Joint work with Marc Culler

Volume is a powerful topological invariant for hyperbolic 3-manifolds.

But connections with more classical invariants remain elusive.

We have proved

Theorem. M^3 closed, orientable, hyperbolic.

If vol(M) < 3.08 then

\[ \text{rank } H_1(M; \mathbb{Z}_2) \leq 6. \]
The SnapPea census (Weeks-Hodgson) gives two examples with \( \text{vol} < 3 \)
and \( \text{rank } H_1(M; \mathbb{Z}_2) = 3 \). None with bigger rank.

I'll sketch a proof that

\[ \text{vol} (M) < 3.08 \Rightarrow \text{rank } H_1(M; \mathbb{Z}_2) \leq 10 \]

(just posted on ArXiv)
Theorem (Anderson, Canary, Culler, S.)

If $M^3$ is closed, orb'ble, hyperbolic

* $\pi_1(M)$ 3-free, 3-tame

then $\text{vol}(M) > 3.08$.

$M = \mathbb{H}^3/\Gamma \quad \Gamma \simeq \pi_1(M)$

$\Gamma$ 3-free if every subgroup
of rank $\leq 3$ is free

$\Gamma$ 3-tame if every subgroup $\Delta$
of rank $\leq 3$ is tame

$\mathbb{H}^3/\Delta$

i.e. homeo. to int $M$ for some
cpct 3-mfld w/ bdy $N$. 
Thm (Agol; Calegari-Gabai)

Every finitely generated torsion-free discrete subgroup of Isom ($\mathbb{H}^3$) is tame.

So $\pi_1(M)$ 3-Free $\Rightarrow$ vol($M$) > 3.08.

\[ \therefore \text{In proving our theorem we may assume} \]

$\pi_1(M) \text{ not 3-Free}$

Fix a non-free subgroup $\Delta \leq \pi_1(M)$

of rank $\leq 3$
We may assume (for a contradiction) that
\[ \text{rank } H_1(M; \mathbb{Z}_2) \geq 11 \]

This implies
\[ |\pi_1(M); \Delta| = \infty \]

by

**Thm. (S-Wagreich)** If $M^3$ is closed, orb\'ble

\[ \Delta \leq \pi_1(M) \text{ a subgp of rank } r \]

\[ \text{rank } H_1(M; \mathbb{Z}_p) \geq r + 2 \text{ for some prime } p \]

then
\[ |\pi_1(M); \Delta| = \infty. \]
Any subgroup of rank ≤ 2 in $\Delta$ is free because of

Thm. (Jaco-S.) $M^3$ closed, or'ble, hyperbolic.

⇒ any infinite-index subgroup of $\pi_1(M)$ with rank ≤ 2 is free.

It follows that $\Delta$ is freely indecomposable.

This raises

**Question.** What does the existence of an infinite-index freely indecomposable rank-3 subgroup of $\pi_1(M^3)$ say about the topology and geometry of $M^3$?
\( \tilde{\mathcal{M}} = \text{covering space defined by } \Delta \)

\( N = \text{compact core of } \tilde{\mathcal{M}} \)

\[ |\pi_1(\tilde{\mathcal{M}}); \Delta| = \infty \Rightarrow \partial N \neq \emptyset \]

\( \Delta \text{ freely indecomposable } \Rightarrow \partial N \text{ incompressible } \)

\( \mathcal{M} \text{ hyperbolic } \Rightarrow \text{no torus components in } \partial N \)

\( \text{rk } \Delta = 3 \Rightarrow \text{total genus } (\partial N) \leq 3 \Rightarrow \text{connected} \)

\( \text{genus } (\partial N) = 3 \Rightarrow \Delta \text{ free } \)

So \( \pi_1(\mathcal{M}) \) contains a genus-2 surface gp

Obvious way for this to happen:

\( \mathcal{M} \supset \text{genus-2 incompressible surface} \)
IF $S \subset M$ is a genus-2 incompressible surface, a regular neighborhood of $S$ in $M$ is an $I$-bundle $W$ with incompressible boundary and $\chi(W) = -2$.

**Topological Theorem (Culler-S.)**

- $M^3$ closed, orientable, hyperbolic
- Assume rank $H_1(M; \mathbb{Z}_2) \geq 11$
- Assume $\pi_1(M)$ has a freely indecomposable subgroup of rank 3 normal
- Then $M$ contains a book of $I$-bundles $W$ with $\partial W$ incompressible and $\chi(W) = -2$. 
A book of \( I \)-bundles is a 3-mfd \( W \) with a decomposition

\[
W = B \cup P
\]

Where

- \( P \) is an \( I \)-bundle over a possibly disconnected surface, each component of Euler characteristic \( \leq 0 \)
- \( B \) is a disjoint union of solid tori
- \( \text{int } P \cap \text{int } B = \emptyset \)
- Each component of \( P \cap B \) is an annulus which is vertical in \( P \) and \( \not\subset \emptyset \) in \( B \)

\( W \) is normal if each component of \( P \) has Euler characteristic \( \leq 0 \).
Corollary. If \( M^3 \) closed, orientable, hyperbolic, \( H_1(M; \mathbb{Z}_2) \) has rank \( > 11 \) and \( \pi_1(M) \) has a freely indecomposable rank-3 subgroup, then \( M \) contains an incompressible surface of genus 2 or a separating incompressible surface of genus 3.

