CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

Symmetry and reduction in Poisson geometry

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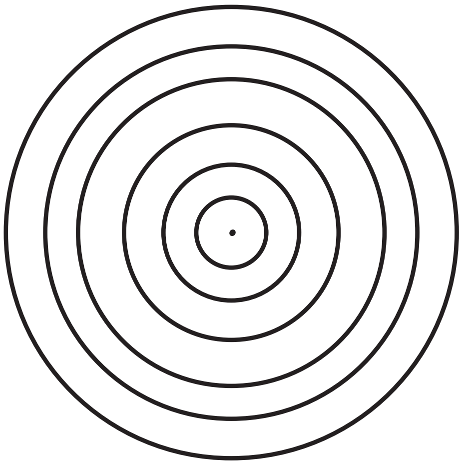
Reduction is…

- Algebra: a procedure to pass to the quotient in the Hamiltonian category
- Geometry: a way to construct new Poisson and symplectic manifolds
- Applied dynamics: a systematic method to eliminate variables using symmetries **and/or** conservation laws
- Theoretical dynamics: a way to get intuition on the dynamical behavior of symmetric systems
- Numerics: is it worth it?

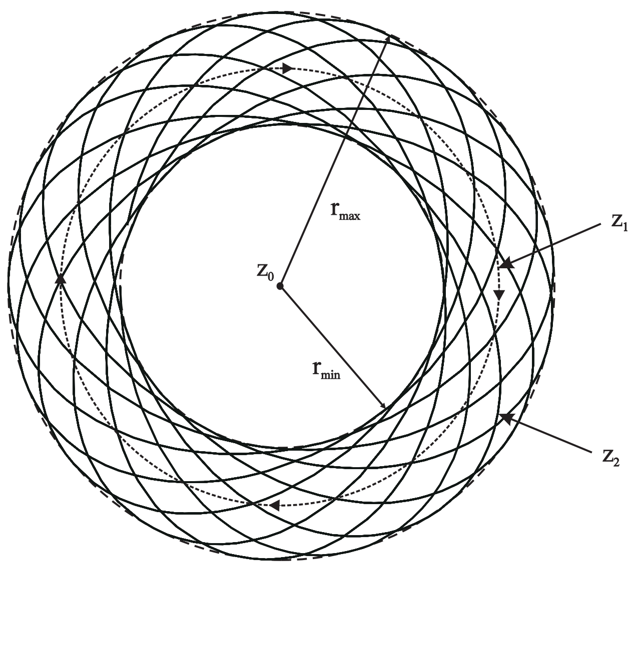
Example: The Weinstein-Moser Theorem

- Weinstein-Moser: if $\mathbf{d}^2 h(m) > 0$ then $1/2 \dim M$ periodic orbits at each neighboring energy level.



- Relative Weinstein-Moser (JPO (2003)): Relative periodic orbits around stable relative at neighboring energy-momentum-isotropy levels

$$\frac{1}{2}\left(\dim U^K - \dim(N(K)/K) - \dim\left(N(K)/K\right)_\lambda\right)$$



We will focus on

- Poisson and symplectic category

Leave aside

- Lagrangian side: different philosophy.
- Singular cotangent bundle reduction, nonholonomic reduction, reduction of Dirac manifolds and implicitly defined Hamiltonian systems, Sasakian, Kähler, hyperkähler, contact manifolds….

References

- Look at review
- Symplectic: Marsden, Weinstein (1974), Sjamaar, Lerman (1991)
- Poisson: Marsden, Ratiu (1986), JPO, Ratiu (1998)

Structure of the course

- **Lecture I:** Introduction. Preliminaries on:
  - Symmetries/group actions
  - Poisson and symplectic manifolds
- **Lecture II**: Poisson reduction.
- **Lecture III:** Momentum maps. Normal forms.
- **Lecture IV:** Symplectic reduction. Regular and singular.

Symmetry/Group actions

Definition. M a manifold and G a Lie group. A ***left action*** of G on M is a smooth mapping $\Phi : G \times M \to M$ such that

(i) $\Phi(e, z) = z$, for all $z \in M$ and

(ii) $\Phi(g, \Phi(h, z)) = \Phi(gh, z)$ for all $g, h \in G$ and $z \in M$.

We will often write

$$g \cdot z := \Phi(g, z) := \Phi_g(z) := \Phi^z(g).$$

and

$$A_G := \{\Phi_g \mid g \in G\} \subset \mathrm{Diff}(M).$$

The triple (M, G, Φ) is called a G–***space*** or a G–***manifold***.

Examples of group actions.

- **Translation and conjugation.** The ***left (right) translation*** $L_g : G \to G$, (R_g) $h \mapsto gh$, induces a left (right) action of G on itself.

- The ***inner automorphism*** $\mathrm{AD}_g \equiv I_g : G \to G$, given by $I_g := R_{g^{-1}} \circ L_g$ defines a left action of G on itself called ***conjugation***.
- **Adjoint and coadjoint action.** The differential at the identity of the conjugation mapping defines a linear left action of G on $\mathfrak{g}$ called the ***adjoint representation*** of G on $\mathfrak{g}$

$$\mathrm{Ad}_g := T_e I_g : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

If $\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of Ad_g, then the map

$$\begin{aligned} \Phi : G \times \mathfrak{g}^* &\longrightarrow \quad \mathfrak{g}^* \\ (g, \nu) &\longmapsto \mathrm{Ad}^*_{g^{-1}} \nu, \end{aligned}$$

defines also a linear left action of G on $\mathfrak{g}^*$ called the ***coadjoint representation*** of G on $\mathfrak{g}^*$.

- **Group representation.** If the manifold M is a vector space V and G acts linearly on V, that is, $\Phi_g \in \mathrm{GL}(V)$ for all $g \in G$, where $\mathrm{GL}(V)$ denotes the group of all linear automorphisms of V, then the action is said to be a ***representation*** of G on V. For example, the adjoint and coadjoint actions of G defined above are representations.
- **Tangent lifts of group actions.** The map Φ induces a natural action on the tangent bundle TM of M by

  $$g \cdot v_m := T_m \Phi_g \cdot v_m,$$

  where $g \in G$ and $v_m \in T_m M$.
- **Cotangent lifts of group actions.** Let $\Phi : G \times M \to M$ be a smooth Lie group action on the manifold M. The map Φ induces a natural action on the cotangent bundle T^*M of M by

  $$g \cdot \alpha_m := T^*_{g \cdot m} \Phi_{g^{-1}} \cdot \alpha_m$$

  where $g \in G$ and $\alpha_m \in T^*_m M$.

The ***infinitesimal generator*** $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$ is the vector field on M defined by

$$\xi_M(m) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(m) = T_e\Phi^m \cdot \xi.$$

The infinitesimal generators are complete vector fields. The flow of ξ_M equals $(t, m) \mapsto \exp t\xi \cdot m$. Moreover, the map $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a ***Lie algebra antihomomorphism***, that is,

(i) $(a\xi + b\eta)_M = a\xi_M + b\eta_M$,

(ii) $[\xi, \eta]_M = -[\xi_M, \eta_M]$.

Let $\mathfrak{g}$ be a Lie algebra and M a smooth manifold. A ***right (left) Lie algebra action*** of $\mathfrak{g}$ on M is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \longmapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \longmapsto \xi_M(m) \in TM$ is smooth. Given a Lie group action, we will refer to the Lie algebra action induced by its infinitesimal generators as the ***associated Lie algebra action***.

Stabilizers and orbits. The ***isotropy subgroup*** or ***stabilizer*** of an element m in the manifold M acted upon by the Lie group G is the closed subgroup

$$G_m := \{g \in G \,|\, \Phi_g(m) = m\} \subset G$$

whose Lie algebra $\mathfrak{g}_m$ equals

$$\mathfrak{g}_m = \{\xi \in \mathfrak{g} \,|\, \xi_M(m) = 0\}. \qquad (1)$$

The ***orbit*** $\mathcal{O}_m$ of the element $m \in M$ under the group action Φ is the set

$$\mathcal{O}_m \equiv G \cdot m := \{\Phi_g(m) \,|\, g \in G\}.$$

The isotropy subgroups of the elements in a group orbit are related by the expression

$$G_{g \cdot m} = gG_m g^{-1} \text{ for all } g \in G.$$

The notion of orbit allows the introduction of an equivalence relation in the manifold M, namely, two elements $x, y \in M$ are equivalent if and only if they are in the same G–orbit, that is, if there exists an element $g \in G$ such that $\Phi_g(x) = y$.

The space of classes with respect to this equivalence relation is usually referred to as the ***space of orbits*** and, depending on the context, it is denoted by the symbols M/G or M/A_G.

The action is

- ***Transitive*** if there is only one orbit.
- ***Free*** if the isotropy of every element in M consists only of the identity element.
- ***Proper*** whenever the map $\Theta : G \times M \to M \times M$ defined by

  $$\Theta(g, z) = (z, \Phi(g, z))$$

  is proper. Equivalent to the following condition: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in M, there exists a convergent subsequence $\{g_{n_k}\}$ in G.

Examples of proper actions: compact group actions, $SE(n)$ acting on $\mathbb{R}^n$, Lie groups acting on themselves by translation.

Proper actions

$\Phi : G \times M \to M$ be a proper action of the Lie group G on the manifold M. Then:

(i) The isotropy subgroups G_m are compact.

(ii) The orbit space M/G is a Hausdorff topological space. (Even when M and G are not Hausdorff.)

(iii) If the action is free, M/G is a smooth manifold, and the canonical projection $\pi : M \to M/G$ defines on M the structure of a smooth left principal G–bundle.

(iv) If all the isotropy subgroups of the elements of M under the G–action are conjugate to a given one H then M/G is a smooth manifold and $\pi : M \to M/G$ defines the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H.

