Covering Spaces
of 3-orbifolds

Marc Lackenby

University of Oxford
Conjecture:

Any closed orientable hyperbolic 3-orbifold with non-empty singular locus has large fundamental group.
LARGE GROUPS

A GROUP IS LARGE IF SOME FINITE INDEX SUBGROUP ADMITS A SURJECTIVE HOMOMORPHISM ONTO A NON-ABELIAN FREE GROUP.

LARGENESS $\Rightarrow$

1. \( v_b, = \infty \) WHERE

   \[ v_b,(G) = \sup \left\{ v_i(G_i) : G_i \leq G \right\} \]

2. SUPER-EXPONENTIAL SUBGROUP GROWTH

3. LINEAR GROWTH OF \( p \)-HOMOLOGY
BASIC PRINCIPLE

EASIER

3-MANIFOLDS
WITH NON-EMPTY
BOUNDARY

---

HARDER

CLOSED
3-MANIFOLDS

3-ORBIFOLDS
WITH NON-EMPTY
SINGULAR LOCUS

THEOREM: [COOPER-LONG-REID]

LET M BE A COMPACT ORIENTABLE
IRREDUCIBLE 3-MANIFOLD WITH NON-EMPTY
BOUNDARY. THEN EITHER M = B^3, S^1 x D^2,
T^3 x I OR K^3 x I, OR IF M IS LARGE.
Theorem 1: Let $M$ be a compact orientable 3-manifold, and let $K$ be a knot in $M$ such that $M - K$ has a finite-volume hyperbolic structure. For any $n \in \mathbb{N}$, let $M(K, n)$ be the orbifold obtained by placing a singularity along $K$ of order $n$. Then if $M(K, n)$ is large for infinitely many $n$. 
The Evidence

II: Subgroup Growth

For a finitely generated group $G$, let $S_n(G)$ be the number of subgroups with index $\leq n$, 'subgroup growth function'.

Two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ have the same growth type if

$\exists K > 1 \ s.t \ f(n)^{1/K} \leq g(n) \leq f(n)^K$

$G$ large $\Rightarrow S_n(G)$ has growth type $2^{n \log n}$
Let $O$ be a compact orientable 3-orbifold with non-empty singular locus and a finite volume hyperbolic structure.

**Theorem 2**: $\pi_1 O$ has at least exponential subgroup growth.

In fact, the subnormal subgroup growth has exponential growth type.

**Alternative formulation**: Any lattice in $\text{PSL}(2, \mathbb{C})$ with torsion has at least exponential subgroup growth.
Contrast this with what is known for hyperbolic 3-manifolds:

**Theorem [Lubotzky]**

Let \( M \) be a closed hyperbolic 3-manifold. Then \( \forall \varepsilon > 0, \exists \kappa > 0 \) s.t.

\[
5^n (\pi, M) \geq 2^k (\log n)^{3-\varepsilon}
\]
The Evidence

III: Linear Growth of $p$-Homology

Let $p$ be a prime.

Let $\mathbb{F}_p$ be the field of order $p$.

For a group $G$,

Let $d_p(G) = \dim H_1(G; \mathbb{F}_p)$

We say that a collection $\{G_i\}$ of finite index subgroups has linear growth of $p$-homology if

$$\inf \frac{d_p(G_i)}{[G_i:G]} > 0$$

Very strong condition.

$G$ large $\Rightarrow$ $G$ has a nested collection of finite index subgroups with linear growth of $p$-homology.
Theorem 3: \([L]\) \(\pi_1 \mathcal{O}\) has a nested collection of finite index subgroups with linear growth of \(p\)-homology, for some prime \(p\).

Let \(\{\mathcal{O}_i\}\) be the corresponding covering spaces. (We may in fact take these to be manifolds.)

