Symmetry and reduction in Poisson geometry

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Reduction is...

• Algebra: a procedure to pass to the quotient in the Hamiltonian category
• Geometry: a way to construct new Poisson and symplectic manifolds
• Applied dynamics: a systematic method to eliminate variables using symmetries and/or conservation laws
• Theoretical dynamics: a way to get intuition on the dynamical behavior of symmetric systems
• Numerics: is it worth it?
Example: The Weinstein-Moser Theorem

- Weinstein-Moser: if $d^2 h(m) > 0$ then $1/2 \dim M$ periodic orbits at each neighboring energy level.

- Relative Weinstein-Moser (JPO (2003)): Relative periodic orbits around stable relative at neighboring energy-momentum-isotropy levels

$$\frac{1}{2} \left( \dim U^K - \dim (N(K)/K) - \dim (N(K)/K)_\lambda \right)$$
We will focus on

- Poisson and symplectic category

Leave aside

- Lagrangian side: different philosophy.
- Singular cotangent bundle reduction, nonholonomic reduction, reduction of Dirac manifolds and implicitly defined Hamiltonian systems, Sasakian, Kähler, hyperkähler, contact manifolds....

References

- Look at review
Structure of the course

• **Lecture I:** Introduction. Preliminaries on:
  – Symmetries/group actions
  – Poisson and symplectic manifolds

• **Lecture II:** Poisson reduction.

• **Lecture III:** Momentum maps. Normal forms.

• **Lecture IV:** Symplectic reduction. Regular and singular.
Symmetry/Group actions

Definition. \( M \) a manifold and \( G \) a Lie group. A \textbf{left action} of \( G \) on \( M \) is a smooth mapping \( \Phi : G \times M \to M \) such that

(i) \( \Phi(e, z) = z \), for all \( z \in M \) and

(ii) \( \Phi(g, \Phi(h, z)) = \Phi(gh, z) \) for all \( g, h \in G \) and \( z \in M \).

We will often write

\[ g \cdot z := \Phi(g, z) := \Phi_g(z) := \Phi^z(g). \]

and

\[ A_G := \{ \Phi_g \mid g \in G \} \subset \text{Diff}(M). \]

The triple \((M, G, \Phi)\) is called a \textbf{\( G \)-space} or a \textbf{\( G \)-manifold}.

Examples of group actions.

- Translation and conjugation. The \textbf{left (right) translation} \( L_g : G \to G \), \( (R_g) \) \( h \mapsto gh \), induces a left (right) action of \( G \) on itself.
• The **inner automorphism** \( \text{AD}_g \equiv I_g : G \to G \), given by \( I_g := R_{g^{-1}} \circ L_g \) defines a left action of \( G \) on itself called \textit{conjugation}.

• **Adjoint and coadjoint action.** The differential at the identity of the conjugation mapping defines a linear left action of \( G \) on \( \mathfrak{g} \) called the \textit{adjoint representation} of \( G \) on \( \mathfrak{g} \)

\[
\text{Ad}_g := T_eI_g : \mathfrak{g} \longrightarrow \mathfrak{g}.
\]

If \( \text{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^* \) is the dual of \( \text{Ad}_g \), then the map

\[
\Phi : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \\
(g, \nu) \mapsto \text{Ad}_{g^{-1}}^* \nu,
\]

defines also a linear left action of \( G \) on \( \mathfrak{g}^* \) called the \textit{coadjoint representation} of \( G \) on \( \mathfrak{g}^* \).
• **Group representation.** If the manifold $M$ is a vector space $V$ and $G$ acts linearly on $V$, that is, $\Phi_g \in \text{GL}(V)$ for all $g \in G$, where $\text{GL}(V)$ denotes the group of all linear automorphisms of $V$, then the action is said to be a *representation* of $G$ on $V$. For example, the adjoint and coadjoint actions of $G$ defined above are representations.

• **Tangent lifts of group actions.** The map $\Phi$ induces a natural action on the tangent bundle $TM$ of $M$ by

$$g \cdot v_m := T_m \Phi_g \cdot v_m,$$

where $g \in G$ and $v_m \in T_m M$.

• **Cotangent lifts of group actions.** Let $\Phi : G \times M \rightarrow M$ be a smooth Lie group action on the manifold $M$. The map $\Phi$ induces a natural action on the cotangent bundle $T^*M$ of $M$ by

$$g \cdot \alpha_m := T^*_g m \Phi_{g^{-1}} \cdot \alpha_m$$

where $g \in G$ and $\alpha_m \in T^*_m M$. 


The infinitesimal generator $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$ is the vector field on $M$ defined by

$$\xi_M(m) := \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp t\xi}(m) = T_e \Phi^m \cdot \xi.$$ 

The infinitesimal generators are complete vector fields. The flow of $\xi_M$ equals $(t, m) \mapsto \exp t\xi \cdot m$. Moreover, the map $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a **Lie algebra antihomomorphism**, that is,

(i) $(a\xi + b\eta)_M = a\xi_M + b\eta_M,$

(ii) $[\xi, \eta]_M = -[\xi_M, \eta_M].$

Let $\mathfrak{g}$ be a Lie algebra and $M$ a smooth manifold. A **right (left) Lie algebra action** of $\mathfrak{g}$ on $M$ is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \mapsto \xi_M(m) \in TM$ is smooth. Given a Lie group action, we will refer to the Lie algebra action induced by its infinitesimal generators as the **associated Lie algebra action**.
Stabilizers and orbits. The *isotropy subgroup* or *stabilizer* of an element \( m \) in the manifold \( M \) acted upon by the Lie group \( G \) is the closed subgroup

\[ G_m := \{ g \in G \mid \Phi_g(m) = m \} \subset G \]

whose Lie algebra \( \mathfrak{g}_m \) equals

\[ \mathfrak{g}_m = \{ \xi \in \mathfrak{g} \mid \xi_M(m) = 0 \} \]. \hspace{1cm} (1)

The *orbit* \( \mathcal{O}_m \) of the element \( m \in M \) under the group action \( \Phi \) is the set

\[ \mathcal{O}_m \equiv G \cdot m := \{ \Phi_g(m) \mid g \in G \} \].

The isotropy subgroups of the elements in a group orbit are related by the expression

\[ G_{g \cdot m} = g G_m g^{-1} \text{ for all } g \in G \].

The notion of orbit allows the introduction of an equivalence relation in the manifold \( M \), namely, two elements \( x, y \in M \) are equivalent if and only if they are in the same \( G \)-orbit, that is, if there exists an element \( g \in G \) such that \( \Phi_g(x) = y \).
The space of classes with respect to this equivalence relation is usually referred to as the *space of orbits* and, depending on the context, it is denoted by the symbols $M/G$ or $M/A_G$.

The action is

- **Transitive** if there is only one orbit.

- **Free** if the isotropy of every element in $M$ consists only of the identity element.

- **Proper** whenever the map $\Theta : G \times M \to M \times M$ defined by
  \[ \Theta(g, z) = (z, \Phi(g, z)) \]
  is proper. Equivalent to the following condition: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in $M$, there exists a convergent subsequence $\{g_{n_k}\}$ in $G$.

Examples of proper actions: compact group actions, $SE(n)$ acting on $\mathbb{R}^n$, Lie groups acting on themselves by translation.
Proper actions

Φ : G × M → M be a proper action of the Lie
group G on the manifold M. Then:

(i) The isotropy subgroups G_m are compact.

(ii) The orbit space M/G is a Hausdorff topological space. (Even when M and G are not Haus-
dorff.)

(iii) If the action is free, M/G is a smooth manifold, and the canonical projection π : M → M/G defines on M the structure of a smooth left principal G–bundle.

(iv) If all the isotropy subgroups of the elements of M under the G–action are conjugate to a given one H then M/G is a smooth manifold and π : M → M/G defines the structure of a smooth locally trivial fiber bundle with structure group N(H)/H and fiber G/H.

(v) If the manifold M is paracompact then there exists a G–invariant Riemannian metric on it.

(vi) If the manifold M is paracompact then smooth G–invariant functions separate the G–orbits.
Tubes and Slices

Twisted product. Let $G$ be a Lie group and $H \subset G$ a subgroup. Suppose that $H$ acts on the left on the manifold $A$. The **twisted action** of $H$ on the product $G \times A$ is defined by

$$h \cdot (g, a) = (gh, h^{-1} \cdot a).$$

This action is free and proper by the freeness and properness of the action on the $G$–factor. The **twisted product** $G \times_H A$ is defined as the orbit space $(G \times A)/H$ corresponding to the twisted action.