But if \( M \) contains such a surface and \( Vol(M) < 3.66 \) we can use a recent result due to Agol, Storm and W. Thurston to show \( rk H_1(M; \mathbb{Z}_2) \leq 7 \). This will prove our geometrical theorem.
An incompressible surface $S \subset M^3$ is called a fibroid if each component of the manifold obtained splitting $M$ along $S$ is a book of $I$-bundles.

**Theorem** (Agol-Storm-Thurston)

$M^3$ closed, hyperbolic, orible

$S \subset M$ incompressible

If $S$ is not a fibroid then $\text{vol}(M) > 3.66$

Elementary argument show that if $S \subset M$ is a fibroid and genus $S = 2$

or genus $S \geq 3$ and $S$ is separating

then $\text{rank } H_1(M; \mathbb{Z}_2) \leq 7$. 
Proof of top. thm. follows the plan of Shapiro-Whitehead proof of Dehn's Lemma.

$\Delta$ a rank-3 indecomposable gp

An injection $\Delta \to \pi_1(M)$ is induced by a map

$K \to M \quad K$ a polyhedron with $\pi_1(K) = \Delta$

We construct a tower
$N_{n} \subset M_{n}$

$N_{n-1} \subset M_{n-1}$

$\vdots$

$N_{2} \subset M_{2}$

$N_{1} \subset M_{1}$

$N_{0} \subset M_{0} = M$
Here the $p_i$ are degree-2 coverings $N_i$ is a compact submanifold of $M_i$ with incompressible boundary (possibly $\partial N_i = \emptyset$). $\tilde{f}^*$ is a surjection $H_1(K; \mathbb{Z}_2) \to H_1(N_n; \mathbb{Z}_2)$ and the diagram commutes up to homotopy.

Furthermore, for any $i$ such that $M_i$ is closed we have $\text{rank } H_1(M_i; \mathbb{Z}_2) > 10$ and likewise for $M_i$. Since $H_1(N_n; \mathbb{Z}_2) \leq 3$ we must have $\partial N_n \neq \emptyset$. 
One can show $dN_n$ is a genus-2 surface (cf. compact core argument)

So $M_n$ is incompressible genus-2 surface

In particular $M_n$ contains a normal book of $I$-bundles with incompressible bdry and $N^{1,1}$

Now show if $M_j$ contains such a book of $I$-bundles, so does $N_{j-1}$ and hence $M_{j-1}$

This will prove the theorem
Induction step is much easier in the special case where $\mathcal{M}_j$ contains a genus-2 incompressible surface $S$.

After an isotopy, we may assume $S$ and $\tau S$ meet transversally in non-trivial simple closed curves where $\tau : \mathcal{M}_j \to \mathcal{M}_j$ is the deck transformation. Then a regular nbd of $\tau S \cap S$ is a book of I-bundles $\tilde{W}$, and $p_j (\tilde{W})$ is a book of I-bundles in $\mathcal{N}_{j-1}$. Call it $W$.

Using atoroidality, we can find a normal book of I-bundles $W^+$ with $W \subset W^+ \subset \mathcal{N}_{j-1}$. We have $\chi (W^+) = -2$. If $\partial W^+$ is incompressible, induction step is complete.
Here are three typical ways $\partial W^+$ can fail to be incompressible:

1. $\partial W^+$ compresses to a non-empty incompressible surface $T$. In this case $T$ must have genus 2 and its nbhd is the required nbhd of $T$-bundles.

2. $W^+$ is a handlebody. This leads to a contradiction, because the image of $\pi_1(W^+)$ in $\pi_1(N_{\rho^0})$ contains $(\rho^0)_* (\pi_1(S))$, a genus-2 surface group.
3) \(N_{j-1} - W^+\) is a handlebody. In this case \(N_{j-1}\) is closed and hence

\[H_1(N_{j-1}, \mathbb{Z}_2)\] has rank \(\geq 10\). But

\[H_1(W^+, \mathbb{Z}_2) \to H_1(N_{j-1}, \mathbb{Z}_2)\] is surjective in this case. An elementary argument shows that a normal book of \(I\)-bundles \(W^+\) with \(\gamma(W^+) = -2\) has \(H_1(W^+, \mathbb{Z}_2)\) of rank \(\leq 5\). So \(H_1(N_{j-1}, \mathbb{Z}_2)\) has rank \(\leq 5\), a contradiction.

For this special case of the induction, 10 could be replaced by 6, but the general case is more complicated.