c small. The network becomes more and more densely connected as ξ increases further. But when ξ decreases, we observe that the giant component remains stable also beyond the transition point ($\xi < \xi_2$). Only at a second point ξ_1 does the network lose stability and the population gets back to an unconnected state. There is a whole interval $[\xi_1, \xi_2]$ where both a dense-network phase and one with a nearly empty network coexist. This behaviour is typical of first-order phase transitions. The coexistence region $[\xi_1, \xi_2]$ shrinks as η increases.

In loose words, the model shows that the continuous struggle of agents' continuous search must be strong enough to offset volatility if a dense and effective social network is to be preserved. On the other hand, search can be effective only in a densely networked society. So information diffusion and a dense network of interactions are two elements of a feedback self-reinforcing loop. As a result, the system displays a discontinuous phase transition and hysteresis, enjoying some resistance to a moderate deterioration of the underlying environmental conditions. Such a resilience can be interpreted as consequence of the buffer effects and enhanced flexibility enjoyed by a society that has accumulated high levels of social capital.

These features are captured by a mean field theory which is in good qualitative agreement with numerical simulation results (see Ref. [14]). This theory highlights the particular role that clustering plays in the dynamics of the model. Indeed search is particularly effective when clustering is low whereas it is suppressed in a high clustered society. The average clustering coefficient q – defined as the fraction of pairs of neighbours of i who are also neighbours among themselves[1] – shows a non-trivial behaviour. In the unconnected phase, q increases with ξ as expected. In this phase, q is close to one because the expansion of the network is mostly carried out through global search, and local search

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[1] The averaging is done only over nodes with at least two neighbours.
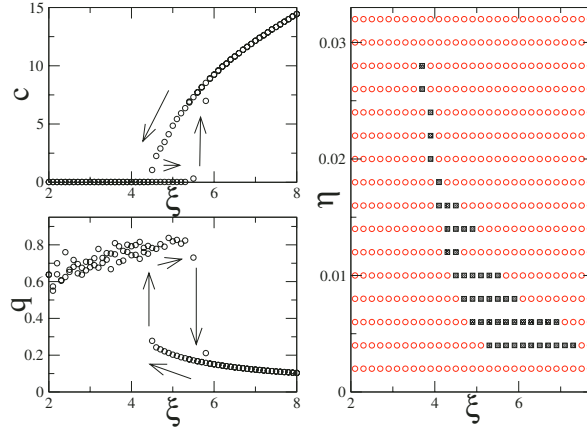
FIGURE 1. Average degree c (top) and clustering coefficient q (bottom) from numerical simulations with $\eta/\lambda = 0.01$ for populations of size $n = 1000$. Here and in all other figures, runs were equilibrated for a time $t_{eq} = 3000/\lambda$ before taking averages for a further $3000/\lambda$. The network was started in both the low connected and high connected state for each value of ξ. For the central coexistence region, the two distinct points for each ξ represent the two different starting configurations. The arrows show the hysteretic region, the rightmost arrows indicating ξ_2. The right hand graph shows the phase diagram, black squares denote the coexistence region, red circles the regions in which only the low (lower left) or high (upper right) phases are stable.

quickly saturates all possibilities of new connections. On the other hand, in the dense-network phase, q takes relatively small values. This makes local search very effective. Remarkably we find that *q decreases* with ξ in this phase, which is rather counterintuitive: *increasing the rate ξ at which bonds between neighbours form through local search, the density q of these bonds decreases*. In fact, similar behaviour is found if, fixing ξ and η, the volatility rate λ decreases.

These conclusions rest on two basic assumptions, which might be unrealistic in practical cases. The first is that links have always a positive effect on the formation of other links. Indeed, "negative" links (i.e. animosity) may have an important effect in inhibiting link formation. If one of the my possible friends has a negative relation with a friend of mine, I might not wish to form the link with him/her, because this would increase the "frustration" of my social neighbourhood. It is indeed a well accepted fact in social science [17] that social relationships evolve in such a way as to decrease frustration.

The second assumption of the model, is that agents are treated equivalently in the global search process. In many real cases, agents are located in a geometrical space and this influences the likelihood with which they establish new links among themselves. Notice that a dependence of the link formation rate on proximity in space has arguably strong consequences on clustering, which is a key aspect of the model.

In both cases, as we shall see, the inclusion of these effects enhances the non-linear effect and result in an even wider region of coexistence.

THE EFFECT OF NEGATIVE LINKS

Here we extend the model to include the effect of negative links. In addition to the long-range search, introduction of friends, and decay of links, we also include negative links. These links model the effect of animosity between nodes. Thus, when two nodes i and j are introduced, before they form a positive link they check through all their neighbours to see if any of them have a negative link with their prospective neighbour. They are in effect using their contacts to check the 'references' of their prospective neighbour. If there are one or more negative links (or if i and j already have a negative link) then the new connection is not formed.

Negative links themselves are formed by the 'souring' of existing positive links at a rate γ. In other words, every link is positive when it is created, but it may turn to negative at rate γ. Once formed, negative links decay at rate λ^- which we set equal to λ for simplicity except when stated otherwise.