Tube. Let $M$ be a manifold and $G$ a Lie group acting properly on $M$. Let $m \in M$ and denote $H := G_m$. A **tube** around the orbit $G \cdot m$ is a $G$–equivariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

where $U$ is a $G$–invariant neighborhood of $G \cdot m$ and $A$ is some manifold on which $H$ acts.
Slice Theorem. $G$ a Lie group acting properly on $M$ at the point $m \in M$, $H := G_m$. There exists a tube

$$\varphi : G \times_H B \longrightarrow U$$

about $G \cdot m$. $B$ is an open $H$–invariant neighborhood of 0 in a vector space $H$–equivariantly isomorphic to $T_m M/T_m(G \cdot m)$ on which $H$ acts linearly by

$$h \cdot (v + T_m(G \cdot m)) := T_m \Phi_h \cdot v + T_m(G \cdot m).$$

Dynamical consequences. $G$-invariant vector fields $X$ can be locally decomposed as

$$X = X_T + X_N$$

Geometric consequences. Isotropy, fixed point, and orbit type spaces are submanifolds:

$$M_{(H)} = \{ z \in M \mid G_z \in (H) \},$$

$$M^H = \{ z \in M \mid H \subset G_z \},$$

$$M_H = \{ z \in M \mid H = G_z \}.$$
Structure Theorems

Principal Orbit Theorem: $M$ connected. The subset $M^{\text{reg}} \cap M$ is connected, open, and dense in $M$. $M/G$ contains only one principal orbit type, which is a connected open and dense subset of it.

The Stratification Theorem: Let $M$ be a smooth manifold and $G$ a Lie group acting properly on it. The connected components of the orbit type manifolds $M(H)$ and their projections onto orbit space $M(H)/G$ constitute a Whitney stratification of $M$ and $M/G$, respectively. This stratification of $M/G$ is minimal among all Whitney stratifications of $M/G$.

Theorem. Let $G$ be a Lie group acting properly on the smooth manifold $M$ and $m \in M$ a point with isotropy subgroup $H := G_m$. Then

$$((T_m(G \cdot m))^{\circ})^H = \{d f(m) \mid f \in C^\infty(M)^G\}.$$
Symmetry Reduction

- $M$ a $G$–manifold. $X \in \mathfrak{X}(M)^G$. Flow $F_t$.
- $H$–isotropy type submanifold $M_H$:
  $$M_H := \{ m \in M \mid G_m = H \}$$
  preserved by the flow $F_t$ and $N(H)$–invariant.
- $\pi_H : M_H \rightarrow M_H/(N(H)/H)$
  $i_H : M_H \hookrightarrow M$.
- Reduced vector field:
  $$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$
  with flow $F_t^H$ given by
  $$F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H.$$
- Linear compact actions and Hilbert’s Theorem.
Symplectic manifolds

A *symplectic manifold* is a pair \((M, \omega)\), where \(M\) is a manifold and \(\omega \in \Omega^2(M)\) is a closed non–degenerate two–form on \(M\), that is,

- \(d\omega = 0\)
- for every \(m \in M\), the map 
  \[ v \in T_mM \mapsto \omega(m)(v, \cdot) \in T^*_mM \]
  is a linear isomorphism

If \(\omega\) is allowed to be degenerate, \((M, \omega)\) is called a *presymplectic manifold*. A *Hamiltonian dynamical system* is a triple \((M, \omega, h)\), where \((M, \omega)\) is a symplectic manifold and \(h \in C^\infty(M)\) is the *Hamiltonian function* of the system. By non–degeneracy of the symplectic form \(\omega\), to each Hamiltonian system one can associate a *Hamiltonian vector field* \(X_h \in \mathfrak{X}(M)\), defined by the equality

\[ i_{X_h}\omega = dh. \]
Example  Let $V$ be a vector space and $V^*$ its dual. Let $Z = V \times V^*$. The canonical symplectic form $\Omega$ on $Z$ is defined by

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.$$ 

Example  Let $Q$ be a smooth manifold and $T^*Q$ its cotangent bundle. Let $\pi_Q : T^*Q \to Q$ be the projection and $\Theta$ the one–form on $T^*Q$ defined by

$$\Theta(\beta) \cdot v_\beta := \langle \beta, T_\beta \pi_Q \cdot v_\beta \rangle,$$

where $\beta \in T^*Q$ and $v_\beta \in T_\beta(T^*Q)$. The canonical symplectic form $\Omega$ on the cotangent bundle $T^*Q$ is defined by $\Omega = -d\Theta$.

Darboux theorem  Locally

$$\omega|_U = \sum_{i=1}^{n} dq^i \wedge dp_i.$$
In canonical coordinates, $X_h$ is determined by the well-known \textit{Hamilton equations},

\[
\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}.
\]

The \textbf{Poisson bracket} of $f, g \in C^\infty(M)$ is the function $\{f, g\} \in C^\infty(M)$ defined by

\[
\{f, g\}(z) = \omega(z)(X_f(z), X_g(z)).
\]

In canonical coordinates, the Poisson bracket takes the form

\[
\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right).
\]
Poisson manifolds

- $(M, \{\cdot, \cdot\})$ Poisson manifold. $(C^\infty(M), \{\cdot, \cdot\})$ Lie algebra such that
  \[
  \{fg, h\} = f\{g, h\} + g\{f, h\}
  \]
- **Casimirs** elements in the center of algebra.
- Derivations and vector fields. Hamiltonian vector fields
  \[
  X_h[f] = \{f, h\}
  \]
- **Example: The Lie-Poisson bracket** The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with respect to the $\pm$–**Lie–Poisson** brackets $\{\cdot, \cdot\}_\pm$ defined by
  \[
  \{f, g\}_\pm(\mu) := \pm \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle
  \]
  $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ is defined by
  \[
  \langle \nu, \frac{\delta f}{\delta \mu} \rangle := Df(\mu) \cdot \nu,
  \]
  for any $\nu \in \mathfrak{g}^*$. Given $h \in C^\infty(\mathfrak{g}^*)$
  \[
  X_h(\mu) = \mp \text{ad}^*_{\delta h/\delta \mu} \mu, \quad \mu \in \mathfrak{g}^*.
  \]
The Poisson tensor. The derivation property of the Poisson bracket implies that for any two functions $f, g \in C^\infty(M)$, the value of the bracket $\{f, g\}(z)$ on $f$ only through $df(z)$ which allows us to define a contravariant antisymmetric two–tensor $B \in \Lambda^2(T^*M)$ by

$$B(z)(\alpha_z, \beta_z) = \{f, g\}(z),$$

with $df(z) = \alpha_z$ and $dg(z) = \beta_z$. This tensor is called the Poisson tensor of $M$. The vector bundle map $B^\# : T^*M \to TM$ naturally associated to $B$ is defined by

$$B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\#(\beta_z) \rangle.$$

Its range $D := B^\#(T^*M) \subset TM$ is called the characteristic distribution. For any point $m \in M$, the dimension of $D(m)$ as a vector subspace of $T_mM$ is called the rank of the Poisson manifold $(M, \{\cdot, \cdot\})$ at the point $m$. 
The Weinstein coordinates of a Poisson manifold. Let \((M, \{\cdot, \cdot\})\) be a \(m\)-dimensional Poisson manifold and \(z_0 \in M\) a point where the rank of \((M, \{\cdot, \cdot\})\) equals \(2n\), \(0 \leq 2n \leq m\). There exists a chart \((U, \varphi)\) of \(M\) whose domain contains the point \(z_0\) and such that the associated local coordinates, denoted by

\[
(q^1, \ldots, q^n, p_1, \ldots, p_n, z_1, \ldots, z_{m-2n}),
\]
satisfy

\[
\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, z_k\} = \{p_i, z_k\} = 0,
\]
and \(\{q^i, p_j\} = \delta^i_j\), for all \(i, j, k\), \(1 \leq i, j \leq n\), \(1 \leq k \leq m - 2n\).

For all \(k, l\), \(1 \leq k, l \leq m - 2n\), the Poisson bracket \(\{z_k, z_l\}\) is a function of the local coordinates \(z^1, \ldots, z^{m-2n}\) exclusively, and vanishes at \(z_0\). Hence, the restriction of the bracket \(\{\cdot, \cdot\}\) to the coordinates \(z^1, \ldots, z^{m-2n}\) induces a Poisson structure that is usually referred to as the \textit{transverse Poisson structure} of \((M, \{\cdot, \cdot\})\) at \(m\).
A smooth mapping \( \varphi : (M_1, \{\cdot, \cdot\}_1) \to (M_2, \{\cdot, \cdot\}_2) \) is \textbf{canonical} or \textbf{Poisson} if for all \( g, h \in C^\infty(M_2) \) we have
\[
\varphi^*\{g, h\}_2 = \{\varphi^*g, \varphi^*g\}_1.
\]

In the symplectic category, \( \varphi : (M_1, \omega_1) \to (M_2, \omega_2) \) \textbf{canonical} or \textbf{symplectic} if
\[
\varphi^*\omega_2 = \omega_1.
\]

- Symplectic maps are immersions.
- A diffeomorphism \( \varphi : M_1 \to M_2 \) between two symplectic manifolds \( (M_1, \omega_1) \) and \( (M_2, \omega_2) \) is symplectic if and only if it is Poisson.
- If the symplectic map \( \varphi : M_1 \to M_2 \) is not a diffeomorphism it may not be a Poisson map.

Let \( (S, \{\cdot, \cdot\}_S) \) and \( (M, \{\cdot, \cdot\}_M) \) be two Poisson manifolds such that \( S \subset M \) and the inclusion \( i_S : S \hookrightarrow M \) is an immersion. \( (S, \{\cdot, \cdot\}_S) \) is a \textbf{Poisson submanifold} of \( (M, \{\cdot, \cdot\}_M) \) if \( i_S \) is a canonical map.
An immersed submanifold \( Q \) of \( M \) is called a **quasi Poisson submanifold** of \((M, \{\cdot, \cdot\}_M)\) if for any \( q \in Q \), any open neighborhood \( U \) of \( q \) in \( M \), and any \( f \in C^\infty_M(U) \) we have

\[
X_f(i_Q(q)) \in T_qi_Q(T_qQ),
\]

where \( i_Q : Q \hookrightarrow M \) is the inclusion and \( X_f \) is the Hamiltonian vector field of \( f \) on \( U \) with respect to the restricted Poisson bracket \( \{\cdot, \cdot\}_U^M \). Any Poisson submanifold is quasi Poisson. The converse is not true.