(v) If the manifold M is paracompact then there exists a G–invariant Riemannian metric on it.

(vi) If the manifold M is paracompact then smooth G–invariant functions separate the G–orbits.

Tubes and Slices

Twisted product. Let G be a Lie group and $H \subset G$ a subgroup. Suppose that H acts on the left on the manifold A. The ***twisted action*** of H on the product $G \times A$ is defined by

$$h \cdot (g, a) = (gh, h^{-1} \cdot a).$$

This action is free and proper by the freeness and properness of the action on the G–factor. The ***twisted product*** $G \times_H A$ is defined as the orbit space $(G \times A)/H$ corresponding to the twisted action.

Tube. Let M be a manifold and G a Lie group acting properly on M. Let $m \in M$ and denote $H := G_m$. A ***tube*** around the orbit $G \cdot m$ is a G–equivariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

where U is a G–invariant neighborhood of $G \cdot m$ and A is some manifold on which H acts.

Slice Theorem. G a Lie group acting properly on M at the point $m \in M$, $H := G_m$. There exists a tube

$$\varphi : G \times_H B \longrightarrow U$$

about $G \cdot m$. B is an open H–invariant neighborhood of 0 in a vector space H–equivariantly isomorphic to $T_m M / T_m(G \cdot m)$ on which H acts linearly by

$$h \cdot (v + T_m(G \cdot m)) := T_m \Phi_h \cdot v + T_m(G \cdot m).$$

Dymanical consequences. G-invariant vector fields X can be locally decomposed as

$$X = X_T + X_N$$

Geometric consequences. ***Isotropy***, ***fixed point***, and ***orbit type spaces*** are submanifolds:

$$\begin{aligned} M_{(H)} &= \{z \in M \mid G_z \in (H)\}, \\ M^H &= \{z \in M \mid H \subset G_z\}, \\ M_H &= \{z \in M \mid H = G_z\}. \end{aligned}$$

Structure Theorems

Principal Orbit Theorem: M connected. The subset $M^{reg} \cap M$ is connected, open, and dense in M. M/G contains only one principal orbit type, which is a connected open and dense subset of it.

The Stratification Theorem: Let M be a smooth manifold and G a Lie group acting properly on it. The connected components of the orbit type manifolds $M_{(H)}$ and their projections onto orbit space $M_{(H)}/G$ constitute a Whitney stratification of M and M/G, respectively. This stratification of M/G is minimal among all Whitney stratifications of M/G.

Theorem. Let G be a Lie group acting properly on the smooth manifold M and $m \in M$ a point with isotropy subgroup $H := G_m$. Then

$$((T_m(G \cdot m))^\circ)^H = \{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\}.$$

Symmetry Reduction

- M a G–manifold. $X \in \mathfrak{X}(M)^G$. Flow F_t.
- H–isotropy type submanifold M_H:

$$M_H := \{m \in M \mid G_m = H\}$$

preserved by the flow F_t and $N(H)$–invariant.

- $\pi_H : M_H \to M_H/(N(H)/H)$
$i_H : M_H \hookrightarrow M$.
- Reduced vector field:

$$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$

with flow F_t^H given by

$$F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H.$$

- Linear compact actions and Hilbert's Theorem.

Symplectic manifolds

A ***symplectic manifold*** is a pair (M, ω), where M is a manifold and $\omega \in \Omega^2(M)$ is a closed non–degenerate two–form on M, that is,

- $\mathbf{d}\omega = 0$
- for every $m \in M$, the map

  $$v \in T_mM \mapsto \omega(m)(v, \cdot) \in T_m^*M$$

  is a linear isomorphism

If ω is allowed to be degenerate, (M, ω) is called a ***presymplectic manifold***. A ***Hamiltonian dynamical system*** is a triple (M, ω, h), where (M, ω) is a symplectic manifold and $h \in C^\infty(M)$ is the ***Hamiltonian function*** of the system. By non–degeneracy of the symplectic form ω, to each Hamiltonian system one can associate a ***Hamiltonian vector field*** $X_h \in \mathfrak{X}(M)$, defined by the equality

$$\mathbf{i}_{X_h}\omega = \mathbf{d}h.$$

Example Let V be a vector space and V^* its dual. Let $Z = V \times V^*$. The **canonical symplectic form** Ω on Z is defined by

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.$$

Example Let Q be a smooth manifold and T^*Q its cotangent bundle. Let $\pi_Q : T^*Q \to Q$ be the projection and Θ the one–form on T^*Q defined by

$$\Theta(\beta) \cdot v_\beta := \langle \beta, T_\beta \pi_Q \cdot v_\beta \rangle,$$

where $\beta \in T^*Q$ and $v_\beta \in T_\beta(T^*Q)$. The **canonical symplectic form** Ω on the cotangent bundle T^*Q is defined by $\Omega = -\mathbf{d}\Theta$.

Darboux theorem Locally

$$\omega|_U = \sum_{i=1}^{n} \mathbf{d}q^i \wedge \mathbf{d}p_i.$$

In canonical coordinates, X_h is determined by the well–known ***Hamilton equations***,

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}.$$

The ***Poisson bracket*** of $f, g \in C^\infty(M)$ is the function $\{f,\, g\} \in C^\infty(M)$ defined by

$$\{f,\, g\}(z) = \omega(z)(X_f(z),\, X_g(z)).$$

In canonical coordinates, the Poisson bracket takes the form

$$\{f,\, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \right) .$$

Poisson manifolds

- $(M, \{\cdot,\cdot\})$ Poisson manifold. $(C^\infty(M), \{\cdot,\cdot\})$ Lie algebra such that

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

- ***Casimirs*** elements in the center of algebra.
- Derivations and vector fields. Hamiltonian vector fields

$$X_h[f] = \{f, h\}$$

- **Example: The Lie-Poisson bracket** The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with respect to the $\pm$–***Lie–Poisson*** brackets $\{\cdot,\cdot\}_\pm$ defined by

$$\{f, g\}_\pm(\mu) := \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right] \right\rangle$$

$\dfrac{\delta f}{\delta \mu} \in \mathfrak{g}$ is defined by

$$\langle \nu, \frac{\delta f}{\delta \mu} \rangle := Df(\mu) \cdot \nu,$$

for any $\nu \in \mathfrak{g}^*$. Given $h \in C^\infty(\mathfrak{g}^*)$

$$X_h(\mu) = \mp \mathrm{ad}^*_{\delta h/\delta \mu}\mu, \quad \mu \in \mathfrak{g}^*.$$

The Poisson tensor. The derivation property of the Poisson bracket implies that for any two functions $f,\, g \in C^\infty(M)$, the value of the bracket $\{f,\, g\}(z)$ on f only through $\mathbf{d}f(z)$ which allows us to define a contravariant antisymmetric two–tensor $B \in \Lambda^2(T^*M)$ by

$$B(z)(\alpha_z,\, \beta_z) = \{f,\, g\}(z),$$

with $\mathbf{d}f(z) = \alpha_z$ and $\mathbf{d}g(z) = \beta_z$. This tensor is called the ***Poisson tensor*** of M. The vector bundle map $B^\sharp : T^*M \to TM$ naturally associated to B is defined by

$$B(z)(\alpha_z,\, \beta_z) = \langle \alpha_z,\, B^\sharp(\beta_z)\rangle.$$

Its range $D := B^\sharp(T^*M) \subset TM$ is called the ***characteristic distribution***. For any point $m \in M$, the dimension of $D(m)$ as a vector subspace of T_mM is called the ***rank*** of the Poisson manifold $(M, \{\cdot, \cdot\})$ at the point m.

The Weinstein coordinates of a Poisson manifold. Let $(M, \{\cdot,\cdot\})$ be a m–dimensional Poisson manifold and $z_0 \in M$ a point where the rank of $(M, \{\cdot,\cdot\})$ equals $2n$, $0 \le 2n \le m$. There exists a chart (U, φ) of M whose domain contains the point z_0 and such that the associated local coordinates, denoted by

$$(q^1, \dots, q^n, p_1, \dots, p_n, z_1, \dots, z_{m-2n}),$$

satisfy

$$\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, z_k\} = \{p_i, z_k\} = 0,$$

and $\{q^i, p_j\} = \delta^i_j$, for all i, j, k, $1 \le i, j \le n$, $1 \le k \le m - 2n$.

For all k, l, $1 \le k, l \le m - 2n$, the Poisson bracket $\{z_k, z_l\}$ is a function of the local coordinates $z^1, \dots, z^{m-2n}$ exclusively, and vanishes at z_0. Hence, the restriction of the bracket $\{\cdot,\cdot\}$ to the coordinates $z^1, \dots, z^{m-2n}$ induces a Poisson structure that is usually referred to as the ***transverse Poisson structure*** of $(M, \{\cdot,\cdot\})$ at m.

A smooth mapping $\varphi : (M_1, \{\cdot, \cdot\}_1) \to (M_2, \{\cdot, \cdot\}_2)$ is ***canonical*** or ***Poisson*** if for all $g, h \in C^\infty(M_2)$ we have

$$\varphi^*\{g, h\}_2 = \{\varphi^* g, \varphi^* g\}_1 .$$

In the symplectic category, $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ ***canonical*** or ***symplectic*** if

$$\varphi^* \omega_2 = \omega_1 .$$

- Symplectic maps are immersions.
- A diffeomorphism $\varphi : M_1 \to M_2$ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is symplectic if and only if it is Poisson.
- If the symplectic map $\varphi : M_1 \to M_2$ is not a diffeomorphism it may not be a Poisson map.

Let $(S, \{\cdot, \cdot\}^S)$ and $(M, \{\cdot, \cdot\}^M)$ be two Poisson manifolds such that $S \subset M$ and the inclusion $i_S : S \hookrightarrow M$ is an immersion. $(S, \{\cdot, \cdot\}^S)$ is a ***Poisson submanifold*** of $(M, \{\cdot, \cdot\}^M)$ if i_S is a canonical map.