This additional mechanism has two effects on the network: firstly, positive links now disappear at a rate $\lambda + \gamma$ rather than λ as before. Secondly, the rate of introduction of nodes through mutual friends (the ξ process) is reduced. Since it is the nonlinearity of the ξ process that produces the coexistence region, one might expect this to have important effects on the size and location of the coexistence region.

Figure 2 shows plots for four values of γ. As γ is increased, the value ξ_2 above which the low connected state becomes unstable increases markedly – indeed for $\gamma = 0.1$ the value of ξ at which the transition occurs (for the times of $3000 + 3000$ studied here) is around $\xi = 20$. Also, the average degree of the network in the connected region decreases and the value ξ_1 below which the connected region collapses moves slightly up. The overall effect is that the coexistence region gets larger and moves slightly to higher values of ξ when γ increases.
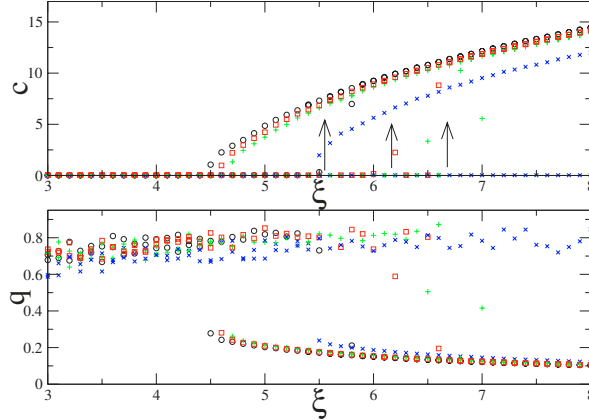

FIGURE 2. Average degree c (top) and clustering coefficient q (bottom) from numerical simulations with $\eta/\lambda = 0.01$ for populations of size $n = 1000$. Original case $\gamma = 0$ (black circle), $\gamma = 0.01$ (red square), $\gamma = 0.02$ (green plus), and $\gamma = 0.1$ (blue cross). The arrows indicate the approximate locations of ξ_2 for each value of γ.

More dramatic effect occurs for large values of γ. Figure 3 shows that the system may enter into a regime where the network undergoes successive rises and crashes due to the spread of animosity. This behavior also sets in if negative links are much more stable

than positive ones ($\lambda^- \ll \lambda$, lower panel of Fig. 3). Then once a connected society is formed, its network of relationships gets slowly poisoned with long lasting negative links, which inhibit the formation of other positive links.

We believe that the occurrence of such time-dependent behaviours is intimately related to the phase coexistence of the original system. Here the low connectivity state is unstable, over some mean waiting time, to the formation of the highly connected state. However the highly connected state is also not stable once a sufficiently large number of links have turned to negative links. The system thus alternates between the two states, but not in a periodic manner due to the stochastic nature of the process.
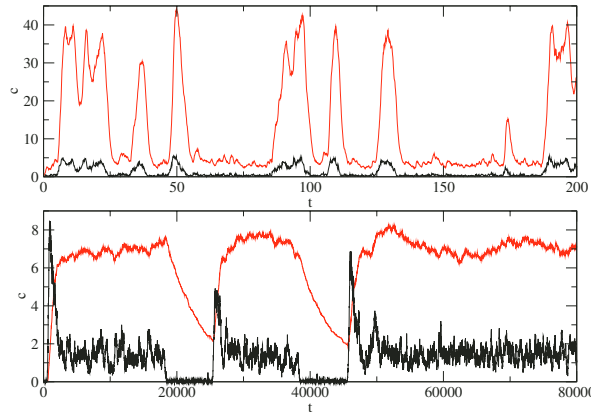
FIGURE 3. Average positive degree c (black) and negative degree (red) plotted against time. From numerical simulations with populations of size $n = 100$. For the upper graph the parameters are: $\lambda = \lambda^- = 1$, $\eta = 1$, $\gamma = 10$, $\xi = 400$. For the lower graph the parameters are: $\lambda = 1$, $\lambda^- = 2 \times 10^{-4}$, $\eta = 10^{-2}$, $\gamma = 10^{-3}$, $\xi = 8$. Note that in this case $\lambda \gg \lambda^-$.

THE EFFECT OF GEOMETRY

We now consider another important effect not considered in the original model, that of the physical space in which the agents live. We introduce a modified version of the model which accounts for the fact that agents embedded in space are more likely to make random acquaintances with other agents who are geographically near to them.

We modify the original model in the following way: We embed the agents on a one-dimensional periodic lattice of length n, with agent i being placed at a distance i from the origin. When creating long-range links (the η process), we select site i at random and then site j with a probability $P(d_{ij})$ which decays with the distance d_{ij} between i and j on the lattice.[2] We studied distributions of the form $P(d) \propto d^{-\alpha}$ ($\alpha > 0$) decays with distance.