Given two symplectic manifolds \((M, \omega)\) and \((S, \omega_S)\) such that \( S \subset M \) and the inclusion \( i : S \hookrightarrow M \) is an immersion, the manifold \((S, \omega_S)\) is a **symplectic submanifold** of \((M, \omega)\) when \( i \) is a symplectic map. Symplectic submanifolds of a symplectic manifold \((M, \omega)\) are in general neither Poisson nor quasi Poisson manifolds of \( M \). The only quasi Poisson submanifolds of a symplectic manifold are its open sets which are, in fact, Poisson submanifolds.
**Symplectic Foliation Theorem.** Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold and \(D\) the associated characteristic distribution. \(D\) is a smooth and integrable generalized distribution and its maximal integral leaves form a generalized foliation decomposing \(M\) into initial submanifolds \(\mathcal{L}\), each of which is symplectic with the unique symplectic form that makes the inclusion \(i : \mathcal{L} \hookrightarrow M\) into a Poisson map, that is, \(\mathcal{L}\) is a Poisson submanifold of \((M, \{\cdot, \cdot\})\).

**Example** Let \(\mathfrak{g}^*\) with the Lie–Poisson structure. The symplectic leaves of the Poisson manifolds \((\mathfrak{g}^*, \{\cdot, \cdot\}_{\pm})\) coincide with the connected components of the orbits of the elements in \(\mathfrak{g}^*\) under the coadjoint action. In this situation, the symplectic form for the leaves is given by the **Kostant–Kirillov–Souriau (KKS) expression**

\[
\omega_{\mathcal{O}}^{\pm}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle.
\]
Canonical symmetries

• \((M, \{\cdot, \cdot\})\) Poisson manifold. \(G\) acts canonically on \(M\) when
  \[
  \Phi^*_g\{f, h\} = \{\Phi^*_g f, \Phi^*_g h\}
  \]

• Easy Poisson reduction: \((M, \{\cdot, \cdot\})\) Poisson manifold, \(G\) Lie group acting canonically, freely, and properly on \(M\). The orbit space \(M/G\) is a Poisson manifold with bracket
  \[
  \{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m),
  \]

• Reduction of Hamiltonian dynamics: \(h \in C^\infty(M)^G\) reduces to \(\overline{h} \in C^\infty(M/G)\) given by \(\overline{h} \circ \pi = h\) such that
  \[
  X_{\overline{h}} = T\pi \circ X_h
  \]

• What about the symplectic leaves?
How do we do it?

• Consider $\mathbb{R}^6$ with bracket

$$\{f, g\} = \sum_{i=1}^{6} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i}$$

• $S^1$–action given by

$$\Phi: \quad S^1 \times \mathbb{R}^6 \rightarrow \mathbb{R}^6$$

$$(e^{i\phi}, (x, y)) \mapsto (R_{\phi}x, R_{\phi}y),$$

• Hamiltonian of the spherical pendulum

$$h = \frac{1}{2} <y, y> + <x, e_3>$$

• Impose constraint $<x, x> = 1$

• Angular momentum: $J(x, y) = x_1y_2 - x_2y_1$.

Hilbert basis of the algebra of $S^1$–invariant polynomials is given by

$$\sigma_1 = x_3 \quad \sigma_3 = y_1^2 + y_2^2 + y_3^2 \quad \sigma_5 = x_1^2 + x_2^2$$

$$\sigma_2 = y_3 \quad \sigma_4 = x_1y_1 + x_2y_2 \quad \sigma_6 = x_1y_2 - x_2y_1.$$ 

Semialgebraic relations

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \quad \sigma_3 \geq 0, \quad \sigma_5 \geq 0.$$
Hilbert map
\[ \sigma : \mathbb{T}^3 \rightarrow \mathbb{R}^6 \]
\[ (x, y) \mapsto (\sigma_1(x, y), \ldots, \sigma_6(x, y)) \].

The \( S^1 \)-orbit space \( \mathbb{T}^3/S^1 \) can be identified with the semialgebraic variety \( \sigma(\mathbb{T}^3) \subset \mathbb{R}^6 \), defined by these relations.

\( TS^2 \) is a submanifold of \( \mathbb{R}^6 \) given by
\[ TS^2 = \{(x, y) \in \mathbb{R}^6 | <x, x> = 1, <x, y> = 0\} \].

\( TS^2 \) is \( S^1 \)-invariant. \( TS^2/S^1 \) can be thought of the semialgebraic variety \( \sigma(TS^2) \) defined by the previous relations and

\[ \sigma_5 + \sigma_1^2 = 1 \quad \sigma_4 + \sigma_1 \sigma_2 = 0, \]

which allow us to solve for \( \sigma_4 \) and \( \sigma_5 \), yielding
\[ TS^2/S^1 = \sigma(TS^2) = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_6) \in \mathbb{R}^4 | \sigma_1^2 \sigma_2^2 + \sigma_6^2 = (1 - \sigma_1^2)(\sigma_3 - \sigma_2^2), |\sigma_1| \leq 1, \sigma_3 \geq 0\}. \]
If $\mu \neq 0$ then $(TS^2)_\mu$ appears as the graph of the smooth function

$$\sigma_3 = \frac{\sigma_2^2 + \mu^2}{1 - \sigma_1^2}, \quad |\sigma_1| < 1.$$  

The case $\mu = 0$ is singular and $(TS^2)_0$ is not a smooth manifold.

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Reduced Hamiltonian

$$H = \frac{1}{2}\sigma_3 + \sigma_1$$
Poisson reduction by distributions

- Reduction of the 4-tuple \((M, \{\cdot, \cdot\}, D, S)\)
- \(S\) encodes conservation laws and \(D\) invariance properties
- \(S\) submanifold of \((M, \{\cdot, \cdot\})\). \(D_S := D \cap TS\)
- \((M, \{\cdot, \cdot\})\) Poisson manifold, \(D \subset TM\). \(D\) is Poisson if
  \[
  df|_D = dg|_D = 0 \Rightarrow d\{f, g\}|_D = 0
  \]
- When does the bracket on \(M\) induce a bracket on \(S/D_S\)?
• The functions $C^\infty_{S/D_S}$ are characterized by the following property: $f \in C^\infty_{S/D_S}(V)$ if and only if for any $z \in V$ there exists $m \in \pi_{D_S}^{-1}(V)$, $U_m$ open neighborhood of $m$ in $M$, and $F \in C^\infty_M(U_m)$ such that
  \[
  f \circ \pi_{D_S} \big|_{\pi_{D_S}^{-1}(V) \cap U_m} = F \big|_{\pi_{D_S}^{-1}(V) \cap U_m}.
  \]
  $F$ is a \textbf{local extension} of $f \circ \pi_{D_S}$ at the point $m \in \pi_{D_S}^{-1}(V)$.

• $C^\infty_{S/D_S}$ has the $(D, D_S)$–\textbf{local extension property} when the local extensions of $f \circ \pi_{D_S}$ can always be chosen to satisfy
  \[
  dF(n)|_{D(n)} = 0.
  \]

• $(M, \{\cdot, \cdot\}, D, S)$ is \textbf{Poisson reducible} when $(S/D_S, C^\infty_{S/D_S}, \{\cdot, \cdot\}^{S/D_S})$ is a well defined Poisson manifold with
  \[
  \{f, g\}_V^{S/D_S}(\pi_{D_S}(m)) := \{F, G\}(m),
  \]
  $F, G$ are local $D$–invariant extensions of $f \circ \pi_{D_S}$ and $g \circ \pi_{D_S}$.
Theorem (Marsden, Ratiu (1986)/ JPO, Ratiu (1998) (singular)).

\((M, \{\cdot, \cdot\}, D, S)\) is Poisson reducible if and only if

\[ B^\#(D^\circ) \subset TS + D. \]

Examples

Coisotropic submanifolds:

\[ B^\#((TS)^\circ) \subset TS \]

Dirac’s first class constraints (Bojowald, Strobl (2002)).

If \(S\) be an embedded coisotropic submanifold of \(M\) and \(D := B^\#((TS)^\circ)\) then \((M, \{\cdot, \cdot\}, D, S)\) is Poisson reducible.

Appear in the context of integrable systems as the level sets of integrals in involution.
Cosymplectic manifolds and Dirac’s formula

An embedded submanifold $S \subset M$ is called cosymplectic when

(i) $B^\#((TS)^\circ) \cap TS = \{0\}$.

(ii) $T_s S + T_s \mathcal{L}_s = T_s M$,

for any $s \in S$ and $\mathcal{L}_s$ the symplectic leaf of $(M, \{\cdot, \cdot\})$ containing $s \in S$. The cosymplectic submanifolds of a symplectic manifold $(M, \omega)$ are its symplectic submanifolds (a.k.a. second class constraints). In this case

$$TM|_S = B^\#((TS)^\circ) \oplus TS$$

**Theorem** (Weinstein (1983)) $S$ cosymplectic. Let $D := B^\#((TS)^\circ) \subset TM|_S$. Then

(i) $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible.

