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Symmetry Reduction in Symplectic and Poisson Geometry

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Abstract. We present a quick review of several reduction techniques for symplectic and Poisson manifolds using local and global symmetries compatible with these structures. Reduction based on the standard momentum map (symplectic or Marsden–Weinstein reduction) and on generalized distributions (the optimal momentum map and optimal reduction) is emphasized. Reduction of Poisson brackets is also discussed and it is shown how it defines induced Poisson structures on cosymplectic and coisotropic submanifolds.

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1. Introduction

The use of symmetries in the quantitative and qualitative study of dynamical systems has a long history that goes back to the founders of mechanics. In most cases, the symmetries of a system are used to implement a procedure generically known under the name of *reduction* that restricts the study of its dynamics to a system of smaller dimension. This procedure is also used in a purely geometric context to construct new nontrivial symplectic or Poisson manifolds.

Most of the reduction methods presented in this paper can be seen as a generalization systematizing the techniques of elimination of variables found in classical mechanics. These procedures consist basically of two steps. First, one restricts the dynamics to flow invariant submanifolds of the system in question. Sometimes, these invariant manifolds appear as the level sets of a *momentum map* induced by the symmetry of the system. The construction of these momentum maps and the interplay between symmetry and conservation laws is one of the main topics of this presentation. The second step consists in projecting the restricted dynamics onto the symmetry orbit quotients of the spaces constructed in the first step. This passage to the quotient generally yields spaces that are not smooth manifolds, which explains why this procedure is sometimes called *singular reduction*.

Here we provide a self-contained, quick, and general overview of some of the reduction techniques found in the literature. The results presented here are not original, even though many of them cannot be found in journals; they appear for the first time in [44]. The proofs are omitted to keep the size of this review within a reasonable length. This allows the reader to gain a panoramic overview of these methods without being distracted by technical details. These are extremely important when a deeper understanding is desired but are avoidable in a first contact with the subject. All the proofs can be found in the original papers cited in the text or in our monograph [44].

2. Symmetry Reduction

The word *reduction* appears in the mathematics and physics literature in a variety of contexts.

2.1. THE CASE OF GENERAL VECTOR FIELDS

Let M be a smooth manifold and G be a Lie group acting properly on M. Let $X \in \mathfrak{X}(M)^G$ be a G-equivariant vector field on M and F_t be the corresponding (necessarily equivariant) flow. For any isotropy subgroup H of the G-action on M, the *H-isotropy type submanifold* M_H defined by

$$M_H := \{m \in M \mid G_m = H\} \tag{2.1}$$

is preserved by the flow F_t. The symbol G_m denotes the isotropy subgroup of the element $m \in M$. This property is known as the *law of conservation of isotropy*. The properness of the action guarantees that G_m is compact and that the (connected components of) M_H are embedded submanifolds of M for any closed subgroup H of G. The manifolds M_H are, in general, not closed in M. Moreover, the quotient group $N(H)/H$ (where $N(H)$ denotes the normalizer of H in G) acts freely and properly on M_H. Hence, if π_H: $M_H \to M_H/(N(H)/H)$ denotes the projection onto orbit space and i_H: $M_H \hookrightarrow M$ is the injection, the vector field X induces a unique vector field X^H on the quotient $M_H/(N(H)/H)$ defined by the expression

$$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$

whose flow F_t^H is given by $F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H$. We will refer to $X^H \in \mathfrak{X}(M_H/(N(H)/H))$ as the *H-isotropy type reduced vector field* corresponding to X.

This reduction technique has been widely exploited in specific examples (see [6, 13, 14]). When the symmetry group G is compact and we are dealing with a linear action the construction of the quotient $M_H/(N(H)/H)$ can be implemented in a very explicit and convenient manner by using the invariant polynomials of the action and the theorems of Hilbert, Schwarz, and Mather. Apart from the already cited works, the papers [7, 17–19] all use this method in concrete examples.

2.2. THE HAMILTONIAN CASE

Let (M, ω) be a symplectic manifold and G a connected Lie group, with Lie algebra $\mathfrak{g}$, acting freely and properly by symplectomorphisms on (M, ω). Assume that this action admits an associated equivariant momentum map $\mathbf{J}: M \to \mathfrak{g}^*$. If G is compact or semisimple this always holds. Recall that $\mathbf{J}$ is defined by the condition that for any element $\xi \in \mathfrak{g}$, the Hamiltonian vector field $X_{\mathbf{J}^\xi}$ associated to the function $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$ satisfies $X_{\mathbf{J}^\xi} = \xi_M$, where ξ_M is the infinitesimal generator vector field given by $\xi \in \mathfrak{g}$.

The Marsden–Weinstein reduction theorem [30] states that for any regular value $\mu \in \mathbf{J}(M) \subset \mathfrak{g}^*$ of $\mathbf{J}$, the quotient $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form ω_μ uniquely determined by the equality $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where G_μ is the isotropy subgroup of the element $\mu \in \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$, i_μ: $\mathbf{J}^{-1}(\mu) \hookrightarrow M$ the canonical injection, and π_μ: $\mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ the projection onto the orbit space.

In terms of dynamics, the interest of this construction is given by the fact that for any G-invariant Hamiltonian $h \in C^\infty(M)^G$, the corresponding Hamiltonian flow F_t leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant (*Noether's Theorem*) and commutes with the G-action, so it induces a flow F_t^μ on M_μ, uniquely determined by the identity $\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu$. The flow F_t^μ is Hamiltonian on (M_μ, ω_μ), with Hamiltonian function $h_\mu \in C^\infty(M_\mu)$ defined by the relation $h_\mu \circ \pi_\mu = h \circ i_\mu$. The function h_μ is called the *reduced Hamiltonian*.

Symplectic reduction is a very powerful tool that has been involved in many developments in symplectic geometry and in the study of Hamiltonian dynamical systems with symmetry [1]. Nevertheless, there are situations in which the just described reduction procedure does not work or is not efficient enough. For instance, the following situations can occur:

- The symmetry of the system does not admit a momentum map. This problem has been solved in some situations with the introduction of other types of momentum maps [2, 8, 12, 15, 32].
- The action is not free and therefore the symplectic quotient M_μ is not a smooth manifold. In the presence of a momentum map this situation has been treated in [4, 5, 9, 37, 46].
- The symmetry group is discrete and therefore the momentum map does not provide any conservation law.
- The phase space the system is not a symplectic but a Poisson manifold [28, 42].

3. Conservation Laws via Generalized Distributions

The optimal momentum map has been introduced in [43] as an approach, based on generalized distributions, to the problem of finding and describing the conservation laws associated to a canonical symmetry.

Unlike the standard momentum map, this object is related to global rather than to infinitesimal symmetries. One of the main goals behind its study consists in capturing the conservation laws that cannot be detected by the previously described momentum map. Even the generalized momentum maps alluded to above become trivial when the Lie algebra of the symmetry group is zero. This eliminates discrete symmetries from the general reduction scheme, a case that is very important in applications.

Another particularly convenient feature of the optimal momentum map is its generality. The construction presented previously (and other similar methods) is very symplectic in nature. This can be generalized to the Poisson setting, but there the existence of the momentum map becomes even more problematic. As will be shown in this section, the optimal momentum map *always exists* for *any* canonical group action on a Poisson manifold.

The use of the term 'optimal' is justified by the following property: the level sets of this map are the smallest possible submanifolds of phase space that are preserved by the flows of Hamiltonian vector fields of G-invariant functions. To be more specific, recall that the Hamiltonian vector field associated to an invariant Hamiltonian is automatically equivariant and therefore satisfies the law of conservation of the isotropy, discussed in Section 2. Thus, the isotropy type manifolds are invariant under its flow. This conservation law cannot be detected either by the standard momentum map discussed previously, or by its various generalizations mentioned above.

3.1. GENERALIZED FOLIATIONS AND DISTRIBUTIONS

To explain all of this, we quickly review generalized foliations and distributions. We begin with the notion of initial submanifold that naturally appears in this context. Let M and N be smooth manifolds and assume that $N \subset M$ as sets. Then N is called an *initial submanifold* of M if the inclusion map i: $N \hookrightarrow M$ is an immersion satisfying the following condition: for any smooth manifold P and any map g: $P \rightarrow N$, g is smooth if and only if $i \circ g$: $P \rightarrow M$ is smooth. By its very definition, the smooth manifold structure that makes N into an initial submanifold of M is unique. As we shall see below, initial submanifolds are very much relevant for generalized foliations.

A *generalized foliation* on M is a partition $\Phi = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ of this manifold into disjoint connected sets, called *leaves*, such that each point $z \in M$ has a *generalized foliated chart*, defined as a pair $(U, \varphi : U \rightarrow W \subset \mathbb{R}^m)$ with $z \in U$ and such that for each leaf $\mathcal{L}_\alpha$ there is a natural number $n \leq m$, called the *dimension* of $\mathcal{L}_\alpha$, and a subset $A_\alpha \subset \mathbb{R}^{m-n}$ such that

$$\varphi(U \cap \mathcal{L}_\alpha) = \{(z_1, \ldots, z_m) \in W \mid (z_{n+1}, \ldots, z_m) \in A_\alpha\}.$$

Each element $(z^i_{n+1}, \ldots, z^i_m) \in A_\alpha$ determines a connected component $(U \cap \mathcal{L}_\alpha)^i$ of $U \cap \mathcal{L}_\alpha$, that is, $\varphi((U \cap \mathcal{L}_\alpha)^i) = \{(z_1, \ldots, z_n, z^i_{n+1}, \ldots, z^i_m) \in W\}$. Notice that, unlike in the case of standard foliations, the number n may change from leaf to leaf. The generalized foliated charts induce on the leaves a smooth manifold structure relative to which they are initial submanifolds of M. Recall that even in the case of the usual foliations, that is, the dimension n is constant on M, the leaves are rarely embedded; they are, however, initial manifolds.

