On hyperbolic and spherical volumes for knot and link cone-manifolds and polyhedra

Alexander Mednykh
Sobolev Institute of Mathematics
Novosibirsk State University
Novosibirsk, Russia
On volume of tetrahedron in hyperbolic and spherical spaces

A. D. Mednykh
Novosibirsk, Russia

The calculation of the volume of a polyhedron in 3-dimensional space $E^3$, $H^3$ or $S^3$ is a very old and difficult problem. The first known result in this direction belongs to Tartaglia (1494) who found a formula for the volume of Euclidean tetrahedron. Now this formula is known as Cayley-Menger determinant. More precisely, let

\[ T = \begin{array}{ccc}
\text{d} & \text{d} & \text{d} \\
\text{d} & \text{d} & \text{d} \\
\text{d} & \text{d} & \text{d}
\end{array} \]

be an Euclidean tetrahedron with edge lengths $d_{ij}, 1 \leq i < j \leq 4$. Then $V = \text{Vol}(T)$ is given by

\[
288 V^2 = \begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & d_1^2 & d_2^2 \\
1 & d_1^2 & 0 & d_3^2 \\
1 & d_1^2 & d_2^2 & 0
\end{vmatrix}
\]

Note that $V$ is a root of quadratic equation whose coefficients are integer polynomials in $d_{ij}, 1 \leq i < j \leq 4$. 
Surprisingly, but this result can be generalized on any Euclidean polyhedron in the following way.

**Theorem (I. Kh. Sabitov, 1996)** Let $P$ be an Euclidean polyhedron. Then $V = \text{Vol}(P)$ is a root of an even degree algebraic equation whose coefficients are integer polynomials on edge lengths of $P$ depending on combinatorial type of $P$ only.

**Example.**

\[ P_1 = \text{cube} \quad \text{and} \quad P_2 = \text{ellipsoid} \]

are of the same combinatorial type.

Hence, $V_1 = \text{Vol}(P_1)$ and $V_2 = \text{Vol}(P_2)$ are roots of the same algebraic equation

\[ a_0 V^{2n} + a_1 V^{2n-2} + \cdots + a_n V^0 = 0. \]

(All edge lengths are taken to be 1.)
Cauchy theorem (1813) states that if the faces of a convex polyhedron are made of metal plates and the polyhedron edges are replaced by hinges, the polyhedron would be rigid.

In spite of this, there are non-convex polyhedra which are flexible.

Bricard, 1897 (self-intersecting flexible octahedron)

Connelly, 1978 (the first example of true flexible polyhedron)

The smallest example is given by Steffen (14 triangular faces and 9 edges)

--- mountain fold
----- valley fold

The Steffen flexible polyhedron
Very important consequence of Sabitov's theorem is a positive solution of Bellows Conjecture proposed by Dennis Sullivan.

**Theorem (R. Connelly, J. Sabitov, A. Walg, 1997)**

All flexible polyhedra keep a constant volume as they are flexed.

It was shown by Victor Alexandrov (Novosibirsk) that the Bellows Conjecture fails in the spherical space $S^3$. In the hyperbolic space $H^3$ the problem is still open.

Any analog of Sabitov's theorem is unknown in both spaces $S^3$ and $H^3$. 
The volume of a biorthogonal tetrahedron (orthoscheme) was calculated by Lobachevsky and Bolai in $H^3$ and by Schläfli in $S^3$.

Since Lobachevsky's formula is widely known we restrict ourselves by Bolai's and Schläfli's results.

