LC circuit example and references for lecture 1

Guido Blankenstein
SYSTeMS, Ghent University, Belgium
Guido.Blankenstein@UGent.be

July 6, 2005

1 LC circuit example

Consider the LC circuit in Fig. 1, consisting of two inductors L_1 and L_2 and two capacitors C_1 and C_2. The graph of the circuit is given in Fig. 2. A maximal tree Γ of the graph is given by, for instance, $\Gamma = \{C_1\}$. The corresponding co-tree (i.e., the branches which, when added to the tree, produce a loop) are then given by $\Sigma = \{C_2, L_1, L_2\}$.

Denote the currents and voltages corresponding to the elements by: i_{C_1} and v_{C_1} for C_1; i_{C_2} and v_{C_2} for C_2; i_{L_1} and v_{L_1} for L_1; i_{L_2} and v_{L_2} for L_2. According to standard network theory we can write

$$i_\Gamma = P i_\Sigma, \quad v_\Sigma = -P^T v_\Gamma, \tag{1}$$

for some matrix P. That is, the currents in the tree can be expressed as linear fuctions of the currents in the co-tree and, dually, the voltages in the co-tree can be expressed as linear functions of the voltages in the tree. Kirchhoff's current law for the network in Fig. 1 yields

$$i_{C_1} + i_{C_2} - i_{L_1} + i_{L_2} = 0. \tag{2}$$

Alternatively, the incoming currents (note the orientation!) at each node of the graph in Fig. 2 should sum to zero. Kirchhoff's voltage laws yield

$$v_{C_1} - v_{C_2} = 0, \quad v_{C_1} + v_{L_1} = 0, \quad v_{C_1} - v_{L_2} = 0. \tag{3}$$

Alternatively, the voltages over every loop in the graph should sum to zero (again, note the orientation). Now the currents and voltages can be written as in Eq. (1):

$$i_{C_1} = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} i_{C_2} \\ i_{L_1} \\ i_{L_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_{C_2} \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} v_{C_1}. \tag{4}$$

We can write

$$i_{C_1} = \dot{q}_{C_1}, \quad v_{C_1} = \frac{\partial H}{\partial q_{C_1}} \tag{5}$$

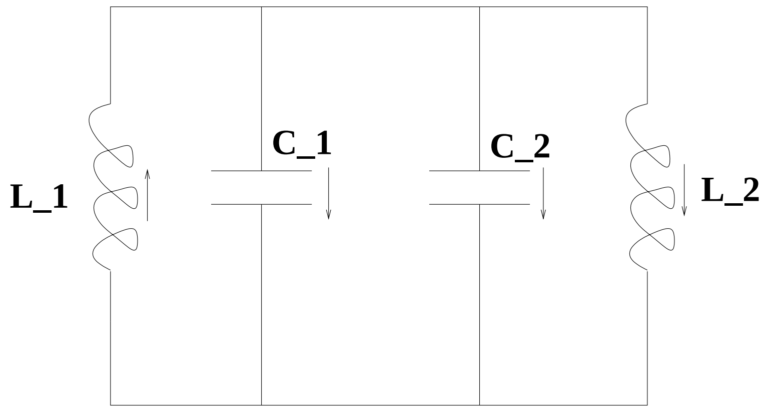

Figure 1: *LC* circuit.

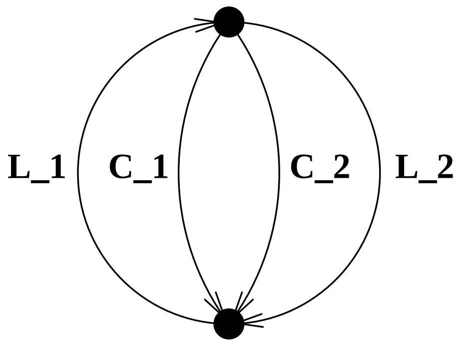

Figure 2: Graph of the circuit.

and

$$(i_{C_2}, i_{L_1}, i_{L_2}) = \left(\dot{q}_{C_2}, \frac{\partial H}{\partial \phi_{L_1}}, \frac{\partial H}{\partial \phi_{L_2}}\right), \quad (v_{C_2}, v_{L_1}, v_{L_2}) = \left(\frac{\partial H}{\partial q_{C_2}}, \dot{\phi}_{L_1}, \dot{\phi}_{L_2}\right), \tag{6}$$

where

$$H(q_{C_1}, q_{C_2}, \phi_{L_1}, \phi_{L_2}) = \frac{q_{C_1}^2}{2C_1} + \frac{q_{C_2}^2}{2C_2} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2} \tag{7}$$

is the total electromagnetic energy in the circuit (the energy variables q_{C_i} denote the charge of the capacitor C_i and ϕ_{L_i} the flux of the inductor L_i, $i = 1, 2$).

The (general) circuit's dynamics can be written as (cf. lecture 1)

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \partial H/\partial \phi_\Gamma \\ \dot{q}_\Gamma \\ \dot{\phi}_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & -P_{21}^T & -P_{11}^T & 0 \\ P_{21} & 0 & 0 & P_{22} \\ P_{11} & 0 & 0 & P_{12} \\ 0 & -P_{22}^T & -P_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_\Sigma \\ \dot{\phi}_\Gamma \\ \partial H/\partial q_\Gamma \\ \partial H/\partial \phi_\Sigma \end{pmatrix}. \tag{8}$$

In this example there are no inductors in the tree, hence ϕ_Γ is absent. Therefore, we can eliminate the second row and the second column of the skew-symmetric matrix in (8). The matrices P_{11} and P_{12} are given by

$$P_{11} = -1, \quad P_{12} = \begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{9}$$