A *generalized distribution* D on M is a subset of the tangent bundle TM such that, for any point $m \in M$, the fiber $D(m) := D \cap T_m M$ is a vector subspace of $T_m M$. The dimension of $D(m)$ is called the *rank* or the *dimension* of the distribution D at the point m. A *differentiable section* of D is a differentiable vector field X defined on an open subset U of M, such that for any point $m \in U$, $X(m) \in D(m)$. An immersed connected submanifold N of M is said to be an *integral manifold* of the distribution D if, for every $z \in N$, $T_z i(T_z N) \subset D(z)$, where $i: N \hookrightarrow M$ is the injection. The integral submanifold N is said to be of *maximal dimension* at a point $z \in N$ if $T_z i(T_z N) = D(z)$. The generalized distribution D is *differentiable* if, for every point $m \in M$ and for every vector $v \in D(m)$, there exists a differentiable section X of D, defined on an open neighborhood U of m, such that $X(m) = v$. The generalized distribution D is *completely integrable* if, for every point $m \in M$, there exists an integral manifold of D everywhere of maximal dimension which contains m. The generalized distribution D is *involutive* if it is invariant under the (local) flows associated to differentiable sections of D. This definition of involutivity is more general than the usual one encountered in the Frobenius theorem and it only coincides with it when the dimension of $D(m)$ is the same for any $m \in M$, that is, precisely when D is a vector subbundle of TM. There are various characterizations of the complete integrability of a distribution, the most common being the *Stefan–Sussmann Theorem*: *D is completely integrable if and only if it is involutive*.

Let D be an integrable generalized distribution. Then for every point $m \in M$ there exists a unique connected integral manifold $\mathcal{L}_m$ of D that contains m and which is maximal in the following sense: it is everywhere of maximal dimension and if there is any other connected integral manifold $\mathcal{L}'$ of maximal dimension that intersects $\mathcal{L}_m$, then $\mathcal{L}'$ is an open submanifold of $\mathcal{L}_m$. The submanifold $\mathcal{L}_m$ is called the *maximal integral manifold* or the *accessible set* of D containing m. The maximal integral manifolds of D are always initial submanifolds of M and constitute a generalized foliation Φ_D of M. We shall denote by $M/D := M/\Phi_D$ the leaf space of Φ_D. The term 'accessible set' is justified by the fact that the maximal integral manifold $\mathcal{L}_m$ of D containing the point m coincides with the set of points that can be reached by applying to m finite compositions of flows of the (locally defined) differentiable sections that span D. This immediately leads to the concept of pseudogroups of transformations to which we turn next.

3.2. PSEUDOGROUPS AND THE EXTENSION PROPERTY

Recall that a *monoid* is a set with an associative operation which contains a two-sided identity element (which is hence unique). A *pseudogroup* is a submonoid A of a given monoid such that each element has an inverse in A. In particular, the set of all local diffeomorphisms of a manifold is not just a monoid but a pseudogroup. A useful property of pseudogroups of local diffeomorphisms of M is that they have orbits that partition the manifold. The orbit through $m \in M$ of the pseudogroup of transformations A is defined by

$$A \cdot m := \{\varphi(m) \mid \varphi \in A, m \text{ is in the domain of } \varphi\}.$$

Endowing the space of orbits M/A of a pseudogroup A of local diffeomorphisms with the quotient topology, makes the canonical projection $M \to M/A$ both continuous and open. A pseudogroup A of local diffeomorphisms of the manifold M is called *integrable* if its orbits form a generalized foliation of M. In particular, the orbits of an integrable pseudogroup are initial submanifolds of M. The generalized distribution D_A associated to the pseudogroup A is defined by the condition that $D_A(m)$ equals the tangent space to the A-orbit through $m \in M$ at m.

The pseudogroup A of local diffeomorphisms of M is said to have the *extension property* if any A-invariant function $f \in C^\infty(U)^A$ defined on any A-invariant open subset U has the following feature: for any $z \in U$, there is an A-invariant open neighborhood $V \subset U$ of z and an A-invariant smooth function $F \in C^\infty(M)^A$ such that $f|_V = F|_V$.

The group of (global) diffeomorphisms associated to a proper Lie group action has the extension property.

3.3. POLAR PSEUDOGROUPS

If $(M, \{\cdot,\cdot\})$ is a Poisson manifold, denote by $\mathcal{P}_L(M)$ the pseudogroup of all local Poisson diffeomorphisms of M and by $\mathcal{P}(M)$ the group of Poisson diffeomorphisms of M. It turns out that the optimal momentum map presented later on in this section has much to do with the notion of *polarity* introduced in [38].

If $A \subset \mathcal{P}_L(M)$ is a pseudogroup of local Poisson diffeomorphisms of M, denote by F_A the set of Hamiltonian vector fields associated to all the elements of $C^\infty(U)^A$ (A-invariant functions in $C^\infty(U)$), for all open A-invariant subsets U of M, that is,

$$F_A = \left\{X_f \mid f \in C^\infty(U)^A, \text{with } U \subset M \text{ open and } A\text{-invariant}\right\}.$$

The distribution $\mathcal{D}_{F_A}$ associated to the family F_A, that is,

$$\mathcal{D}_{F_A}(m) := \left\{X_f(m) \mid f \in C^\infty(U)^A, \text{ with } U \subset M \text{ open and } A\text{-invariant}, m \in U\right\}$$

for every $m \in U$, is called the *polar distribution* defined by A. Any generating family of vector fields for $\mathcal{D}_{F_A}$ is called a *polar family* of A. The family F_A is the *standard polar family* of A. The *polar pseudogroup* of A is defined by

$$A' := \{F^1_{t_1} \circ \cdots \circ F^n_{t_n} \mid n \in \mathbb{N} \text{ and } F^k_{t_k} \text{ is a local flow of some } X_{f_k} \in F_A, 1 \leq k \leq n\}.$$

For example, if $A \subset \mathcal{P}_L(M)$ a pseudogroup of local Poisson diffeomorphisms of M that has the extension property, then the family $\{X_f \mid f \in C^\infty(M)^A\}$ is a polar family.

A very important property of the polar distribution of a group of Poisson diffeomorphisms is that it is automatically Poisson and integrable.

PROPOSITION 3.1. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $A \subset \mathcal{P}(M)$ a group of Poisson diffeomorphisms of M. Then the following hold.*

(i) *The polar pseudogroup A' acts canonically and is integrable.*
(ii) *Any element of A commutes with any element of A'.*
(iii) *Any element $\varphi \in A'$ induces a local diffeomorphism $\bar{\varphi}$ of the presheaf space $(M/A, C^\infty_{M/A})$, uniquely determined by the relation $\bar{\varphi} \circ \pi_A = \pi_A \circ \varphi$, where π_A: $M \to M/A$ is the projection. In other words, the standard polar pseudogroup A' acts on the presheaf space $(M/A, C^\infty_{M/A})$.*
(iv) *The group A acts naturally on the orbit space M/A'. More specifically, for any $\phi \in A$, there is a diffeomorphism $\bar{\phi}$ of the quotient space $(M/A', C^\infty_{M/A'})$ uniquely determined by the relation $\bar{\phi} \circ \pi_{A'} = \pi_{A'} \circ \phi$, where $\pi_{A'}$: $M \to M/A'$ is the projection.*

3.4. PRESHEAF SPACES

We elaborate now on the meaning of the smoothness statements in parts (iii) and (iv).

Let $\mathcal{F}$ be a presheaf of functions defined on the topological space P. The pair $(P, \mathcal{F})$ is called a *presheaf space*. In all that follows it is assumed that $\mathcal{F}(U)$ is an algebra of continuous real valued functions on U for every open set $U \subset P$.

Let $(P_1, \mathcal{F}_1)$ and $(P_2, \mathcal{F}_2)$ be two presheaf spaces. The continuous map f: $(P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ is said to be *smooth* if for any open set $U \subset P_2$ we have $f^*\mathcal{F}_2(U) \subset \mathcal{F}_1(f^{-1}(U))$, where $f^*s := s \circ f$ for any $s \in \mathcal{F}_2(U)$. A bijective smooth map between presheaf spaces whose inverse is also smooth is called a *diffeomorphism*.

Let $\mathfrak{R}$ be an equivalence relation on the presheaf space $(M, \mathcal{F}_M)$ and π: $M \to M/\mathfrak{R}$ the canonical projection. The presheaf $\mathcal{F}_M$ on M naturally induces the *quotient presheaf* $\mathcal{F}_{M/\mathfrak{R}}$ on $M/\mathfrak{R}$ by

$$\mathcal{F}_{M/\mathfrak{R}}(U) := \{f \text{ function on } U \,|\, f \circ \pi|_{\pi^{-1}(U)} \in \mathcal{F}_M(\pi^{-1}(U))\}.$$

If $\mathcal{F}_M$ is a sheaf, then so is $\mathcal{F}_{M/\mathfrak{R}}$.

If M is a smooth manifold, the map that assigns to each open set the smooth functions on it is a sheaf denoted by C^∞_M. If A is a pseudogroup of local diffeomorphisms acting on M, then it defines an equivalence relation on M whose classes are the A-orbits. Thus the previous construction yields the quotient presheaf $C^\infty_{M/A}$ on M/A given on any open set $U \subset M$ by

$$C^\infty_{M/A}(U) := \{f \in C^\infty(U) \mid f \circ \pi|_{\pi^{-1}(U)} \in C^\infty_M(\pi^{-1}(U))\},$$

where π: $M \to M/A$ is the canonical projection. The words 'smooth' and 'diffeomorphism' in parts **(iii)** and **(iv)** of Proposition 3.1 need to be understood in terms of these definitions.

Let M be a topological space and $\mathcal{F}_M$ a presheaf of functions on M. Let $S \subset M$ be a subset of M endowed with a given topology $\mathcal{T}$ that does not necessarily coincide with the subspace topology. The presheaf $\mathcal{F}_M$ induces naturally the presheaf $\mathcal{F}_{S,M}$ of

Whitney smooth functions on $(S, \mathcal{T})$ which is defined in the following way: for each open subset V of S the set of functions $\mathcal{F}_{S,M}(V)$ equals all functions on V having the property that for any $z \in V$ there is a open neighborhood U_z of z in M and a function $F \in \mathcal{F}_M(U_z)$ such that $f|_{U_z \cap V} = F|_{U_z \cap V}$. The function F is called a *local extension* of f at z.

