CYCLIC BRANCHED COVERS OF PRIME KNOTS

joint with N. BOILEAU

ICTP TRIESTE - 23rd JUNE 2005
Nakanishi - Sakuma.

\[ M \rightarrow n\text{-fold cyclic branched cover of } K \text{ and } K' \]
\[ \cong \mathbb{Z}_n \otimes \mathbb{Z}_n \text{-fold branched cover of } \mathbb{D} \]

\[ (S^3, K) \quad (S^3, K') \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \] 

\[ (S^3, L = \overline{K \cup K'}) \]
\[ \overline{K}, \overline{K'} \text{ trivial} \]
\[ (\ell_K(\overline{K}, \overline{K'}), n) = 1 \]

N.B. \( \mathbb{Z}_n \otimes \mathbb{Z}_n = \langle h, h' \rangle \) acts on \( M \)

\( h \) deck transformation for \( K \)
\( h' \) deck transformation for \( K' \)

\( h' \) (resp. \( h \)) induces an \( n \)-axial symmetry of \( K \) (resp. \( K' \)) with trivial quotient

\[ n = 3 \quad 9^2_{35} \]
Theorem (Zimmermann)

If \( k \) hyperbolic, \( n \geq 3 \) then

\( k \) and \( k' \) are related by Nakamura and Sakuma's standard abelian construction.

ie. \( (S^3, k) \xrightarrow{\mathcal{R}} (S^3, k') \xrightarrow{\mathcal{H}} (S^3, L = \mathcal{R} \cup \mathcal{R}') \)
Consequences:

1. Three cyclic branched covers of orders $\geq 3$ determine a hyperbolic knot.

2. A hyperbolic knot has at most one $n$-twin if $n \geq 3$.

Definition: $k'$ is an $n$-twin of $k$ if $k$ and $k'$ have the same $n$-fold cyclic branched cover.
Proof:

1. If $k$ is hyperbolic $\Rightarrow$ the symmetries of $k$ commute (not necessarily an axis of symmetry for $k$ = trivial knot $\Rightarrow$ $A_k k =$ Hopf link.

2. If $k$ is hyperbolic, Smith's conjecture $\Rightarrow$ symmetries of $k$ of order $\geq 3$ are unique.

N.B. $\times$ hyperbolicity is only used to prove:

1. Commutativity of symmetries $\Rightarrow$

2. Uniqueness of symmetries $\Rightarrow$

$\times$ In 1. reason by contradiction and show that $k$ is trivial.
Q: Is the standard abolition situation the only possible one?

No, need to require:

- \( k \) is prime

  e.g. \( k \) non-invertible then \( k = \lambda + k' \) and \( k' = k' + k' \) have the same \( n \)-fold cyclic branched cover for all \( n \geq 3 \)

- \( n \geq 3 \)

  e.g. for \( n = 3 \) one has Conway mutation...
New Construction (Boileau - P.):

* $L = L_1 \cup L_2 \cup \ldots \cup L_m$, $m \geq 3$, $L$ hyperbolic
* $L_1 \cup L_k$ and $L_2 \cup L_k$ Hopf for all $k \geq 3$
* $L_3 \cup \ldots \cup L_m$ trivial link
* $L_1$ and $L_2$ non-exchangeable
* $\langle lk(L_1, L_2), n \rangle = 1$, e.g. $\langle lk(L_1, L_2), 1 \rangle = 1$.

$N' = (S^3, K)$, $(S^3, K') = N'$

\[ (S^3, L) = (L_1 \setminus \bar{K}) \cup (L_2 \setminus \bar{K'}) \cup L_3 \cup \ldots \cup L_m) \]
\[ (S^3, K) = (N' \setminus \sim L_3 \cup \ldots \cup L_m) \cup \bigcup_{i=3}^{m} E_i \quad \text{k' image of k} \]
\[ (S^3, K') = (N' \setminus \sim L_3 \cup \ldots \cup L_m) \cup \bigcup_{i=3}^{m} E_i \quad \text{k image of k'} \]
\[ E_i = \text{knot exterior} \]
Theorem 1. (Boileau, P.)