An immersed submanifold Q of M is called a ***quasi Poisson submanifold*** of $(M, \{\cdot,\cdot\}^M)$ if for any $q \in Q$, any open neighborhood U of q in M, and any $f \in C^\infty_M(U)$ we have

$$X_f(i_Q(q)) \in T_q i_Q(T_q Q),$$

where $i_Q : Q \hookrightarrow M$ is the inclusion and X_f is the Hamiltonian vector field of f on U with respect to the restricted Poisson bracket $\{\cdot,\cdot\}^M_U$. Any Poisson submanifold is quasi Poisson. The converse is not true.

Given two symplectic manifolds (M, ω) and (S, ω_S) such that $S \subset M$ and the inclusion $i : S \hookrightarrow M$ is an immersion, the manifold (S, ω_S) is a ***symplectic submanifold*** of (M, ω) when i is a symplectic map. Symplectic submanifolds of a symplectic manifold (M, ω) are in general neither Poisson nor quasi Poisson manifolds of M. The only quasi Poisson submanifolds of a symplectic manifold are its open sets which are, in fact, Poisson submanifolds.

Symplectic Foliation Theorem. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and D the associated characteristic distribution. D is a smooth and integrable generalized distribution and its maximal integral leaves form a generalized foliation decomposing M into initial submanifolds $\mathcal{L}$, each of which is symplectic with the unique symplectic form that makes the inclusion $i : \mathcal{L} \hookrightarrow M$ into a Poisson map, that is, $\mathcal{L}$ is a Poisson submanifold of $(M, \{\cdot, \cdot\})$.

Example Let $\mathfrak{g}^*$ with the Lie–Poisson structure. The symplectic leaves of the Poisson manifolds $(\mathfrak{g}^*, \{\cdot, \cdot\}_\pm)$ coincide with the connected components of the orbits of the elements in $\mathfrak{g}^*$ under the coadjoint action. In this situation, the symplectic form for the leaves is given by the ***Kostant–Kirillov–Souriau (KKS)*** expression

$$\omega^{\pm}_{\mathcal{O}}(\nu)(\xi_{\mathfrak{g}^*}(\nu),\, \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu,\, [\xi,\, \eta]\rangle.$$

Canonical symmetries

- $(M, \{\cdot, \cdot\})$ Poisson manifold. G acts canonically on M when
$$\Phi_g^*\{f, h\} = \{\Phi_g^* f, \Phi_g^* h\}$$
- Easy Poisson reduction: $(M, \{\cdot, \cdot\})$ Poisson manifold, G Lie group acting canonically, freely, and properly on M. The orbit space M/G is a Poisson manifold with bracket
$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m),$$
- Reduction of Hamiltonian dynamics: $h \in C^\infty(M)^G$ reduces to $\overline{h} \in C^\infty(M/G)$ given by $\overline{h} \circ \pi = h$ such that
$$X_{\overline{h}} = T\pi \circ X_h$$
- What about the symplectic leaves?

How do we do it?

- Consider $\mathbb{R}^6$ with bracket

$$\{f, g\} = \sum_{i=1} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$$

- S^1–action given by

$$\begin{array}{rccc} \Phi : & S^1 \times \mathbb{R}^6 & \longrightarrow & \mathbb{R}^6 \\ & (e^{i\phi}, (\mathbf{x}, \mathbf{y})) & \longmapsto & (R_\phi \mathbf{x}, R_\phi \mathbf{y}), \end{array}$$

- Hamiltonian of the spherical pendulum

$$h = \frac{1}{2} < y, y > + < x, e_3 >$$

- Impose constraint $< x, x >= 1$
- Angular momentum: $\mathbf{J}(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$.

Hilbert basis of the algebra of S^1–invariant polynomials is given by

$$\begin{array}{lll} \sigma_1 = x_3 & \sigma_3 = y_1^2 + y_2^2 + y_3^2 & \sigma_5 = x_1^2 + x_2^2 \\ \sigma_2 = y_3 & \sigma_4 = x_1 y_1 + x_2 y_2 & \sigma_6 = x_1 y_2 - x_2 y_1. \end{array}$$

Semialgebraic relations

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \qquad \sigma_3 \geq 0, \qquad \sigma_5 \geq 0.$$

Hilbert map

$$\begin{aligned} \sigma : \ T\mathbb{R}^3 &\longrightarrow \mathbb{R}^6 \\ (\mathbf{x}, \mathbf{y}) &\longmapsto (\sigma_1(\mathbf{x}, \mathbf{y}), \ldots, \sigma_6(\mathbf{x}, \mathbf{y})). \end{aligned}$$

The S^1–orbit space $T\mathbb{R}^3/S^1$ can be identified with the semialgebraic variety $\sigma(T\mathbb{R}^3) \subset \mathbb{R}^6$, defined by these relations.

TS^2 is a submanifold of $\mathbb{R}^6$ given by

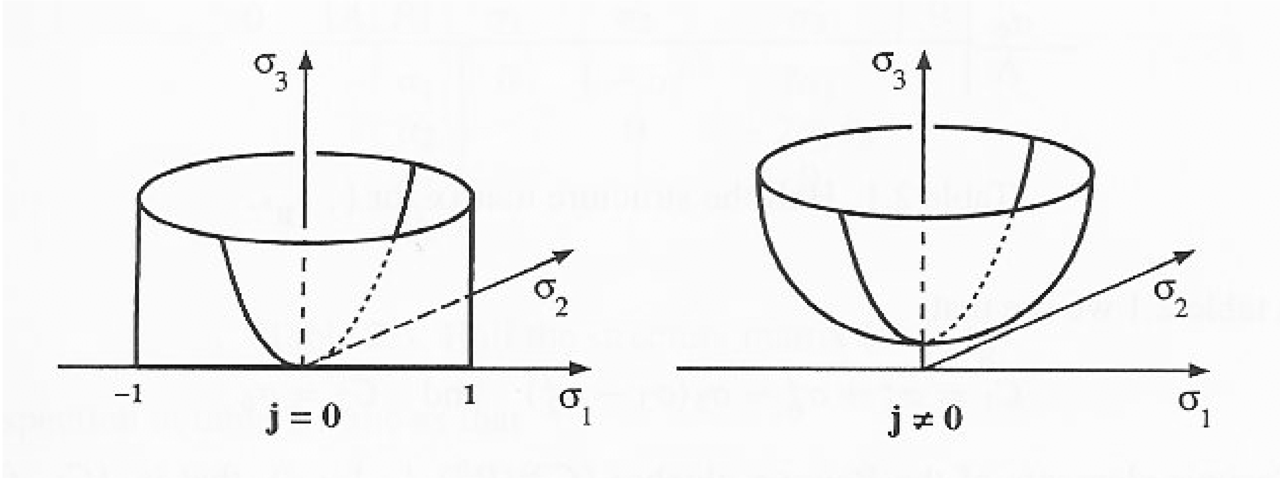
$TS^2 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 \mid < \mathbf{x}, \mathbf{x} >= 1, \ < \mathbf{x}, \mathbf{y} >= 0\}.$

TS^2 is S^1–invariant. TS^2/S^1 can be thought of the semialgebraic variety $\sigma(TS^2)$ defined by the previous relations and

$$\sigma_5 + \sigma_1^2 = 1 \qquad \sigma_4 + \sigma_1\sigma_2 = 0,$$

which allow us to solve for σ_4 and σ_5, yielding

$$\begin{aligned} TS^2/S^1 = \sigma(TS^2) = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_6) \in \mathbb{R}^4 \mid \\ \sigma_1^2\sigma_2^2 + \sigma_6^2 = (1 - \sigma_1^2)(\sigma_3 - \sigma_2^2), \\ |\sigma_1| \leq 1, \sigma_3 \geq 0\}. \end{aligned}$$



If $\mu \neq 0$ then $(TS^2)_\mu$ appears as the graph of the smooth function

$$\sigma_3 = \frac{\sigma_2^2 + \mu^2}{1 - \sigma_1^2}, \qquad |\sigma_1| < 1.$$

The case $\mu = 0$ is singular and $(TS^2)_0$ is not a smooth manifold.

$\{\cdot,\cdot\}^{TS^2/S^1}$	σ_1	σ_2	σ_3	σ_6
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	0
σ_2	$-(1-\sigma_1^2)$	0	$-2\sigma_1\sigma_3$	0
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_3$	0	0
σ_6	0	0	0	0

Reduced Hamiltonian

$$H = \frac{1}{2}\sigma_3 + \sigma_1$$

Poisson reduction by distributions

- Reduction of the 4-tuple $(M, \{\cdot, \cdot\}, D, S)$
- S encodes conservation laws and D invariance properties
- S submanifold of $(M, \{\cdot, \cdot\})$. $D_S := D \cap TS$
- $(M, \{\cdot, \cdot\})$ Poisson manifold, $D \subset TM$. D is ***Poisson*** if

$$\mathbf{d}f|_D = \mathbf{d}g|_D = 0 \Rightarrow \mathbf{d}\{f, g\}|_D = 0$$

- When does the bracket on M induce a bracket on S/D_S?

