Minimal Hyperbolic
3-Manifolds and Character Varieties

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• 3-mflds: compact, connected, orientable, irred.

Definition

• Say $V^3$ dominates $W^3$, written

$$V \geq W$$

if there exists a non-zero degree map

$$f : (V, \partial V) \to (W, \partial W)$$

• Say $V$ strictly dominates $W$ if $V \geq W$ but $V \not\cong W$.

Examples

(1) $\partial V \cong S^1 \times S^1 \Rightarrow V \geq S^1 \times D^2$

(2) $V, W$ closed and $\pi_1(W)$ finite implies $V \geq W$.

Thus “≥” not a partial order (eg. $S^3 \geq P^3 \geq S^3$).

If $f : V \geq W$, Gromov’s inequality

$$\|V\| \geq d(f)\|W\|$$

implies that it is a partial order when restricted to

$$\{V : V \text{ has at least one hyperbolic piece } \}$$
**Definition**

$V$ is *minimal* if $V \succeq W \Rightarrow \begin{cases} |\pi_1(W)| < \infty & \text{if } \partial V = \emptyset \\ W \cong S^1 \times D^2 & \text{if } \partial V \cong S^1 \times S^1 \end{cases}$

Before we discuss examples, recall that

- The $PSL_2(\mathbb{C})$ *representation variety* of a finitely generated group $\Gamma$ is the set

  $$R_{PSL_2}(\Gamma) = \text{Hom}(\Gamma, PSL_2(\mathbb{C}))$$

  endowed with the compact-open topology. It is a complex, affine algebraic set in a natural way.

- $PSL_2(\mathbb{C})$ acts on $R_{PSL_2}(\Gamma)$ by conjugation.

- The $PSL_2(\mathbb{C})$ *character variety* of $\Gamma$ is the set

  $$X_{PSL_2}(\Gamma) = R_{PSL_2}(\Gamma) // PSL_2(\mathbb{C})$$

  In other words, $\mathbb{C}[X_{PSL_2}(\Gamma)] = \mathbb{C}[R_{PSL_2}(\Gamma)]^{PSL_2(\mathbb{C})}$.

- $R_{PSL_2}(V) = R_{PSL_2}(\pi_1(V)), X_{PSL_2}(V) = X_{PSL_2}(\pi_1(V))$

- If $V^3$ is small, then

  $$\dim_{\mathbb{C}}(X_{PSL_2}(V)) = \begin{cases} 0 & \text{if } \partial V = \emptyset \\ 1 & \text{if } \partial V \cong S^1 \times S^1 \end{cases}$$
Examples

1) \( \partial V \cong S^1 \times S^1 \)

(a) punctured torus bundles whose monodromies are not proper powers.

(b) hyperbolic twist knot exteriors \( M \)

Idea There is only one non-trivial curve in \( X_{PSL_2}(M) \) (Burde) and \( M \) covers only itself.

(c) the exterior \( M \) of the \((-2,3,n)\) pretzel knot is minimal iff \( n \not\equiv 0 \pmod{3} \).

Idea There is only one non-trivial curve in \( X_{PSL_2}(M) \) if \( n \not\equiv 0 \pmod{3} \) (Mattman). When \( n \equiv 0 \pmod{3} \) there are two, the extra one coming from a strict domination \( M \geq T_{2,3} \). Also, \( M \) covers only itself.

(d) the exterior of the \( \frac{p}{q} \) rational knot is not minimal in general (Ohtsuki-Riley, Sakuma). If \( M_{\frac{p}{q}} \geq W \), then \( W \cong M_{\frac{p'}{q'}} \) for some \( p' \) which divides \( p \). Can use \( PSL_2(\mathbb{C}) \) character variety methods to show that if \( M_{\frac{p}{q}} \geq M_{\frac{p'}{q'}} \), then either \( M_{\frac{p}{q}} = M_{\frac{p'}{q'}} \) or \( p > p' \). In particular

\[ p \text{ prime } \Rightarrow M_{\frac{p}{q}} \text{ minimal.} \]
2) $\partial V = \emptyset$ (Reid-Wang)

(a) $V = M \cup M'$ where $M = M'$ trefoil exteriors glued together so that $\mu = \phi'$ and $\phi = \mu'$.