**Theorem (Bolai).** The volume of a hyperbolic orthoscheme $K_{\alpha\beta\gamma}$ is given by the formula

$$\text{vol}(K_{\alpha\beta\gamma}) = \frac{\tan\gamma}{2\tan\alpha} \int_0^\infty \frac{z \sinh z \, dz}{(\cosh z - 1)^{1/2} \sinh^2 z - 1}.$$
Theorem (Schläfli)

The volume of a spherical orthoscheme with essential dihedral angles $A$, $B$, and $C$ is given by the formula

$$V = \frac{1}{4} S(A, B, C),$$

where

$$S(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z) = S(x, y, z)$$

and

$$D = \sqrt{\cos^2 x \cos^2 z - \cos^2 y}.$$
Theorem (Lobachevsky, Coxeter)

The volume of a hyperbolic orthoscheme with essential dihedral angles $A$, $B$, and $C$ is given by the formula

$$V = \frac{c}{4} S(A, B, C)$$

where $S(A, B, C)$ is the Schlafli function.

The structure of the Schlafli function

$$S(x, y, z) = S(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z)$$

is slightly complicated but it is naturally divided into four elementary pieces.
Theorem (Coxeter, 1935)

In the hyperbolic case $\cos^2 x \cos^2 \theta > 0$ the Schlafli function and the Lobachevsky function are related by the formula

$$
\hat{S}(x, y, z) = -\Delta(x, \theta) + \Delta(y, \theta) - \Delta(z, \theta) + \Delta(\theta)
$$

where $\Delta(x, \theta) = \Lambda(x + \theta) - \Lambda(x - \theta)$,

$$
\Lambda(x) = -s_x \ln 2 \min \{1, x \} dt
$$

is the Lobachevsky function and

$$
\tan \theta = \frac{\min x \min y \min z}{\sqrt{\cos^2 x \cos^2 \theta - \cos^2 y \cos^2 z}}.
$$

Theorem (Derevnin, Mednykh, 2002)

In the spherical case $\cos^2 x \cos^2 \theta \leq 0$ the Schlafli function satisfies

$$
\hat{S}(x, y, z) = -\delta(x, \theta) + \delta(y, \theta) - \delta(z, \theta) + \delta(\theta)
$$

where $\delta(x, \theta) = \int_0^\theta \ln \left(1 - \cos^2 x \cos^2 2t\right) \frac{dt}{\cos 2t}$

and

$$
\tan \theta = \frac{\min x \min y \min z}{\sqrt{\cos^2 x \cos^2 \theta - \cos^2 y \cos^2 z}}.
$$
The function $\delta(x, \theta)$ is considered as a spherical analog of the function $\Lambda(x+\theta) - \Lambda(x-\theta)$ and satisfies the following properties:

(i) $\delta(x, \theta)$ is continuous for all $(x, \theta) \in \mathbb{R}^2$ and differentiable for $x = \frac{\theta}{2} + k\pi$, $k \in \mathbb{Z}$.

(1) $\delta(x, 0) = \pi^2/4 - |\pi/2 - x\pi|$, $0 \leq x < \pi$.

(2c) Let $\check{\delta}(x, \theta) = \delta(x, \theta) + (2\theta/\pi - 1)\delta(x, \theta)$.

Then

a) $\check{\delta}(x, \theta)$ is even and $\pi$-periodic on $x$.

b) $\check{\delta}(x, \theta)$ is odd and $\pi$-periodic on $\theta$.

c) $|\delta(x, \theta)| \leq \pi^2/4$ and $\delta(\pi/2, 3\pi/2) = \pi^2/4$. 
Consider an ideal hyperbolic tetrahedron with all vertices on the infinity.

\[ T(A,B,C) : \]

\[ \text{Opposite dihedral angles of tetrahedron are equal to each other and } A + B + C = \pi. \]

**Theorem (Milnor, 1982)**

\[ \text{Vol } T(A,B,C) = \Lambda(A) + \Lambda(B) + \Lambda(C), \]

where

\[ \Lambda(x) = -\int_0^x \log |2\sin t| \, dt \]

is the Lobachevsky function.

More complicated case with only one vertex on the infinity was investigated by **Vinberg (1993)**.
Despite of these partial results, a formula for the volume of an arbitrary hyperbolic tetrahedron has been unknown until very recently. The general algorithm for obtaining such a formula was indicated by Hsiang (1988), and the complete solution of the problem was given by Yu. Cho and H. Kim (1999), Y. Murakami, M. Yano (2001), and A. Ushijima (2002).