- The functions C^∞_{S/D_S} are characterized by the following property: $f \in C^\infty_{S/D_S}(V)$ if and only if for any $z \in V$ there exists $m \in \pi^{-1}_{D_S}(V)$, U_m open neighborhood of m in M, and $F \in C^\infty_M(U_m)$ such that
$$f \circ \pi_{D_S}|_{\pi^{-1}_{D_S}(V)\cap U_m} = F|_{\pi^{-1}_{D_S}(V)\cap U_m}.$$
F is a ***local extension*** of $f \circ \pi_{D_S}$ at the point $m \in \pi^{-1}_{D_S}(V)$.
- C^∞_{S/D_S} has the (D, D_S)–***local extension property*** when the local extensions of $f \circ \pi_{D_S}$ can always be chosen to satisfy
$$\mathbf{d}F(n)|_{D(n)} = 0.$$
- $(M, \{\cdot,\cdot\}, D, S)$ is ***Poisson reducible*** when $(S/D_S, C^\infty_{S/D_S}, \{\cdot,\cdot\}^{S/D_S})$ is a well defined Poisson manifold with
$$\{f, g\}^{S/D_S}_V(\pi_{D_S}(m)) := \{F, G\}(m),$$
F, G are local D–invariant extensions of $f \circ \pi_{D_S}$ and $g \circ \pi_{D_S}$.

Theorem (Marsden, Ratiu (1986)/ JPO, Ratiu (1998) (singular)).
$(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible if and only if

$$B^\sharp(D^\circ) \subset TS + D.$$

Examples

Coisotropic submanifolds:

$$B^\sharp\left((TS)^\circ\right) \subset TS$$

Dirac's first class constraints (Bojowald, Strobl (2002)).
If S be an embedded coisotropic submanifold of M and $D := B^\sharp((TS)^\circ)$ then $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible.
Appear in the context of integrable systems as the level sets of integrals in involution.

Cosymplectic manifolds and Dirac's formula

An embedded submanifold $S \subset M$ is called ***cosymplectic*** when

(i) $B^\sharp((TS)^\circ) \cap TS = \{0\}$.

(ii) $T_sS + T_s\mathcal{L}_s = T_sM$,

for any $s \in S$ and $\mathcal{L}_s$ the symplectic leaf of $(M, \{\cdot, \cdot\})$ containing $s \in S$. The cosymplectic submanifolds of a symplectic manifold (M, ω) are its symplectic submanifolds (a.k.a. ***second class constraints***). In this case

$$TM|_S = B^\sharp((TS)^\circ) \oplus TS$$

Theorem(Weinstein (1983)) S cosymplectic. Let $D := B^\sharp((TS)^\circ) \subset TM|_S$. Then

(i) $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible.

(ii) The corresponding quotient manifold equals S and the reduced bracket $\{\cdot, \cdot\}^S$ is given by

$$\{f, g\}^S(s) = \{F, G\}(s),$$

$F, G \in C^\infty_M(U)$ are local D–invariant extensions of f and g.

(iii) The Hamiltonian vector field X_f of an arbitrary function $f \in C^{\infty}_{S,M}(V)$ can be written as

$$Ti \cdot X_f = \pi_S \circ X_F \circ i, \qquad (1)$$

where $F \in C^{\infty}_{M}(U)$ is an arbitrary local extension of f and $\pi_S : TM|_S \to TS$ is the projection induced by the Whitney sum decomposition $TM|_S = B^{\sharp}((TS)^{\circ}) \oplus TS$ of $TM|_S$.

(v) The symplectic leaves of $(S, \{\cdot,\cdot\}^S)$ are the connected components of the intersections $S \cap \mathcal{L}$, with $\mathcal{L}$ a symplectic leaf of $(M, \{\cdot,\cdot\})$. Any symplectic leaf of $(S, \{\cdot,\cdot\}^S)$ is a symplectic submanifold of the symplectic leaf of $(M, \{\cdot,\cdot\})$ that contains it.

(vi) Let $\mathcal{L}_s$ and $\mathcal{L}_s^S$ be the symplectic leaves of $(M, \{\cdot,\cdot\})$ and $(S, \{\cdot,\cdot\}^S)$, respectively, that contain the point $s \in S$. Let $\omega_{\mathcal{L}_s}$ and $\omega_{\mathcal{L}_s^S}$ be the corresponding symplectic forms. Then $B^{\sharp}(s)((T_sS)^{\circ})$ is a symplectic subspace of $T_s\mathcal{L}_s$

and

$$B^\sharp(s)((T_sS)^\circ) = \left(T_s\mathcal{L}_s^S\right)^{\omega_{\mathcal{L}_s}(s)}. \qquad (2)$$

(vii) Let $B_S \in \Lambda^2(T^*S)$ be the Poisson tensor associated to $(S, \{\cdot,\cdot\}^S)$. Then

$$B_S^\sharp = \pi_S \circ B^\sharp|_S \circ \pi_S^*, \qquad (3)$$

where $\pi_S^* : T^*S \to T^*M|_S$ is the dual of $\pi_S : TM|_S \to TS$.

Formula (3) gives in local coordinates Dirac's formula:

$$\{f,g\}^S(s) = \{F,G\}(s) - \sum_{i,j=1}^{n-k} \{F,\psi^i\}(s)C_{ij}(s)\{\psi^j,G\}(s)$$

The momentum map

- (M, ω) symplectic manifold, G acting canonically
- Momentum map $\mathbf{J} : M \to \mathfrak{g}^*$

$$\mathbf{J}^\xi := < \mathbf{J}, \xi >, \quad \mathbf{i}_{\xi_M}\omega = \mathbf{dJ}^\xi$$

with

$$\xi_M(m) = \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot m$$

- Noether's Theorem: the fibers of $\mathbf{J}$ are preserved by the Hamiltonian flows associated to G-invariant Hamiltonians.

Example: linear momentum. Take the phase space of the N–particle system, that is, $T^*\mathbb{R}^{3N}$. The additive group $\mathbb{R}^3$ acts on it by

$$\boldsymbol{v} \cdot (\boldsymbol{q}_i, \boldsymbol{p}^i) = (\boldsymbol{q}_i + \boldsymbol{v}, \boldsymbol{p}^i)$$

$$\begin{array}{rcl} \mathbf{J} : T^*\mathbb{R}^{3N} & \longrightarrow & \mathrm{Lie}(\mathbb{R}^3) \simeq \mathbb{R}^3 \\ (\boldsymbol{q}_i, \boldsymbol{p}^i) & \longmapsto & \sum_{i=1}^N \boldsymbol{p}_i. \end{array}$$

Example: angular momentum. Let SO(3) act on $\mathbb{R}^3$ and then, by lift, on $T^*\mathbb{R}^3$, that is, $A \cdot (\boldsymbol{q}, \boldsymbol{p}) = (A\boldsymbol{q}, A\boldsymbol{p})$.

$$\begin{aligned} \mathbf{J} : \; T^*\mathbb{R}^3 &\longrightarrow \mathfrak{so}(\;)^* \simeq \mathbb{R}^3 \\ (\boldsymbol{q}, \boldsymbol{p}) &\longmapsto \quad \boldsymbol{q} \times \boldsymbol{p}. \end{aligned}$$

which is the classical ***angular momentum***.

Example: lifted actions on cotangent bundles. Let G be a Lie group acting on the manifold Q and then by lift on its cotangent bundle T^*Q.

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle,$$

for any $\alpha_q \in T^*Q$ and any $\xi \in \mathfrak{g}$.

Example: symplectic linear actions. Let (V, ω) be a symplectic linear space and let G be a subgroup of the linear symplectic group, acting naturally on V.

$$\langle \mathbf{J}(v), \xi \rangle = \frac{1}{2}\omega(\xi_V(v), v).$$

Properties of the momentum map

- Regularity of the action is equivalent to the regularity of the momentum map

$$\text{range}\, T_m\mathbf{J} = (\mathfrak{g}_m)^0$$

- $\ker T_m\mathbf{J} = (\mathfrak{g}\cdot m)^\omega$.
- Existence:

$$\begin{array}{rcl} \rho : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] & \longrightarrow & H^1(M,\mathbb{R}) \\ [\xi] & \longmapsto & [\mathbf{i}_{\xi_M}\omega] \end{array}$$

- Equivariance: When $(\mathfrak{g},[\cdot,\cdot]) \to (C^\infty(M),\{\cdot,\cdot\})$ defined by $\xi \mapsto \mathbf{J}^\xi$, $\xi\in\mathfrak{g}$, is a Lie algebra homomorphism, that is,

$$\mathbf{J}^{[\xi,\eta]} = \{\mathbf{J}^\xi,\, \mathbf{J}^\eta\}, \quad \xi,\eta\in\mathfrak{g}.$$

  Answer: iff

$$T_z\mathbf{J}\cdot\xi_M(z) = -\operatorname{ad}^*_\xi \mathbf{J}(z),$$

  A momentum map that satisfies this relation in called ***infinitesimally equivariant***.

- $\mathbf{J}$ is G–***equivariant*** when

$$\operatorname{Ad}^*_{g^{-1}} \circ \mathbf{J} = \mathbf{J}\circ\Phi_g,$$

- If G is compact $\mathbf{J}$ can be chosen G-equivariant

Equivariance

Define the ***non equivariance one–cocycle*** associated to $\mathbf{J}$ as the map

$$\begin{aligned} \sigma : G &\longrightarrow \mathfrak{g}^* \\ g &\longmapsto \mathbf{J}(\Phi_g(z)) - \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(z)). \end{aligned}$$

Then:

(i) The definition of σ does not depend on the choice of $z \in M$;

(ii) The mapping σ is a $\mathfrak{g}^*$–valued one–cocycle on G with respect to the coadjoint representation of G on $\mathfrak{g}^*$.

We define the ***affine action*** of G on $\mathfrak{g}^*$ with cocycle σ by

$$\begin{aligned} \Theta : G \times \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ (g, \mu) &\longmapsto \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g). \end{aligned}$$

Θ determines a left action of G on $\mathfrak{g}^*$. The momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ is equivariant with respect to the symplectic action Φ on M and the affine action Ψ on $\mathfrak{g}^*$.