The dynamics of the circuit can thus be written as

$$\begin{pmatrix} \frac{\partial H}{\partial q_{C_2}} \\ \dot{q}_{C_1} \\ \dot{\phi}_{L_1} \\ \dot{\phi}_{L_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_{C_2} \\ \frac{\partial H}{\partial q_{C_1}} \\ \frac{\partial H}{\partial \phi_{L_1}} \\ \frac{\partial H}{\partial \phi_{L_2}} \end{pmatrix}. \tag{10}$$

Hence,

$$\mathbb{J}_{11} = 0, \quad \mathbb{J}_{12} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{11}$$

and

$$\mathbb{J}_{21} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{J}_{22} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

Eq. (10) can be written out to obtain

$$\frac{\partial H}{\partial q_{C_1}} - \frac{\partial H}{\partial q_{C_2}} = 0, \tag{13}$$

$$\dot{q}_{C_1} = -\dot{q}_{C_2} + \frac{\partial H}{\partial \phi_{L_1}} - \frac{\partial H}{\partial \phi_{L_2}}, \tag{14}$$

$$\dot{\phi}_{L_1} = -\frac{\partial H}{\partial q_{C_1}}, \tag{15}$$

$$\dot{\phi}_{L_2} = \frac{\partial H}{\partial q_{C_1}}. \tag{16}$$

This is a set of differential and algebraic equations. Eq. (13) is an *algebraic constraint*, corresponding to the capacitor loop C_1–C_2 in the circuit, i.e., $v_{C_1} - v_{C_2} = 0$. Eqs. (15) and (16) imply that $\phi_{L_1} + \phi_{L_2}$ is a *conserved quantity* of the system. This corresponds to the inductor loop L_1–L_2, i.e., $v_{L_1} + v_{L_2} = 0$.

In order to find the canonical coordinates of the system, first define the variables

$$y = x_2 - \mathbb{J}_{21} x_1 \quad \text{and} \quad z = x_1, \tag{17}$$

where $x_1 = (q_\Sigma, \phi_\Gamma)$ and $x_2 = (q_\Gamma, \phi_\Sigma)$. For this example this yields

$$y_1 = q_{C_1} + q_{C_2}, \quad y_2 = \phi_{L_1}, \quad y_3 = \phi_{L_2}, \quad z = q_{C_2}. \tag{18}$$

In these coordinates the Hamiltonian becomes

$$\tilde{H}(y_1, y_2, y_3, z) = \frac{(y_1 - z)^2}{2C_1} + \frac{z^2}{2C_2} + \frac{y_2^2}{2L_1} + \frac{y_3^2}{2L_2} \tag{19}$$

and the system can be written as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{H}}{\partial y_1} \\ \frac{\partial \tilde{H}}{\partial y_2} \\ \frac{\partial \tilde{H}}{\partial y_3} \end{pmatrix}, \qquad 0 = \frac{\partial \tilde{H}}{\partial z}. \tag{20}$$

Canonical coordinates for the skew-symmetric matrix in (20) are

$$\xi_1 = y_1, \quad \xi_2 = \frac{1}{2}(y_2 - y_3), \quad \xi_3 = \frac{1}{2}(y_2 + y_3), \tag{21}$$

in which the Hamiltonian becomes

$$\hat{H}(\xi_1, \xi_2, \xi_3, z) = \frac{(\xi_1 - z)^2}{2C_1} + \frac{z^2}{2C_2} + \frac{(\xi_2 + \xi_3)^2}{2L_1} + \frac{(\xi_3 - \xi_2)^2}{2L_2}. \tag{22}$$

In the *canonical coordinates* (ξ, z) the implicit Hamiltonian system takes the canonical form

$$\dot{\xi}_1 = \frac{\partial \hat{H}}{\partial \xi_2}, \tag{23}$$

$$\dot{\xi}_2 = -\frac{\partial \hat{H}}{\partial \xi_1}. \tag{24}$$

$$\dot{\xi}_3 = 0, \tag{25}$$

$$0 = \frac{\partial \hat{H}}{\partial z}. \tag{26}$$

Note that the canonical coordinates are related to the original energy variables of the circuit by

$$\xi_1 = q_{C_1} + q_{C_2}, \quad \xi_2 = \frac{1}{2}(\phi_{L_1} - \phi_{L_2}), \quad \xi_3 = \frac{1}{2}(\phi_{L_1} + \phi_{L_2}), \quad z = q_{C_2}. \tag{27}$$

The system (23)–(26) is an implicit Hamiltonian system in canonical form. The underlying geometric structure is that of a *Dirac structure*. One observes that the system has conserved quantities (25) as well as algebraic constraints (26). As such it *combines* properties of Poisson systems (i.e., (23)–(25)) and pre-symplectic systems (i.e., (23),(24),(26)). The conserved quantity (25) corresponds to the inductor loop L_1–L_2 in the circuit. The algebraic constraint (26) corresponds to the capacitor loop C_1–C_2 in the circuit.

2 References

- Some interesting papers on the modeling of physical systems can be found in the lecture notes by Peter Breedveld on

  `http://www-lar.deis.unibo.it/euron-geoplex-sumsch/lectures_1.html`.

- Notes on Port-Hamiltonian systems modeling (including LC circuits) can be found in the lecture notes by Arjan van der Schaft and Bernhard Maschke on the website mentioned above.

- The modeling of LC circuits using Dirac structures, and its construction such as used in this example, was first described in:

  A.M. Bloch and P.E. Crouch, Representations of Dirac structures on vector spaces and nonlinear LC circuits, In: H. Hermes, G. Ferraya, R. Gardner and H. Sussmann, editors, *Proc. of Symposia in Pure Mathematics, Differential Geometry and Control Theory*, vol. 64, pp. 103–117, 1999.