In these papers the volume of tetrahedron is expressed as an analytic formula involving 15 dilogarithm or Lobadatsky functions whose arguments depend on the dihedral angles of the tetrahedron and on some additional parameter which is a root of some complicated quadratic equation with complex coefficients.

A geometrical meaning of the obtained formula was recognized by G. Leibon (2002) from the viewpoint of the Regge symmetry. An excellent exposition of these ideas and a complete geometric proof of the volume formula was given by Y. Mohanty (2003).
We suggest the following integral formula for the volume of a tetrahedron.

Let \( T = T(A, B, C, D, E, F) \) be a hyperbolic tetrahedron with dihedral angles \( A, B, C, D, E, F \):

We set:
\[
\begin{align*}
V_1 &= A + B + C, & V_2 &= A + E + F, \\
V_3 &= B + D + F, & V_4 &= C + D + E \\
3 & \text{(for vertices)} \\
H_1 &= A + B + D + E, & H_2 &= A + C + D + F \\
H_3 &= B + C + E + F, & H_4 &= 0 \\
& \text{(for Hamiltonian cycles)}
\end{align*}
\]

Theorem (Derevnin-Mednykh, 2003)
The volume of a hyperbolic tetrahedron is given by the formula:

\[
\text{Vol}(T) = -\frac{1}{4} \int \log \prod_{i=1}^{4} \frac{\cos \frac{V_i}{2}}{\sin \frac{H_i}{2}} \, dz,
\]

where \( z_1 \) and \( z_2 \) are roots of the integrand, given by formulas:

\[
z_1 = \arctan \frac{K_2}{K_1}, \quad z_2 = \arctan \frac{K_4}{K_3},
\]

and

\[
\begin{align*}
K_1 &= -\Sigma_{i=1}^{4} (\cos(S - H_i)) + \cos(S - V_i)) \\
K_2 &= \Sigma_{i=1}^{4} (\sin(S - H_i)) + \sin(S - V_i)) \\
K_3 &= 2 \left( \sin A \sin D + \sin B \sin E + \sin C \sin F \right) \\
K_4 &= \sqrt{K_1^2 + K_2^2 - K_3^2} \\
S &= A + B + C + D + E + F.
\end{align*}
\]
Recall that the Digamma function is defined by

\[ \text{Li}_2(x) = - \int_0^x \frac{\log(1-t)}{t} \, dt. \]

We set

\[ \ell(z) = \text{Li}_2(e^{i\pi z}). \]

The following result is a consequence of the above theorem.

**Theorem (Murakami-Yano, 2001)**

\[ \text{Vol}(T) = \frac{1}{2} \text{Im}(U(z_1,T) - U(z_2,T)), \]

where

\[ U(z, T) = \frac{1}{2} (\ell(z) + \ell(A+B+D+E+z) + \ell(A+C+D+F+z) + \ell(B+C+E+F+z) - \ell(\pi+A+B+C+z) - \ell(\pi+A+E+F+z) - \ell(\pi+B+D+F+z) - \ell(\pi+D+E+z)). \]

**Remark.** Since

\[ \text{Im}(\ell(z)) = \text{Im}(\text{Li}_2(e^{i\pi z})) = 2\Lambda \left( \frac{\pi z}{2} \right) \]

the volume function can be expressed in terms of 16 Lobachevsky functions.

\[ \Lambda(x) = - \int_0^x \log \left| 2 \sinh t \right| \, dt. \]
A tetrahedron $T = T(A, B, C, D, E, F)$ is called to be symmetric if $A = D$, $B = E$, $C = F$.  