The affine orbits $\mathcal{O}_\mu$ are also symplectic with G-invariant symplectic structure given by

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu),\, \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu,\, [\xi,\, \eta]\rangle \mp \Sigma(\xi,\, \eta),$$

where the infinitesimal non equivariance cocycle $\Sigma \in Z^2(\mathfrak{g}, \mathbb{R})$ is given by

$$\begin{aligned} \Sigma :\ \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{R} \\ (\xi,\, \eta) &\longmapsto \Sigma(\xi,\, \eta) = \mathbf{d}\widehat{\sigma}_\eta(e) \cdot \xi, \end{aligned}$$

with $\widehat{\sigma}_\eta : G \to \mathbb{R}$ defined by $\widehat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

Reduction Lemma:

$$\mathfrak{g}_\mu \cdot m = \mathfrak{g} \cdot m \cap \ker T_m\mathbf{J} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^\omega.$$

Momentum maps and isotropy type manifolds The free, proper, and canonical action of $L^m := N(G_m)^m/G_m$ on $M^m_{G_m}$ has a momentum map $\mathbf{J}_{L^m} : M^m_{G_m} \rightarrow (\mathrm{Lie}(L^m))^*$ given by

$$\mathbf{J}_{L^m}(z) := \Lambda(\mathbf{J}|_{M^m_{G_m}}(z) - \mu), \quad z \in M^m_{G_m}.$$

In this expression $\Lambda : (\mathfrak{g}_m^\circ)^{G_m} \rightarrow (\mathrm{Lie}(L^m))^*$ denotes the natural L^m–equivariant isomorphism given by

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} \exp t\xi G_m \right\rangle = \langle \beta, \xi \rangle,$$

for any $\beta \in (\mathfrak{g}_m^\circ)^{G_m}$, $\xi \in \mathrm{Lie}(N(G_m)^m) = \mathrm{Lie}(N(G_m))$.
The non equivariance one–cocycle $\tau : M^m_{G_m} \rightarrow (\mathrm{Lie}(L^m))^*$ of the momentum map $\mathbf{J}_{L^m}$ is given by the map

$$\tau(l) = \Lambda(\sigma(n) + n \cdot \mu - \mu).$$

Convexity

$\mathbf{J} : M \to \mathfrak{g}^*$ coadjoint equivariant. G, M compact. The intersection of the image of $\mathbf{J}$ with a Weyl chamber is a *compact and convex polytope*. This polytope is referred to as the ***momentum polytope***.

Delzant's theorem proves that the symplectic toric manifolds are classified by their momentum polytopes. A ***Delzant polytope*** in $\mathbb{R}^n$ is a convex polytope that is also:

(i) Simple: there are n edges meeting at each vertex.

(ii) Rational: the edges meeting at a vertex p are of the form $p + tu_i$, $0 \leq t < \infty$, $u_i \in \mathbb{Z}^n$, $i \in \{1, \dots, n\}$.

(iii) Smooth: the vectors $\{u_1, \dots, u_n\}$ can be chosen to be an integral basis of $\mathbb{Z}^n$.

Delzant's Theorem can be stated by saying that

$$\begin{array}{ccc} \{\text{symplectic toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes} \\ (M, \omega, \mathbb{T}^n, \mathbf{J} : M \to \mathbb{R}^n) & \longmapsto & \mathbf{J}(M) \end{array}$$

is a bijection.

The cylinder valued momentum map

- Condevaux, Dazord, Molino [1988] Géometrie du moment. UCB, Lyon.
- $M \times \mathfrak{g}^* \longrightarrow M$, M connected
- $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$

$$\langle \alpha(m,\mu) \cdot (v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M}\omega)(m) \cdot v_m - \langle \nu, \xi \rangle$$

- α has zero curvature $\Rightarrow \mathcal{H}$ discrete
- $\widetilde{M}$ holonomy bundle $\Leftrightarrow$ horizontal leaf

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathbf{K}}} & \mathfrak{g}^* \\ \downarrow \widetilde{p} & & \downarrow \pi_C \\ M & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\widetilde{\mathcal{H}} \end{array}$$

- Standard momentum map exists $\Leftrightarrow \mathcal{H} = \{0\}$
- $\mathbf{K}$ always exists and it is a smooth momentum
- $\ker(T_m\mathbf{K}) = \left(\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot m \right)^\omega$
- $\mathrm{range}\,(T_m\mathbf{K}) = T_\mu \pi_C\left((\mathfrak{g}_m)^\circ\right)$

Equivariance

$\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ cylinder valued momentum map associated to a G–action on (M, ω)

- $\mathcal{H}$ is Ad*-invariant: $\mathrm{Ad}^*_{g^{-1}}(\mathcal{H}) \subset \mathcal{H}$, $g \in G$. If G is connected, $\mathcal{H}$ is pointwise fixed.
- There exists a unique action

$$\mathcal{A}d^* : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \to \mathfrak{g}^*/\overline{\mathcal{H}}$$

such that for any $g \in G$

$$\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$$

Define

$$\sigma(g, m) := \mathbf{K}(g \cdot m) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(m)$$

- If M is connected $\sigma : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ does not depend on M.
- $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ is a group-valued one-cocyle, that is

$$\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h)$$

The map

$$\begin{aligned}\Theta : G \times \mathfrak{g}^*/\overline{\mathcal{H}} &\longrightarrow \mathfrak{g}^*/\overline{\mathcal{H}} \\ (g, \mu + \overline{\mathcal{H}}) &\longmapsto \mathcal{A}d^*_{g^{-1}}(\mu + \overline{\mathcal{H}}) + \sigma(g)\end{aligned}$$

is a group action such that

$$\mathbf{K}(g \cdot m) = \Theta_g(\mathbf{K}(m))$$

Reduction Lemma

$$\mathfrak{g}_{\mu+\overline{\mathcal{H}}} \cdot m = \ker T_m\mathbf{K} \cap \mathfrak{g} \cdot m$$

Corollary: if $\mathcal{H}$ is closed then

$$\mathfrak{g}_{\mu+\overline{\mathcal{H}}} \cdot m = (\mathfrak{g} \cdot m)^\omega \cap \mathfrak{g} \cdot m$$

Cylinder and Lie group valued momentum maps

McDuff, Ginzburg, Huebschmann,Jeffrey, Huebschmann, Alekseev, Malkin, and Meinreken (1998)

$(\cdot,\cdot)$ bilinear symmetric non degenerate form on $\mathfrak{g}$. $\mathbf{J}: M \to G$ is a G–***valued momentum map*** for the $\mathfrak{g}$–action on M whenever

$$\mathbf{i}_{\xi_M}\omega(m)\cdot v_m = \left(T_m(L_{\mathbf{J}(m)^{-1}}\circ \mathbf{J})(v_m),\xi\right)$$

Any cylinder valued momentum map associated to an Abelian Lie algebra action whose corresponding holonomy group is closed can be understood as a Lie group valued momentum map.

Proposition $f:\mathfrak{g}\to\mathfrak{g}^*$ isomorphism given by $\xi \longmapsto (\xi,\cdot)$, $\xi\in\mathfrak{g}$ and $\mathcal{T}:=f^{-1}(\mathcal{H})$. f induces an Abelian group isomorphism $\bar{f}:\mathfrak{g}/\mathcal{T}\to\mathfrak{g}^*/\mathcal{H}$ by $\bar{f}(\xi+\mathcal{T}):=(\xi,\cdot)+\mathcal{H}$. Suppose that $\mathcal{H}$ is closed in $\mathfrak{g}^*$ and define $\mathbf{J}:=\bar{f}^{-1}\circ\mathbf{K}:M\to\mathfrak{g}/\mathcal{T}$, where $\mathbf{K}$ is a cylinder valued momentum map for the $\mathfrak{g}$–action. Then $\mathbf{J}:M\to\mathfrak{g}/\mathcal{T}$ is a $\mathfrak{g}/\mathcal{T}$–valued momentum map for the action of the Lie algebra $\mathfrak{g}$ of $(\mathfrak{g}/\mathcal{T},+)$ on (M,ω).

Lie group valued momentum maps produce closed holonomy groups

Theorem $\mathcal{H} \subset \mathfrak{g}^*$ holonomy group associated to the $\mathfrak{g}$–action. $f : \mathfrak{g} \to \mathfrak{g}^*$, $\bar{f} : \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}$, and $\mathcal{T} := f^{-1}(\mathcal{H})$ as before. Let G be a connected Abelian Lie group whose Lie algebra is $\mathfrak{g}$ and suppose that there exists a G–valued momentum map $\mathbf{A} : M \to G$ associated to the $\mathfrak{g}$–action whose definition uses the form $(\cdot, \cdot)$.

(i) If $\exp : \mathfrak{g} \to G$ is the exponential map, then

$$\mathcal{H} \subset f(\ker \exp).$$

(ii) $\mathcal{H}$ is closed in $\mathfrak{g}^*$.

Let $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K} : M \to \mathfrak{g}/\mathcal{T}$, where $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ is a cylinder valued momentum map for the $\mathfrak{g}$–action on (M, ω). If $f(\ker \exp) \subset \mathcal{H}$ then $\mathbf{J} : M \to \mathfrak{g}/\mathcal{T} = \mathfrak{g}/\ker\exp \simeq G$ is a G–valued momentum map that differs from $\mathbf{A}$ by a constant in G.

Conversely, if $\mathcal{H} = f(\ker \exp)$ then $\mathbf{J} : M \to \mathfrak{g}/\ker\exp \simeq G$ is a G–valued momentum map.