(ii) The corresponding quotient manifold equals $S$ and the reduced bracket $\{\cdot, \cdot\}_S$ is given by

$$\{f, g\}_S(s) = \{F, G\}(s),$$

$F, G \in C_M^\infty(U)$ are local $D$–invariant extensions of $f$ and $g$. 

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(iii) The Hamiltonian vector field $X_f$ of an arbitrary function $f \in C^\infty_{S,M}(V)$ can be written as

$$Ti \cdot X_f = \pi_S \circ X_F \circ i,$$

where $F \in C^\infty_M(U)$ is an arbitrary local extension of $f$ and $\pi_S : TM|_S \rightarrow TS$ is the projection induced by the Whitney sum decomposition $TM|_S = B^\#((TS)^\circ) \oplus TS$ of $TM|_S$.

(v) The symplectic leaves of $(S, \{\cdot, \cdot\}_S)$ are the connected components of the intersections $S \cap \mathcal{L}$, with $\mathcal{L}$ a symplectic leaf of $(M, \{\cdot, \cdot\})$. Any symplectic leaf of $(S, \{\cdot, \cdot\}_S)$ is a symplectic submanifold of the symplectic leaf of $(M, \{\cdot, \cdot\})$ that contains it.

(vi) Let $\mathcal{L}_s$ and $\mathcal{L}_s^S$ be the symplectic leaves of $(M, \{\cdot, \cdot\})$ and $(S, \{\cdot, \cdot\}_S)$, respectively, that contain the point $s \in S$. Let $\omega_{\mathcal{L}_s}$ and $\omega_{\mathcal{L}_s^S}$ be the corresponding symplectic forms. Then $B^\#(s)((T_sS)^\circ)$ is a symplectic subspace of $T_s \mathcal{L}_s$.
and

\[ B \mathring{\#}(s)((T_s S) \circ) = \left( T_s \mathcal{L}_S^S \right)^{\omega \mathcal{L}_S(s)}. \tag{2} \]

(vii) Let \( B_S \in \Lambda^2(T^*S) \) be the Poisson tensor associated to \((S, \{\cdot, \cdot\}_S)\). Then

\[ B \mathring{\#} = \pi_S \circ B \mathring{\#}|_S \circ \pi^*_S, \tag{3} \]

where \( \pi^*_S : T^*S \rightarrow T^*M|_S \) is the dual of \( \pi_S : TM|_S \rightarrow TS \).

Formula (3) gives in local coordinates Dirac’s formula:

\[
\{f, g\}^S(s) = \{F, G\}(s) \\
- \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s) C_{ij}(s) \{\psi^j, G\}(s)
\]
The momentum map

- \((M, \omega)\) symplectic manifold, \(G\) acting canonically
- Momentum map \(J : M \rightarrow g^*\)

\[ J^\xi := \langle J, \xi \rangle, \quad i_{\xi_M} \omega = dJ^\xi \]

with

\[ \xi_M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp t\xi \cdot m \]

- Noether’s Theorem: the fibers of \(J\) are preserved by the Hamiltonian flows associated to \(G\)-invariant Hamiltonians.

Example: linear momentum. Take the phase space of the \(N\)-particle system, that is, \(T^*\mathbb{R}^{3N}\). The additive group \(\mathbb{R}^3\) acts on it by

\[ \mathbf{v} \cdot (q_i, p^i) = (q_i + \mathbf{v}, p^i) \]

\[ J : T^*\mathbb{R}^{3N} \rightarrow \text{Lie}(\mathbb{R}^3) \cong \mathbb{R}^3 \]

\( (q_i, p^i) \mapsto \sum_{i=1}^{N} p_i \).
Example: angular momentum. Let $\text{SO}(3)$ act on $\mathbb{R}^3$ and then, by lift, on $T^*\mathbb{R}^3$, that is, $A \cdot (q, p) = (Aq, Ap)$.

$$J: T^*\mathbb{R}^3 \longrightarrow \mathfrak{so}(\mathbb{R}^3)^* \cong \mathbb{R}^3$$

$$(q, p) \longmapsto q \times p.$$ which is the classical angular momentum.

Example: lifted actions on cotangent bundles. Let $G$ be a Lie group acting on the manifold $Q$ and then by lift on its cotangent bundle $T^*Q$.

$$\langle J(\alpha q), \xi \rangle = \langle \alpha q, \xi_Q(q) \rangle,$$

for any $\alpha q \in T^*Q$ and any $\xi \in \mathfrak{g}$.

Example: symplectic linear actions. Let $(V, \omega)$ be a symplectic linear space and let $G$ be a subgroup of the linear symplectic group, acting naturally on $V$.

$$\langle J(v), \xi \rangle = \frac{1}{2} \omega(\xi_V(v), v).$$
Properties of the momentum map

• Regularity of the action is equivalent to the regularity of the momentum map
  \[ \text{range } T_m J = (g_m)^0 \]
• \( \ker T_m J = (g \cdot m) \omega \).
• Existence:
  \[ \rho : g/[g, g] \longrightarrow H^1(M, \mathbb{R}) \]
  \[ [\xi] \longrightarrow [i_{\xi M} \omega] \]
• Equivariance: When \((g, [\cdot, \cdot]) \rightarrow (C^\infty(M), \{\cdot, \cdot\})\) defined by \(\xi \mapsto J^\xi\), \(\xi \in g\), is a Lie algebra homomorphism, that is,
  \[ J[\xi, \eta] = \{J^\xi, J^\eta\}, \quad \xi, \eta \in g. \]
  Answer: iff
  \[ T_z J \cdot \xi_M(z) = - \text{ad}^*_\xi J(z), \]
  A momentum map that satisfies this relation in called \textit{infinitesimally equivariant}.
• \( J \) is \textit{G-equivariant} when
  \[ \text{Ad}^*_{g^{-1}} \circ J = J \circ \Phi_g, \]
• If \( G \) is compact \( J \) can be chosen \( G \)-equivariant
Define the non equivalence one–cocycle associated to \( J \) as the map
\[
\sigma : G \longrightarrow g^* \\
g \longmapsto J(\Phi_g(z)) - \text{Ad}_{g^{-1}}^*(J(z)).
\]

Then:

(i) The definition of \( \sigma \) does not depend on the choice of \( z \in M \);

(ii) The mapping \( \sigma \) is a \( g^* \)–valued one–cocycle on \( G \) with respect to the coadjoint representation of \( G \) on \( g^* \).

We define the affine action of \( G \) on \( g^* \) with cocycle \( \sigma \) by
\[
\Theta : G \times g^* \longrightarrow g^* \\
(g, \mu) \longmapsto \text{Ad}_{g^{-1}}^* \mu + \sigma(g).
\]

\( \Theta \) determines a left action of \( G \) on \( g^* \). The momentum map \( J : M \rightarrow g^* \) is equivariant with respect to the symplectic action \( \Phi \) on \( M \) and the affine action \( \Psi \) on \( g^* \).
The affine orbits $O_\mu$ are also symplectic with $G$-invariant symplectic structure given by
\[
\omega_{\pm}^O_\mu(\nu)(\xi_{g^*(\nu)}, \eta_{g^*(\nu)}) = \pm \langle \nu, [\xi, \eta] \rangle \mp \Sigma(\xi, \eta),
\]
where the infinitesimal non equivariance cocycle $\Sigma \in Z^2(g, \mathbb{R})$ is given by
\[
\Sigma : g \times g \longrightarrow \mathbb{R} \\
(\xi, \eta) \longmapsto \Sigma(\xi, \eta) = d\widehat{\sigma}_\eta(e) \cdot \xi,
\]
with $\widehat{\sigma}_\eta : G \rightarrow \mathbb{R}$ defined by $\widehat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

**Reduction Lemma:**
\[
g_\mu \cdot m = g \cdot m \cap \ker T_mJ = g \cdot m \cap (g \cdot m)^\omega.
\]
Momentum maps and isotropy type manifolds The free, proper, and canonical action of 
\( L^m := N(G_m)^m / G_m \) on \( M^m_{G_m} \) has a momentum map \( J_{L^m} : M^m_{G_m} \to (\text{Lie}(L^m))^\ast \) given by

\[
J_{L^m}(z) := \Lambda(J|_{M^m_{G_m}}(z) - \mu), \quad z \in M^m_{G_m}.
\]

In this expression \( \Lambda : (\mathfrak{g}_m^0)^G_m \to (\text{Lie}(L^m))^\ast \) denotes the natural \( L^m \)-equivariant isomorphism given by

\[
\left\langle \Lambda(\beta), \frac{d}{dt} \bigg|_{t=0} \exp t\xi G_m \right\rangle = \langle \beta, \xi \rangle,
\]

for any \( \beta \in (\mathfrak{g}_m^0)^G_m, \xi \in \text{Lie}(N(G_m)^m) = \text{Lie}(N(G_m)) \).