Let f: $(M, \mathcal{F}_M) \to (N, \mathcal{F}_N)$ be a smooth function and S and T two topological subspaces of M and N, respectively, such that $f(S) \subset T$. Then the map $\bar{f}$: $(S, \mathcal{F}_{S,M}) \to (T, \mathcal{F}_{T,N})$ constructed by restricting the domain and range of f to S and T, respectively, is also smooth.

If $\mathfrak{R}$ is a regular equivalence relation on the smooth manifold M then the quotient topological space $M/\mathfrak{R}$ is a smooth manifold and the canonical projection π: $M \to M/\mathfrak{R}$ is a surjective submersion. Let $C^\infty_{M/\mathfrak{R}}$ denote the presheaf of smooth functions on the manifold $M/\mathfrak{R}$. At the same time, the presheaf C^∞_M of smooth functions on M induces a quotient presheaf of functions on $M/\mathfrak{R}$, denoted by $C^\infty_{M/\mathfrak{R},\pi}$. The fact that π is a submersion implies that

$$C^\infty_{M/\mathfrak{R}} = C^\infty_{M/\mathfrak{R},\pi}. \tag{3.1}$$

An equivalence relation $\mathfrak{R}$ on the topological space M with a presheaf of functions $\mathcal{F}_M$ can be used to define another presheaf on the topological space of saturated open sets. An open subset of M is said to be $\mathfrak{R}$-*invariant* or $\mathfrak{R}$-*saturated* if it is the union of $\mathfrak{R}$-equivalence classes. The $\mathfrak{R}$-saturated sets of M form a topology for M, in general strictly weaker than the original topology. The presheaf $\mathcal{F}^{\mathfrak{R}}_M$ of $\mathfrak{R}$-*invariant* or $\mathfrak{R}$-*saturated functions* associates to each $\mathfrak{R}$-invariant open subset U the set

$$\mathcal{F}^{\mathfrak{R}}_M(U) := \{f \in \mathcal{F}_M(U) \mid f \text{ is constant on the equivalence classes of } \mathfrak{R}\}.$$

Let $S \subset M$ be a subset of M endowed with a given topology $\mathcal{T}$ and restrict the equivalence relation $\mathfrak{R}$ to S. Consider the presheaf $(\mathcal{F}_{S,M})^{\mathfrak{R}}$ of $\mathfrak{R}$-invariant functions on S and the restriction $(\mathcal{F}^{\mathfrak{R}}_M)_{S,M}$ to S of the presheaf $\mathcal{F}^{\mathfrak{R}}_M$ on $\mathfrak{R}$-invariant functions of M. A presheaf of much importance later on is the intersection of these two, denoted by $\mathcal{F}^{\mathfrak{R}}_{S,M}$, that is,

$$\mathcal{F}^{\mathfrak{R}}_{S,M} := (\mathcal{F}_{S,M})^{\mathfrak{R}} \cap (\mathcal{F}^{\mathfrak{R}}_M)_{S,M}.$$

The presheaf $(\mathcal{F}_{S,M})^{\mathfrak{R}}$ limits the domain of $\mathcal{F}^{\mathfrak{R}}_{S,M}$ to $\mathfrak{R}$-invariant open sets of $(S, \mathcal{T})$. To be more explicit, for any such set V, $\mathcal{F}^{\mathfrak{R}}_{S,M}(V)$ consists of $\mathfrak{R}$-invariant functions f defined on V with the property that for any $z \in V$ there exists an open $\mathfrak{R}$-saturated neighborhood U_z of z in M and a function $F \in \mathcal{F}^{\mathfrak{R}}_M(U_z)$ such that $f|_{U_z \cap V} = F|_{U_z \cap V}$. We will refer to $\mathcal{F}^{\mathfrak{R}}_{S,M}$ as the *presheaf of Whitney invariant functions* on S induced by $\mathcal{F}^{\mathfrak{R}}_M$.

PROPOSITION 3.2. *Let M be a topological space with a presheaf $\mathcal{F}_M$ of functions on it. Let $\mathfrak{R}$ be an equivalence relation on M and S an $\mathfrak{R}$-invariant subset of M endowed with a given topology $\mathcal{T}$. If $\mathcal{F}_{S/\mathfrak{R},M/\mathfrak{R}}$ is the presheaf of Whitney smooth functions on $S/\mathfrak{R}$ considered as a subset of $M/\mathfrak{R}$, then $\mathcal{F}_{S/\mathfrak{R},M/\mathfrak{R}} \subset \mathcal{F}^{\mathfrak{R}}_{S/\mathfrak{R}}$, where $\mathcal{F}^{\mathfrak{R}}_{S/\mathfrak{R}}$ is the quotient presheaf on $S/\mathfrak{R}$ corresponding to $\mathcal{F}^{\mathfrak{R}}_{S,M}$. If the projection π: $M \to M/\mathfrak{R}$ is an open map then $\mathcal{F}_{S/\mathfrak{R},M/\mathfrak{R}} = \mathcal{F}^{\mathfrak{R}}_{S/\mathfrak{R}}$.*

3.5. THE OPTIMAL MOMENTUM MAP

If Φ: $G \times M \to M$ is a canonical Lie group action on the Poisson manifold $(M, \{\cdot,\cdot\})$, denote by $A_G := \{\Phi_g | g \in G\} \subset \mathcal{P}(M)$ the associated group of Poisson diffeomorphisms and by A'_G the polar pseudogroup. The *optimal momentum map* $\mathcal{J}$: $M \to M/A'_G$ is defined as the projection of M onto the orbit space M/A'_G of the pseudogroup A'_G, polar to A_G that, by Proposition 3.1, is integrable. We will refer to the quotient M/A'_G as the *momentum space* of $\mathcal{J}$.

Notation. To simplify the notation, we shall use in the sequel interchangeably the symbol A'_G to denote both the standard polar pseudogroup to A_G and the polar distribution. It will be always clear from the context which notion is used. Moreover, we will denote by $A'_G \cdot m$ the orbit of the polar pseudogroup through $m \in M$ and by $A'_G(m)$ the polar distribution evaluated at m.

3.6. THE OPTIMAL MOMENTUM MAP FOR PROPER ACTIONS

If the G-action on M is proper, the subgroup A_G has the extension property. In this case, it can be shown that the optimal momentum map can be defined as the projection $\mathcal{J}$: $M \to M/\mathcal{D}_F$ onto the leaf space of the integrable distribution spanned by the family of vector fields

$$F := \left\{ X_f \mid f \in C^\infty(M)^G \right\} \tag{3.2}$$

and that the polar pseudogroup A'_G is a subgroup of the global diffeomorphisms group of M.

A particular case of the situation presented above is the case of a compact Lie group G acting canonically and linearly on a Poisson vector space $(V, \{\cdot,\cdot\})$. Let $\mathcal{B} := \{\sigma_1, \ldots, \sigma_n\}$ be a Hilbert basis for this action. By the Schwarz–Mather Theorem, any G-invariant function can be written as $f(\sigma_1, \ldots, \sigma_n)$, for some $f \in C^\infty(\mathbb{R}^n)$, so the chain rule guarantees that the distribution spanned by the family F in (3.2) is the same as the one spanned by the *finite* family $\{X_{\sigma_1}, \ldots, X_{\sigma_n}\}$.

Let us compute a few examples of optimal momentum maps.

EXAMPLE 1. As already remarked, a canonical Lie group action on a Poisson manifold does not necessarily preserve its symplectic leaves. Here is a simple example. Endow $(\mathbb{R}^3, \{\cdot,\cdot\})$ with the Poisson bracket defined by the Poisson tensor whose matrix in standard Euclidean coordinates is

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

If $f \in C^\infty(\mathbb{R}^3)$, the associated Hamiltonian vector field is given by

$$X_f(x, y, z) = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right)\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial z}. \tag{3.3}$$

This shows that the Casimir functions are all of the form $f(x, y, z) := g(x + z)$, with $g \in C^\infty(\mathbb{R})$ and that the symplectic leaves are hence given by the planes $x + z =$ constant.

Let the additive group $(\mathbb{R}, +)$ act on $\mathbb{R}^3$ by $\lambda \cdot (x, y, z) := (x + \lambda, y, z)$, for any $\lambda \in \mathbb{R}$ and any $(x, y, z) \in \mathbb{R}^3$. It is clear that this action preserves the Poisson bracket but, obviously, the symplectic leaves $x + z =$ constant are not invariant under this group action. In spite of this, we shall compute below the optimal momentum map which produces a conservation law associated to this symmetry.

As the $(\mathbb{R}, +)$-action is proper, we can use the distribution in (3.2) to define the corresponding optimal momentum map. Notice first that the invariant functions $f \in C^\infty(M)^{\mathbb{R}}$ in this case are all of the form $f(x, y, z) \equiv \bar{f}(y, z)$, with $\bar{f} \in C^\infty(\mathbb{R}^2)$ arbitrary. The expression (3.3) of the Hamiltonian vector fields defined by this bracket shows that the $A'_{\mathbb{R}}$-orbits on $\mathbb{R}^3$ coincide with those of the $\mathbb{R}^2$-action on $\mathbb{R}^3$ given by $(\mu, \nu) \cdot (x, y, z) := (x + \mu, y + \nu, z - \mu)$, for any $(\mu, \nu) \in \mathbb{R}^2$ and any $(x, y, z) \in \mathbb{R}^3$. Therefore, M/A'_G can be identified with $\mathbb{R}$ and the associated optimal momentum map $\mathcal{J}\colon \mathbb{R}^3 \to \mathbb{R}$ is given by $\mathcal{J}(x, y, z) = x + z$. A straightforward verification shows that the Hamiltonian flow associated to any invariant function $f(x, y, z) \equiv \bar{f}(y, z)$ preserves the level sets of $\mathcal{J}$. Note that $\mathcal{J}$ is a Casimir function of the Poisson manifold $(\mathbb{R}^3, \{\cdot, \cdot\})$.

