Roots of 3-mfds, ...

\[ \mathcal{G} = \{ T_1, T_2, \ldots \} : \text{a set of moves on } M^3. \]

Def: \( M^3 \) is a root of \( M^3 \) if:

1. \( M \xrightarrow{T_i} M_0 \)
2. \( M_0 \) admits no \( T_i \)

Q1: Does \( M_0 \) exist?

Q2: Is \( M_0 \) unique?

\[ \begin{cases} \text{Theorems 1 - 10 (for diff. 6)} \\ \forall M \text{ the root exists and is unique} \end{cases} \]
Existence: Yes, if $T_i$ simplify manifolds. (w.r.t. a complexity function)

Uniqueness:

diamond trick:
Example.

\( \sigma = \{ T_s \} : \text{spherical compressions} \)

Given \( SCM \), we cut \( M \) along \( S \) and add two balls.

\[
\begin{align*}
M & \quad \overset{T_s}{\longrightarrow} \quad O \quad B \quad B \\
M & \quad \overset{T_s}{\longrightarrow} \quad C
\end{align*}
\]

Theorem 1. For \( \sigma = \{ T_s \} \), the root exists and is unique.
Theorem (Kneser–Milnor)

1. \( \forall M = \#i M_i \# n (S^2 \times S^1) \)

   (\( M_i \) are irreducible)

2. \( M_i \) and \( n \) are determined by \( M \)

1) belongs to Kneser:

Any sequence of T-moves stops after \( \leq N(M) \) steps.

\[ \Rightarrow \exists C_{\text{kneser}}(M) \]

(kneser complexity of \( M \)).
Relation to roots:

\[ M = \# M_i \# \langle n_2 s \times s' \rangle \iff \]

\( U M_i \) is a root of \( M \)

Proof of (2): the root is unique:

Suppose not. Let \( M_j \) be a minimal manifold having two roots (minimal with respect to the Kneser complexity)
CASE $A \cap B = \emptyset$

\[ T_B T_A(M) = T_A T_B(M) \]

Remark: $A$ may be trivial
Case $\mathcal{A} \cap \mathcal{B} \neq \emptyset$

$\#(C_i \cap B) = \emptyset,$

$\#(C_i \cap A) < \#(B \cap A)$

$D$ is an innermost disc in $A$
bounded by $\mathcal{A} \cap \mathcal{B}$

$C_i$ is nontrivial
$\sigma = \{ T_{D}, T_{S} \}$: Theorem 3: $\exists$!

Roots of cobordisms:

\[ \sigma = \{ T_5, T_D, T_A \} \]

\[ A \subset (M, \partial_- M, \partial_+ M) \text{ incompressible annulus; } \partial_- A \subset \partial_- M \]

\[ \partial_+ A \subset \partial_+ M \]

Cut \( M \) along \( A \) and attach two plates:

\[ \Rightarrow \]

[Diagram showing the cutting and attaching process]
Theorem 4. $\exists !
Cobordisms \( \rightsquigarrow \) virtual knots

virtual knot = knot in F \times I

up to destabilization:

Theorem 5. Any virtual knot has a unique minimal genus representative

G. Kuperberg, Algebraic & Geometric Topology, 3 (2003), 587-591

"What is a virtual link?"
cobordisms \rightarrow manifolds

\sigma = \{ T_5, T_D, T_A \}

A is incompressible and

* either boundary incompressible

or \partial A, \partial -A are in different components of AM.

Theorem 6. For \sigma ** the root \exists!

Theorem 6. For \sigma * the root \exists!

False!
Example.

\[ (p', q') \rightarrow T^2 \times I \rightarrow (p, q) \rightarrow Q \]

\[ Q \rightarrow (p', q') \rightarrow \text{(p', q')-curve} \]

\[ \text{m} \times 1 \rightarrow (p, q) \rightarrow \text{(p, q)-curve} \]

\[ Q_{p'q'} \cup Q \]
Knotted graph =
\((M, G)\), where \(G\) is a graph
in \(M\). Compact!

\[ S' = \{ T_s \} \& \]

Cut along \(S\) and
take cones over \(S^+\)
\((S^+, S^- \cap G)\).
Restrictions:
1. $S$ should be incompressible

\[ \emptyset \cup S = \emptyset \]

\[ \emptyset \cap G = \emptyset \Rightarrow \emptyset ' \cap G = \emptyset \text{ or } \emptyset '' \cap G = \emptyset \]
2. $\#(S \cap G)$ should be bounded

3. $\#(S \cap G) \leq B$
Accept: $S$ is incompressible

$\#(\text{snG}) \leq 3$

Theorem 7.

$\exists$ Kneser works:

$\diamondsuit$ works

If $\text{snG} = \emptyset$

$\#(\mathcal{Anc}) < \#(A \cap B)$
May assume: $\mathcal{P}' \cap G \neq \emptyset$
$\mathcal{P}'' \cap G \neq \emptyset$

Since $3 = 1 + 2$, then
$2 = 1 + 1$

$\#(\mathcal{P}' \cap G) = 1$ or $\#(\mathcal{P}'' \cap G) = 1$.
There always exists $\mathcal{P}$ with $\mathcal{P} \cap G = \emptyset$. 
The root is unique up to addition/removing trivial pairs.
Partial cases:

Theorem 8: $\exists!$ $f$ if $G$ is a link.

Theorem 9: $\exists!$ $f$ if $G$ is a knot.
Orbifolds

\((M^3, G, \text{coloring by natural numbers})\)

\(n > 1\)

\(m = 3, 4, 5\)
\[ \sigma = \{ T_5 \} \text{ except } \]

\[ S \cap G = \ast \]

\[ (k_1, m, n) \neq (2, 2, p) \text{ or } (2, 3, q) \text{ with } q = 3, 4, 5 \]

\[ \text{Th} \{ 10 \} \text{ The root } \exists ! \]