The non equivariance one–cocycle \( \tau : M^m_{G_m} \to (\text{Lie}(L^m))^\ast \) of the momentum map \( J_{L^m} \) is given by the map

\[
\tau(l) = \Lambda(\sigma(n) + n \cdot \mu - \mu).
\]
Convexity

\( J : M \to g^* \) coadjoint equivariant. \( G, M \) compact. The intersection of the image of \( J \) with a Weyl chamber is a \textit{compact and convex polytope}. This polytope is referred to as the \textit{momentum polytope}.

Delzant’s theorem proves that the symplectic toric manifolds are classified by their momentum polytopes. A \textit{Delzant polytope} in \( \mathbb{R}^n \) is a convex polytope that is also:

(i) \textbf{Simple}: there are \( n \) edges meeting at each vertex.

(ii) \textbf{Rational}: the edges meeting at a vertex \( p \) are of the form \( p + t u_i, 0 \leq t < \infty, u_i \in \mathbb{Z}^n, i \in \{1, \ldots, n\} \).

(iii) \textbf{Smooth}: the vectors \( \{u_1, \ldots, u_n\} \) can be chosen to be an integral basis of \( \mathbb{Z}^n \).

Delzant’s Theorem can be stated by saying that

\[
\{\text{symplectic toric manifolds}\} \longrightarrow \{\text{Delzant polytopes} (M, \omega, \mathbb{T}^n, J : M \to \mathbb{R}^n) \} \longleftrightarrow \ J(M)
\]

is a bijection.
The cylinder valued momentum map


- $M \times \mathfrak{g}^* \longrightarrow M$, $M$ connected

- $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$

\[
\langle \alpha(m, \mu) \cdot (v_m, \nu), \xi \rangle := (i_{\xi_M} \omega)(m) \cdot v_m - \langle \nu, \xi \rangle
\]

- $\alpha$ has zero curvature $\Rightarrow \mathcal{H}$ discrete

- $\tilde{M}$ holonomy bundle $\Leftrightarrow$ horizontal leaf

\[
\begin{array}{ccc}
\tilde{M} & \overset{\tilde{K}}{\longrightarrow} & \mathfrak{g}^* \\
\tilde{p} | & | & | \\
M & \overset{K}{\longrightarrow} & \mathfrak{g}^*/\tilde{\mathcal{H}} \end{array}
\]

- Standard momentum map exists $\Leftrightarrow \mathcal{H} = \{0\}$

- $K$ always exists and it is a smooth momentum

- $\ker(T_m K) = \left( (\text{Lie}(\mathcal{H}))^\circ \cdot m \right)^\omega$

- $\text{range } (T_m K) = T_\mu \pi_C ((\mathfrak{g}_m)^\circ)$
Equivariance

\( \mathbf{K} : M \to g^*/\mathcal{H} \) cylinder valued momentum map associated to a \( G \)-action on \((M, \omega)\)

- \( \mathcal{H} \) is \( \text{Ad}^* \)-invariant: \( \text{Ad}_{g^{-1}}^*(\mathcal{H}) \subset \mathcal{H}, \ g \in G \).
  - If \( G \) is connected, \( \mathcal{H} \) is pointwise fixed.

- There exists a unique action \( \text{Ad}^* : G \times g^*/\mathcal{H} \to g^*/\mathcal{H} \)
  such that for any \( g \in G \)
  \[ \text{Ad}^*_{g^{-1}} \circ \pi_C = \pi_C \circ \text{Ad}^*_g \]

Define

\[ \sigma(g, m) := \mathbf{K}(g \cdot m) - \text{Ad}^*_{g^{-1}} \mathbf{K}(m) \]

- If \( M \) is connected \( \sigma : G \times M \to g^*/\mathcal{H} \) does not depend on \( M \).

- \( \sigma : G \to g^*/\mathcal{H} \) is a group-valued one-cocycle, that is
  \[ \sigma(gh) = \sigma(g) + \text{Ad}^*_{g^{-1}} \sigma(h) \]
The map
\[ \Theta : G \times g^*/H \to g^*/H \]
\[ (g, \mu + H) \mapsto \text{Ad}_{g^{-1}}^*(\mu + H) + \sigma(g) \]
is a group action such that
\[ K(g \cdot m) = \Theta_g(K(m)) \]

**Reduction Lemma**

\[ g_{\mu + \overline{H}} \cdot m = \ker T_m K \cap g \cdot m \]

**Corollary**: if \( \mathcal{H} \) is closed then
\[ g_{\mu + \overline{H}} \cdot m = (g \cdot m)^{\omega} \cap g \cdot m \]
Cylinder and Lie group valued momentum maps

McDuff, Ginzburg, Huebschmann, Jeffrey, Huebschmann, Alekseev, Malkin, and Meinreken (1998) \((\cdot, \cdot)\) bilinear symmetric non degenerate form on \(\mathfrak{g}\). \(J : M \to G\) is a \(G\)-valued momentum map for the \(\mathfrak{g}\)-action on \(M\) whenever

\[
i_{\xi_M} \omega(m) \cdot v_m = \left( T_m(L_{J(m)}^{-1} \circ J)(v_m), \xi \right)
\]

Any cylinder valued momentum map associated to an Abelian Lie algebra action whose corresponding holonomy group is closed can be understood as a Lie group valued momentum map.

**Proposition** \(f : \mathfrak{g} \to \mathfrak{g}^*\) isomorphism given by \(\xi \longmapsto (\xi, \cdot)\), \(\xi \in \mathfrak{g}\) and \(\mathcal{T} := f^{-1}(\mathcal{H})\). \(f\) induces an Abelian group isomorphism \(\tilde{f} : \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}\) by \(\tilde{f}(\xi + \mathcal{T}) := (\xi, \cdot) + \mathcal{H}\). Suppose that \(\mathcal{H}\) is closed in \(\mathfrak{g}^*\) and define \(J := \tilde{f}^{-1} \circ K : M \to \mathfrak{g}/\mathcal{T}\), where \(K\) is a cylinder valued momentum map for the \(\mathfrak{g}\)-action. Then \(J : M \to \mathfrak{g}/\mathcal{T}\) is a \(\mathfrak{g}/\mathcal{T}\)-valued momentum map for the action of the Lie algebra \(\mathfrak{g}\) of \((\mathfrak{g}/\mathcal{T}, +)\) on \((M, \omega)\).
Lie group valued momentum maps produce closed holonomy groups

**Theorem** $\mathcal{H} \subset g^*$ holonomy group associated to the $g$–action. $f : g \to g^*$, $\bar{f} : g/\mathcal{T} \to g^*/\mathcal{H}$, and $\mathcal{T} := f^{-1}(\mathcal{H})$ as before. Let $G$ be a connected Abelian Lie group whose Lie algebra is $g$ and suppose that there exists a $G$–valued momentum map $A : M \to G$ associated to the $g$–action whose definition uses the form $(\cdot, \cdot)$.

(i) If $\exp : g \to G$ is the exponential map, then

$$\mathcal{H} \subset f(\ker \exp).$$

(ii) $\mathcal{H}$ is closed in $g^*$.

Let $J := \bar{f}^{-1} \circ K : M \to g/\mathcal{T}$, where $K : M \to g^*/\mathcal{H}$ is a cylinder valued momentum map for the $g$–action on $(M, \omega)$. If $f(\ker \exp) \subset \mathcal{H}$ then $J : M \to g/\mathcal{T} = g/\ker \exp \simeq G$ is a $G$–valued momentum map that differs from $A$ by a constant in $G$.

Conversely, if $\mathcal{H} = f(\ker \exp)$ then $J : M \to g/\ker \exp \simeq G$ is a $G$–valued momentum map.
The optimal momentum map

Problems with the traditional momentum map:

- Possible non existence of $\mathbf{J}$:
  
  1. $S^1$ acting on $\mathbb{T}^2$ by
     \[ e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\phi+\theta_1)}, e^{i\theta_2}). \]
     Lie group valued momentum maps. Dirac [1926], McDuff [1988], Alekseev et al. [1997].
  
  2. $(\mathbb{R}^3, \{\cdot, \cdot\})$ with Poisson tensor
     \[ B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \]
     $(\mathbb{R}, +)$ acts on $\mathbb{R}^3$ by $\lambda \cdot (x, y, z) := (x + \lambda, y, z)$. NO MOMENTUM MAP!!

- Singular case not optimal (finite groups). Does not see law of conservation of isotropy.
  \[ \mathbf{J}^{-1}(\mu) \quad \text{versus} \quad \mathbf{J}^{-1}(\mu) \cap M_H \]
• JPO, Ratiu [2002]

• $G$ acts on $(M, \{\cdot, \cdot\})$ via $\Phi : G \times M \to M$.

• $A_G := \{\Phi_g : M \to M \mid g \in G\} \subset \mathcal{P}(M)$.

• $A'_G := \{X_f(m) \mid f \in C^\infty(M)^G\}$.

• The canonical projection

$$\mathcal{J} : M \to M/A'_G$$

is the **optimal momentum map** associated to the $G$-action on $M$. 
• \( \mathcal{J} \) always defined:

1. \( S^1 \) on \( \mathbb{T}^2 \)
   \[
   \mathcal{J} : \mathbb{T}^2 \rightarrow S^1 \\
   (e^{i\theta_1}, e^{i\theta_2}) \mapsto e^{i\theta_2}.
   \]

2. \( \mathbb{R} \) on \( \mathbb{R}^3 \)
   \[
   \mathcal{J} : \mathbb{R}^3 \rightarrow \mathbb{R} \\
   (x, y, z) \mapsto x + z. \quad \blacklozenge
   \]

• Why momentum map?