**Theorem (Derevmin-Mednykh-Pashkevich 2004)**

Let $T = T(A, B, C, A, B, C)$ be a symmetric hyperbolic tetrahedron. Then

$$\frac{\pi}{2} \sqrt{\text{Vol}(T) = 2 \int (\arcsin(a \cos t) + \arcsin(b \cos t)) \theta + \arcsin(c \cos t) - \arcsin(c \cos t)) \frac{dt}{\sin 2t},}$$

where $\theta \in (0, \frac{\pi}{2})$ is defined by

$$\tan \theta = \frac{1 - a^2 - b^2 - c^2 - 2abc}{(1 - a b + d)(1 - a b - c)(1 + a b - c)(1 + a b + c)}$$

with $a = \cos A$, $b = \cos B$, and $c = \cos C$.

**Remark.** Let $l_A, l_B, l_C$ are the lengths of the edges of $T$ with dihedral angles $A, B, C$ respectively. Then

$$\frac{\sin A}{\sinh l_A} = \frac{\sin B}{\sinh l_B} = \frac{\sin C}{\sinh l_C} = \tan \theta.$$
Duality formula for the volume of spherical tetrahedron.

Let $T$ be a spherical tetrahedron with dihedral angles $A_{ij}, 1 \leq i < j \leq 4$ and edge lengths $a_{ij}, 1 \leq i < j \leq 4$, respectively.

We define dual tetrahedron $T^*$ to be with dihedral angles $A^*_{ij} = \pi - A_{ij}$ and edge lengths $a^*_{ij} = \pi - A_{ij}, 1 \leq i < j \leq 4$.

Middle curvature $K = K(T)$ is defined by formula

$$K = \sum_{1 \leq i < j \leq 4} \frac{(\pi - A_{ij}) a_{ij}}{2} = \sum_{1 \leq i < j \leq 4} \frac{a^*_{ij} - a_{ij}}{2}.$$

We note that $K(T) = K(T^*)$.

Denote by $V$ and $V^*$ the volumes of tetrahedra $T$ and $T^*$, respectively.
Theorem (Mednykh, 2005)

\[ V + V^* + K = \pi^2. \]

Proof. By the Schl"afli formula we get

\[ dV = \sum_{ij} a_{ij} dA_{ij} = -\sum_{ij} a_{ij} d a_{ij}^* \]
and

\[ dV^* = \sum_{ij} a_{ij}^* dA^* = -\sum_{ij} a_{ij}^* d a_{ij}. \]

Hence,

\[ 2(dV + dV^*) = -\sum_{ij} (a_{ij} d a_{ij}^* + a_{ij}^* d a_{ij}) \]

\[ = -2dK. \]

We have

\[ 2(dV + dV^* + dK) = 0 \quad \text{and} \quad V + V^* + K = \text{const} = C. \]

By taking \( A_{ij} = a_{ij} = \frac{\pi}{2}, 1 \leq i < j \leq 4 \)

we obtain

\[ C = V + V^* + K = \frac{\pi^2}{8} + \frac{\pi^2}{8} + \frac{6\pi^2}{8} = \pi^2. \]
Sine and cosine rules for hyperbolic tetrahedron

Let \( T = T(A, B, C, D, E, F) \) be a hyperbolic tetrahedron with dihedral angles \( A, B, C, D, E, F \) and edge lengths \( a, b, c, d, e, f \) respectively.

Consider two Gram matrices

\[
G = \begin{pmatrix}
1 - \cos A & -\cos B & -\cos F \\
-\cos A & 1 - \cos C & -\cos E \\
-\cos B & -\cos C & 1 - \cos D \\
-\cos F & -\cos E & -\cos D & 1
\end{pmatrix}
\]

and

\[
G^* = \begin{pmatrix}
-1 & -\sin B & -\sin E & -\sin C \\
-\sin B & -1 & -\sin F & -\sin D \\
-\sin E & -\sin F & -1 & -\sin A \\
-\sin C & -\sin D & -\sin A & -1
\end{pmatrix}
\]
Starting volume calculation for tetrahedra we rediscover the following classical result:

**Theorem (Sine Rule, E. d'Ovidio (1872), J.L. Coolidge (1909), W. Fenchel (1989))**

\[
\frac{\sin A \sin D}{\sin B \sin E} = \frac{\sin B \sin E}{\sin C \sin F} = \frac{\sin C \sin F}{\sin D \sin A} = \sqrt{\frac{\det G}{\det G^*}}
\]

The following result seems to be new or at least well-forgotten.