EXAMPLE 2. The following example is classical: a free and canonical action of a compact Lie group on a compact symplectic manifold that does not admit a standard momentum map. Consider the two torus $\mathbb{T}^2 = \{(e^{i\theta_1}, e^{i\theta_2})\}$ with the symplectic form $\omega := d\theta_1 \wedge d\theta_2$. The circle $S^1 = \{e^{i\phi}\}$ acts canonically on $\mathbb{T}^2$ by $e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta_1 + \phi)}, e^{i\theta_2})$ but does not admit a standard momentum map $\mathbf{J}$: $\mathbb{T}^2 \to \mathbb{R}$. We shall compute below the optimal momentum map for this action.

The properness of the action allows us to use again the leaf space of the distribution (3.2). It is easy to see that in this case, every S^1-invariant function $f \in C^\infty(\mathbb{T}^2)^{S^1}$ can be written as $f(e^{i\theta_1}, e^{i\theta_2}) = g(e^{i\theta_2})$ for some arbitrary $g \in C^\infty(S^1)$. The Hamiltonian vector field associated to any of these invariant functions is given by $X_f = \partial g/\partial\theta_2\ \partial/\partial\theta_1$. Since g is an arbitrary function on the circle, we can identify the quotient M/A'_G with the second circle S^1 in the torus $\mathbb{T}^2$. The optimal momentum map $\mathcal{J}\colon \mathbb{T}^2 \to S^1$ is therefore given by $\mathcal{J}(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_2}$. In this case, the optimal momentum map is S^1-valued and coincides with the Lie group valued momentum map defined in [2].

3.7. THE MOMENTUM SPACE

In both examples the momentum space M/A'_G is a smooth manifold. This is a very rare occurrence. The quotient space M/A'_G carries, in general, a rather complicated topology that has not yet been fully explored. Even if the canonical G-action on the Poisson manifold M is such that the quotient topological space $M/G = M/A_G$ is a smooth manifold with the projection $\pi\colon M \to M/G$ a surjective submersion, the associated momentum space M/A'_G could be an extremely nonsmooth topological space with very unpleasant properties. The only general statements known today about the optimal momentum map $\mathcal{J}\colon M \to M/A'_G$ is that it is a continuous and open map.

EXAMPLE 3. Endow $M := \mathbb{T}^2 \times \mathbb{T}^2$ with the symplectic form $\omega := \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 + \sqrt{2}\,\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2$, where $(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \mathrm{e}^{i\psi_1}, \mathrm{e}^{i\psi_2}) \in M$, and consider the canonical free circle action given by

$$\mathrm{e}^{i\phi} \cdot (\mathrm{e}^{i\theta_1}, e^{i\theta_2}, \mathrm{e}^{i\psi_1}, \mathrm{e}^{i\psi_2}) := (\mathrm{e}^{i(\theta_1+\phi)}, \mathrm{e}^{i\theta_2}, \mathrm{e}^{i(\psi_1+\phi)}, \mathrm{e}^{i\psi_2}).$$

Thus M/A_{S^1} is a smooth manifold and the projection $\pi_{A_{S^1}}\colon M \to M/A_{S^1}$ is a surjective submersion. The polar distribution A'_{S^1} does not behave the same way. Indeed,

$$C^\infty(M)^{S^1} = \{f \in C^\infty(M) \mid f(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \mathrm{e}^{i\psi_1}, e^{i\psi_2}) = g(\mathrm{e}^{i\theta_2}, \mathrm{e}^{i\psi_2}, \mathrm{e}^{i(\theta_1-\psi_1)}) \text{ for some } g \in C^\infty(\mathbb{T}^3)\},$$

so by looking at the flows of Hamiltonian vector fields of functions in $C^\infty(M)^{S^1}$, one immediately sees that the leaves of A'_{S^1} fill densely the manifold M and that the leaf space M/A'_{S^1} can be identified with the leaf space $\mathbb{T}^2/\mathbb{R}$ of a Kronecker (irrational) foliation of a two-torus $\mathbb{T}^2$.

EXAMPLE 4. Consider on $\mathbb{C}^3$ the standard symplectic form

$$\omega((z_1, z_2, z_3), (z'_1, z'_2, z'_3)) = -\mathrm{Im}\,\langle (z_1, z_2, z_3), (z'_1, z'_2, z'_3)\rangle$$

and let SU(3) act naturally on $\mathbb{C}^3$. This action is canonical and linear and therefore has a standard associated momentum map. The polar distribution $A'_{\mathrm{SU}(3)}$ is spanned by the Hamiltonian vector fields associated to the elements of a Hilbert basis of invariant polynomials. In this case, the polynomial

$$f(z_1, z_2, z_3) = \tfrac{1}{2}\Big(|z_1|^2 + |z_2|^2 + |z_3|^2\Big)$$

constitutes such a basis. The Hamiltonian flow of X_f is given by

$$F_t(z_1, z_2, z_3) = (z_1\mathrm{e}^{-it}, z_2\mathrm{e}^{-it}, z_3\mathrm{e}^{-it}).$$

Therefore, the momentum space $\mathbb{C}^3/A'_{\mathrm{SU}(3)}$ coincides with $\mathbb{C}^3/S^1$, where S^1 acts on $\mathbb{C}^3$, by

$$\mathrm{e}^{i\phi} \cdot (z_1, z_2, z_3) = (\mathrm{e}^{i\phi}z_1, \mathrm{e}^{i\phi}z_2, \mathrm{e}^{i\phi}z_3). \tag{3.4}$$

This quotient space can be identified with $(\mathbb{CP}(2) \times \mathbb{R}^+) \cup \{*\}$, where $\{*\}$ denotes a singleton or, said differently, with the cone $C(\mathbb{CP}(2))$ based on $\mathbb{CP}(2)$. Indeed, if $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3/S^1$ is the canonical projection and $\mathbf{z} = (z_1, z_2, z_3)$, then the mapping that assigns $\pi(z_1, z_2, z_3)$ to $([\mathbf{z}/\|\mathbf{z}\|], \|\mathbf{z}\|)$ if $\mathbf{z} \neq 0$, and to $*$ if $\mathbf{z} = \mathbf{0}$, provides the needed identification (the symbol $[\mathbf{z}/\|\mathbf{z}\|]$ denotes the element $\pi(\mathbf{z}/\|\mathbf{z}\|) \in \mathbb{CP}(2)$). The optimal momentum map $\mathcal{J}\colon \mathbb{C}^3 \to (\mathbb{CP}(2) \times \mathbb{R}^+) \cup \{*\}$ has the expression

$$\mathcal{J}(\mathbf{z}) = \begin{cases} \left(\left[\frac{\mathbf{z}}{\|\mathbf{z}\|}\right] \|\mathbf{z}\|\right), & \text{if } \mathbf{z} \neq 0\ , \\ *, & \text{if } \mathbf{z} = 0. \end{cases}$$

3.8. THE MOMENTUM SPACE AS A G-TOPOLOGICAL SPACE

By Proposition 3.1(iv) there is a smooth G-action on M/A'_G (smooth in the sense of presheaf spaces) given by

$$g \cdot \mathcal{J}(m) := \mathcal{J}(g \cdot m), \quad \text{for any } g \in G,\ m \in M. \tag{3.5}$$

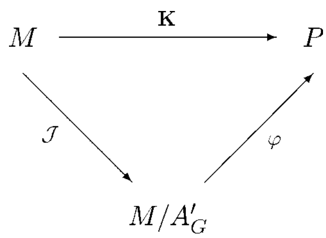
This is the unique G-action on M/A'_G that makes the optimal momentum map G-equivariant and it coincides with the usual smooth G-action on the leaf space of any distribution spanned by G-equivariant vector fields.

3.9. THE UNIVERSALITY PROPERTY

The optimal momentum map $\mathcal{J}$: $M \to M/A'_G$ associated to a canonical G-action on a Poisson manifold $(M, \{\cdot,\cdot\})$ satisfies *Noether's condition*, that is, $\mathcal{J}$ is constant along the flow of any Hamiltonian vector field defined by a G-invariant function. Indeed, due to the integrability of the polar distribution A'_G (see Proposition 3.1), the Stefan–Sussmann Theorem implies that the level sets of the optimal momentum map, that is, the leaves of the polar distribution, coincide with the orbits of the polar pseudogroup. More specifically, if $m \in M$ is such that $\mathbf{J}(m) = \rho \in M/A'_G$, then $\mathcal{J}^{-1}(\rho) = A'_G \cdot m$. As the polar pseudogroup consists of finite compositions of flows of Hamiltonian vector fields associated to all the possible invariant Hamiltonians, Noether's condition for $\mathcal{J}$ follows immediately.

In addition, $\mathcal{J}$ has the following universality property. Below, by 'momentum map' on M we mean any map $\mathbf{K}$: $M \to S$ whose target space is some set S such that $\mathbf{J}$ satisfies the Noether condition stated above. If S has additional topological or smooth structure, one requires that $\mathbf{K}$ is a map in the same category.

THEOREM 3.3 (Universality of the optimal momentum map). *The optimal momentum map is a universal object in the category of Hamiltonian symmetric systems with a momentum map. More specifically, if* $(M, \{\cdot,\cdot\}, G, \mathbf{K}$: $M \to P)$ *is any Hamiltonian G-space with momentum map* $\mathbf{K}$: $M \to P$ and $\mathcal{J}$: $M \to M/A'_G$ *is the optimal momentum map defined using the canonical G-action on M, then there exists a unique map* φ: $M/A'_G \to P$ *such that the following diagram commutes:*



If $\mathbf{K}$ *is smooth and G-equivariant with respect to some presheaf of functions on P and some G-action on P, then* φ *is also smooth and G-equivariant.*

3.10. COMPARISON BETWEEN THE OPTIMAL AND STANDARD MOMENTUM MAPS

Let G be a Lie group acting properly and canonically on the symplectic manifold (M, ω). The polar distribution A'_G can be explicitly determined in this case.

