On hyperbolic and spherical volumes for knot and link cone-manifolds and polyhedra

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On volume of tetrahedron in hyperbolic and spherical spaces

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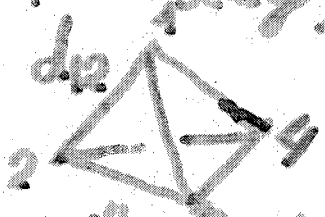
The calculation of the volume of a polyhedron in 3-dimensional space E^3, H^3 or S^3 is a very old and difficult problem. The first known result in this direction belongs to **Tartaglia (1494)** who found a formula for the volume of Euclidean tetrahedron. Now this formula is known as Cayley-Menger determinant.

More precisely, let

$T =$ 

be an Euclidean tetrahedron with edge lengths $d_{ij}, 1 \le i < j \le 4$. Then $V = \mathrm{Vol}(T)$ is given by

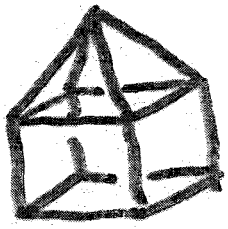
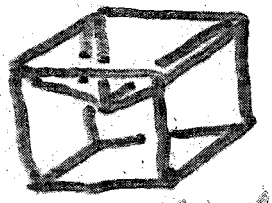
$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Note that V is a root of quadratic equation whose coefficients are integer polynomials in $d_{ij}, 1 \le i < j \le 4$.

Surprisely, but this result can be generalized on any Euclidean polyhedron in the following way.

Theorem (I. Kh. Sabitov, 1996) Let P be an Euclidean polyhedron. Then $V = \mathrm{Vol}(P)$ is a root of ~~an~~ even degree algebraic equation whose coefficients are integer polynomials in edge lengths of P depending on combinatorial type of P only.

Example.

$P_1 =$  and $P_2 =$ 

are of the same combinatorial type.

Hence, $V_1 = \mathrm{Vol}(P_1)$ and $V_2 = \mathrm{Vol}(P_2)$ are roots of the same algebraic equation

$$a_0 V^{2n} + a_1 V^{2n-2} + \dots + a_n V^0 = 0 .$$

(All edge lengths are taken to be 1)

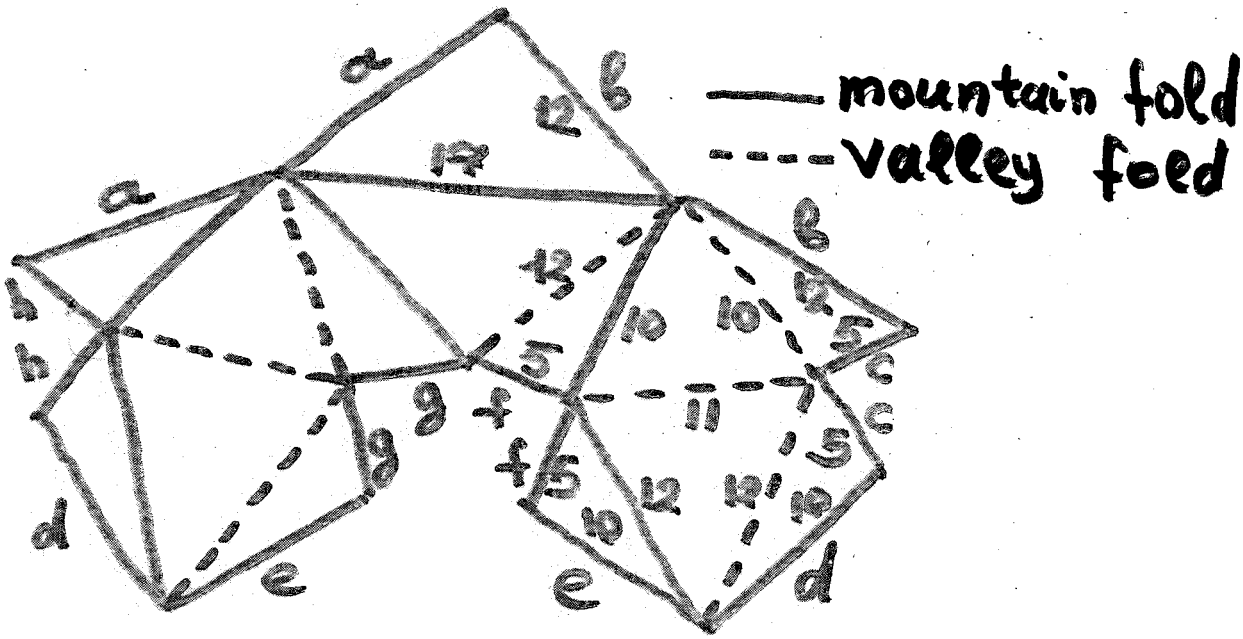
Cauchy theorem (1813) states that if the faces of a convex polyhedron are made of metal plates and the polyhedron edges are replaced by hinges, the polyhedron would be rigid.

In spite of this there are non-convex polyhedra which are **flexible**.

Bricard, 1897 (self-intersecting flexible octahedron)

Connelly, 1978 (the first example of true flexible polyhedron)

The smallest example is given by **Steffen** (14 triangular faces and 9 edges)



The Steffen flexible polyhedron

Very important consequence of Sabitov's theorem is a positive solution of **Bellows Conjecture** proposed by Dennis Sullivan.