**Noether’s Theorem:** \( \mathcal{J} \) is universal. Let \( F_t \) flow of \( X_h, h \in C^\infty(M)^G \) then
\[
\mathcal{J} \circ F_t = \mathcal{J}
\]

• Why optimal?

**Theorem:** \( G \) acting properly on \( (M, \omega) \) with associated momentum map \( \mathbf{J} : M \rightarrow g^* \).
Then:
\[
\mathcal{A}_G'(m) = \ker T_m \mathbf{J} \cap T_m M_{G_m}.
\]

Hence, the level sets of \( \mathcal{J} \) are
\[
\mathcal{J}^{-1}(\rho) = \mathbf{J}^{-1}(\mu) \cap M_H,
\]
Symplectic reduction

• Marsden, Weinstein (1974): free proper action implies $J^{-1}(\mu)/G\mu$ “canonically” symplectic
  
  $\pi^*_\mu \omega_\mu = i^*_\mu \omega$

  where $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G\mu$ and $i_\mu : J^{-1}(\mu) \hookrightarrow M$

• Reduction of dynamics: $h \in C^\infty(M)^G$. The flow $F_t$ of $X_h$ leaves $J^{-1}(\mu)$ invariant and commutes with the $G$–action, so it induces a flow $F^\mu_t$ on $M_\mu$ defined by
  
  $\pi_\mu \circ F_t \circ i_\mu = F^\mu_t \circ \pi_\mu$

  The flow $F^\mu_t$ on $(M_\mu, \omega_\mu)$ is Hamiltonian with associated reduced Hamiltonian function $h_\mu \in C^\infty(M_\mu)$ defined by
  
  $h_\mu \circ \pi_\mu = h \circ i_\mu$

  The triple $(M_\mu, \omega_\mu, h_\mu)$ is called the reduced Hamiltonian system.

• Reduces the search of relative equilibria and relative periodic orbits to equilibria and periodic orbits.
Reconstruction of dynamics: Assume that an integral curve $c_\mu(t)$ of the reduced Hamiltonian system $X_{h_\mu}$ on $(M_\mu, \omega_\mu)$ is known. Let $m_0 \in J^{-1}(\mu)$ be given. Can one determine from this data the integral curve of the Hamiltonian system $X_h$ with initial condition $m_0$? In other words, can one reconstruct the solution of the given system knowing the corresponding reduced solution?

Pick a smooth curve $d(t)$ in $J^{-1}(\mu)$ such that $d(0) = m_0$ and $\pi_\mu(d(t)) = c_\mu(t)$. Then, if $c(t)$ denotes the integral curve of $X_h$ with $c(0) = m_0$, we can write $c(t) = g(t) \cdot d(t)$ for some smooth curve $g(t)$ in $G_\mu$ determined in two steps:

- **Step 1:** find a smooth curve $\xi(t)$ in $g_\mu$
  \[ \xi(t)_M(d(t)) = X_h(d(t)) - \dot{d}(t); \]

- **Step 2:** with $\xi(t) \in g_\mu$ determined above, solve the non–autonomous differential equation on $G_\mu$
  \[ \dot{g}(t) = T_{eL}g(t)\xi(t), \quad \text{with} \quad g(0) = e. \]
Coadjoint orbits as reduced spaces. Take $M = T^*G$, where $G$ is a Lie group with Lie algebra $\mathfrak{g}$, the $G$–action being the cotangent lift of left translation, and the associated momentum map $J_L : \alpha_g \in T^*G \mapsto T^*_e R_g(\alpha_g) \in \mathfrak{g}^*$ which is right invariant. For each $\mu \in \mathfrak{g}^*$ we can form the symplectic point reduced space $((T^*G)_\mu, \omega_\mu)$. Recall also that the momentum map for the lift of right translations is left invariant and is given by $J_R : \alpha_g \in T^*G \mapsto T^*_e L_g(\alpha_g) \in \mathfrak{g}^*$.

The momentum map $J_R : T^*G \to \mathfrak{g}^*$ induces for each $\mu \in \mathfrak{g}^*$ a symplectic diffeomorphism $\bar{J}_R : ((T^*G)_\mu, \omega_\mu) \to (\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu})$ given by $\bar{J}_R([T^*_g R_{g^{-1}}\mu]) = \text{Ad}^*_g \mu$.

Cotangent bundles. $G$ acts on $Q$ freely and properly. The map

$$\varphi_0 : ((T^*Q)_0, (\Omega_Q)_0) \to \left(T^*(Q/G), \Omega_{Q/G}\right)$$

given by $\varphi_0([\alpha_q])(T_q \rho(v_q)) := \alpha_q(v_q)$, with $\alpha_q \in J^{-1}(0)$, $v_q \in T_q Q$, is a symplectomorphism.
We now study the symplectic reduced space

$$((T^*Q)_\mu, \omega_\mu).$$

Let $$\mu' := \mu|_{\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$$ the restriction of $$\mu$$ to $$\mathfrak{g}_\mu,$$ and consider the $$G_\mu$$–action on $$Q$$ and its lift to $$T^*Q.$$ An equivariant momentum map of this action is the map $$J^\mu : T^*Q \to \mathfrak{g}_\mu$$ obtained by restricting $$J.$$ Assume there is a $$G_\mu$$–invariant one-form $$\alpha_\mu$$ on $$Q$$ with values in $$(J^\mu)^{-1}(\mu').$$

For $$\xi \in \mathfrak{g}_\mu$$ and $$q \in Q,$$ the identity $$(\text{i}_{\xi_Q} \alpha_\mu)(q) = \alpha_\mu(q)(\xi_Q(q)) = \langle J(\alpha_\mu(q)), \xi \rangle = \langle \mu', \xi \rangle$$ shows that $$\text{i}_{\xi_Q} \alpha_\mu$$ is a constant function on $$Q.$$ Therefore, for $$\xi \in \mathfrak{g}_\mu,$$ this implies $$\text{i}_{\xi_Q} d\alpha_\mu = \mathcal{L}_{\xi_Q} \alpha_\mu - \text{di}_{\xi_Q} \alpha_\mu = 0,$$ since $$\mathcal{L}_{\xi_Q} \alpha_\mu = 0$$ by $$G_\mu$$-invariance of $$\alpha_\mu.$$ It follows that there is a unique two-form $$\beta_\mu$$ on $$Q_\mu$$ such that $$\rho_\mu^* \beta_\mu = d\alpha_\mu.$$ Since $$\rho_\mu$$ is a submersion, $$\beta_\mu$$ is closed, but need not be exact. Let $$B_\mu = \pi_{Q_\mu}^* \beta_\mu$$ where $$\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$$ is the cotangent bundle projection.
Embedding cotangent bundle reduction theorem

Under the above hypotheses, the map

\[ \varphi_\mu : ((T^*Q)_\mu, (\Omega_Q)_\mu) \to (T^*Q_\mu, \Omega_{Q_\mu} - B_\mu), \]

given by \( \varphi_\mu([\alpha_q])(T_q\rho_\mu(v_q)) := (\alpha_q - \alpha_\mu(q))(v_q), \)
for \( \alpha_q \in J^{-1}(\mu), v_q \in T_qQ, \) is a symplectic embedding onto a vector subbundle of \( T^*Q_\mu. \) The map \( \varphi_\mu \) is onto \( T^*Q_\mu \) if and only if \( g = g_\mu. \) The additional summand \( B_\mu \) in the symplectic structure of \( T^*Q_\mu \) is called a magnetic term.
Symplectic orbit reduction

(i) The set $M_{\mathcal{O}_\mu} := J^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i_{\mathcal{O}_\mu}^* \omega = \pi_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu} + J_{\mathcal{O}_\mu}^* \omega_\mathcal{O}_\mu^+,$$

where $J_{\mathcal{O}_\mu}$ is the restriction of $J$ to $J^{-1}(\mathcal{O}_\mu)$ and $\omega_\mathcal{O}_\mu^+$ is the $+$–symplectic structure on the affine orbit $\mathcal{O}_\mu$. The maps $i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \hookrightarrow M$ and $\pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \rightarrow M_{\mathcal{O}_\mu}$ are natural injection and the projection, respectively. The pair $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the symplectic orbit reduced space.

(ii) These are, up to connected components, the symplectic leaves of $(M/G, \{\cdot, \cdot\}_{M/G})$.

(iii) Same dynamical statements that we have for the point reduced spaces.
Cylinder Valued Regular Reduction Theorem: $G$ acts freely and properly. If $\mathcal{H}$ is closed then $K^{-1}([\mu])/G[\mu]$ is symplectic with form given by

$$\pi^*[\mu]\omega[\mu] = i^*_\mu\omega$$

If $\mathcal{H}$ is not closed the theorem is false in general. See presentation of Ratiu in the conference for the general case.
Optimal reduction

\((M, \{\cdot, \cdot\})\) Poisson manifold, \(G\) acts properly and canonically on \(M\). Then, for any \(\rho \in M/A'_G\),

- \(\mathcal{J}^{-1}(\rho)\) is an initial submanifold of \(M\).

- The isotropy subgroup \(G_\rho \subset G\) of \(\rho\) is an (immersed) Lie subgroup of \(G\).