**Theorem (Cosine Rule, M. Pashkevich - A. Mednykh (2005))**

\[
\frac{\cos A \cos D - \cos B \cos E}{\cos C \cos F} = \frac{\sqrt{\det G}}{\det G^*}
\]

Both theorems are obtained as a consequence of the Jacobian identities relating complementary minors of matrices \(G\) and \(G^*\).
2.2. The Borromean cone-manifold

\[ B(\alpha, \beta, \gamma) \]

**Th.6. (The Tangent Rule)**

Let \( B(\alpha, \beta, \gamma) \) be a hyperbolic Borromean cone-manifold with cone angles \( \alpha, \beta, \gamma < \pi \) and the singular geodesics' lengths \( l_\alpha, l_\beta, l_\gamma \). Then

\[
\frac{\tan \frac{\alpha}{2}}{\tanh \frac{\ell_\alpha}{4}} = \frac{\tan \frac{\beta}{2}}{\tanh \frac{\ell_\beta}{4}} = \frac{\tan \frac{\gamma}{2}}{\tanh \frac{\ell_\gamma}{4}} = T.
\]

where \( T \) is a positive number defined by

\[
T^2 = K + \sqrt{K^2 + L^2M^2N^2},
\]

\[
L = \tan \frac{\alpha}{2}, \quad M = \tan \frac{\beta}{2}, \quad N = \tan \frac{\gamma}{2},
\]

and

\[
K = (L^2 + M^2 + N^2 + 1)/2.
\]

The proof is based on the results of

\[ \text{[References]} \]
Theorem 6. (Derevynin - M, 2000)

Let $B(\alpha, \beta, \gamma)$ be a spherical Borromean cone-manifold with cone angles $\pi < \alpha, \beta, \gamma < 2\pi$ and singular geodesic lengths $l_\alpha, l_\beta, l_\gamma$. Then

$$\frac{\tan \frac{\alpha}{4}}{\tan \frac{\beta}{4}} = \frac{\tan \frac{\beta}{4}}{\tan \frac{\gamma}{4}} = \frac{\tan \frac{\gamma}{4}}{\tan \frac{\alpha}{4}} = T,$$

where $T$ is a negative root of equation

$$T^2 = -K + \sqrt{K^2 + L^2 M^2 N^2},$$

$L = \tan \frac{\pi}{2}$, $M = \tan \frac{\beta}{2}$, $N = \tan \frac{\gamma}{2}$

and $K = (L^2 + M^2 - N^2 + 1)/2$

Note that the existence of spherical structure on $B(\alpha, \beta, \gamma)$, $\pi < \alpha, \beta, \gamma < 2\pi$, was shown by P. Díaz (1999)
Th.12 (Derevytnin-M, 2001)
The volume of a spherical Borromean rings cone-manifold $B(\alpha, \beta, \gamma)$, $\pi < \alpha, \beta, \gamma < 2\pi$ is given by the formula
\[
V(\alpha, \beta, \gamma) = 2\left( \delta(\frac{\alpha}{2}, \theta) + \delta(\frac{\beta}{2}, \theta) + \delta(\frac{\gamma}{2}, \theta) - 2\delta(\frac{\pi}{2}, \theta) - \delta(0, \theta) \right),
\]
where
\[
\delta(\alpha, \theta) = \int_0^\pi \frac{\log(1 - \cos 2\alpha \cos 2\tau)}{\cos 2\tau} d\tau,
\]
and $\theta$, $\frac{\pi}{2} < \theta < \pi$ is defined by
\[
\tan^2 \theta = -K + \sqrt{K^2 + 4L^2M^2N^2}, \quad K = (L^2 + M^2 + N^2 + 1)/2, \quad L = \tan \frac{\pi}{2}, \quad M = \tan \frac{\beta}{2}, \quad N = \tan \frac{\gamma}{2}.
\]

