A Combination Theorem for Convex Hyperbolic Manifolds

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1. Idea of Convex Combination Thm

Let $M$ be connected hyperbolic $n$-manifold that is the union of two convex hyperbolic $n$-submanifolds. If $M$ has a $K$ thickening with the same topology, then $M$ can be thickened to be convex. 

(Can take $K = 6$)
2. Properties of convex hyp manifolds

Let $M$ be hyperbolic $n$-manifold.

Then $\exists$ two maps:

$\text{dev}: \tilde{M} \to \mathbb{H}^n$ developing map
$\text{hol}: \pi_1(M) \to \text{Isom}(\mathbb{H}^n)$ holonomy

s.t.

$\text{dev}(g \cdot x) = \text{hol}(g) \cdot \text{dev}(x)$

Definition: $M$ is convex if $\text{dev}$ is 1-1 and $\text{dev}(\tilde{M}) \subseteq \mathbb{H}^n$ is convex.

$\iff$ Any two points of $\tilde{M}$ can be joined by a geodesic arc

$\iff$ Every path in $M$ is homotopic rel endpts to a geodesic in $M$
$M$ convex $\Rightarrow M = \frac{\text{dev}(\bar{M})}{\text{hol}(\pi_1(M))}$

Example: hyperbolic annulus

$A = \frac{e}{\epsilon}$

$e$ - geodesic length $\ell$

$\epsilon$ - thickening of geodesic loop

$\text{dev}(\bar{A})$

$A = \frac{\text{dev}(\bar{A})}{\text{hol}(\pi_1(A))}$
3. Manifolds with convex thickenings

$M$ has convex thickening if $\exists \ N \text{ convex s.t. } M \subset N$ and $\text{incl}_*: \pi_1(M) \to \pi_1(N)$ is an isomorphism

$\iff \ \text{dev}: \tilde{M} \to \mathbb{H}^n \text{ is injective}$

$\iff \ \text{hol}: \pi_1(M) \to \text{Isom}(\mathbb{H}^n) \text{ is discrete and 1-1}$

Then can take $N = \mathbb{H}^n / \text{hol}(\pi_1(M))$

$\Rightarrow M \ 'corresponds' \ to \ a \ discrete \ subgp \ of \ \text{Isom}(\mathbb{H}^n)$. 
4. Two dimensional example

\[ M_1 = M_2 = \text{hyperbolic annulus} \]

\[ M = M_1 \cup M_2 \]

= punctured torus \( T(l, \theta, \xi) \)
Case 1: $M = M_1 \cup M_2$ has no convex thickening if $I$ is too short

$hol: \pi_1(M) \to \mathbb{H}^2$ is not discrete and 1-1
Case 2: If $l$ is long enough, then $M \times M, u M_2$ has a convex thickening:

\[
\begin{align*}
\text{hol} : \pi_1(M) \rightarrow \text{Isom}(\mathbb{H}^2) & \text{ is 1-1} \\
\text{hol}(\pi_1(M)) & \text{ is discrete}
\end{align*}
\]

*Note*: If $l$ is too short (case 1), we can lengthen $l$ by gluing covers $M_1', M_2'$ to obtain case 2.
5. **Convex Combination Theorem**

There is a constant $K (\geq 6)$ such that if:

1) \( Y = Y_1 \cup Y_2 \) is connected hyp. $n$-manifold
   where \( Y_1, Y_2 \) are convex $n$-submanifolds

2) \( M = M_1 \cup M_2 \) is connected hyp. $n$-manifold
   where \( M_1, M_2 \) are convex $n$-submanifolds

3) \( M \subset Y \) and \( Y \) is a thickening of \( M \)

4) \( \forall p \in M \), \( \exp_p: T_p^K M \to Y \) is defined
   \( (T_p^K M = \text{tangent vectors of length} \leq K) \)

5) **No bumping**: For \( p \in M \setminus \text{int}(M_1 \cap M_2) \)
   we have \( \exp_p(T_p^K M) \subset Y_i \) \((i=1,2)\)

6) Every component of \( Y_i \cap Y_2 \) contains a point of \( M_1 \cup M_2 \)

Then \( M \) has a convex thickening. Also if \( Y \) finite vol \( \Rightarrow M \) geom. finite
6. Corollary: Virtual Amalgam Theorem

Let $\Gamma \subseteq \text{Isom}(\mathbb{H}^n)$ be a discrete subgp $A, B \subseteq \Gamma$ geom. finite subgps

(\text{+ technical condition if parabolics})

Then

$\exists A' \subseteq A, B' \subseteq B$ finite index

such that

$$G' = \langle A', B' \rangle = A' * B'$$

$A' \cap B'$
7. **Multiple immersed boundary slopes**

M compact 3-manifold, $T = \text{torus} \subset \partial M$

**Def:** A slope $\alpha$ on $T$ is a multiple immersed boundary slope, MIIBS, if there exists a compact surface $S$ and $f : S \rightarrow M$ such that

1) $f_* : \pi_1(S) \rightarrow \pi_1(M)$ is injective

2) $f$ is not homotopic rel $\partial S$ into $\partial M$

3) $\forall \beta \in \partial S$, $f_*([\beta]) = n[\alpha]$ (for $n > 0$)

**Theorem:** If $\text{int}(M)$ has complete hyp. structure, then every slope on $T$ is a MIIBS.
Note: False for SFS

Example:

MIBS:

$(0,1), (6,1)$

Idea of Proof:

1. Culler-Shalen \( \Rightarrow \) \( \exists \) Q.F. surfaces \( F_1, F_2 \subseteq M \) s.t.

\( \partial F_1 \) = parallel copies of \( \pm \alpha_1 \)

\( \partial F_2 \) = parallel copies of \( \pm \alpha_2 \)

\( [\alpha_1] \neq \pm [\alpha_2] \) in \( H_1(T) \)
Intuitively: cut and cross-join copies of $F_1$, $F_2$ to get slopes on $T$:

![Diagram showing cut and cross-join of $\alpha_1$ and $\alpha_2$]

Problem: $F_1 \cup F_2$ will probably not be $\Pi_1$-injective

Taking covers $\tilde{F}_1$, $\tilde{F}_2$ and using convex combination theorem, we obtain convex $3$-manifold $X$ and isometric immersion

$f: X \hookrightarrow M$

that times desired slopes.
\( \pi_1 \) injectivity follows from

Lemma: \( f : \pi_1(X) \to \pi_1(M) \) is 1-1