THEOREM 3.4. *Let G be a Lie group acting properly and canonically on the symplectic manifold* (M, ω). *Then for any* $m \in M$

$$A'_G(m) = (\mathfrak{g} \cdot m)^{\omega} \cap T_m M^m_{G_m}, \tag{3.6}$$

where $M^m_{G_m}$ *is the connected component of the isotropy type submanifold* M_{G_m} *that contains the point m.*

Using this information one can compare $\mathcal{J}$: $M \to M/A'_G$ and a given standard momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$. We shall return to the relationships below when discussing singular reduction. In the next corollary it is assumed that the symplectic group action has an associated standard momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$ with *nonequivariance one-cocycle* σ: $G \to \mathfrak{g}^*$, that is, $\sigma(g) := \mathbf{J}(g \cdot m) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(m)$ for any $g \in G$ and $m \in M$. The fact that σ does not depend on $m \in M$ is a consequence of the connectivity of M. Denote by Θ: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ the affine action of G on $\mathfrak{g}^*$ defined by σ, that is, $\Theta(g, v) := \mathrm{Ad}^*_{g^{-1}} v + \sigma(g)$ for any $g \in G$ and $v \in \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a value of $\mathbf{J}$; G_μ will denote the isotropy subgroup of μ with respect to the affine action Θ.

COROLLARY 3.5. *Let G be a Lie group acting properly and canonically on the connected symplectic manifold* (M, ω) *and admitting a standard momentum map* $\mathbf{J}$: $M \longrightarrow \mathfrak{g}^*$ *with nonequivariance one-cocycle* σ: $G \to \mathfrak{g}^*$. *Let* $\mathcal{J}$: $M \to M/A'_G$ *be the optimal momentum map. Then, for any* $m \in M$ *such that* $\mathbf{J}(m) = \mu$ *and* $\mathcal{J}(m) = \rho$, *we have*

$$\mathcal{J}^{-1}(\rho) = (\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m, \tag{3.7}$$

where $(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$ *denotes the connected component of* $\mathbf{J}^{-1}(\mu) \cap M^m_{G_m}$ *that contains the point m.*

The isotropy subgroup G_ρ *of the point* $\rho \in M/A'_G$ *with respect to the action (3.5) equals* $G_\rho = N_{G_\mu}(G_m)^{c(m)}$, *where* $N_{G_\mu}(G_m)^{c(m)}$ *is the closed subgroup of* $N_{G_\mu}(G_m) := N(G_m) \cap G_\mu$ *consisting of the elements in* $N_{G_\mu}(G_m)$ *that leave the connected component* $(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$ *of* $\mathbf{J}^{-1}(\mu) \cap M^m_{G_m}$ *invariant;* $N(G_m)$ *denotes the normalizer of* G_m *in G.*

The standard momentum map has the following remarkable property. Let (M, ω) be a connected symplectic manifold and G a Lie group acting on M in a canonical and proper fashion. Suppose that this action has an associated (not necessarily equivariant) momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$. Then for any $m \in M$, the intersection $\mathbf{J}^{-1}(\mathbf{J}(m)) \cap M^m_{G_m}$ is an embedded submanifold of M.

Even though, in general, the level sets of the optimal momentum map are just initial submanifolds of M, Corollary 3.5 and the result above imply that in the symplectic case and if a standard momentum map exists, the level sets $\mathcal{J}^{-1}(\rho)$ are embedded submanifolds of M.

4. The Optimal Momentum Map and Groupoids

In this short section we show that, in some sense, the optimal momentum map can be interpreted as the moment map of a natural groupoid action. The results in this section are admittedly incomplete because the investigation of the relationship between the optimal momentum map and groupoids begun in [33] and [52] has not been yet totally clarified and is the subject of ongoing research.

4.1. GROUPOIDS

We recall here the minimal necessary background on groupoids for our developments. We refer to [3, 23, 34] and references therein for further information.

A *groupoid* $G \rightrightarrows X$ over the set X, the *base*, is a set G, the *total space*, together with the following structure maps:

(i) α, β: $G \to X$; α is the *target* and β is the *source* map. An element $g \in G$ is thought of as an arrow from $\beta(g)$ to $\alpha(g)$ in X.

(ii) The *set of composable pairs* is defined as

$$G^{(2)} := \{(g,h) \in G \times G \mid \beta(g) = \alpha(h)\}.$$

There is a *product map* m: $G^{(2)} \to G$ that satisfies

$$\alpha(m(g,h)) = \alpha(g), \qquad \beta(m(g,h)) = \beta(h),$$

and

$$m(m(g,h),k) = m(g, m(h,k)), \text{for any } g,h,k \in G.$$

One writes usually gh for $m(g,h)$.

(iii) An injection ϵ: $X \to G$, called the *identity section*, such that $\epsilon(\alpha(g))g = g = g\epsilon(\beta(g))$. In particular, $\alpha \circ \epsilon = \beta \circ \epsilon$ is the identity map on X.

(iv) An *inversion map* i: $G \to G$, also denoted by $i(g) = g^{-1}$, $g \in G$, such that $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

If the total space and the base of a groupoid $G \rightrightarrows X$ are smooth manifolds, the target and source maps are surjective submersions, the multiplication, the inversion, and the identity section are smooth maps, then $G \rightrightarrows X$ is a called a *Lie groupoid*.

Given the groupoid $G \rightrightarrows X$, a subset $H \subset G$ is a *subgroupoid* of G when it is closed under multiplication and inversion. Under those circumstances H is a groupoid over $\alpha(H) = \beta(H) \subset X$. If $\alpha(H) = \beta(H) = X$, $H \rightrightarrows X$ is called a *wide subgroupoid* of G.

Any group is a groupoid over a set with just one element. Any set X can be endowed with a *trivial groupoid* structure over itself by taking for the source and target maps the identity. The Cartesian product $X \times X$ of any set X is a groupoid over X by taking as target and source maps the projection on the first and second factors, respectively. The product is given by $(x,y)(y,z) = (x,z)$, $x,y,z \in X$, the identity section is $\epsilon(x) = (x,x)$, and $(x,y)^{-1} = (y,x)$. This is usually called the *pair* or *coarse groupoid*. Let us give a few examples that are not trivial. Several of them will be important in the ensuing discussion on the optimal momentum map.

EXAMPLE 1 Given two groupoids G_1 and G_2 over the sets X_1 and X_2, respectively, there is a naturally defined *product groupoid* $G_1 \times G_2 \rightrightarrows X_1 \times X_2$ by taking the Cartesian product of the target and the source maps.

EXAMPLE 2 (The groupoid associated to a pseudogroup of transformations). Let M be a smooth manifold and A a pseudogroup of local diffeomorphisms of M. Define $\bar{A}$ by

$$\bar{A} = \{\bar{\varphi} : M \to M \mid \varphi \in A,\ \bar{\varphi}(x) := \varphi(x) \text{ for } x \text{ in the domain of } \varphi \text{ and } \bar{\varphi}(x) := x \text{ if not}\}$$

The product $M \times \bar{A}$ is a groupoid over M if one defines the structure maps α, β: $M \times \bar{A} \to M$ by $\alpha(x,\bar{\varphi}) = \bar{\varphi}(x)$ and $\beta(x,\bar{\varphi}) = x$, the product by $m((x,\bar{\varphi}),(y,\bar{\psi})) := (y, \overline{\varphi \circ \psi})$, the identity section by $\epsilon(x) = (x, \mathrm{id}_M)$, and the inversion by $(x,\bar{\varphi})^{-1} = (\bar{\varphi}(x), \overline{\varphi^{-1}})$, where $\bar{\varphi}, \bar{\psi} \in \bar{A}$. This groupoid $M \times \bar{A} \rightrightarrows M$ is called the *transformation groupoid* associated to the pseudogroup A. Note that if A consists of global diffeomorphisms of M, then $\bar{A} = A$.

EXAMPLE 3 (The action groupoid). An important particular case of the previous example is obtained when one takes $A = A_G = \{\Phi_g \mid g \in G\}$, where Φ: $G \times M \to M$ is a smooth Lie group action. The resulting groupoid $G \times M \rightrightarrows M$ is called the *action groupoid* . Since this example is crucial later on, we specify now the structure maps:

$$\alpha(g,m) := g \cdot m, \beta(g,m) := m, \epsilon(m) := (e,m),$$

$$m((g, h \cdot n),(h,n)) := (gh, n), \quad \text{and} \quad (g,m)^{-1} := (g^{-1}, g \cdot m),$$

for any $g,h \in G$ and $m,n \in M$.

EXAMPLE 4 (The cotangent bundle of a Lie group as a groupoid). Let G be a Lie group, T^*G its cotangent bundle, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. If we identify T^*G with $G \times \mathfrak{g}^*$ using right translations we can use the previous example to endow T^*G with a groupoid structure over $\mathfrak{g}^*$ by taking the following structure maps: for any $(g,\mu) \in G \times \mathfrak{g}^*$ let

$$\alpha(g,\mu) := \mathrm{Ad}^*_{g^{-1}}\mu, \qquad \beta(g,\mu) := \mu, \qquad \epsilon(\mu) = (e,\mu),$$

$$m((g, \mathrm{Ad}^*_{h^{-1}}\mu),(h,\mu)) = (gh,\mu), \quad \text{and} \quad (g,\mu)^{-1} = (g^{-1}, \mathrm{Ad}^*_{g^{-1}}\mu).$$

$T^*G \rightrightarrows \mathfrak{g}^*$ is a symplectic groupoid.

EXAMPLE 5 (The Baer groupoid). Let G be a group and $\mathfrak{S}(G)$ be the set of subgroups of G. Let $\mathfrak{B}(G)$ be the set of cosets of elements in $\mathfrak{S}(G)$. It is not necessary to specify if $\mathfrak{B}(G)$ is the set of right or left cosets since they coincide: indeed, for any $g \in G$ and any $H \in \mathfrak{S}(G)$ we have $gH = (gHg^{-1})g$.