Remark. The function $\delta(\alpha, \theta)$ can be considered as a spherical analog of the function
\[
\Delta(\alpha, \theta) = \Lambda(\alpha + \theta) - \Lambda(\alpha - \theta).
\]
Then the main result of R. Kellerhals (1989) for hyperbolic volume can be obtained from the above theorem by replacing $\delta(\alpha, \theta)$ to $\Delta(\alpha, \theta)$ and $K$ to $-K$. 
Bozzonean Rings cone-manifold and Lambert cube

\[ S^3 \left( \frac{0}{2} \right) \approx S^3 \left( \frac{\pi}{2} \right) = 8 \times L \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right) \]

\[ \text{Vol } B(\alpha, \beta, \gamma) = 8 \times \text{Vol } L \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right) \]

Recall that \( B(\alpha, \beta, \gamma) \) is
(i) hyperbolic if \( 0 < \alpha, \beta, \gamma < \pi \) (Andreaz, Rivin...)
(ii) Euclidean if \( \alpha = \beta = \gamma = \pi \)
(iii) spherical if \( \pi \leq \alpha, \beta, \gamma < 2\pi \) (R. Diaa)
( convex dodecahedron )
(iv) spherical if \( \pi < \alpha, \beta, \gamma < 3\pi \cdot \alpha, \beta, \gamma \neq 2\pi \)
(Derezyn, M.)

Note: convex

Icosahedron by B. Grünbaum
Volume calculation for \( L(\alpha; \beta, \gamma) \): 

(The main idea)

1. Existence: \( L(\alpha; \beta, \gamma) \):
   \[
   \begin{cases}
   0 \leq \alpha \leq \beta \leq \gamma < \frac{\pi}{2} & \text{H}^3 \\
   \alpha = \beta = \gamma < \frac{\pi}{2} & \text{E}^3 \\
   0 \leq \alpha, \beta, \gamma < \pi & \text{S}^3
   \end{cases}
   \]

2. Schl"afli's formula for \( V = \int \omega \in L(\alpha; \beta, \gamma) \)
   \[
   dV = \frac{1}{2} (l_\alpha d\alpha + l_\beta d\beta + l_\gamma d\gamma), \quad \kappa = \pm 1, 0
   \]
   In particular, in hyperbolic case:
   \[
   \begin{cases}
   \frac{\partial V}{\sqrt{2}} = -l_\beta, \quad \frac{\partial V}{\sqrt{2}} = -l_\gamma, \quad \frac{\partial V}{\sqrt{2}} = -l_\alpha \\
   V\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 0
   \end{cases}
   \]

3. Trigonometrical and algebraic identities

Proposition 1. Let \( L(\alpha; \beta, \gamma) \) be a hyperbolic Lambert cube with essential edge lengths \( l_\alpha, l_\beta, l_\gamma \). Then

(a) Tangent Rule
   \[
   \tan \alpha = \frac{\tan \beta}{\tanh l_\alpha} = \frac{\tan \gamma}{\tanh l_\beta} = T \quad \text{(R. Kellerhals)}
   \]

(b) Sine-Cosine Rule (3 different relations)
   \[
   \begin{cases}
   \sin \alpha = \frac{\sin B}{\sinh l_\alpha}, \quad \sin \beta = \frac{\sin C}{\sinh l_\beta}, \quad \sin \gamma = \frac{\sin A}{\sinh l_\gamma} \\
   \cosh l_\alpha = \frac{\sinh B}{\sin \alpha}, \quad \cosh l_\beta = \frac{\sinh C}{\sin \beta}, \quad \cosh l_\gamma = \frac{\sinh A}{\sin \gamma}
   \end{cases}
   \]