Idea All representations to $PSL_2(\mathbb{C})$ have finite cyclic image. Thus $V$ is minimal in $\mathcal{M}$ where

$$\mathcal{M} = \{ V : V \text{ Haken or admits a geometric structure} \}$$

(b) $V = M(\frac{1}{2})$ where $M$ is the exterior of the figure 8 knot.

Idea All irreducible representations to $PSL_2(\mathbb{C})$ are faithful and there are exactly two conjugacy classes of discrete representations. Further, $V$ covers only itself. Thus $V$ minimal in $\mathcal{M}$.

**Theorem 1** Only finitely many surgeries on a hyperbolic twist knot (e.g. the figure 8 knot) can yield a manifold which is not minimal in the class $\mathcal{M}$. 
Theorem 2  Suppose that $M$ is a small hyperbolic $3$-manifold with torus boundary such that

- $M$ does not strictly dominate any hyperbolic $3$-manifold.
- there is a slope $\alpha_0$ on $\partial M$ such that $M(\alpha_0)$ does not dominate any hyperbolic $3$-manifold.

Suppose as well that for any essential surface $S_0$ in $M$, either

(a) $|\partial S_0| \leq 2$, or
(b) for each norm curve $X_0 \subset X_{PSL_2}(M)$, there is a character $\chi_\rho \in X_0$ which restricts to an irreducible character on $\pi_1(S_0)$.

Then there are only finitely many slopes $\alpha$ on $\partial M$ such that the manifold $M(\alpha)$ can strictly dominate a closed hyperbolic $3$-manifold.

Examples

1) hyperbolic twist knot exteriors.
2) 2-bridge knot exteriors $M_{\ell_1}$ with $p$ prime.
3) $(-2,3,n)$ pretzel knot exterior.
4) many pctrd torus bundles (eg. RSS examples).

Remark  First two conditions are necessary.
**Idea of proof**

For $n > 0$, consider non-zero degree maps

$$f_n : M(\alpha_n) \to V_n, V_n \text{ closed hyperbolic}$$

where $\alpha_n \neq \alpha_m$ for $n \neq m$ and $f_n$ not a htpy equiv

- define $\rho_n \in R_{PSL_2}(M)$ by

$$\pi_1(M(\alpha_n)) \xrightarrow{(f_n)^\#} \pi_1(V_n) \xrightarrow{\psi_n \simeq} \Gamma_n$$

- hypotheses $\Rightarrow \ker(\rho_n|\pi_1(\partial M)) = \langle \alpha_n \rangle$

- set $\chi_n = \chi_{\rho_n} \in X_{PSL_2}(M)$

- characters distinct and $\dim X_{PSL_2}(M) = 1$ so WLOG

$$\chi_n \in X_0 \subset X_{PSL_2}(M)$$

where $X_0$ is an affine curve.

Two cases to consider:

1. $\{\chi_n\}$ subconverges to some $\chi_{\rho_0} \in X_0$
2. $\{\chi_n\}$ subconverges to an ideal point of $X_0$
Case 1. \(\{\chi_n\}\) subconverges to some \(\chi_{\rho_0} \in X_0\)

- WLOG \(\lim \rho_n = \rho_0\)
- set \(\Gamma_n := \rho_n(\pi_1(M))\) \((n \geq 0)\) and note \(\Gamma_n\) torsion-free, cocpct, non-elem, Kleinian for \(n \geq 1\)
- Jørgensen-Marden ⇒
  
  (a) \(\Gamma_0\) torsion-free, non-elem, Kleinian

and after passing to a subsequence we may assume that

  (b) \(\{\Gamma_n\}\) converges geometrically to torsion-free, non-elem, Kleinian \(\Gamma\) containing \(\Gamma_0\).

  (c) \(\lim_n V_n = V := \mathbb{H}^3/\Gamma\) (Gromov-Hausdorff)

  (d) \(\exists\) homoms \(\theta_n : \Gamma \to \Gamma_n\) such that \(\rho_n = \theta_n \circ \rho_0\)

Note

(c) ⇒ \(\text{vol}(V) = \lim \text{vol}(V_n) \leq \lim \text{vol}(M(\alpha_n)) = \text{vol}(M)\)

(d) ⇒ \(\rho_0|_{\pi_1(\partial M)}\) injective ⇒ \(V\) has cusps and \(V_n\) obtained by Dehn filling \(V\) for \(n \gg 0\)

Can construct map \(g : (M, \partial M) \to (V, \partial V)\) realizing \(\rho_0\) and show has non-zero degree. Then WLOG, \(g\) a homeomorphism and can show that for infinitely many \(n\), \(f_n\) is induced (up to htpy) by \(g\). For such \(n\), \(f_n : M(\alpha_n) \xrightarrow{\sim} V_n\), which contradicts our hypotheses.
Case 2. \( \{\chi_n\} \) sbconverges to an ideal point \( x_0 \) of \( X_0 \)

Let \( S_0 \subset M \) be a conn. essential surface associated to \( x_0 \).

