Rigidity of cone-3-manifolds

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Hartmut Weiß (LMU München)
cone-3-manifold of curvature $\kappa$: metric space locally modelled on the $\kappa$-cone over a spherical cone-surface $\cong S^2$

$\Sigma$

disk transverse to $\Sigma$

sector in $M^2_\kappa$  =  disk with cone-point

generalizes notion of geometric orbifold, where cone-angles are of the form $2\pi/n$, $n \geq 2$
cone-angles $\leq 2\pi \Rightarrow$ curvature bounded below by $\kappa$ in the triangle comparison sense

cone-angles $\leq \pi \Rightarrow$ singular locus $\Sigma$ a trivalent graph

topological type of $C :=$ homeomorphism type of the pair $(C, \Sigma)$

**local rigidity** holds :\(\iff\) the deformation space of hyperbolic (spherical) cone-manifold structures is locally parametrized by the vector of cone-angles

**global rigidity** holds :\(\iff\) the isometry type of $C$ is determined by the topological type of $C$ and the vector of cone-angles
1. The hyperbolic case

**Theorem** [Kojima]: Global rigidity holds for hyperbolic cone-3-manifolds with cone-angles $\leq \pi$ and singular locus a link.

**Proof:**

Decrease cone-angles to 0 using

**Theorem** [Hodgson-Kerckhoff]: Local rigidity holds for closed hyperbolic cone-3-manifolds with cone-angles $\leq 2\pi$ and singular locus a link.

and the techniques used in the proof of the cyclic case of the Orbifold Theorem.

Then use Mostow rigidity for complete hyperbolic 3-manifolds of finite volume to deduce global rigidity.
**Theorem [W.]:** Global rigidity holds for closed hyperbolic cone-3-manifolds with cone-angles $\leq \pi$ in the general case.

**Proof:**

Follow the same strategy as Kojima: decrease cone-angles to 0, then use Mostow rigidity.

cone-angles $0 \Leftrightarrow$ complete hyperbolic 3-manifold of finite volume, possibly with totally geodesic boundary consisting of thrice punctured spheres

Geometry of links of singular vertices changes:

$\alpha + \beta + \gamma > 2\pi \Leftrightarrow$ spherical $S^2(\alpha, \beta, \gamma)$

$\alpha + \beta + \gamma = 2\pi \Leftrightarrow$ horospherical $E^2(\alpha, \beta, \gamma)$

$\alpha + \beta + \gamma < 2\pi \Leftrightarrow$ hyperspherical $H^2(\alpha, \beta, \gamma)$
Local deformation theory:

**Theorem [W.]:** Local rigidity holds for hyperbolic cone-3-manifolds of finite volume with cone-angles $\leq \pi$, at most finitely many ends which are (smooth or singular) cusps with compact cross-sections $\neq E^2(\pi, \pi, \pi, \pi)$, and possibly with totally geodesic hyperbolic turnover boundary.

**Proof:**

Let $M = C \setminus \Sigma$ be the smooth part and \[ \mathcal{E} = \mathfrak{so}(TM) \oplus TM = \tilde{M} \times_{Ad_{hol}} \mathfrak{sl}_2(\mathbb{C}) \]

the flat bundle of infinitesimal isometries.

Step 1: Prove $H^1_{L^2}(M, \mathcal{E}) = 0$ using analysis on manifolds with conical singularities (Cheeger, Brüning-Seeley).

Step 2: Analyze the variety of representations of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$ near the holonomy of a hyperbolic cone-manifold structure.
Study of degenerations:

Geometry of hyperbolic cone-manifolds with $diam(C') \geq D > 0$ and cone-angles $\leq \alpha < \pi$ according to Boileau, Leeb and Porti:

**thin parts:** $\exists$ a short list of local models for the thin part of $C'$ (smooth Margulis tubes, tubes around closed singular geodesics, umbilic tubes with turnover cross-sections)

**thickness:** $\exists r = r(D, \alpha) > 0$ such that $C'$ contains an embedded smooth standard ball of radius $r$.

thickness $\Rightarrow$ no collapse

**finiteness:** $vol(C') < \infty \Rightarrow C'$ has at most finitely many ends, all of which are (smooth or singular) cusps with compact cross-sections, i.e. $T^2$ or $E^2(\alpha, \beta, \gamma)$. 
Finishing the proof (the essential step):

Given a family of hyperbolic cone-3-manifolds \((C_t)_{t \in (t_\infty, 1]}\) with cone-angles \((t_\alpha_1, \ldots, t_\alpha_N)\) and \(C_1 = C\), show that this family extends to the closed interval \([t_\infty, 1]\)!

Schläfli’s formula: \(\text{vol}(C_t) \nearrow\) as \(t \searrow t_\infty\)

\[\Rightarrow \text{diam}(C_t) \geq D\]

Kojima’s straightening argument: \(\text{vol}(C_t) \leq V\)

Boileau, Leeb and Porti: The only possible degenerations are tubes around closed (smooth or singular) geodesics opening into rank-2 cusps.

These cusps can be closed via hyperbolic Dehn surgery (in the setting of hyperbolic cone-3-manifolds).
2. The finite-volume case

The same proof yields the following result in the finite-volume case:

**Theorem [W.]:** Global rigidity holds for hyperbolic cone-3-manifolds of finite volume with cone-angles $\leq \pi$, at most finitely many ends which are (smooth or singular) cusps with compact cross-sections $\neq E^2(\pi,\pi,\pi,\pi)$, and possibly with totally geodesic hyperbolic turnover boundary.

Remark: If cone-angles are $< \pi$, by the finiteness result of Boileau, Leeb and Porti, this is the general finite-volume case.
3. The spherical case

**Theorem [W.]:** Global rigidity holds for closed spherical cone-3-manifolds with cone-angles $\leq \pi$ which are not Seifert fibered.

Proof:

Use the spherical version of local rigidity, i.e.

**Theorem [W.]:** Local rigidity holds for closed spherical cone-3-manifolds with cone-angles $\leq \pi$ which are not Seifert fibered.

and the fact that spherical cone-3-manifolds don’t collapse according to Boileau, Leeb and Porti to deform cone-angles to $\pi$.

Global rigidity follows from

**Theorem [de Rham]:** A spherical structure on a closed 3-orbifold is unique.