(c) \[
   \frac{T^2-A^2}{1+A^2} = \frac{T^2-B^2}{1+B^2} = \frac{T^2-C^2}{1+C^2} = 1,
   \quad \text{(HLM, Topology)}
   \]

\( A = \tan \alpha, \ B = \tan \beta, \ C = \tan \gamma \)

\( \Rightarrow (\alpha^2+1) (\beta^2+1) (\gamma^2+1) (A^2+B^2+C^2+1) T^2 + A^2 B^2 C^2 = 0 \)

Remark. (cc) \( \Rightarrow (c) \) and \( (\alpha \& (cc)) \Rightarrow (ccc) \).

3. Integral formula for volume

Proposition 2. Hyperbolic volume of \( L(\alpha; \beta, \gamma) \) is given by

\[
V = \frac{4}{\sqrt{T}} \int \frac{\log \left( \frac{T^2-A^2}{1+A^2} \frac{T^2-B^2}{1+B^2} \frac{T^2-C^2}{1+C^2} \right)}{1+T^2} dT
\]

where \( T \) is a positive root of equation (ccc)

Proof. By direct calculation we have:

\[
\frac{\partial W}{\partial x} = \frac{\partial W}{\partial A} \frac{\partial A}{\partial x} = -\frac{1}{2} \arctan \frac{A}{T} = -\frac{1}{2} \frac{l_\alpha}{l_\alpha}
\]

In similar way \( \frac{\partial W}{\partial l_\beta} = -\frac{l_\beta}{2} \) and \( \frac{\partial W}{\partial l_\gamma} = -\frac{l_\gamma}{2} \).

By convergence of the integral \( W\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 0 \).

Hence, \( W = V = \int \omega \in L(\alpha; \beta, \gamma) \)
Let $T(\alpha, \beta, \gamma)$ be a spherical orthoscheme such that
\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \]
Then
\[ \text{Vol } T(\alpha, \beta, \gamma) = \frac{1}{4} \left( \beta^2 - \left( \frac{\pi}{2} - \alpha \right)^2 - \left( \frac{\pi}{2} - \gamma \right)^2 \right). \]

Derevnyuk-M (2002)

Let $L(\alpha, \beta, \gamma)$ be a spherical Lambert cube such that
\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \]
Then
\[ \text{Vol } L(\alpha, \beta, \gamma) = \frac{1}{4} \left( \frac{\pi^2}{2} - \left( \frac{\pi}{2} - \alpha \right)^2 - \left( \frac{\pi}{2} - \beta \right)^2 - \left( \frac{\pi}{2} - \gamma \right)^2 \right). \]

Rational Volume Problem.

Let $P$ be a spherical polyhedron whose dihedral angles are in $\pi \mathbb{Q}$.
Then $\text{Vol}(P) \in \pi^2 \mathbb{Q}$. 
Examples

1. (c) \( \cos \frac{2 \pi}{3} + \cos \frac{2 \pi}{3} + \cos \frac{3 \pi}{4} = 1 \)

(c) \( \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \Rightarrow \text{Lambert cube } L \left( \frac{2 \pi}{3}, \frac{2 \pi}{3}, \frac{3 \pi}{4} \right) \text{ is spherical} \)

(c) By the above formula we obtain

\[ \text{Vol} \ L \left( \frac{2 \pi}{3}, \frac{2 \pi}{3}, \frac{3 \pi}{4} \right) = \frac{55}{276} \pi^2 \]

2. P - Coxeter polyhedron in \( S^3 \)

(= all dihedral angles are \( \frac{\pi}{n} \) for some \( n \in \mathbb{N} \))

Then

(c) The Coxeter group \( \Delta(P) \) generated by reflections on faces of \( P \) is finite

(c) \[ \text{Vol}(P) = \frac{\text{Vol}(S^3)}{\# \Delta(P)} = \frac{2 \pi^2}{\# \Delta(P)} = \frac{2 \pi^2}{\# \Delta(P)} \in \mathbb{Q} \pi^2 \]