Theorem (R. Connelly, I. Sabitov, A. Walz, 1997)

All flexible polyhedra keep a constant volume as they are flexed.

It was shown by Victor Alexandrov (Novosibirsk) that the Bellows Conjecture fails in the spherical space S^3. In the hyperbolic space H^3 the problem is still open.

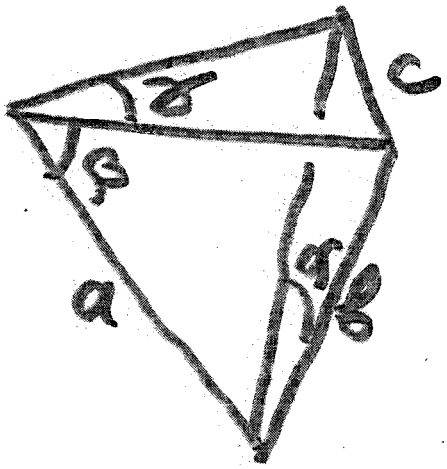
Any analog of Sabitov's theorem is unknown in both spaces S^3 and H^3.

The volume of a biorthogonal tetrahedron (orthoscheme) was calculated by **Lobachevsky** and **Bolai** in H^3 and by **Schläfli** in S^3.

Since Lobachevsky formula is widely known we restrict ourself by Bolai's and Schläfli's results

Theorem (Bolai). The volume of a hyperbolic orthoscheme

$K_{\alpha,\beta,\gamma}$:

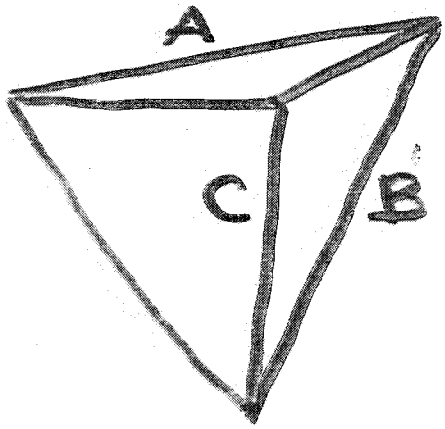


is given by the formula

$$\mathrm{vol}(K_{\alpha,\beta,\gamma}) = \frac{\tan\gamma}{2\tan\beta}\int_0^c \frac{z\sinh z\,dz}{\left(\frac{\cosh z}{\cos^2\alpha}-1\right)\sqrt{\frac{\cosh^2 z}{\cos^2\gamma}-1}}.$$

Theorem (Schläfli)

The volume of a spherical orthoscheme with essential dihedral angles A, B, and C



S^3

is given by the formula

$$V = \frac{1}{4} S(A, B, C),$$

where

$$S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) = \hat{S}(x, y, z)$$

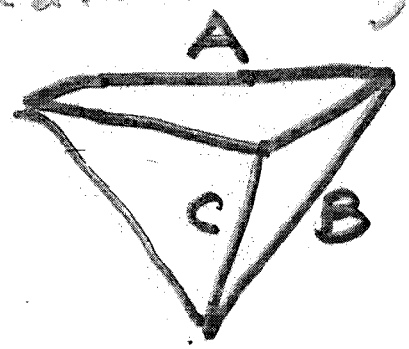
$$= \sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z}\right)^m \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2}$$

$$- x^2 + y^2 - z^2$$

and $D = \sqrt{\cos^2 x \cos^2 z - \cos^2 y}$.

Theorem (Lobachevsky, Coxeter)

The volume of a hyperbolic ortho scheme with essential dihedral angles A, B, and C



H^3

is given by the formula

$$V = \frac{i}{4} S(A, B, C)$$

where $S(A, B, C)$ is the Schläfli function.

The structure of the Schläfli function

$$\mathfrak{S}(x, y, z) = S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right)$$

is slightly complicated but it is naturally divided in four elementary pieces.

Theorem (Coxeter, 1935)

In the hyperbolic case $\cos^2 x \cos^2 z < \cos^2 y$ the Schläfli function and the Lobachevsky function are related by the formula

$$i\hat{S}(x,y,z) = -\Delta(x,\theta) + \Delta(y,\theta) - \Delta(z,\theta) + \Delta(0,\theta)$$

where $\Delta(x,\theta) = \Lambda(x+\theta) - \Lambda(x-\theta)$,

$$\Lambda(x) = -\int_0^x \log|2\sin t|\,dt$$

is the Lobachevsky function and

$$\tan\theta = \frac{\sin x \sin z}{\sqrt{\cos^2 y - \cos^2 x \cos^2 z}}.$$

Theorem (Derevnin, Mednykh, 2002)

In the spherical case $\cos^2 x \cos^2 z > \cos^2 y$ the Schläfli function satisfies

$$\hat{S}(x,y,z) = -\delta(x,\theta) + \delta(y,\theta) - \delta(z,\theta) + \delta(0,\theta)$$

where $\delta(x,\theta) = \int_\theta^{\frac{\pi}{2}} \log(1-\cos 2x \cos 2t)\frac{dt}{\cos 2t}$

and

$$\tan\theta = \frac{\sin x \sin z}{\sqrt{\cos^2 x \cos^2 z - \cos^2 y}}.$$

The function $\delta(x,\theta)$ is considered as a spherical analog of the function $\Lambda(x+\theta)-\Lambda(x-\theta)$ and satisfies the following properties