The set $\mathfrak{B}(G)$ is a groupoid over $\mathfrak{S}(G)$, called the *Baer groupoid*, by choosing $\alpha(D) = Dg^{-1}, \beta(D) = g^{-1}D$ for $g \in D$. The set of composable pairs $(\mathfrak{B}(G))^{(2)}$ is given by

$$(\mathfrak{B}(G))^{(2)} := \{(D_1, D_2) \in \mathfrak{B}(G) \times \mathfrak{B}(G) \mid g_1^{-1}D_1 = D_2 g_2^{-1}, \text{ for any } g_1 \in D_1, g_2 \in D_2\}.$$

The groupoid product defined on $(\mathfrak{B}(G))^{(2)}$ is given by $m(D_1, D_2) := D_1 D_2 = \{gh \mid g \in D_1, h \in D_2\}$. If $D \in \mathfrak{B}(G)$ define $D^{-1} := g^{-1}Dg^{-1}$, for any $g \in D$. The identity section is given by the inclusion map.

4.2. GROUPOID ACTIONS

Let $G \rightrightarrows X$ be a groupoid over X, M a set, and $J: M \to X$ a map. Define the fiber product

$$G \times_J M := \{(g, m) \in G \times M \mid \beta(g) = J(m)\}.$$

A (left) *groupoid action* of G on M with *moment map* $J: M \to X$ is a mapping $\Psi: G \times_J M \to M$, denoted also by $\Psi(g, m) = g \cdot m$, that satisfies the following properties:

(i) $J(g \cdot m) = \alpha(g)$,
(ii) $gh \cdot m = g \cdot (h \cdot m)$,
(iii) $(\epsilon(J(m))) \cdot m = m$,

for any $g, h \in G$ and $m \in M$. Notice that (i) guarantees that in (ii) each side of the equality is defined if the other is.

Two immediate examples are the following. A groupoid $G \rightrightarrows X$ acts on G by left multiplication with moment map α. G also acts on X with moment map id_X, where $g \cdot \beta(g) := \alpha(g)$. We shall give below two nontrivial examples.

EXAMPLE 6 (The G-action groupoid acts on G-spaces). Let G be a group acting on two sets M and N and let $J: M \to N$ be any equivariant map with respect to these two actions. The map J naturally induces an action of the product groupoid $G \times N \rightrightarrows N$ on the set M. Indeed,

$$(G \times N) \times_J M = \{((g, J(m)), m) \mid g \in G, m \in M\} \subset (G \times N) \times M.$$

The action $\Psi: (G \times N) \times_J M \to M$ with moment J is defined by $J(((g, J(m)), m) := g \cdot m$.

EXAMPLE 7 (The Baer groupoid acts on G-spaces) ([52]). Let G be a Lie group, M a G-space, and $B: M \to \mathfrak{S}(G)$ the map that assigns to each point $m \in M$ its isotropy subgroup $G_m \in \mathfrak{S}(G)$. Define $\mathfrak{B}(G) \times_B M := \{(gG_m, m) \in \mathfrak{B}(G) \times M \mid m \in M\}$. The map $\mathfrak{B}(G) \times_B M \to M$ given by $(gG_m, m) \mapsto g \cdot m$ defines an action of the Baer groupoid $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$ on the G-space M with moment map B. Notice that the level sets of the moment map are the isotropy type subsets M_H of of the G-action on M.

EXAMPLE 8 (Action groupoids and momentum maps). Let (M, ω) be a connected symplectic manifold acted canonically upon by a Lie group G. Suppose that this action admits a standard momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$ with nonequivariance one-cocycle σ: $G \to \mathfrak{g}^*$. Let Θ: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the affine action on $\mathfrak{g}^*$ constructed with this cocycle, that is, $g \cdot \mu := \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g)$ for $g \in G$ and $\mu \in \mathfrak{g}^*$, and $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ the associated action groupoid. Since the momentum map $\mathbf{J}$ is equivariant with respect to the G-action on M and the affine G-action on $\mathfrak{g}^*$, it naturally induces an action of the groupoid $T^*G \simeq G \times \mathfrak{g}^*$ (Example 4) on M whose associated moment map is $\mathbf{J}$ itself (see Example 6).

The same remark can be made regarding the optimal momentum map $\mathcal{J}$: $M \to M/A'_G$ associated to a G-canonical action on the Poisson manifold $(M, \{\cdot, \cdot\})$. In this case the groupoid in question is the action groupoid $G \times M/A'_G \rightrightarrows M/A'_G$ associated to the G-action on M/A'_G (see Proposition 3.1). This groupoid acts naturally on M with associated moment map $\mathcal{J}$.

4.3. A GROUPOID MODEL FOR THE OPTIMAL MOMENTUM MAP

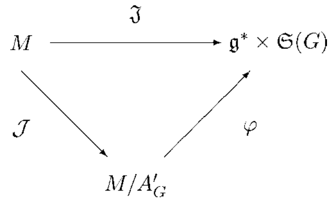
With this background we can now link the concept of optimal momentum map to groupoid moments. The expression (3.7) suggests that if the given G-action admits a standard momentum map, the level sets of the optimal momentum map can be 'parametrized', up to connected components, by the isotropy subgroups of the group action and the momentum values.

Let (M, ω) be a connected symplectic manifold acted canonically upon by a Lie group G and suppose that this action admits a standard momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$ with nonequivariance one-cocycle σ: $G \to \mathfrak{g}^*$. Let $T^*G \simeq G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ be the action groupoid associated to the affine action of G on $\mathfrak{g}^*$ and $\mathfrak{B}(G) \rightrightarrows \mathfrak{S}(G)$ the Baer groupoid of G (Example 5). Let $T^*G \times \mathfrak{B}(G) \rightrightarrows \mathfrak{g}^* \times \mathfrak{S}(G)$ be the product groupoid and $\Gamma \rightrightarrows \mathfrak{g}^* \times \mathfrak{S}(G)$ the wide subgroupoid defined by

$$\Gamma := \{((g, \mu), gH) \mid g \in G,\ \mu \in \mathfrak{g}^*,\ H \in \mathfrak{S}(G)\}.$$

It can be easily verified that $\Gamma \rightrightarrows \mathfrak{g}^* \times \mathfrak{S}(G)$ acts naturally on M with moment map $\mathfrak{J}$: $M \to \mathfrak{g}^* \times \mathfrak{S}(G)$ given by $\mathfrak{J}(m) = (\mathbf{J}(m), G_m)$.

The moment map $\mathfrak{J}$ has the Noether property and encodes through its two components the conservation of the standard momentum and the law of conservation of the isotropy which was one of the guiding principles behind the introduction of the optimal momentum map. Indeed, both objects are closely related since the universality property of the optimal momentum map (Theorem 3.3) implies that there exists a unique map φ: $M/A'_G \to \mathfrak{g}^* \times \mathfrak{S}(G)$ such that the diagram



commutes. Recall that the map φ is defined by the equality

$$\varphi(\mathcal{J}(m)) = \mathfrak{J}(m) = (\mathbf{J}(m), G_m), \quad m \in M.$$

This expression, together with (3.7), allows us to injectively embed the quotient space M/A'_G into $\mathfrak{g}^* \times \mathfrak{S}(G)$ provided that both the isotropy orbit type manifolds M_{G_m} as well as the intersections $\mathbf{J}^{-1}(\mu) \cap M_{G_m}$ are connected. Indeed, let $m, m' \in M$ be such that $\varphi(\mathcal{J}(m)) = \varphi(\mathcal{J}(m'))$ or, equivalently, $(\mathbf{J}(m), G_m) = (\mathbf{J}(m'), G_{m'})$. Expression (3.7) in Corollary 3.5 together with the connectedness hypotheses implies that there is a unique element $\rho \in M/A'_G$ such that $m, m' \in \mathcal{J}^{-1}(\rho)$, which establishes that φ is injective.

5. Optimal Reduction

In this section we present and comment on the reduction procedure using the optimal momentum map. As it will be seen, this approach overcomes the difficulties posed by the use of the standard momentum map raised at the end of Section 2 and unifies the different approaches to reduction discussed in that section. The reader interested in the proofs of the following results is encouraged to check with the original papers [39, 40] or with [44].

5.1. OPTIMAL POINT REDUCTION

The analogue of the Marsden–Weinstein reduction theorem in the optimal momentum setting is the following.

THEOREM 5.1 (Optimal point reduction by Poisson actions). *Let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold and G a Lie group acting canonically and properly on M. Let $\mathcal{J}\colon M \to M/A'_G$ be the optimal momentum map associated to this action. For any $\rho \in M/A'_G$ whose isotropy subgroup G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$ (which is an initial submanifold of M as the leaf of the integrable generalized distribution defined by the pseudogroup A'_G), we have*

(i) *The orbit space $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth symplectic regular quotient manifold with symplectic form ω_ρ defined by*

$$\pi_\rho^* \omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m), \tag{5.1}$$

for any $m \in \mathcal{J}^{-1}(\rho)$ and any $f, h \in C^\infty(M)^G$, where $\pi_\rho : \mathcal{J}^{-1}(\rho) \to M_\rho$ is the canonical projection. The pair (M_ρ, ω_ρ) is the optimal reduced space *of $(M, \{\cdot,\cdot\})$ at ρ.*

(ii) *Let $h \in C^\infty(M)^G$. The flow F_t of X_h leaves $\mathcal{J}^{-1}(\rho)$ invariant, commutes with the G-action, and therefore induces a flow F_t^ρ on M_ρ uniquely determined by the relation $\pi_\rho \circ F_t \circ i_\rho = F_t^\rho \circ \pi_\rho$, where $i_\rho : \mathcal{J}^{-1}(\rho) \hookrightarrow M$ is the inclusion.*

(iii) *The flow F_t^ρ in (M_ρ, ω_ρ) is Hamiltonian with the Hamiltonian function $h_\rho \in C^\infty(M_\rho)$ given by the equality $h_\rho \circ \pi_\rho = h \circ i_\rho$.*

(iv) *Let $k \in C^\infty(M)^G$ be another G-invariant function on M and $\{\cdot,\cdot\}_\rho$ the Poisson bracket associated to the symplectic form ω_ρ on M_ρ. Then $\{h,k\}_\rho = \{h_\rho, k_\rho\}_\rho$.*

Note that the hypotheses of this theorem do not require the existence of a standard momentum map associated to the action. The theorem is general enough to include the Poisson case. Moreover, there are no assumptions on the freeness of the action and the theorem still provides valuable information when the symmetry group is discrete, even $\{e\}$. Indeed, in this case the distribution A'_G coincides with the characteristic distribution of the Poisson manifold. The level sets of the optimal momentum map, and thereby the symplectic quotients M_ρ, are exactly the symplectic leaves of the Poisson manifold $(M, \{\cdot,\cdot\})$. Thus, applying Theorem 5.1(i) for $G = \{e\}$, one obtains the structure theorem for Poisson manifolds, that is, its stratification into symplectic leaves.