Notice that the local search process ξ can only connect members of a community of already connected agents. It is only by the η process that such a community can reach agents further away. Hence we expect that a sharp decay of $P(d)$ with distance has

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[2] Notice that because of periodic boundary conditions, if $i < j$ then $d_{ij} = \min(j - i, i - j + L)$.
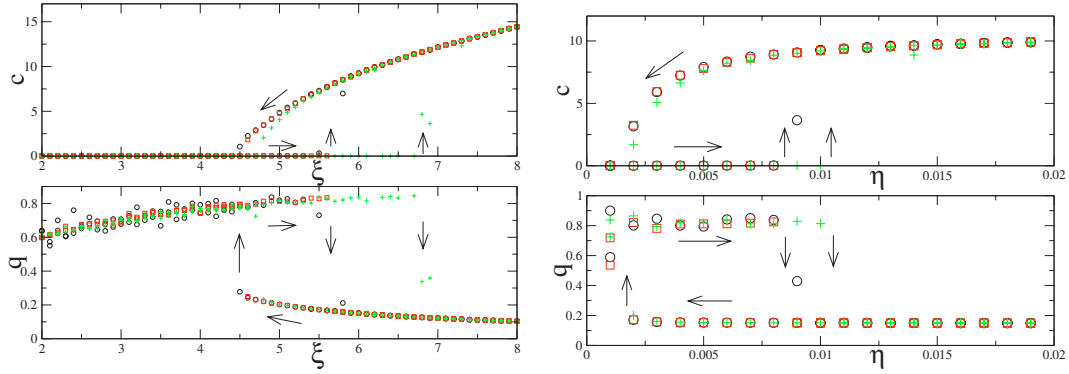
FIGURE 4. Average degree c (top) and clustering coefficient q (bottom) from numerical simulations for populations of size $n = 1000$. The left-hand plots show results for $\eta/\lambda = 0.01$, plotted against ξ. The right-hand plots show results for $\xi/\lambda = 6$, plotted against η. The points are: original case $\alpha = 0$ (black circle), $\alpha = 1$ (red square), $\alpha = 2$ (green plus). The arrows indicate the points at which transitions occur and the directions in which the system moves within the hysteretic region. Notice that the coexistence region is extended for $\alpha = 2$.

strong effects on the η process, which is the limiting factor in the nucleation of a dense network,[3] thus increasing the stability of the low density phase. Figures 4 confirm this expectation for the case $P(d) \sim d^{-\alpha}$ with $\alpha = 1,2$. The main change occurs for $\alpha = 2$ where the stability of the low connectivity phase and hence the coexistence region is significantly extended. Notice also that well inside the dense network phase there is no significant effect. This confirms that this phase is sustained by the local search process alone: once a global network spanning the whole system is formed, geometry has no effect.

ACKNOWLEDGMENTS

REFERENCES

1. Albert, R., and A.-L., B., *Rev. Mod. Phys.*, **74**, 47 (2002).
2. Dorogovtsev, S. N., and Mendes, J. F. F., *Adv. Phys.*, **51**, 1079 (2002).
3. Watts, D. J., and Strogatz, S. H., *Nature*, **393**, 440 (1998).

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[3] Furthermore, this modification has also the effect of reducing the rate at which links are formed by the η process. This is because links between close agents are more likely to exist already and cannot be added again.

4. Dorogovtsev, S. N., Mendes, J. F. F., and Samukhin, A. N., *Phys. Rev. E*, **64**, 066110 (2001).
5. Farkas, I. J., Derényi, I., Barabási, A.-L., and Vicsek, T., *Phys. Rev. E*, **64**, 026704 (2001).
6. Goh, K.-I., Kahng, B., and Kim, D., *Phys. Rev. E*, **64**, 051903 (2001).
7. Dorogovtsev, S., Goltsev, A., and Mendes, J., *Eur. Phys. J. B*, **38**, 177–182 (2004).
8. Ehrhardt, G. C. M. A., and Marsili, M., *J. Stat. Mech.* (2005).
9. Pastor-Satorras, R., and Vespignani, A., *Phys. Rev. Lett.*, **86**, 3200 (2001).
10. Putnam, R., *Making Democracy Work*, Princeton University Press, Princeton, 1993.
11. Granovetter, M., *Getting a Job: A Study of Contacts and Careers*, Chicago University Press, Chicago, 1995.
12. Topa, G., *Rev. of Econ. Studies*, **68**, 261–295 (2001).
13. Coleman, J. S., *Amer. J. Soc.*, **94**, S95 (1988).
14. Marsili, M., Vega-Redondo, F., and Slanina, F., *Proceedings of the National Academy of Sciences*, **101**, 1439–1443 (2004).
15. Ehrhardt, G. C. M. A., Marsili, M., and Vega-Redondo, F., *arxiv/physics*, **0504124** (2005).
16. Bianconi, G., Marsili, M., and Vega-Redondo, F., *Physica A*, **346**, 116–122 (2005).
17. Hummon, N. P., and Doreian, P., *Social Networks*, **25**, 17–49 (2003).