(i) $\delta(x,\theta)$ is continuous for all $(x,\theta)\in\mathbb{R}^2$ and differentiable for $x\neq\frac{\pi}{2}+k\pi, k\in\mathbb{Z}$

(ii) $\delta(x,0)=\pi^2/4-|\pi^2/2-x\pi|$, $0\leq x\leq\pi$

(iii) Let $\tilde{\delta}(x,\theta)=\delta(x,\theta)+(2\theta/\pi-1)\delta(x,0)$

Then

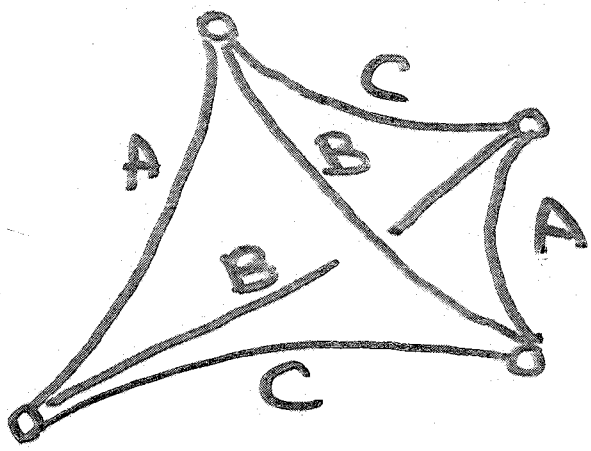
a) $\tilde{\delta}(x,\theta)$ is even and π-periodic on x

b) $\tilde{\delta}(x,\theta)$ is odd and π-periodic on θ

c) $|\tilde{\delta}(x,\theta)|\leq\pi^2/4$ and $\tilde{\delta}(\pi/2,3\pi/4)=\pi^2/4$.

Consider an ideal hyperbolic tetrahedron with all vertices on the infinity

T(A,B,C):



Opposite dihedral angles of tetrahedron are equal to each other and $A+B+C=\pi$.

Theorem (Milnor, 1982)

$$\mathrm{Vol}\, T(A,B,C) = \Lambda(A)+\Lambda(B)+\Lambda(C),$$

where

$$\Lambda(x) = -\int_0^x \log|2\sin t|\, dt$$

is the Lobachevsky function.

More complicated case with only one vertex on the infinity was investigated by Vinberg (1993).

Despite of these partial results, a formula for the volume of **an arbitrary hyperbolic tetrahedron** has been unknown until very recently. The general algorithm for obtaining such a formula was indicated by **Hsiang (1988)**, and the complete solution of the problem was given by

Yu. Cho and H. Kim (1999),
J. Murakami, M. Yano (2001), and
A. Ushijima (2002).

In these papers the volume of tetrahedron is expressed as an analytic formula involving 16 Dilogarithm or Lobachevsky functions whose arguments depend on the dihedral angles of the tetrahedron and on some additional parameter which is a root of some complicated quadratic equation with complex coefficients.

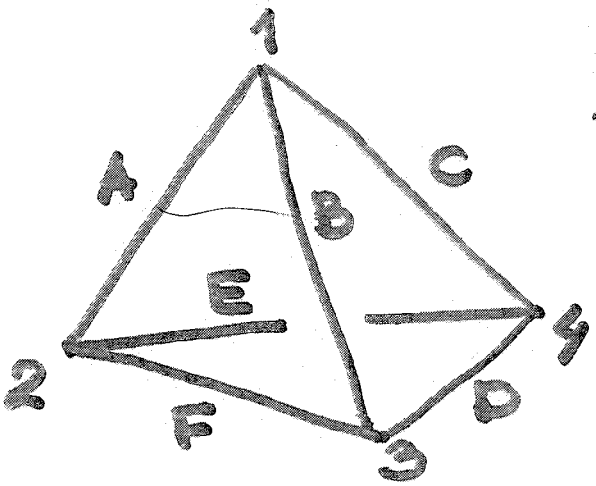
A geometrical meaning of the obtained formula was recognized by **G. Leibon (2002)** from the viewpoint of the Regge symmetry.

An excellent exposition of these ideas and a complete geometric proof of the volume formula was given by

Y. Mohanty (2003)

We suggest the following integral formula for the volume of tetrahedron.

Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron with dihedral angles $A\ B\ C\ D\ E\ F$:



We set

$V_1 = A+B+C$, $V_2 = A+E+F$,
$V_3 = B+D+F$, $V_4 = C+D+E$
(for vertices)

$H_1 = A+B+D+E$, $H_2 = A+C+D+F$
$H_3 = B+C+E+F$, $H_4 = 0$
(for Hamiltonian cycles).