The very definition of the polar distribution implies that for any $\rho \in M/A'_G$ there is a unique symplectic leaf $\mathcal{L}_\rho$ of the Poisson manifold $(M, \{\cdot,\cdot\})$ such that $\mathcal{J}^{-1}(\rho) \subset \mathcal{L}_\rho$. Let $i_{\mathcal{L}_\rho}\colon \mathcal{J}^{-1}(\rho) \hookrightarrow \mathcal{L}_\rho$ be the inclusion of $\mathcal{J}^{-1}(\rho)$ into the symplectic leaf $(\mathcal{L}_\rho, \omega_{\mathcal{L}_\rho})$ of $(M, \{\cdot,\cdot\})$ that contains it. As $\mathcal{L}_\rho$ is an initial submanifold of M, the injection $i_{\mathcal{L}_\rho}$ is a smooth map. The form ω_ρ can also be written in terms of the symplectic structure of the leaf $\mathcal{L}_\rho$ as $\pi_\rho^*\omega_\rho = i_{\mathcal{L}_\rho}^*\omega_{\mathcal{L}_\rho}$. However, this does *not* imply that the previous theorem could be obtained by just performing symplectic optimal reduction on each symplectic leaf of the Poisson manifold, because these leaves are not G-manifolds, in general. As we already noted, Poisson actions are not necessarily leaf preserving.

Let us apply optimal reduction to the case of a proper G-action on a connected symplectic manifold (M, ω) admitting a not necessarily equivariant momentum map $\mathbf{J}\colon M \to \mathfrak{g}^*$. Corollary 3.5 relates the level sets of $\mathbf{J}$ and of the optimal momentum map $\mathcal{J}$, namely, if $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$, $\mathcal{J}(m) = \rho \in M/A'_G$, then $\mathcal{J}^{-1}(\rho) = (\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$, where $(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$ denotes the connected component of $\mathbf{J}^{-1}(\mu) \cap M^m_{G_m}$ that contains the point m. In addition, if $N_{G_\mu}(G_m)^{c(m)}$ is the closed subgroup of $N_{G_\mu}(G_m)$ that leaves the connected component $(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$ of $\mathbf{J}^{-1}(\mu) \cap M^m_{G_m}$ invariant, then the isotropy subgroup G_ρ of the point $\rho \in M/A'_G$ with respect to the action (3.5) equals $G_\rho = N_{G_\mu}(G_m)^{c(m)}$ and $N_{G_\mu}(G_m)^{c(m)}/G_m$ acts freely and properly on $(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m})^m$. Thus the optimal reduced space at ρ equals

$$\frac{\mathcal{J}^{-1}(\rho)}{G_\rho} = \frac{\left(\mathbf{J}^{-1}(\mu) \cap M^m_{G_m}\right)^m}{N_{G_\mu}(G_m)^{c(m)}/G_m}. \tag{5.2}$$

This shows the following:

- If there is a Lie group acting freely, properly, canonically, and this action has an associated momentum map, then the optimal reduced spaces coincide (up to connected components) with the Marsden–Weinstein reduced spaces discussed in Section 2.
- If in the previous setup we drop the freeness hypothesis, the optimal reduced spaces coincide with the singular reduced spaces of [5, 37, 44, 46], a topic that will be discussed in the next section.
- If the group G is discrete, the optimal reduced spaces are (up to connected components) the quotient manifolds $M_H/(N(H)/H)$ which, by the theorem, are symplectic.

Regarding the last point notice that, as we mentioned in Section 2, the quotients $M_H/(N(H)/H)$ are the spaces traditionally involved in the reduction of symmetric vector fields on manifolds. That reduction scheme can actually be obtained by following an approach identical to the one presented in Theorem 5.1 by replacing the distribution A'_G by the object that naturally generalizes it in the category of G-manifolds. Indeed, let M be a smooth manifold acted properly upon by a Lie group G and let $\mathfrak{X}(M)^G$ be the set of G-equivariant vector fields on M. It can be proved (see [41, 44]) that the generalized distribution defined by

$$D(m) = \{X(m) \mid X \in \mathfrak{X}(M)^G\}, \quad m \in M,$$

is integrable. Moreover, if $\mathcal{J}\colon M \to M/D$ is the projection onto the leaf space of the distribution D, we have for any $\rho \in M/D$

$$\mathcal{J}^{-1}(\rho)/G_\rho = M^m_{G_m}/\Big(N(G_m)^{c(m)}/G_m\Big),$$

where $m \in \mathcal{J}^{-1}(\rho)$ and G_ρ is the isotropy subgroup of $\rho \in M/D$ with respect to the unique G-action on M/D that makes $\mathcal{J}$ equivariant. This expression shows that the distribution theoretical approach to reduction unifies the apparently disconnected procedures introduced in Section 2.

Theorem 5.1 has a properness hypothesis on the G_ρ-action on $\mathcal{J}^{-1}(\rho)$, something that was not present in the classical Marsden–Weinstein reduction theorem. In that case, the properness of the G-action automatically implies the properness of the restricted coadjoint isotropy group action on the level set of the momentum map. In the case of optimal reduction, the properness of the G_ρ-action on $\mathcal{J}^{-1}(\rho)$ is a real hypothesis. From the reduction point of view the existence of a standard momentum map could be interpreted as an extra integrability property of the polar distribution that makes its integrable leaves imbedded (and not just initial) submanifolds of M and their isotropy subgroups automatically closed.

Here is an example due to Montaldi and Tokieda of a proper G-action with a nonproper G_ρ-action on $\mathcal{J}^{-1}(\rho)$. Consider Example 3 in Section 3, that is, $M := \mathbb{T}^2 \times \mathbb{T}^2$ with the symplectic form $\omega := \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 + \sqrt{2}\,\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2$. Let $\mathbb{T}^2$ act canonically on M by

$$(e^{i\phi_1}, e^{i\phi_2}) \cdot (e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) := (e^{i(\theta_1+\phi_1)}, e^{i(\theta_2+\phi_2)}, e^{i(\psi_1+\phi_1)}, e^{i(\psi_2+\phi_2)}).$$

Since the two-torus is compact this action is necessarily proper. Moreover, as $\mathbb{T}^2$ acts freely, the corresponding orbit space $M/A_{\mathbb{T}^2}$ is a smooth manifold such that the projection $\pi_{A_{\mathbb{T}^2}}: M \to M/A_{\mathbb{T}^2}$ is a surjective submersion. As in Example 3 of Section 3, the polar distribution behaves badly. Indeed, $C^\infty(M)^{\mathbb{T}^2}$ consists of all the functions f of the form $f \equiv f(e^{i(\theta_1-\psi_1)}, e^{i(\theta_2-\psi_2)})$. An inspection of the Hamiltonian flows associated to such functions readily shows that the leaves of $A'_{\mathbb{T}^2}$, that is, the level sets of the optimal momentum map $\mathcal{J}$, are the products of two leaves of an irrational foliation in a two-torus. Moreover, it can be checked that for any $\rho \in M/A'_{\mathbb{T}^2}$, the isotropy subgroup $\mathbb{T}^2_\rho$ is the product of two discrete subgroups of S^1, each of which fill densely the circle. This density property immediately implies that the $\mathbb{T}_\rho$-action on $\mathcal{J}^{-1}(\rho)$ is not proper.

Let $\mathcal{J}: M \to M/A'_G$ be the optimal momentum map corresponding to a proper G-action on (M, ω). Fix $\rho \in M/A'_G$ a momentum value of $\mathcal{J}$ and let $H \subset G$ be the unique G-isotropy subgroup such that $\mathcal{J}^{-1}(\rho) \subset M_H$ and $G_\rho \subset H$. The normalizer $N(H)$ of H in G acts naturally on M_H. This action induces a free action of the quotient group $L := N(H)/H$ on M_H. Let M_H^ρ be the unique connected component of M_H that contains $\mathcal{J}^{-1}(\rho)$ and let L^ρ be the closed subgroup of L that leaves it invariant. Obviously, L^ρ can be written as $L^\rho = N(H)^\rho/H$ for some closed subgroup $N(H)^\rho$ of $N(H)$.

The subset M_H^ρ is a symplectic embedded submanifold of M on which the group L^ρ acts freely and canonically. Denote by $\mathcal{J}_{L^\rho}: M_H^\rho \to M_H^\rho/A'_{L^\rho}$ the associated optimal momentum map.

PROPOSITION 5.2 (Optimal Sjamaar's Principle). *With the notation just introduced, we have the following*:

(i) *Let* $i_H^\rho : M_H^\rho \hookrightarrow M$ *be the inclusion. Then* $T_z i_H^\rho\big(A'_{L^\rho}(z)\big) = A'_G(z)$ *for any* $z \in M_H^\rho$.
(ii) *Let* $z \in \mathcal{J}^{-1}(\rho)$ *be such that* $\mathcal{J}_{L^\rho}(z) =: \sigma \in M_H^\rho/A'_{L^\rho}$. *Then* $\mathcal{J}^{-1}(\rho) = \mathcal{J}_{L^\rho}^{-1}(\sigma)$.
(iii) $L^\rho_\sigma = G_\rho/H$.
(iv) $(M_H^\rho)_\sigma = \mathcal{J}_{L^\rho}^{-1}(\sigma)/L^\rho_\sigma = \mathcal{J}^{-1}(\rho)/(G_\rho/H) = \mathcal{J}^{-1}(\rho)/G_\rho = M_\rho$. *Moreover, if* G_ρ *acts properly on* $\mathcal{J}^{-1}(\rho)$ *this equality is true when we consider* M_ρ *and* $(M_H^\rho)_\sigma$ *as symplectic spaces, that is,* $(M_\rho, \omega_\rho) = ((M_H^\rho)_\sigma, (\omega|_{M_H^\rho})_\sigma)$.