The optimal momentum map

Problems with the traditional momentum map:

- Possible non existence of $\mathbf{J}$:
  1. S^1 acting on $\mathbb{T}^2$ by
  $$e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\phi+\theta_1)}, e^{i\theta_2}).$$
  Lie group valued momentum maps. Dirac [1926], McDuff [1988], Alekseev et al. [1997].
  2. $(\mathbb{R}^3, \{\cdot, \cdot\})$ with Poisson tensor
  $$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$
  $(\mathbb{R}, +)$ acts on $\mathbb{R}^3$ by $\lambda \cdot (x, y, z) := (x + \lambda, y, z)$. NO MOMENTUM MAP!!
- Singular case not optimal (finite groups). Does not see law of conservation of isotropy.
$$\mathbf{J}^{-1}(\mu) \qquad \text{versus} \qquad \mathbf{J}^{-1}(\mu) \cap M_H$$

- JPO, Ratiu [2002]
- G acts on $(M, \{\cdot, \cdot\})$ via $\Phi : G \times M \to M$.
- $A_G := \{\Phi_g : M \to M \mid g \in G\} \subset \mathcal{P}(M)$.
- $A'_G := \{X_f(m) \mid f \in C^\infty(M)^G\}$.
- The canonical projection

$$\mathcal{J} : M \to M/A'_G$$

is the **optimal momentum map** associated to the G-action on M.

- $\mathcal{J}$ always defined:

  1. S^1 on $\mathbb{T}^2$

$$\begin{array}{rccc} \mathcal{J}: & \mathbb{T}^2 & \longrightarrow & S^1 \\ & (e^{i\theta_1},\, e^{i\theta_2}) & \longmapsto & e^{i\theta_2}. \end{array}$$

  2. $\mathbb{R}$ on $\mathbb{R}^3$

$$\begin{array}{rccc} \mathcal{J}: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & x+z. \quad \blacklozenge \end{array}$$

- Why momentum map?

  Noether's Theorem: $\mathcal{J}$ is universal. Let F_t flow of X_h, $h \in C^\infty(M)^G$ then

$$\mathcal{J} \circ F_t = \mathcal{J}$$

- Why optimal?

  Theorem: G acting properly on $(M,\, \omega)$ with associated momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Then:

$$A'_G(m) = \ker T_m\mathbf{J} \cap T_m M_{G_m}.$$

  Hence, the level sets of $\mathcal{J}$ are

$$\mathcal{J}^{-1}(\rho) = \mathbf{J}^{-1}(\mu) \cap M_H,$$

Symplectic reduction

- Marsden, Weinstein (1974): free proper action implies $\mathbf{J}^{-1}(\mu)/G_\mu$ "canonically" symplectic
$$\pi_\mu^*\omega_\mu = i_\mu^*\omega$$
where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ and $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$
- Reduction of dynamics: $h \in C^\infty(M)^G$. The flow F_t of X_h leaves $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the G–action, so it induces a flow F_t^μ on M_μ defined by
$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu.$$
The flow F_t^μ on (M_μ, ω_μ) is Hamiltonian with associated ***reduced Hamiltonian function*** $h_\mu \in C^\infty(M_\mu)$ defined by
$$h_\mu \circ \pi_\mu = h \circ i_\mu.$$
The triple $(M_\mu, \omega_\mu, h_\mu)$ is called the ***reduced Hamiltonian system***.
- Reduces the search of relative equilibria and relative periodic orbits to equilibria and periodic orbits.

Reconstruction of dynamics: Assume that an integral curve $c_\mu(t)$ of the reduced Hamiltonian system X_{h_μ} on (M_μ, ω_μ) is known. Let $m_0 \in \mathbf{J}^{-1}(\mu)$ be given. Can one determine from this data the integral curve of the Hamiltonian system X_h with initial condition m_0? In other words, can one ***reconstruct*** the solution of the given system knowing the corresponding reduced solution?

Pick a smooth curve $d(t)$ in $\mathbf{J}^{-1}(\mu)$ such that $d(0) = m_0$ and $\pi_\mu(d(t)) = c_\mu(t)$. Then, if $c(t)$ denotes the integral curve of X_h with $c(0) = m_0$, we can write $c(t) = g(t) \cdot d(t)$ for some smooth curve $g(t)$ in G_μ determined in two steps:

- **Step 1:** *find a smooth curve* $\xi(t)$ *in* $\mathfrak{g}_\mu$

$$\xi(t)_M(d(t)) = X_h(d(t)) - \dot{d}(t);$$

- **Step 2:** *with* $\xi(t) \in \mathfrak{g}_\mu$ *determined above, solve the non–autonomous differential equation on* G_μ

$$\dot{g}(t) = T_e L_{g(t)} \xi(t), \qquad \textit{with} \qquad g(0) = e.$$

Coadjoint orbits as reduced spaces. Take $M = T^*G$, where G is a Lie group with Lie algebra $\mathfrak{g}$, the G–action being the cotangent lift of *left* translation, and the associated momentum map $\mathbf{J}_L : \alpha_g \in T^*G \mapsto T_e^*R_g(\alpha_g) \in \mathfrak{g}^*$ which is right invariant. For each $\mu \in \mathfrak{g}^*$ we can form the symplectic point reduced space $((T^*G)_\mu, \omega_\mu)$. Recall also that the momentum map for the lift of *right* translations is left invariant and is given by $\mathbf{J}_R : \alpha_g \in T^*G \mapsto T_e^*L_g(\alpha_g) \in \mathfrak{g}^*$.

The momentum map $\mathbf{J}_R : T^*G \to \mathfrak{g}^*$ induces for each $\mu \in \mathfrak{g}^*$ a symplectic diffeomorphism $\overline{\mathbf{J}}_R : ((T^*G)_\mu, \omega_\mu) \to (\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$ given by $\overline{\mathbf{J}}_R([T_g^*R_{g^{-1}}\mu]) = \mathrm{Ad}_g^*\,\mu$.

Cotangent bundles. G acts on Q freely and properly. The map

$$\varphi_0 : \big((T^*Q)_0, (\Omega_Q)_0\big) \to \Big(T^*(Q/G), \Omega_{Q/G}\Big)$$

given by $\varphi_0([\alpha_q])(T_q\rho(v_q)) := \alpha_q(v_q)$, with $\alpha_q \in \mathbf{J}^{-1}(0)$, $v_q \in T_qQ$, is a symplectomorphism.

We now study the symplectic reduced space

$$((T^*Q)_\mu, \omega_\mu).$$

Let $\mu' := \mu|\mathfrak{g}_\mu \in \mathfrak{g}_\mu^*$ the restriction of μ to $\mathfrak{g}_\mu$, and consider the G_μ–action on Q and its lift to T^*Q. An equivariant momentum map of this action is the map $\mathbf{J}^\mu : T^*Q \to \mathfrak{g}_\mu$ obtained by restricting $\mathbf{J}$. Assume there is a G_μ–invariant one-form α_μ on Q with values in $(\mathbf{J}^\mu)^{-1}(\mu')$.

For $\xi \in \mathfrak{g}_\mu$ and $q \in Q$, the identity $(\mathbf{i}_{\xi_Q}\alpha_\mu)(q) = \alpha_\mu(q)(\xi_Q(q)) = \langle \mathbf{J}(\alpha_\mu(q)), \xi \rangle = \langle \mu', \xi \rangle$ shows that $\mathbf{i}_{\xi_Q}\alpha_\mu$ is a constant function on Q. Therefore, for $\xi \in \mathfrak{g}_\mu$, this implies $\mathbf{i}_{\xi_Q}\mathbf{d}\alpha_\mu = \pounds_{\xi_Q}\alpha_\mu - \mathbf{d}\mathbf{i}_{\xi_Q}\alpha_\mu = 0$, since $\pounds_{\xi_Q}\alpha_\mu = 0$ by G_μ-invariance of α_μ. It follows that there is a unique two-form β_μ on Q_μ such that $\rho_\mu^*\beta_\mu = \mathbf{d}\alpha_\mu$. Since ρ_μ is a submersion, β_μ is closed, but need not be exact. Let $B_\mu = \pi_{Q_\mu}^*\beta_\mu$ where $\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$ is the cotangent bundle projection.

Embedding cotangent bundle reduction theorem Under the above hypotheses, the map

$$\varphi_\mu : ((T^*Q)_\mu, (\Omega_Q)_\mu) \to (T^*Q_\mu, \Omega_{Q_\mu} - B_\mu),$$

given by $\varphi_\mu([\alpha_q])(T_q\rho_\mu(v_q)) := (\alpha_q - \alpha_\mu(q))(v_q)$, for $\alpha_q \in \mathbf{J}^{-1}(\mu)$, $v_q \in T_qQ$, is a symplectic embedding onto a vector subbundle of T^*Q_μ. The map φ_μ is onto T^*Q_μ if and only if $\mathfrak{g} = \mathfrak{g}_\mu$. The additional summand B_μ in the symplectic structure of T^*Q_μ is called a ***magnetic term***.

Symplectic orbit reduction

(i) The set $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i^*_{\mathcal{O}_\mu}\omega = \pi^*_{\mathcal{O}_\mu}\omega_{\mathcal{O}_\mu} + \mathbf{J}^*_{\mathcal{O}_\mu}\omega^+_{\mathcal{O}_\mu}, \tag{5}$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ is the restriction of $\mathbf{J}$ to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ and $\omega^+_{\mathcal{O}_\mu}$ is the $+$–symplectic structure on the affine orbit $\mathcal{O}_\mu$. The maps $i_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow M$ and $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ are natural injection and the projection, respectively. The pair $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the ***symplectic orbit reduced space***.

(ii) Thesed are, up to connected components, the symplectic leaves of $(M/G, \{\cdot,\cdot\}_{M/G})$.

(iii) Same dynamical statements that we have for the point reduced spaces.

Cylinder Valued Regular Reduction Theorem: G acts freely and properly. If $\mathcal{H}$ is closed then $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is symplectic with form given by

$$\pi^*_{[\mu]}\omega_{[\mu]} = i^*_\mu \omega$$

If $\mathcal{H}$ is not closed the theorem is **false** in general

See presentation of Ratiu in the conference for the general case.