Theorem (Derevnin-Mednykh, 2003)

The volume of a hyperbolic tetrahedron is given by the formula

$$\mathrm{Vol}(T) = -\frac{1}{4}\int_{z_1}^{z_2} \log \prod_{i=1}^{4} \frac{\cos\frac{V_i+z}{2}}{\sin\frac{H_i+z}{2}}\, dz,$$

where z_1 and z_2 are roots of the integrand given by formulas

$$z_1 = \arctan\frac{K_2}{K_1} - \arctan\frac{K_4}{K_3}, \quad z_2 = \arctan\frac{K_2}{K_1} + \arctan\frac{K_4}{K_3}$$

and

$$K_1 = -\sum_{i=1}^{4}\left(\cos(S-H_i) + \cos(S-V_i)\right)$$

$$K_2 = \sum_{i=1}^{4}\left(\sin(S-H_i) + \sin(S-V_i)\right)$$

$$K_3 = 2\left(\sin A \sin D + \sin B \sin E + \sin C \sin F\right)$$

$$K_4 = \sqrt{K_1^2 + K_2^2 - K_3^2},$$

$$S = A+B+C+D+E+F.$$

Recall that the Dilogarithm function is defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt .$$

We set

$$\ell(z) = \mathrm{Li}_2(e^{iz}) .$$

The following result is a consequence of the above theorem:

Theorem (Murakami-Yano, 2001)

$$\mathrm{Vol}(T) = \frac{1}{2}\,\mathrm{Im}\big(U(z_1,T) - U(z_2,T)\big),$$

where

$$U(z,T) = \frac{1}{2}\big(\ell(z) + \ell(A+B+D+E+z) + \ell(A+C+D+F+z) + \ell(B+C+E+F+z) - \ell(\pi+A+B+C+z) - \ell(\pi+A+E+F+z) - \ell(\pi+B+D+F+z) - \ell(\pi+C+D+E+z)\big).$$

Remark. Since

$$\mathrm{Im}(\ell(z)) = \mathrm{Im}(\mathrm{Li}_2(e^{iz})) = 2\Lambda\left(\frac{z}{2}\right)$$

the volume function can be expressed in terms of 16 Lobachevsky functions

$$\Lambda(x) = -\int_0^x \log|2\sin t|\,dt .$$

A tetrahedron $T = T(A,B,C,D,E,F)$ is called to be symmetric if $A=D$, $B=E$, $C=F$.

Theorem (Derevnin-Mednykh-Pashkevich) 2004

Let $T = T(A,B,C,A,B,C)$ be a symmetric hyperbolic tetrahedron. Then

$$\mathrm{Vol}(T) = 2\int_{\theta}^{\pi/2} \Big(\arcsin(a\cos t) + \arcsin(b\cos t) + \arcsin(c\cos t) - \arcsin(\cos t)\Big)\frac{dt}{\sin 2t},$$

where $\theta \in (0, \pi/2)$ is defined by

$$\tan\theta = \frac{1-a^2-b^2-c^2-2abc}{(1-a+b+c)(1+a-b+c)(1+a+b-c)(-1+b+c+a)}$$

with $a = \cos A$, $b = \cos B$, and $c = \cos C$.

Remark. Let ℓ_A, ℓ_B, ℓ_C are the lengths of the edges of T with dihedral angles A, B, C respectively. Then

$$\frac{\sin A}{\sinh \ell_A} = \frac{\sin B}{\sinh \ell_B} = \frac{\sin C}{\sinh \ell_C} = \tan\theta.$$

Duality formula for the volume of spherical tetrahedron.

Let T be a spherical tetrahedron with dihedral angles A_{ij}, $1 \le i < j \le 4$ and edge lengths a_{ij}, $1 \le i < j \le 4$, respectively.

We define **dual** tetrahedron T^* to be with dihedral angles $A^*_{ij} = \pi - a_{ij}$ and edge lengths $a^*_{ij} = \pi - A_{ij}$, $1 \le i < j \le 4$.

Middle curvature $K = K(T)$ is defined by formula

$$K = \sum_{1 \le i < j \le 4} \frac{(\pi - A_{ij})\, a_{ij}}{2} = \sum_{1 \le i < j \le 4} \frac{a^*_{ij}\, a_{ij}}{2}.$$

We note that

$$K(T) = K(T^*).$$

Denote by V and V^* the volumes of tetrahedra T and T^*, respectively.

Theorem (Mednykh, 2005)

$$V + V^* + K = \pi^2.$$

Proof. By the Schläfli formula we get

$$2dV = \sum_{i<j} a_{ij} dA_{ij} = -\sum_{i<j} a_{ij} d a^*_{ij}$$

and

$$2dV^* = \sum_{i<j} a^*_{ij} dA^* = -\sum_{i<j} a^*_{ij} d a_{ij}$$

Hence,

$$2(dV + dV^*) = -\sum_{i<j} (a_{ij} d a^*_{ij} + a^*_{ij} d a_{ij})$$
$$= -2dK.$$

We have

$$2(dV + dV^* + dK) = 0 \quad \text{and}$$

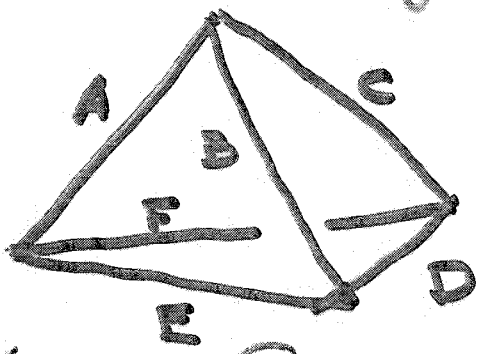
$$V + V^* + K = \text{const} = C.$$

By taking $A_{ij} = a_{ij} = \frac{\pi}{2}$, $1 \le i < j \le 4$ we obtain

$$C = V + V^* + K = \frac{\pi^2}{8} + \frac{\pi^2}{8} + \frac{6\pi^2}{8} = \pi^2.$$

Sine and cosine rules for hyperbolic tetrahedron

Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F and edge lengths $l_A, l_B, l_C, l_D, l_E, l_F$ respectively.