Ex.

\[ \text{Coxeter polyhedron} \quad \text{Coxeter scheme} \quad \text{Coxeter group} \]

\[ \text{Vol}(P) = \frac{2 \pi^2}{\# (D_p \times D_q)} = \frac{\pi^2}{2p q} \]

3. Counterexample ????

Generic polyhedron with dihedral angles \( \approx \Omega \pi \)
\[ \text{Vol}(\alpha) = 2S \int_0^T \log \frac{4(A^2-t^2)}{(1+A^2)(1+B^2)} \frac{dt}{t^2+1}, \quad T > 0 \]

\[ \text{Vol}(\beta) = 2S \int_0^T \log \frac{4(t^2+A^2)(t^2+B^2)}{(1+A^2)(1+B^2)(t^2-2A^2)} \frac{dt}{t^2+1}, \quad T > 0 \]

\[ \text{Vol}(\gamma) = 2S \int_0^T \log \frac{4(t^2-A^2)(t^2-B^2)(t^2+C^2)}{(1+A^2)(1+B^2)(1+C^2)t^2} \frac{dt}{t^2+1}, \quad T > 0 \]

\[ \text{Vol}(\delta) = 2S \int_0^T \log \frac{16(t^2-A^2)(t^2-B^2)(t^2+C^2)}{(1+A^2)(1+B^2)(1+C^2)t^2+4} \frac{dt}{t^2+1}, \quad T > 0 \]

All limits of integration are roots of integrand,

\[ A = \cot \alpha \frac{\pi}{2}, \quad B = \cot \beta \frac{\pi}{2} \]

\[ A = \tan \frac{\alpha}{2}, \quad B = \tan \frac{\beta}{2}, \quad C = \tan \frac{\gamma}{2} \]

\[ \delta = \tan \frac{\alpha+\beta}{2}, \quad \beta = \tan \frac{\alpha-\beta}{2} \]

---

Table of volumes for hyperbolic cone manifolds

\[ S_1(\alpha), S_2(\alpha, \beta), S_3(\alpha, \beta), S_4(\alpha, \beta) \]

and \[ S_2(\alpha, \beta). \]
Problem of recognition
Which polynomial is responsible for volume of link complement?

<table>
<thead>
<tr>
<th>Link</th>
<th>Slope</th>
<th>Polynomial</th>
<th>Volume of $S^3/\text{Link}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4_{1}</td>
<td>5/2</td>
<td>$Q(t)=\frac{1}{4}(1+t^2)$</td>
<td>$2 \int_{0}^{\infty} \log \frac{1}{Q(t)} \frac{dt}{1+t^2} = 2.029...$</td>
</tr>
<tr>
<td>5_{1}</td>
<td>8/3</td>
<td>$P(z)=\frac{1}{2}z^2(1-z)$</td>
<td>$i \int_{1}^{\infty} \log \frac{1}{P(z)} \frac{dz}{z^2-1} = 3.800...$</td>
</tr>
<tr>
<td>6_{2}</td>
<td>10/3</td>
<td>$Q(t)=\frac{1}{16}(1+t^2)^2$</td>
<td>$2 \int_{0}^{\infty} \log \frac{1}{Q(t)} \frac{dt}{1+t^2} = 4.058...$</td>
</tr>
<tr>
<td>6_{3}</td>
<td>12/5</td>
<td>$P(z)=\frac{1}{4}z^2(1-z)^2$</td>
<td>$i \int_{0}^{\infty} \log \frac{1}{P(z)} \frac{dz}{z^2-1} = 4.616...$</td>
</tr>
<tr>
<td>6_{2}</td>
<td>-</td>
<td>$Q(t)=t^8$</td>
<td>$2 \int_{0}^{1} \log \frac{1}{Q(t)} \frac{dt}{1+t^2} = 7.327...$</td>
</tr>
</tbody>
</table>

($t = i \overline{z}$)