Let $K$ be a prime knot, $p$ an odd prime. Assume that $K'$ is a $p$-twin of $K$. Then either $K$ and $K'$ arise from the standard abelian construction or the deck transformation for $K'$ induces a partial axial symmetry of $K$.

i.e. $\exists P \subseteq E(K) = S^3 \setminus \mathcal{N}(K)$, $P$ union of geometric pieces of the JSJ decomposition of $E(K)$ such that

* $P \supseteq O_p : E(K) \setminus UT = O_p$ connected component containing $\text{DECK}(K)$
  
  Te JSJ
  $W(T) = p$
  $T = \text{torus}$

* the deck transformation for $K'$ induces a symmetry $\varphi$ of $P$ such that
  
  $\varphi(O_p) = O_p$
  
  $\text{Fix}(\varphi) \subseteq O_p$, $\text{Fix}(\varphi) \neq \emptyset$
Idea of Proof:

Let $M$ be the $p$-fold cyclic branched cover for $K$ and $K'$. Let $M = \bigcup_{i} V_i$; the JSJ-decomposition of $M$ into geometric pieces.

* The dual JSJ-graph is a tree

Let $h$ (resp. $h'$) be the deck transformation for $K$ (resp. $K'$)

* If $h(V_i) = h'(V_i) = V_i$, then, up to conjugation,
  \[ [h|_{V_i}, h'|_{V_i}] = 1. \]

* If $Fix(h) \subset V_i$ then, up to conjugation,
  \[ h'(V_i) = V_i. \]

Lack of commutativity appears in a well-specified case
Theorem 2 (Boileau P.)

Let $K$ admit three rotational symmetries with pairwise distinct odd prime orders and with trivial quotient.

Then $K$ is the trivial knot.
J. H. C. Whitehead described all possible axial symmetries for a composite knot.

\[ \Rightarrow \] Composite knots cannot admit symmetries with trivial quotient.

\[ \Rightarrow K \text{ must be prime.} \]

**Theorem (Sakuma)**

If \( K \) is **totally prime** (i.e. all its companions are prime) and **pedigreed** (i.e. no companion has winding \#0),

then, up to conjugation, its symmetries of order \( \geq 3 \) commute.

\[ \Rightarrow \] If \( K \) is totally prime and pedigreed,

then \( K \) is trivial.

**Else:**

Use Dehn surgery to construct a new non trivial knot, which is totally prime and pedigreed and satisfies the hypotheses of Theorem 2.
Lemma A (Boileau, P.)

K a prime knot. Then there is at most one odd prime $p$ such that $K$ has a p-twin inducing a partial symmetry. Moreover, if $q^i p$ is an odd prime which is the order of an axial symmetry of $K$ with trivial quotient then $\mathcal{D}^p$ consists of precisely two components (one being $\mathcal{S}(K)$) and $\text{Fix}(\tau) \subseteq E(K) \setminus \mathcal{D}^p$.

NB. The situation of figure can indeed happen.
Lemma B

Let \( \gamma \) and \( \gamma' \) be two axial symmetries of distinct odd prime orders and trivial quotient for a prime knot \( K \).

Then both \( \text{Fix}(\gamma) \) and \( \text{Fix}(\gamma') \) are contained in the geometric component of the JSJ-decomposition for \( E(K) \) adjacent to \( \text{DE}(K) \).
Theorem 3 (Kojima, P.)

Let K be a prime knot.

(a) there are at most two odd prime numbers for which K admits a p-twin.

(b) For any given odd prime p, K admits at most one p-twin.

⇒ three cyclic branched covers of odd prime orders suffice to determine a prime knot.

COMPARE WITH:

Theorem (Kojima)

A prime knot is determined by a cyclic branched cover provided its order is sufficiently large.
(a) By contradiction, assume there are three odd primes.

* If the corresponding $p$-twins all arise from the standard abelian construction then Theorem 2 tells us that $K$ is trivial.
  \[ \Rightarrow \] it is determined by each of its covers (by Smith's conjecture)

* One can thus assume that one (and, by Lemma A, at most 1) $p$-twin induces a partial symmetry.
  Let $\tau$ and $\psi$ be the axial symmetries with trivial quotient induced by the other twins.

  - By Lemma A, $\text{Fix}(\psi)$ and $\text{Fix}(\tau)$ are not
    in the JSJ component containing $\text{DE}(K)$.
  - On the other hand, by Lemma B, $\text{Fix}(\psi)$ and $\text{Fix}(\tau)$ belong to the component containing $\text{DE}(K)$.

  Contradiction!
Theorem (Sakuma)

Up to conjugation, the symmetries of a prime knot are unique provided that their order is odd.

⇒ At most one $p$-twist inducing an axial symmetry of $K$.

Non technical in the other case
(follows from the fact that if two deck transformations coincide on a geometric piece of the cover, then they coincide everywhere)