Optimal reduction

$(M, \{\cdot,\cdot\})$ Poisson manifold, G acts properly and canonically on M. Then, for any $\rho \in M/A'_G$,

- $\mathcal{J}^{-1}(\rho)$ is an initial submanifold of M.
- The isotropy subgroup $G_\rho \subset G$ of is an (immersed) Lie subgroup of G.
- $T_m(G_\rho \cdot m) = T_m(\mathcal{J}^{-1}(\rho)) \cap T_m(G \cdot m)$.
- If G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$ then $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a regular quotient manifold called the **reduced phase space**.
- The canonical projection
$$\pi_\rho : \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)/G_\rho$$
is a submersion.
- **Optimal Symplectic Reduction**: M_ρ is symplectic with ω_ρ given by
$$\pi_\rho^* \omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m),$$
for any $m \in \mathcal{J}^{-1}(\rho)$ and $f, h \in C^\infty(M)^G$.

THE GLOBALLY HAMILTONIAN CASE

$$
\begin{aligned}
M_\rho = \mathbf{J}^{-1}(\mu) \cap M_H / N_{G_\mu}(H) \\
= \mathbf{J}^{-1}(\mu) \cap M_H / (N_{G_\mu}(H)/H) \\
\simeq (\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu})/G_\mu = M_\mu^{(H)}
\end{aligned}
$$

These are Sjamaar and Lerman [1991] reduced spaces.

- Guillemin, Sternberg (1982): Kähler reduction at zero. Kirwan (1984)
- Sjamaar, Lerman (1991), Bates, Lerman (1997): Singular reduction
- Sjamaar (1995): Singular Kähler

Singular reduction

- There is a unique symplectic structure $\omega_\mu^{(H)}$ on $M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z]/G_\mu$ characterized by
$$i_\mu^{(H)\,*}\omega = \pi_\mu^{(H)\,*}\omega_\mu^{(H)}$$
- The symplectic spaces $M_\mu^{(H)}$ stratify $\mathbf{J}^{-1}(\mu)/G_\mu$
- Sjamaar's principle and regularization
$$[\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z]/G_\mu \simeq \mathbf{J}_{L^z}^{-1}(0)/L_0^z$$
- For the record
$$\mathbf{J}_{L^z}^{-1}(0)/L_0^z = [\mathbf{J}^{-1}(\mu) \cap M_H^z]/(N_{G_\mu}(H)^z/H)$$

Stratification in what sense?

- Decomposed space: $R, S \in \mathcal{Z}$ with $R \cap \bar{S} \neq \emptyset$, then $R \subset \bar{S}$
- Stratification: decomposed space with a condition on the set germ of the pieces
- Stratification with smooth structure: there are charts $\phi : U \to \phi(U) \subset \mathbb{R}^n$ from an open set $U \subset P$ to a subset of $\mathbb{R}^n$ such that for every stratum $S \in \mathcal{S}$ the image $\phi(U \cap S)$ is a submanifold of $\mathbb{R}^n$ and the restriction $\phi|_{U \cap S} : U \cap S \to \phi(U \cap S)$ is a diffeomorphism.
- Whitney stratifications
- Cone space: existence of links

$$\psi : U \to (S \cap U) \times CL,$$

Where do the charts come from?

- Marle-Guillemin-Sternberg normal form (1984)
- Hamiltonian G–manifold $(M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*)$ can be locally identified with

$$\left(Y_r := G \times_{G_m} (\mathfrak{m}_r^* \times V_r), \omega_{Y_r}\right)$$

- The momentum map takes the expression

$$\mathbf{J}([g, \rho, v]) = \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(m) + \rho + \mathbf{J}_V(v)) + \sigma(g)$$

- Main observation

$$\mathbf{J}^{-1}(\mu)/G_\mu \simeq \mathbf{J}_{V_r}^{-1}(0)/G_m$$

- The symplectic strata are locally described by the strata obtained (roughly speaking) from the stratification by orbit types of $\mathbf{J}_{V_r}^{-1}(0)$ as a G_m space
- Generalization by Scheerer and Wulff (2001) with local momentum maps and by JPO and Ratiu (2002) using the Chu map

The reconstruction equations

$$X_{\mathfrak{q}} = X_{\mathfrak{h}} = 0$$

$$X_{\mathfrak{m}}(g, \rho, v) = T_e L_g(D_{\mathfrak{m}_r^*}(h \circ \pi)(\rho, v))$$

$$X_{V_r} = B_V^\sharp(D_{V_r}(h \circ \pi)(\rho, v))$$

$$X_{\mathfrak{m}_r^*} = \mathbb{P}_{\mathfrak{m}^*}\left(\mathrm{ad}^*_{D_{\mathfrak{m}_r^*}(h\circ\pi)}\rho\right) + \mathrm{ad}^*_{D_{\mathfrak{m}_r^*}(h\circ\pi)}\mathbf{J}_V(v).$$

Hamiltonian Coverings

$\mathfrak{g}$ acting symplectically on (M, ω). $p_N : N \to M$ is a ***Hamiltonian covering map*** of (M, ω):

(i) p_N is a smooth covering map

(ii) (N, ω_N) is a connected symplectic manifold

(iii) p_N is a symplectic map

(iv) $\mathfrak{g}$ acts symplectically on (N, ω_N) and has a standard momentum map $\mathbf{K}_N : N \to \mathfrak{g}^*$

(v) p_N is $\mathfrak{g}$–equivariant, that is, $\xi_M(p_N(n)) = T_n p_N \cdot \xi_N(n)$, for any $n \in N$ and any $\xi \in \mathfrak{g}$.

The category of Hamiltonian covering maps

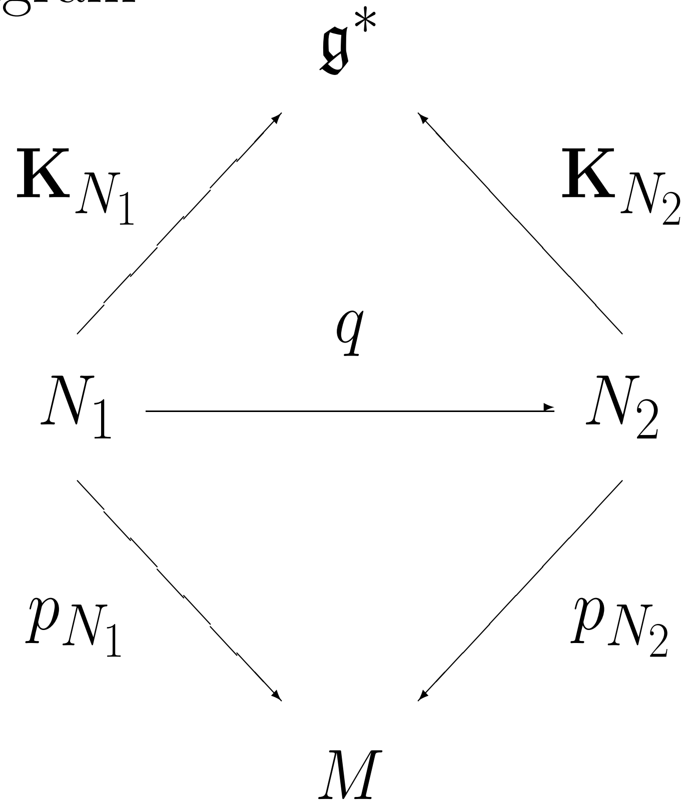
$\mathfrak{g}$ Lie algebra acting symplectically on (M, ω).

- $\mathrm{Ob}(\mathfrak{H}) = \{(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])\}$ with $p_N : N \to M$ a Hamiltonian covering map of (M, ω)
- $\mathrm{Mor}(\mathfrak{H}) = \{q : (N_1, \omega_1) \to (N_2, \omega_2)\}$ with:

  (i) q is a symplectic covering map

  (ii) q is $\mathfrak{g}$–equivariant

  (iii) the diagram



commutes for some $\mathbf{K}_{N_1} \in [\mathbf{K}_{N_1}]$ and $\mathbf{K}_{N_2} \in [\mathbf{K}_{N_2}]$.

Proposition Let (M, ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. Let $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$ be the object in $\mathfrak{H}$ constructed using the universal covering of M.

For any other object $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])$ of $\mathfrak{H}$, there exists a morphism $q : \widehat{M} \to N$ in $\mathrm{Mor}(\mathfrak{H})$.

Any other object in $\mathfrak{H}$ that satisfies the same universality property is isomorphic to $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$.

The holonomy bundles of α are Hamiltonian coverings of $(M, \omega, \mathfrak{g})$.

Proposition The pair $(\widetilde{M}, \omega_{\widetilde{M}} := \widetilde{p}^* \omega)$ is a symplectic manifold on which $\mathfrak{g}$ acts symplectically by

$$\xi_{\widetilde{M}}(m, \mu) := (\xi_M(m), -\Psi(m)(\xi, \cdot)),$$

where $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map. The projection $\widetilde{\mathbf{K}} : \widetilde{M} \to \mathfrak{g}^*$ of $\widetilde{M}$ into $\mathfrak{g}^*$ is a momentum map for this action. The 4–tuple $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ is an object in $\mathfrak{H}$

Theorem $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ is a universal Hamiltonian covered space in $\mathfrak{H}$, that is, given any other object $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])$ in $\mathfrak{H}$, there exists a (not necessarily unique) morphism $q : N \to \widetilde{M}$ in $\mathrm{Mor}(\mathfrak{H})$. Any other object of $\mathfrak{H}$ that satisfies this universality property is isomorphic to $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$.