The Heegaard gradient of \(\{\mathcal{O}_i \to \mathcal{O}\}\) is

\[
\inf_{\text{deg}(\mathcal{O}_i \to \mathcal{O})} \chi^1(\mathcal{O}_i),
\]

where

\[
\chi^1(\mathcal{O}_i) = \min \{ -\chi(F) : F \text{ is a Heegaard surface for } \mathcal{O}_i \}.
\]

Linear growth of \(p\)-homology

\(\Rightarrow\) positive Heegaard gradient
Theorem: Let $M$ be a closed orientable irreducible 3-manifold, and let $(M_i \to M^3)$ be a nested sequence of finite-sheeted regular covering spaces of $M$. Suppose that

1) The Heegaard gradient of $(M_i \to M^3)$ is positive, and

2) $\pi_1 M$ does not have property $(\tau)$ w.r.t. $(\pi_i M_i)$.

Then $M_i$ is Haken $V_i \geq 0$. 
Focus on Thm 2 \& 3

Note: Thm 3 \implies Thm 2.

Let \( G = \pi, 0 \)

Suppose \( \{G_i\} \) has linear growth of \( p \)-homology. Then:

\[ \begin{align*}
N^0 \text{ subgps of } G_i \text{ with index } p \\
\geq |H_i(G_i; \mathbb{F}_p)| - 1 \\
\geq p^{d_p(G_i)} \\
\geq p^{[G:G_i] \times \text{const}}
\end{align*} \]
Let $O$ be closed orientable hyperbolic
3-orbifold with non-empty sing locus.

**Key Lemma:** If the singular locus is a link $L$ and a prime $p$ divides the order of every component of $L$, then $dp(\pi, O) \geq 1|L|$.

**Proof:** $O - \partial(\mathcal{O})$ is a compact orientable 3-manifold $M$.  

$\Rightarrow \quad dp(M) \geq \frac{1}{2} dp(\partial M) = |L|$.  

$\pi, O$ is obtained from $\pi, M$ by killing $p^{th}$ powers of words. This leaves $dp$ unchanged. $\square$
1. Pass to a coveting space \( O' \) where the singular locus is a non-empty link, and where each component has order a prime \( p \).

(Use Selberg's Thm).

2. Pass to a coveting space \( O'' \) with these properties, and where \( d_p(\pi, O'') \geq 11 \).

(Slight generalisation of a result of Lubotzky)

3. Find coveting spaces \( O; \to O'' \) where

\[ |\text{sing } (O;) | \geq \deg m (O; \to O'') \]

\[ \Rightarrow \]

\[ d_p(\pi, O;) \]

This \( \Rightarrow \) linear growth of \( p \)-homology.
Set $\Gamma = \pi, 0^\circ / \langle \pi, \mathbb{D}(C) \rangle$.

Finite index subgroups of $\Gamma$:

$\Rightarrow$ --- --- --- of $\pi, 0^\circ$ containing $\langle \pi, \mathbb{D}(C) \rangle$.

$\Rightarrow$ Finite covers of $0^\circ$ s.t. inverse image of $C$ is a collection of copies of $C$.

**Thm:** [Golod - Shafarevich] If $\Gamma: \langle x, x^2 \rangle$ and $d_\rho(\Gamma)^2 > d_\rho(\Gamma) - 1 \times 1 + 1 \times 1$,

then $\Gamma$ has a nested sequence of finite index subgroups.
THM: [L] Let $M$ be a 3-manifold that is commensurable with a compact orientable finite-volume hyperbolic 3-orbifold with non-empty singular locus. Then:

i) $\pi_1 M$ has exponential subnormal subgroup growth

ii) $M$ has a nested sequence of cones with positive HEEGAARD gradient.

EXAMPLES:

1. MANIFOLDS WITH HEEGAARD GENUS 2

2. ARITHMETIC HYPERBOLIC 3-MANIFOLDS

[Reid]
POSSIBLE APPROACHES TO
THE CONJECTURE

1. IT FOLLOWS FROM A GROUP-THEORETIC
   CONJECTURE OF LUBOTZKY & ZELMANOV
   RELATING TO PROPERTY (τ).

2. THERE IS A LINK WITH THE THEORY
   OF ERROR-CORRECTING CODES.