- \(T_m(G_\rho \cdot m) = T_m(\mathcal{J}^{-1}(\rho)) \cap T_m(G \cdot m)\).

- If \(G_\rho\) acts properly on \(\mathcal{J}^{-1}(\rho)\) then \(M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho\) is a regular quotient manifold called the **reduced phase space**.

- The canonical projection

  \[\pi_\rho : \mathcal{J}^{-1}(\rho) \rightarrow \mathcal{J}^{-1}(\rho)/G_\rho\]

  is a submersion.

- **Optimal Symplectic Reduction:** \(M_\rho\) is symplectic with \(\omega_\rho\) given by

  \[\pi^*_\rho \omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m),\]

  for any \(m \in \mathcal{J}^{-1}(\rho)\) and \(f, h \in C^\infty(M)^G\).
THE GLOBALLY HAMILTONIAN CASE

\[ M_\rho = J^{-1}(\mu) \cap M_H/N_{G_\mu}(H) \]
\[ = J^{-1}(\mu) \cap M_H/(N_{G_\mu}(H)/H) \]
\[ \simeq (J^{-1}(\mu) \cap M_{(H)}^{G_\mu})/G_\mu = M^{(H)}_\mu \]

These are Sjamaar and Lerman [1991] reduced spaces.

Singular reduction

• There is a unique symplectic structure $\omega^{(H)}_{\mu}$ on $M^{(H)}_{\mu} := [J^{-1}(\mu) \cap G_{\mu}M_{H}^z]/G_{\mu}$ characterized by

$$i_{\mu}^{(H)} * \omega = \pi_{\mu}^{(H)} * \omega_{\mu}$$

• The symplectic spaces $M^{(H)}_{\mu}$ stratify $J^{-1}(\mu)/G_{\mu}$

• Sjamaar’s principle and regularization

$$[J^{-1}(\mu) \cap G_{\mu}M_{H}^z]/G_{\mu} \simeq J^{-1}_{L^z}(0)/L^z_0$$

• For the record

$$J^{-1}_{L^z}(0)/L^z_0 = [J^{-1}(\mu) \cap M_{H}^z]/(NG_{\mu}(H)^z/H)$$
Stratification in what sense?

• Decomposed space: \( R, S \in \mathcal{Z} \) with \( R \cap \bar{S} \neq \emptyset \), then \( R \subset \bar{S} \)

• Stratification: decomposed space with a condition on the set germ of the pieces

• Stratification with smooth structure: there are charts \( \phi : U \to \phi(U) \subset \mathbb{R}^n \) from an open set \( U \subset P \) to a subset of \( \mathbb{R}^n \) such that for every stratum \( S \in \mathcal{S} \) the image \( \phi(U \cap S) \) is a submanifold of \( \mathbb{R}^n \) and the restriction \( \phi|_{U \cap S} : U \cap S \to \phi(U \cap S) \) is a diffeomorphism.

• Whitney stratifications

• Cone space: existence of links

\[ \psi : U \to (S \cap U) \times CL, \]
Where do the charts come from?

- Hamiltonian $G$–manifold $(M, \omega, G, J : M \to g^*)$ can be locally identified with

$$(Y_r := G \times_{G_m} (m_r^* \times V_r), \omega_{Y_r})$$

- The momentum map takes the expression

$$J([g, \rho, v]) = \text{Ad}_{g^{-1}}^*(J(m) + \rho + J_V(v)) + \sigma(g)$$

- Main observation

$$J^{-1}(\mu)/G_\mu \simeq J_{V_r}^{-1}(0)/G_{m}$$

- The symplectic strata are locally described by the strata obtained (roughly speaking) from the stratification by orbit types of $J_{V_r}^{-1}(0)$ as a $G_{m}$ space

The reconstruction equations

\[ X_q = X_h = 0 \]
\[ X_m(g, \rho, v) = T_e L_g(D m^*_r(h \circ \pi)(\rho, v)) \]
\[ X_{V_r} = B^\#_V(D_{V_r}(h \circ \pi)(\rho, v)) \]
\[ X_{m^*_r} = \mathbb{P} m^* \left( \text{ad}^*_D m^*_r(h \circ \pi)\rho \right) + \text{ad}^*_D m^*_r(h \circ \pi) J_V(v). \]
Hamiltonian Coverings

g acting symplectically on \((M, \omega)\). \(p_N : N \to M\) is a Hamiltonian covering map of \((M, \omega)\):

(i) \(p_N\) is a smooth covering map

(ii) \((N, \omega_N)\) is a connected symplectic manifold

(iii) \(p_N\) is a symplectic map

(iv) \(g\) acts symplectically on \((N, \omega_N)\) and has a standard momentum map \(K_N : N \to g^*\)

(v) \(p_N\) is \(g\)-equivariant, that is, \(\xi_M(p_N(n)) = T_n p_N \cdot \xi_N(n)\), for any \(n \in N\) and any \(\xi \in g\).
The category of Hamiltonian covering maps

\( \mathfrak{g} \) Lie algebra acting symplectically on \((M, \omega)\).

- \( \text{Ob}(\mathcal{H}) = \{(p_N : N \to M, \omega_N, \mathfrak{g}, [K_N])\} \) with \( p_N : N \to M \) a Hamiltonian covering map of \((M, \omega)\)

- \( \text{Mor}(\mathcal{H}) = \{q : (N_1, \omega_1) \to (N_2, \omega_2)\} \) with:
  1. \( q \) is a symplectic covering map
  2. \( q \) is \( \mathfrak{g} \)-equivariant
  3. the diagram

\[ \begin{array}{ccc}
\mathfrak{g}^* & \downarrow & \mathfrak{g}^* \\
\downarrow & \mathcal{K}_{N_1} & \downarrow \mathcal{K}_{N_2} \\
N_1 & \downarrow q & \downarrow N_2 \\
\downarrow p_{N_1} & \downarrow & \downarrow p_{N_2} \\
M & \end{array} \]

commutes for some \( \mathcal{K}_{N_1} \in [\mathcal{K}_{N_1}] \) and \( \mathcal{K}_{N_2} \in [\mathcal{K}_{N_2}] \).
**Proposition** Let \((M, \omega)\) be a connected symplectic manifold and \(g\) be a Lie algebra acting symplectically on it. Let \((\hat{p} : \hat{M} \rightarrow M, \omega_{\hat{M}}, g, [K])\) be the object in \(\mathcal{H}\) constructed using the universal covering of \(M\).

For any other object \((p_N : N \rightarrow M, \omega_N, g, [K_N])\) of \(\mathcal{H}\), there exists a morphism \(q : \hat{M} \rightarrow N\) in \(\text{Mor}(\mathcal{H})\).

Any other object in \(\mathcal{H}\) that satisfies the same universality property is isomorphic to \((\hat{p} : \hat{M} \rightarrow M, \omega_{\hat{M}}, g, [K])\).
The holonomy bundles of $\alpha$ are Hamiltonian coverings of $(M, \omega, g)$.

**Proposition** The pair $(\widetilde{M}, \omega_{\widetilde{M}} := \tilde{p}^*\omega)$ is a symplectic manifold on which $g$ acts symplectically by

$$\xi_{\widetilde{M}}(m, \mu) := (\xi_M(m), -\Psi(m)(\xi, \cdot)),$$

where $\Psi : M \to Z^2(g)$ is the Chu map. The projection $\tilde{K} : \widetilde{M} \to g^*$ of $\widetilde{M}$ into $g^*$ is a momentum map for this action. The 4–tuple $(\tilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, g, [\tilde{K}])$ is an object in $\mathfrak{H}$

**Theorem** $(\tilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, g, [\tilde{K}])$ is a universal Hamiltonian covered space in $\mathfrak{H}$, that is, given any other object $(p_N : N \to M, \omega_N, g, [K_N])$ in $\mathfrak{H}$, there exists a (not necessarily unique) morphism $q : N \to \tilde{M}$ in $\text{Mor}(\mathfrak{H})$. Any other object of $\mathfrak{H}$ that satisfies this universality property is isomorphic to $(\tilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, g, [\tilde{K}])$. 

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Reduction using the cylinder valued momentum map

First ingredient: a "coadjoint" action

\[ K : M \to \mathfrak{g}^*/\mathcal{H} \] cylinder valued momentum map associated to a \( G \)-action on \((M, \omega)\)

- \( \mathcal{H} \) is \( \text{Ad}^* \)-invariant: \( \text{Ad}^*_{g^{-1}}(\mathcal{H}) \subset \mathcal{H}, \ g \in G \)

- There exists a unique action

\[ \text{Ad}^* : G \times \mathfrak{g}^*/\mathcal{H} \to \mathfrak{g}^*/\mathcal{H} \]

such that for any \( g \in G \)

\[ \text{Ad}^*_{g^{-1}} \circ \pi_C = \pi_C \circ \text{Ad}^*_{g^{-1}} \]

Second ingredient: a non-equivariance cocycle

Define

\[ \sigma(g, m) := K(g \cdot m) - \text{Ad}^*_{g^{-1}}K(m) \]

- If \( M \) is connected \( \sigma : G \times M \to \mathfrak{g}^*/\mathcal{H} \) does not depend on \( M \).