Consider two Gram matrices

$$G = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}$$

and

$$G^* = \begin{pmatrix} -1 & -\operatorname{ch} l_D & -\operatorname{ch} l_E & -\operatorname{ch} l_C \\ -\operatorname{ch} l_D & -1 & -\operatorname{ch} l_F & -\operatorname{ch} l_B \\ -\operatorname{ch} l_E & -\operatorname{ch} l_F & -1 & -\operatorname{ch} l_A \\ -\operatorname{ch} l_C & -\operatorname{ch} l_B & -\operatorname{ch} l_A & -1 \end{pmatrix}$$

Starting volume calculation for tetrahedra we rediscover the following classical result:

Theorem (Sine Rule, E. d'Ovidio (1877), J.L. Coolidge (1909), W. Fenchel (1989))

$$\frac{\sin A \sin D}{\operatorname{sh} \ell_A \operatorname{sh} \ell_D} = \frac{\sin B \sin E}{\operatorname{sh} \ell_B \operatorname{sh} \ell_E} = \frac{\sin C \sin F}{\operatorname{sh} \ell_C \operatorname{sh} \ell_F} = \sqrt{\frac{\det G}{\det G^*}}$$

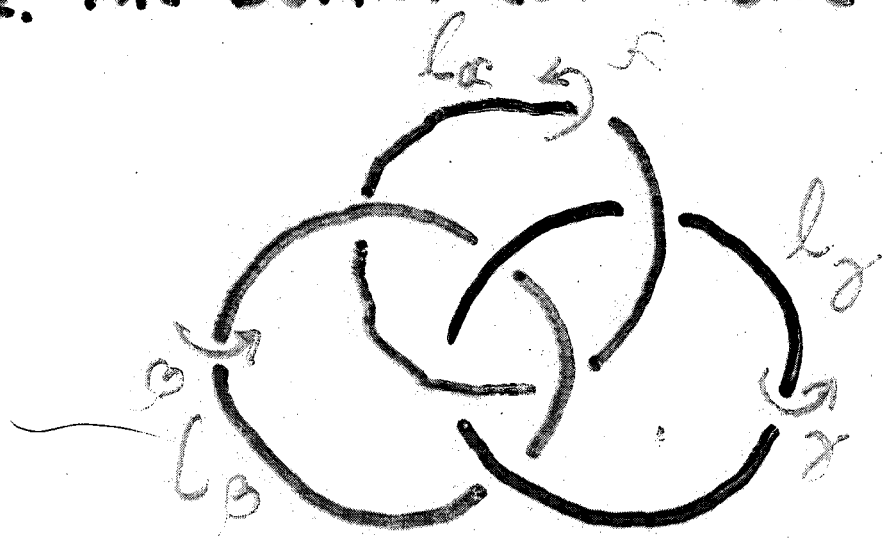
The following result seems to be new or at least well-forgotten

Theorem (Cosine Rule, M. Pashkevich - A. Mednykh (2005))

$$\frac{\cos A \cos D - \cos B \cos E}{\operatorname{ch} \ell_B \operatorname{ch} \ell_E - \operatorname{ch} \ell_A \operatorname{ch} \ell_D} = \sqrt{\frac{\det G}{\det G^*}}.$$

Both Theorems are obtained as a consequence of the Jacobi identities relating complimentary minors of matricies G and G^*.

2.2. The Borromean cone-manifold



$B(\alpha,\beta,\gamma)$

Th. 6. (The Tangent Rule)
Let $B(\alpha,\beta,\gamma)$ be a hyperbolic Borromean cone-manifold with cone angles $0<\alpha,\beta,\gamma<\pi$ and the singular geodesics lengths $l_\alpha, l_\beta, l_\gamma$. Then

$$\frac{\tan\frac{\alpha}{2}}{\tanh\frac{l_\alpha}{4}} = \frac{\tan\frac{\beta}{2}}{\tanh\frac{l_\beta}{4}} = \frac{\tan\frac{\gamma}{2}}{\tanh\frac{l_\gamma}{4}} = T,$$

where T is a positive number defined by $T^2 = K + \sqrt{K^2 + L^2M^2N^2}$,

$L = \tan\frac{\alpha}{2}$, $M = \tan\frac{\beta}{2}$, $N = \tan\frac{\gamma}{2}$,

and $K = (L^2+M^2+N^2+1)/2$.