- Culler-Shalen theory \( \Rightarrow \lim_{n} \chi_n|\pi_1(S_0) = \chi_{\rho} \in X_{PSL_2}(S_0) \)
  where \( \rho \) is reducible.

- WLOG \( \lim_{n} \rho_n|\pi_1(S_0) = \rho \)

- \( \rho_n(\pi_1(S_0)) \) Kleinian \( \Rightarrow \rho_n(\pi_1(S_0)) \) elementary
  \( \Rightarrow \rho_n(\pi_1(S_0)) \) cyclic, loxodromic

Subcase Condition (a) holds (so \( |\partial S_0| \leq 2 \))

Show \( \chi_n \) cannot be bent along \( S_0 \) (\( n \gg 0 \)) so that the restriction of \( \rho_n \) to some complementary component of \( S_0 \) has cyclic image. Then the condition \( |\partial S_0| \leq 2 \) implies that \( \rho_n \) has cyclic image, which is false.

Subcase Condition (b) holds

Since \( \chi_n|\pi_1(S_0) \) reducible for large \( n \), can show that \( \chi|\pi_1(S_0) \) reducible for all \( \chi \in X_0 \). Can also show that \( X_0 \) is a norm curve. This contradicts condition (b).
Theorem 3 Suppose that $M$ is a small hyperbolic 3-manifold with torus boundary such that

- $M$ does not strictly dominate any Seifert manifold besides $S^1 \times D^2$.
- there is a slope $\alpha_0$ on $\partial M$ such that $\pi_1(M(\alpha_0))$ does not admit a surjective homomorphism onto a Fuchsian triangle group.
- the image of $\pi_1(\partial M)$ under any surjective homomorphism of $\pi_1(M)$ onto a Fuchsian triangle group is infinite.

and that either

(a) $H_1(M) \cong \mathbb{Z}$ and $\Delta_M$ is not divisible by a cyclotomic polynomial, or

(b) $H_1(M) \cong \mathbb{Z}$ and $X_{PSL_2}(M(\lambda))$ is finite, or

(c) each non-trivial curve in $X_{PSL_2}(M)$ is a norm curve

Suppose as well that for any essential surface $S_0$ in $M$ and any norm curve $X_0 \subset X_{PSL_2}(M)$, there is a character $\chi_\rho \in X_0$ which restricts to a strictly irreducible character on $\pi_1(S_0)$. Then there are only finitely many slopes $\alpha$ on $\partial M$ such that the manifold $M(\alpha)$ can strictly dominate an $\widetilde{SL}_2$ manifold.
Example A hyperbolic twist knot exterior $M$ satisfies all the hypotheses. Moreover,

- if $M(\alpha)$ dominates a Haken manifold (eg. any $E^3, H^2 \times E^1$, or $Sol$ manifold) or a reducible manifold (eg. an $S^2 \times E^1$ manifold), then $\alpha$ a bound slope. This happens for only finitely many slopes.
- a simple homological argument shows that if some filling of $M$ dominates a non-Haken $Nil$ manifold, then there is a surjective homomorphism $\pi_1(M) \to \Delta(2, 3, 6)$. But this is readily shown to be impossible. Thus Theorem 1 holds.

Addendum If we remove the condition on representations to Fuchsian triangle groups, we can still conclude that there are proper subgroups $L_1, L_2, \ldots, L_k \subset H_1(\partial M)$, none of which contain $\alpha_0$, such that for each slope $\alpha$ in $H_1(\partial M) \setminus (L_1 \cup L_2 \cup \cdots \cup L_k)$, $M(\alpha)$ does not strictly dominate an $\widetilde{SL}_2$ manifold.

Examples
1) 2-bridge knot exteriors $M_{\frac{p}{q}}$ with $p$ prime.
2) $(-2, 3, n)$ pretzel knot exterior, $n \not\equiv 0 \pmod{3}$.
3) many pctrd torus bundles (eg. RSS examples).