Reduction using the cylinder valued momentum map

First ingredient: a ``coadjoint'' action

$\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ cylinder valued momentum map associated to a G–action on (M, ω)

- $\mathcal{H}$ is Ad*-invariant: $\mathrm{Ad}^*_{g^{-1}}(\mathcal{H}) \subset \mathcal{H}$, $g \in G$
- There exists a unique action

$$\mathcal{A}d^* : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \to \mathfrak{g}^*/\overline{\mathcal{H}}$$

  such that for any $g \in G$

$$\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$$

Second ingredient: a non-equivariance cocycle

Define

$$\sigma(g, m) := \mathbf{K}(g \cdot m) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(m)$$

- If M is connected $\sigma : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ does not depend on M.
- $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ is a group-valued one-cocyle, that is

$$\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h)$$

The Reduction Theorem

The map

$$\begin{array}{rcl}\Theta : G \times \mathfrak{g}^*/\overline{\mathcal{H}} & \longrightarrow & \mathfrak{g}^*/\overline{\mathcal{H}} \\ (g, \mu + \overline{\mathcal{H}}) & \longmapsto & \mathcal{A}d^*_{g^{-1}}(\mu + \overline{\mathcal{H}}) + \sigma(g)\end{array}$$

is a group action such that

$$\mathbf{K}(g \cdot m) = \Theta_g(\mathbf{K}(m))$$

Reduction Lemma

$$\mathfrak{g}_{\mu+\overline{\mathcal{H}}} \cdot m = \ker T_m\mathbf{K} \cap \mathfrak{g} \cdot m$$

Corollary: if $\overline{\mathcal{H}}$ is closed then

$$\mathfrak{g}_{\mu+\overline{\mathcal{H}}} \cdot m = (\mathfrak{g} \cdot m)^\omega \cap \mathfrak{g} \cdot m$$

Regular Reduction Theorem: G acts freely and properly. If $\mathcal{H}$ is closed then $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is symplectic with form given by

$$\pi^*_{[\mu]}\omega_{[\mu]} = i^*_\mu\omega$$

If $\mathcal{H}$ is not closed the theorem is **false** in general

Stratification Theorem

Using the symplectic slice theorem the cylinder valued momentum map locally looks like

$$\mathbf{K}(\phi[g,\rho,v]) = \Theta_g(\mathbf{K}(m) + \pi_C(\rho + \mathbf{J}_{V_m}(v)))$$

Reproduce the Bates-Lerman proposition in this setup

$$\begin{aligned} &\mathbf{K}^{-1}([\mu]) \cap Y_0 \\ &\qquad \simeq \{[g,0,v] \in Y_0 \mid g \in G_{[\mu]}, v \in \mathbf{J}_{V_m}^{-1}(0)\} \end{aligned}$$

Stratification Theorem If $\mathcal{H}$ is closed then the quotient $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is a cone space with strata

$$[\mathbf{K}^{-1}([\mu]) \cap G_{[\mu]} M_H^z]/G_{[\mu]} \simeq \mathcal{J}^{-1}(\rho)/G_\rho$$

Sjamaar's principle is missing

Groupoids

A groupoid $G \rightrightarrows X$ with ***base*** X and ***total space*** G:

(i) $\alpha, \beta : G \to X$. α is the ***target*** map and β is the ***source*** map. An element $g \in G$ is thought of as an arrow from $\beta(g)$ to $\alpha(g)$ in X.

(ii) The ***set of composable pairs*** is defined as:

$$G^{(2)} := \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\}.$$

There is a ***product map*** $m : G^{(2)} \to G$ that satisfies $\alpha(m(g, h)) = \alpha(g)$, $\beta(m(g, h)) = \beta(h)$, and $m(m(g, h), k) = m(g, m(h, k))$, for any $g, h, k \in G$.

(iii) An injection $\epsilon : X \to G$, the ***identity section***, such that $\epsilon(\alpha(g))g = g = g\epsilon(\beta(g))$. In particular, $\alpha \circ \epsilon = \beta \circ \epsilon$ is the identity map on X.

(iv) An ***inversion map*** $i : G \to G$, $i(g) = g^{-1}$, $g \in G$, such that $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

Examples

- ***Group:*** $G \rightrightarrows \{e\}$.
- ***The action groupoid:***
  - $\Phi : G \times M \to M$
  - $G \times M \rightrightarrows M$
    - $\alpha(g, m) := g \cdot m$, $\beta(g, m) := m$
    - $\epsilon(m) := (e, m)$
    - $m((g, h \cdot n), (h, n)) := (gh, n)$
    - $(g, m)^{-1} := (g^{-1}, g \cdot m)$
  - The orbits and isotropy subgroups of this groupoid coincide with those of the group action Φ.
- ***The cotangent bundle of a Lie group.***
  - $T^*G \simeq G \times \mathfrak{g}^*$
  - $T^*G \rightrightarrows \mathfrak{g}^*$
    - $\alpha(g, \mu) := \mathrm{Ad}^*_{g^{-1}}\mu$, $\beta(g, \mu) := \mu$
    - $\epsilon(\mu) = (e, \mu)$
    - $m((g, \mathrm{Ad}^*_{h^{-1}}\mu), (h, \mu)) = (gh, \mu)$
    - $(g, \mu)^{-1} = (g^{-1}, \mathrm{Ad}^*_{g^{-1}}\mu)$.

- ***The Baer groupoid*** $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$.
  - $\mathfrak{S}(G)$ set of subgroups of G
  - $\mathfrak{B}(G)$ set of cosets of elements in $\mathfrak{S}(G)$
    - $\alpha, \beta : \mathfrak{B}(G) \to \mathfrak{S}(G)$ are defined by $\alpha(D) = Dg^{-1}$, $\beta(D) = g^{-1}D$ for some $g \in D$.
    - $m(D_1, D_2) := D_1 D_2$.
    - The orbits of $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$ are given by the conjugacy classes of subgroups of G.

Groupoid Actions

$J : M \to X$ a map from M into X and

$G \times_J M := \{(g, m) \in G \times M \mid \beta(g) = J(m)\}$.

A (left) ***groupoid action*** of G on M with ***moment map*** $J : M \to X$ is a mapping

$$\begin{array}{rcl} \Psi : G \times_J M & \longrightarrow & M \\ (g, m) & \longmapsto & g \cdot m := \Psi(g, m), \end{array}$$

that satisfies the following properties:

(i) $J(g \cdot m) = \alpha(g)$,

(ii) $gh \cdot m = g \cdot (h \cdot m)$,

(iii) $(\epsilon(J(m))) \cdot m = m$.

Examples of Actions

(i) A groupoid acts on its total space and on its base. A groupoid $G \rightrightarrows X$ acts on G by multiplication with moment map α. G acts on X with moment map the identity I_X via $g \cdot \beta(g) := \alpha(g)$.

(ii) The G–action groupoid acts on G–spaces. Let G be acting on two sets M and N and let $J : M \to N$ be any equivariant map with respect to those actions. The map J induces an action of the product groupoid $G \times N \rightrightarrows N$ on M. The action is defined by

$$\begin{aligned} \Psi : (G \times N) \times_J M &\longrightarrow M \\ (((g, J(m)), m) &\longmapsto g \cdot m. \end{aligned}$$

(iii) The Baer groupoid acts on G–spaces.
Let G be a Lie group, M be a G–space, and

- $B : M \to \mathfrak{S}(G)$, $m \in M \longmapsto G_m \in \mathfrak{S}(G)$
- $\mathfrak{B}(G) \times_B M := \{(gG_m, m) \in \mathfrak{B}(G) \times M \mid m \in M\}$
- $\mathfrak{B}(G) \times_B M \to M$ given by $(gG_m, m) \mapsto g \cdot m$ defines an action of the Baer groupoid $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$ on the G–space M with moment map B
- The level sets of the moment map are the isotropy type subsets of M

Groupoid model of the optimal momentum map

- $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$, non equivariance one–cocycle $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$.
- $G \times \mathfrak{g}^*/\overline{\mathcal{H}} \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}}$ action groupoid associated to the affine action of G on $\mathfrak{g}^*/\overline{\mathcal{H}}$
- $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$ Baer groupoid of G
- $\left(G \times \mathfrak{g}^*/\overline{\mathcal{H}}\right) \times \mathfrak{B}(G) \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G)$ be the product groupoid and $\Gamma \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G)$ be the wide subgroupoid defined by

$$\Gamma := \{((g, [\mu]), gH) \mid g \in G,\ \mu \in \mathfrak{g}^*/\overline{\mathcal{H}},\ H \in \mathfrak{S}(G)\}.$$

- $\Gamma \rightrightarrows \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G)$ acts naturally on M with moment map

$$\begin{aligned} \mathfrak{J} :\ M &\longrightarrow \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G) \\ m &\longmapsto (\mathbf{K}(m), G_m). \end{aligned}$$

- Action of Γ on M:

$$\begin{array}{rccc} \Psi : & \Gamma \times_{\mathfrak{J}} M & \longrightarrow & M \\ & (((g, \mathbf{K}(m)), gG_m), m) & \longmapsto & g \cdot m. \end{array}$$

By the universality property of the optimal momentum map there exists a unique map $\varphi : M/A'_G \to \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G)$

$$\begin{array}{ccc} M & \xrightarrow{\quad \mathfrak{J} \quad} & \mathfrak{g}^*/\overline{\mathcal{H}} \times \mathfrak{S}(G) \\ & \mathcal{J} \searrow \quad \nearrow \varphi & \\ & M/A'_G & \end{array}$$

If $\mathcal{H}$ is closed

$$\mathfrak{J}^{-1}([\mu], G_m) = \mathbf{K}^{-1}([\mu]) \cap M_{G_m} = \mathcal{J}^{-1}(\rho)$$

Connectedness implies φ injective.