— The proof is based on the results of [illegible] 92 R. Kellerhals 89, Umbera [illegible]

Th. 6' (Derevnin - M, 2000)
Let $B(\alpha,\beta,\gamma)$ be a spherical Borromean cone-manifold with cone angles $\pi < \alpha, \beta, \gamma < 2\pi$ and the singular geodesics lengths $\ell_\alpha, \ell_\beta, \ell_\gamma$. Then

$$\frac{\tan\frac{\alpha}{2}}{\tan\frac{\ell_\alpha}{4}} = \frac{\tan\frac{\beta}{2}}{\tan\frac{\ell_\beta}{4}} = \frac{\tan\frac{\gamma}{2}}{\tan\frac{\ell_\gamma}{4}} = T,$$

where T is a negative root of equation

$$T^2 = -K + \sqrt{K^2 + L^2M^2N^2},$$

$L = \tan\frac{\alpha}{2}$, $M = \tan\frac{\beta}{2}$, $N = \tan\frac{\gamma}{2}$

and $K = (L^2 + M^2 + N^2 + 1)/2$

Note that the existance of spherical structure on $B(\alpha,\beta,\gamma)$, $\pi < \alpha, \beta, \gamma < 2\pi$ was shown by R. Díaz (1999)

Th. 12 (Derevnin - M, 2001)
The volume of a spherical Borromean rings cone-manifold $B(\alpha,\beta,\gamma)$, $\pi < \alpha,\beta,\gamma < 2\pi$ is given by the formula

$$V(\alpha,\beta,\gamma) = 2\left(\delta\left(\frac{\alpha}{2},\theta\right) + \delta\left(\frac{\beta}{2},\theta\right) + \delta\left(\frac{\gamma}{2},\theta\right) - 2\delta\left(\frac{\pi}{2},\theta\right) - \delta(0,\theta)\right),$$

where

$$\delta(\alpha,\theta) = \int_{\theta}^{\frac{\pi}{2}} \log(1-\cos 2\alpha \cos 2\tau)\frac{d\tau}{\cos 2\tau}$$

and θ, $\frac{\pi}{2} < \theta < \pi$ is defined by

$$\tan^2\theta = -K + \sqrt{K^2 + L^2M^2N^2}, \quad K = (L^2+M^2+N^2+1)/2,$$

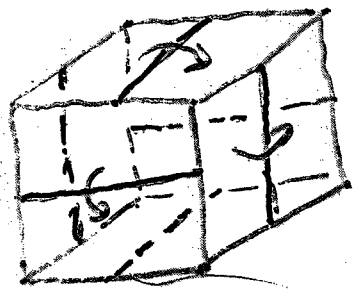
$$L = \tan\frac{\alpha}{2}, \quad M = \tan\frac{\beta}{2}, \quad N = \tan\frac{\gamma}{2}.$$

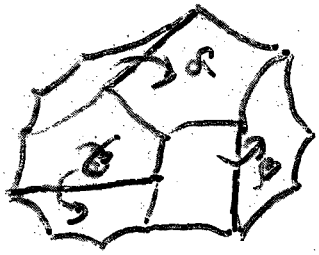
<u>Remark</u>. The function $\delta(\alpha,\theta)$ can be considered as a spherical analog of the function

$$\Delta(\alpha,\theta) = \Lambda(\alpha+\theta) - \Lambda(\alpha-\theta).$$

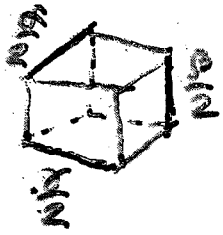
Then the main result of R. Kellerhals (1989) for hyperbolic volume can be obtained from the above Theorem by replacing $\delta(\alpha,\theta)$ to $\Delta(\alpha,\theta)$ and K to $-K$.

Borromean Rings cone-manifold and Lambert cube





$= 8 \times$



$S^3\Big($  $\Big)$ $\qquad S^3\Big($  $\Big) = 8 \times L\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$

$$\mathrm{Vol}\, B(\alpha,\beta,\gamma) = 8\, \mathrm{Vol}\, L\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$$

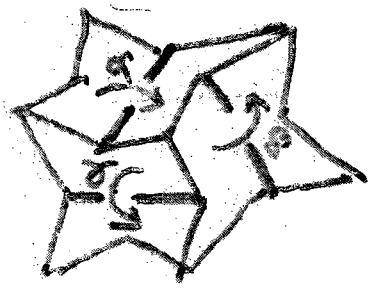
Recall that $B(\alpha,\beta,\gamma)$ is

(i) hyperbolic if $0<\alpha,\beta,\gamma<\pi$ (Andreev, Rivin ...)

(ii) euclidean if $\alpha=\beta=\gamma=\pi$

(iii) spherical if $\pi<\alpha,\beta,\gamma<2\pi$ (R. Diaz)
(convex dodecahedron)

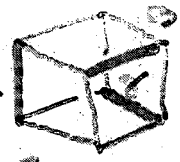
(iv) spherical if $\pi<\alpha,\beta,\gamma<3\pi$
$\alpha,\beta,\gamma \neq 2\pi$



(Derevnin, M.)

non convex
dodecahedron by B. Grünbaum

Volume calculation for $L(\alpha,\beta,\gamma)$: **(The main idea)**



0. Existance $L(\alpha,\beta,\gamma)$: $\begin{cases} 0<\alpha,\beta,\gamma<\frac{\pi}{2}, & H^3 \\ \alpha=\beta=\gamma=\frac{\pi}{2}, & E^3 \\ \frac{\pi}{2}<\alpha,\beta,\gamma<\pi, & S^3 \end{cases}$