- \( \sigma : G \to \mathfrak{g}^*/\mathcal{H} \) is a group-valued one-cocycle, that is

\[ \sigma(gh) = \sigma(g) + \text{Ad}^*_{g^{-1}}\sigma(h) \]
The Reduction Theorem

The map
\[ \Theta : G \times g^*/\mathcal{H} \longrightarrow g^*/\mathcal{H} \]
\[ (g, \mu + \mathcal{H}) \longmapsto \text{Ad}^*_{g^{-1}}(\mu + \mathcal{H}) + \sigma(g) \]
is a group action such that
\[ K(g \cdot m) = \Theta_g(K(m)) \]

Reduction Lemma
\[ g_{\mu + \mathcal{H}} \cdot m = \ker T_mK \cap g \cdot m \]

Corollary: if \( \mathcal{H} \) is closed then
\[ g_{\mu + \mathcal{H}} \cdot m = (g \cdot m)^{\omega} \cap g \cdot m \]

Regular Reduction Theorem: \( G \) acts freely and properly. If \( \mathcal{H} \) is closed then \( K^{-1}([\mu])/G[\mu] \) is symplectic with form given by
\[ \pi^{*}_{[\mu]}\omega_{[\mu]} = \imath_{\mu}^{*}\omega \]

If \( \mathcal{H} \) is not closed the theorem is false in general
Stratification Theorem

Using the symplectic slice theorem the cylinder valued momentum map locally looks like

$$K(\phi[g, \rho, v]) = \Theta_g(K(m) + \pi_C(\rho + J_{V_m}(v)))$$

Reproduce the Bates-Lerman proposition in this setup

$$K^{-1}([\mu]) \cap Y_0 
\approx \{[g, 0, v] \in Y_0 \mid g \in G_{[\mu]}, v \in J_{V_m}^{-1}(0)\}$$

**Stratification Theorem** If $\mathcal{H}$ is closed then the quotient $K^{-1}([\mu])/G_{[\mu]}$ is a cone space with strata

$$[K^{-1}([\mu]) \cap G_{[\mu]}M_H]/G_{[\mu]} \simeq J^{-1}(\rho)/G_{\rho}$$

Sjamaar’s principle is missing
Groupoids

A groupoid $G \to X$ with base $X$ and total space $G$:

(i) $\alpha, \beta : G \to X$. $\alpha$ is the target map and $\beta$ is the source map. An element $g \in G$ is thought of as an arrow from $\beta(g)$ to $\alpha(g)$ in $X$.

(ii) The set of composable pairs is defined as:

$$G^{(2)} := \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\}.$$ 

There is a product map $m : G^{(2)} \to G$ that satisfies $\alpha(m(g, h)) = \alpha(g)$, $\beta(m(g, h)) = \beta(h)$, and $m(m(g, h), k) = m(g, m(h, k))$, for any $g, h, k \in G$.

(iii) An injection $\epsilon : X \to G$, the identity section, such that $\epsilon(\alpha(g))g = g = ge(\beta(g))$. In particular, $\alpha \circ \epsilon = \beta \circ \epsilon$ is the identity map on $X$.

(iv) An inversion map $i : G \to G$, $i(g) = g^{-1}$, $g \in G$, such that $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$. 


Examples

• **Group:** \( G \Rightarrow \{e\} \).

• **The action groupoid:**
  
  \( -\Phi : G \times M \rightarrow M \)
  
  \( -G \times M \Rightarrow M \)
  
  \* \( \alpha(g, m) := g \cdot m, \beta(g, m) := m \)
  
  \* \( \epsilon(m) := (e, m) \)
  
  \* \( m((g, h \cdot n), (h, n)) := (gh, n) \)
  
  \* \( (g, m)^{-1} := (g^{-1}, g \cdot m) \)

  - The orbits and isotropy subgroups of this groupoid coincide with those of the group action \( \Phi \).

• **The cotangent bundle of a Lie group.**

  \( -T^*G \simeq G \times \mathfrak{g}^* \)
  
  \( -T^*G \Rightarrow \mathfrak{g}^* \)
  
  \* \( \alpha(g, \mu) := \text{Ad}^*_{g^{-1}}\mu, \beta(g, \mu) := \mu \)
  
  \* \( \epsilon(\mu) = (e, \mu) \)
  
  \* \( m((g, \text{Ad}^*_{h^{-1}}\mu), (h, \mu)) = (gh, \mu) \)
  
  \* \( (g, \mu)^{-1} = (g^{-1}, \text{Ad}^*_{g^{-1}}\mu) \).
• The Baer groupoid $\mathcal{B}(G) \rightrightarrows \mathcal{S}(G)$.
  
  - $\mathcal{S}(G)$ set of subgroups of $G$
  - $\mathcal{B}(G)$ set of cosets of elements in $\mathcal{S}(G)$
    
    * $\alpha, \beta : \mathcal{B}(G) \to \mathcal{S}(G)$ are defined by $
      \alpha(D) = Dg^{-1}, \beta(D) = g^{-1}D$ for some $g \in D$.
    
    * $m(D_1, D_2) := D_1D_2$.
    
    * The orbits of $\mathcal{B}(G) \rightrightarrows \mathcal{S}(G)$ are given by 
      the conjugacy classes of subgroups of $G$. 

Groupoid Actions

\( J : M \to X \) a map from \( M \) into \( X \) and
\[ G \times_J M := \{ (g, m) \in G \times M \mid \beta(g) = J(m) \}. \]

A (left) groupoid action of \( G \) on \( M \) with moment map \( J : M \to X \) is a mapping
\[ \Psi : G \times_J M \to M \]
\[ (g, m) \mapsto g \cdot m := \Psi(g, m), \]
that satisfies the following properties:

(i) \( J(g \cdot m) = \alpha(g) \),

(ii) \( gh \cdot m = g \cdot (h \cdot m) \),

(iii) \( (\epsilon(J(m))) \cdot m = m \).
Examples of Actions

(i) A groupoid acts on its total space and on its base. A groupoid $G \rightarrow X$ acts on $G$ by multiplication with moment map $\alpha$. $G$ acts on $X$ with moment map the identity $I_X$ via $g \cdot \beta(g) := \alpha(g)$.

(ii) The $G$–action groupoid acts on $G$–spaces. Let $G$ be acting on two sets $M$ and $N$ and let $J : M \rightarrow N$ be any equivariant map with respect to those actions. The map $J$ induces an action of the product groupoid $G \times N \rightarrow N$ on $M$. The action is defined by

$$\Psi : (G \times N) \times J M \longrightarrow M$$

$$(((g, J(m)), m) \longmapsto g \cdot m.$$
(iii) The Baer groupoid acts on $G$–spaces. Let $G$ be a Lie group, $M$ be a $G$–space, and

- $B : M \rightarrow \mathcal{G}(G)$, $m \in M \mapsto G_m \in \mathcal{G}(G)$
- $\mathcal{B}(G) \times_B M := \{(gG_m, m) \in \mathcal{B}(G) \times M \mid m \in M\}$
- $\mathcal{B}(G) \times_B M \rightarrow M$ given by $(gG_m, m) \mapsto g \cdot m$ defines an action of the Baer groupoid $\mathcal{B}(G) \Rightarrow \mathcal{G}(G)$ on the $G$–space $M$ with moment map $B$
- The level sets of the moment map are the isotropy type subsets of $M$
Groupoid model of the optimal momentum map

- $K : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$, non equivariance one-cocycle
  $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$.
- $G \times \mathfrak{g}^*/\overline{\mathcal{H}} \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}}$ action groupoid associated to the affine action of $G$ on $\mathfrak{g}^*/\overline{\mathcal{H}}$.
- $\mathcal{B}(G) \rightrightarrows \mathcal{S}(G)$ Baer groupoid of $G$.
- $(G \times \mathfrak{g}^*/\overline{\mathcal{H}}) \times \mathcal{B}(G) \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathcal{S}(G)$ be the product groupoid and $\Gamma \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathcal{S}(G)$ be the wide subgroupoid defined by $\Gamma := \{((g,[\mu]),gH) \mid g \in G, \mu \in \mathfrak{g}^*/\overline{\mathcal{H}}, H \in \mathcal{S}(G)\}$.
- $\Gamma \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathcal{S}(G)$ acts naturally on $M$ with moment map
  $\mathcal{J} : M \longrightarrow \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathcal{S}(G)$
  $m \longmapsto (K(m), G_m)$.
- Action of $\Gamma$ on $M$:
  $\Psi : \Gamma \times \mathcal{J} M \longrightarrow M$
  $(((g, K(m)), gG_m), m) \longmapsto g \cdot m$. 

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By the universality property of the optimal momentum map there exists a unique map \( \varphi : M/A'_G \rightarrow g^*/\bar{H} \times \mathcal{G}(G) \)

\[
\begin{array}{c}
M \\
\downarrow \mathcal{J} \\
M/A'_G
\end{array}
\quad \xrightarrow{\tilde{\mathcal{J}}} \quad
\begin{array}{c}
g^*/\bar{H} \times \mathcal{G}(G) \\
\downarrow \varphi
\end{array}
\]

If \( \mathcal{H} \) is closed

\[
\tilde{\mathcal{J}}^{-1}([\mu], G_m) = K^{-1}([\mu]) \cap M_{G_m} = \mathcal{J}^{-1}(\rho)
\]

Connectedness implies \( \varphi \) injective.