1. Schläfli formula for $V=\mathrm{Vol}\,L(\alpha,\beta,\gamma)$

$$\kappa\, dV = \frac{1}{2}(l_\alpha d\alpha + l_\beta d\beta + l_\gamma d\gamma), \quad \kappa=\pm 1, 0$$

In particular, in hyperbolic case:

$$\begin{cases} \frac{\partial V}{\partial \alpha} = -\frac{l_\alpha}{2}, \ \frac{\partial V}{\partial \beta} = -\frac{l_\beta}{2}, \ \frac{\partial V}{\partial \gamma} = -\frac{l_\gamma}{2} & (*) \\ V(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = 0 & (**) \end{cases}$$

2. Trigonometrical and algebraic identities

Proposition 1. Let $L(\alpha,\beta,\gamma)$ be a hyperbolic Lambert cube with essential edge lengths $l_\alpha, l_\beta, l_\gamma$. Then

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh l_\alpha} = \frac{\tan\beta}{\tanh l_\beta} = \frac{\tan\gamma}{\tanh l_\gamma} = T \quad \textbf{(R.Kellerhals)}$$

(ii) Sine-Cosine Rule (3 different relation)

$$\frac{\sin\alpha}{\sinh l_\alpha}\,\frac{\sin\beta}{\sinh l_\beta}\,\frac{\cos\gamma}{\cosh l_\gamma} = 1 \quad \textbf{(Derevnin-M)}$$

(iii) $$\frac{T^2-A^2}{1+A^2}\,\frac{T^2-B^2}{1+B^2}\,\frac{T^2-C^2}{1+C^2}\,\frac{1}{T^2} = 1, \quad \textbf{(HLM, Topologj 90)}$$

$A=\tan\alpha$, $B=\tan\beta$, $C=\tan\gamma$

$\Leftrightarrow (T^2+1)(T^4-(A^2+B^2+C^2+1)T^2+A^2B^2C^2)=0$

Remark. (iii) $\Rightarrow$ (i) and (i)&(ii) $\Rightarrow$ (iii).

3. Integral formula for volume

Proposition 2. Hyperbolic volume of $L(\alpha,\beta,\gamma)$ is given by

$$W = \frac{1}{4}\int_T^\infty \log\left(\frac{t^2-A^2}{1+A^2}\,\frac{t^2-B^2}{1+B^2}\,\frac{t^2-C^2}{1+C^2}\,\frac{1}{t^2}\right)\frac{dt}{1+t^2}$$

where T is a positive root of equation (iii)

Proof. By direct calculation we have:

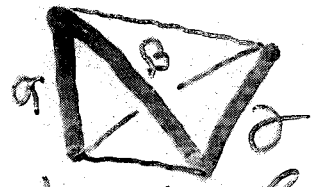
$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A}\,\frac{\partial A}{\partial \alpha} = -\frac{1}{2}\arctan\frac{A}{T} = -\frac{l_\alpha}{2}.$$

In similar way $\frac{\partial W}{\partial \beta} = -\frac{l_\beta}{2}$ and $\frac{\partial W}{\partial \gamma} = -\frac{l_\gamma}{2}$.

By convergence of the integral $W(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})=0$.

Hence, $W = V = \mathrm{Vol}\,L(\alpha,\beta,\gamma)$.

Coxeter (1935).



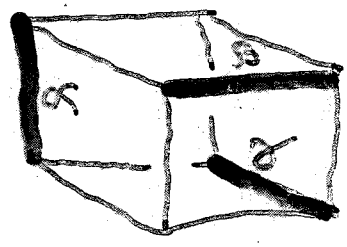
Let $T(\alpha,\beta,\gamma)$ be a spherical orthoscheme such that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Then

$$\mathrm{Vol}\, T(\alpha,\beta,\gamma) = \frac{1}{4}\left(\beta^2 - \left(\frac{\pi}{2}-\alpha\right)^2 - \left(\frac{\pi}{2}-\gamma\right)^2\right).$$

Derevnin-M (2002)



Let $L(\alpha,\beta,\gamma)$ be a spherical Lambert cube such that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Then

$$\mathrm{Vol}\, L(\alpha,\beta,\gamma) = \frac{1}{4}\left(\frac{\pi^2}{2} - \left(\frac{\pi}{2}-\alpha\right)^2 - \left(\frac{\pi}{2}-\beta\right)^2 - \left(\frac{\pi}{2}-\gamma\right)^2\right)$$

Rational Volume Problem.

Let P be a spherical polyhedron whose dihedral angles are in $\pi\mathbb{Q}$

Then $\mathrm{Vol}(P) \in \pi^2\mathbb{Q}$

Examples

1. (i) $\cos^2\frac{2\pi}{3}+\cos^2\frac{2\pi}{3}+\cos^2\frac{3\pi}{4}=1$

(ii) $\frac{\pi}{2}<\frac{2\pi}{3},\ \frac{3\pi}{4}<\pi \Rightarrow$ Lambert cube $L\left(\frac{2\pi}{3},\frac{2\pi}{3},\frac{3\pi}{4}\right)$ is spherical

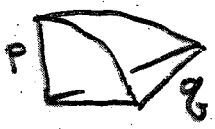
(iii) By the above formula we obtain

$$\mathrm{Vol}\ L\left(\frac{2\pi}{3},\frac{2\pi}{3},\frac{3\pi}{4}\right)=\frac{55}{276}\pi^2$$

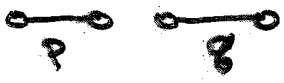
2. P - Coxeter polyhedron in S^3
(= all dihedral angles are $\frac{\pi}{n}$ for some $n\in\mathbb{N}$)

Then
(i) The Coxeter group $\Delta(P)$ generated by reflections on faces of P is finite

(ii)
$$\mathrm{Vol}(P)=\frac{\mathrm{Vol}(S^3)}{\#\Delta(P)}=\frac{2\pi^2}{\#\Delta(P)}\in\mathbb{Q}\pi^2$$

Ex. 

Coxeter polyhedron P



Coxeter scheme

$D_p\times D_q$
Coxeter group

$$\mathrm{Vol}(P)=\frac{2\pi^2}{\#(D_p\times D_q)}=\frac{\pi^2}{2pq}$$

3. Contraexample ?????

Generic Polyhedron with dihedral angles in $\mathbb{Q}\pi$
? ? ? ? ? ? ? ? ? ?

$$\mathrm{Vol}\Big(\ \alpha\ \Big) = 2\int_0^T \log\frac{4(A^2-t^2)}{(1+A^2)(1+t^2)}\,\frac{dt}{t^2+1}\ ,\quad T>0$$

$$\mathrm{Vol}\Big(\ \alpha\ \beta\ \alpha\ \Big) = i\int_{\bar\zeta}^{\zeta} \log\frac{2(z^2+A^2)(z^2+B^2)}{(1+A^2)(1+B^2)(z^2-z^3)}\,\frac{dz}{z^2-1}\ ,\quad \mathrm{Im}\,\zeta>0$$

$$\mathrm{Vol}\Big(\ \beta\ \Big) = i\int_{\bar\zeta}^{\zeta} \log\frac{4(z^2+A^2)(z^2+B^2)}{(1+A^2)(1+B^2)(z-z^2)^2}\,\frac{dz}{z^2-1}\ ,\quad \mathrm{Im}\,\zeta>0$$

$$\mathrm{Vol}\Big(\ \alpha\ \beta\ \gamma\ \Big) = 2\int_T^\infty \log\frac{(t^2-\mathbb{A}^2)(t^2-\mathbb{B}^2)(t^2-\mathbb{C}^2)}{(1+\mathbb{A}^2)(1+\mathbb{B}^2)(1+\mathbb{C}^2)\,t^2}\,\frac{dt}{t^2+1}\ ,\quad T>0$$

$$\mathrm{Vol}\Big(\ \alpha\ \beta\ \Big) = 2\int_T^\infty \log\frac{16(t^2-\hat{A}^2)(t^2-\hat{B}^2)}{(1+\hat{A}^2)(1+\hat{B}^2)(t^2+1)^2}\,\frac{dt}{t^2+1}\ ,\quad T>0$$

All limits of integrations are roots of integrand,

$A=\cot\frac{\alpha}{2}$, $B=\cot\frac{\beta}{2}$,

$\mathbb{A}=\tan\frac{\alpha}{2}$, $\mathbb{B}=\tan\frac{\beta}{2}$, $\mathbb{C}=\tan\frac{\gamma}{2}$,

$\hat{A}=\tan\frac{\alpha+\beta}{4}$, $\hat{B}=\tan\frac{\alpha-\beta}{4}$.

Table of volumes

for hyperbolic cone manifolds

$4_1(\alpha)$, $5^2_1(\alpha,\beta)$, $6^2_3(\alpha,\beta)$, $6^3_2(\alpha,\beta,\gamma)$,

and $6^2_2(\alpha,\beta)$.

Problem of recognition

Which polynomial is responsible for volume of link complement?

Link	Slope	Polynomial	Volume of $S^3 \setminus$ Link
4_1	$5/2$	$Q(t)=\frac{1}{4}(1+t^2)$	$2\int_0^{\sqrt{3}} \log\frac{1}{Q(t)}\frac{dt}{1+t^2}=2.029\ldots$
5_1^2	$8/3$	$P(z)=\frac{1}{2}z^2(1-z)$	$i\int_{1-i}^{1+i} \log\frac{1}{P(z)}\frac{dz}{z^2-1}=3.662\ldots$
6_2^2	$10/3$	$Q(t)=\frac{1}{16}(1+t^2)^2$	$2\int_0^{\sqrt{3}} \log\frac{1}{Q(t)}\frac{dt}{1+t^2}=4.058\ldots$
6_3^2	$12/5$	$P(z)=\frac{1}{4}z^2(1-z)^2$	$i\int_{\bar{3}}^{3} \log\frac{1}{P(z)}\frac{dz}{z^2-1}=4.616\ldots$
6_2^3	—	$Q(t)=t^8$	$2\int_0^1 \log\frac{1}{Q(t)}\frac{dt}{1+t^2}=7.327\ldots$

$(t=iz)$