5.2. OPTIMAL ORBIT REDUCTION

We next turn to another reduction procedure involving the optimal momentum map. If $\rho \in M/A'_G$, denote by $\mathcal{O}_\rho := G \cdot \rho$ the G-orbit of the action (3.5) on M/A'_G. Assume that G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$. It can be shown that $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ has a unique smooth structure relative to which it is an initial submanifold of M. This structure coincides with the one that makes it diffeomorphic to $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$. Consequently, the quotient manifold $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ is naturally diffeomorphic to the symplectic point reduced space because of the sequence of diffeomorphisms

$$\mathcal{J}^{-1}(\mathcal{O}_\rho)/G \simeq G \times_{G_\rho} \mathcal{J}^{-1}(\rho)/G \simeq \mathcal{J}^{-1}(\rho)/G_\rho.$$

The composed diffeomorphism $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G \simeq \mathcal{J}^{-1}(\rho)/G_\rho$ can be explicitly implemented as follows. Let f_ρ: $\mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)$ be the inclusion. Since the inclusion $\mathcal{J}^{-1}(\rho) \hookrightarrow M$ is smooth and $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is initial in M, the map f_ρ is smooth. Also, since f_ρ is (G_ρ, G)-equivariant, it drops to a unique smooth map F_ρ: $\mathcal{J}^{-1}(\rho)/G_\rho \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ that makes the following diagram

$$\begin{array}{ccc} \mathcal{J}^{-1}(\rho) & \xrightarrow{f_\rho} & \mathcal{J}^{-1}(\mathcal{O}_\rho) \\ {\scriptstyle \pi_\rho}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{O}_\rho}} \\ \mathcal{J}^{-1}(\rho)/G_\rho & \xrightarrow{F_\rho} & \mathcal{J}^{-1}(\mathcal{O}_\rho)/G \end{array} \tag{5.3}$$

commutative. It is easy to see that F_ρ is a diffeomorphism. The orbit reduced space $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ can be therefore trivially endowed with a symplectic structure $\omega_{\mathcal{O}_\rho}$ by defining $\omega_{\mathcal{O}_\rho} := (F_\rho^{-1})^* \omega_\rho$. These remarks are summarized in the first points of the following theorem which is the orbit counterpart of Theorem 5.1.

THEOREM 5.3 (Optimal orbit reduction by Poisson actions). *Suppose that G_ρ acts properly on* $\mathcal{J}^{-1}(\rho)$ and let $\mathcal{O}_\rho := G \cdot \rho$.

(i) *There is a unique smooth structure on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ that makes it into an initial submanifold of M.*
(ii) *The restricted G-action on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is smooth and proper and all its isotropy subgroups are conjugate to a given compact isotropy subgroup of the G-action on M.*
(iii) *The quotient $M_{\mathcal{O}_\rho}: = \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ admits a unique smooth structure that makes the projection $\pi_{\mathcal{O}_\rho}$: $\mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ a surjective submersion.*
(iv) *The* optimal orbit reduced space $M_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ *admits a unique symplectic structure $\omega_{\mathcal{O}_\rho}$ that makes it symplectomorphic to the point reduced space M_ρ.*
(v) *Let $h \in C^\infty(M)^G$. The flow F_t of X_h leaves $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ invariant, commutes with the G-action, and therefore induces a flow $F_t^{\mathcal{O}_\rho}$ on $M_{\mathcal{O}_\rho}$ uniquely determined by the relation $\pi_{\mathcal{O}_\rho} \circ F_t \circ i_{\mathcal{O}_\rho} = F_t^{\mathcal{O}_\rho} \circ \pi_{\mathcal{O}_\rho}$, where $i_{\mathcal{O}_\rho}$: $\mathcal{J}^{-1}(\mathcal{O}_\rho) \hookrightarrow M$ is the inclusion.*
(vi) *The flow $F_t^{\mathcal{O}_\rho}$ on $(M_{\mathcal{O}_\rho}, \omega_{\mathcal{O}_\rho})$ is Hamiltonian with the Hamiltonian function $h_{\mathcal{O}_\rho} \in C^\infty(M_{\mathcal{O}_\rho})$ given by the equality $h_{\mathcal{O}_\rho} \circ \pi_{\mathcal{O}_\rho} = h \circ i_{\mathcal{O}_\rho}$.*
(vii) *Let $k \in C^\infty(M)^G$ be another G-invariant function on M and $\{\cdot,\cdot\}_{\mathcal{O}_\rho}$ the Poisson bracket associated to the symplectic form $\omega_{\mathcal{O}_\rho}$ on $M_{\mathcal{O}_\rho}$. Then, $\{h,k\}_{\mathcal{O}_\rho} = \{h_{\mathcal{O}_\rho}, k_{\mathcal{O}_\rho}\}_{\mathcal{O}_\rho}$.*

The counterpart of Proposition 5.2 for orbit reduction is the following statement.

PROPOSITION 5.4. *Assume the notations and hypotheses of Theorem 5.3. Let $H \subset G$ be the unique G-isotropy subgroup such that $\mathcal{J}^{-1}(\rho) \subset M_H$ and $G_\rho \subset H$. Assume that G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$. Let M_H^ρ be the connected component of M_H*

that contains $\mathcal{J}^{-1}(\rho)$, $N(H)^{\rho}$ *the closed subgroup of the normalizer* $N(H)$ *of* H *that leaves* M_H^{ρ} *invariant, and* $L^{\rho} := N(H)^{\rho}/H$.

(i) *Let* $z \in \mathcal{J}^{-1}(\rho)$ *be such that* $\mathcal{J}_{L^{\rho}}(z) =: \sigma \in M_H^{\rho}/A'_{L^{\rho}}$ *and* $\mathcal{N}_{\rho} := N(H)^{\rho} \cdot \rho \subset M/A'_G$. *The set* $\mathcal{J}_{L^{\rho}}^{-1}(L^{\rho} \cdot \sigma) = \mathcal{J}^{-1}(\mathcal{N}_{\rho})$ *is an embedded submanifold of* $\mathcal{J}^{-1}(\mathcal{O}_{\rho})$.

(ii) *The initial submanifold* $\mathcal{J}^{-1}(\mathcal{O}_{\rho})$ *can be written as a disjoint union of its embedded submanifolds:*

$$\mathcal{J}^{-1}(\mathcal{O}_{\rho}) = \coprod_{[g] \in G/N(H)^{\rho}} \mathcal{J}^{-1}(\mathcal{N}_{g \cdot \rho}). \tag{5.4}$$

(iii) *The symplectic quotient* $(\mathcal{J}_{L^{\rho}}^{-1}(L^{\rho} \cdot \sigma)/L^{\rho}, (\omega|_{M_H^{\rho}})_{L^{\rho} \cdot \sigma})$ *is naturally symplectomorphic to the orbit reduced space* $(\mathcal{J}^{-1}(\mathcal{O}_{\rho})/G, \omega_{\mathcal{O}_{\rho}})$. *We will say that* $(\mathcal{J}_{L^{\rho}}^{-1}(L^{\rho} \cdot \sigma)/L^{\rho}, (\omega|_{M_H^{\rho}})_{L^{\rho} \cdot \sigma})$ *is* an orbit regularization *of* $(\mathcal{J}^{-1}(\mathcal{O}_{\rho})/G, \omega_{\mathcal{O}_{\rho}})$.

In Theorem 5.3 we showed that the optimal orbit reduced spaces $\mathcal{J}^{-1}(\mathcal{O}_{\rho})/G$ are symplectic manifolds with the form that makes them symplectomorphic to the point reduced spaces. We now show that the symplectic form $\omega_{\mathcal{O}_{\rho}}$ can be put in relation with the presymplectic structure that one can define on some homogeneous spaces that naturally arise in this context. These are the so called polar reduced spaces that we introduce in the next proposition.

PROPOSITION 5.5. *Let* $(M, \{\cdot,\cdot\})$ *be a smooth Poisson manifold and* G *a Lie group acting canonically and properly on* M. *Let* $\mathcal{J}\colon M \to M/A'_G$ *be the optimal momentum map associated to this action and* $\rho \in M/A'_G$. *Suppose that* G_{ρ} *is closed in* G. *Then the polar distribution* A'_G *restricts to a smooth integrable regular distribution on* $\mathcal{J}^{-1}(\mathcal{O}_{\rho})$, *that we will also denote by* A'_G. *The leaf space* $M'_{\mathcal{O}_{\rho}} := \mathcal{J}^{-1}(\mathcal{O}_{\rho})/A'_G$ *admits a unique smooth structure that makes it into a regular quotient manifold and diffeomorphic to the homogeneous manifold* G/G_{ρ}. *With this smooth structure the projection* $\mathcal{J}_{\mathcal{O}_{\rho}}$: $\mathcal{J}^{-1}(\mathcal{O}_{\rho}) \to \mathcal{J}^{-1}(\mathcal{O}_{\rho})/A'_G$ *is a smooth surjective submersion. We will refer to* $M'_{\mathcal{O}_{\rho}}$ *as the* polar reduced space.

The relation of the polar reduced spaces with orbit reduction is given in the next theorem. For simplicity we formulate this result in the symplectic context. We refer to [40] and [44] for the general Poisson case and examples.

THEOREM 5.6 (Polar reduction of a symplectic manifold). *Let* (M, ω) *be a smooth symplectic manifold and* G *a Lie group acting canonically and properly on* M. *Let* $\mathcal{J}\colon M \to M/A'_G$ *be the optimal momentum map associated to this action and let* $\rho \in M/A'_G$ *be such that* G_{ρ} *is closed in* G. *There is a unique presymplectic form* $\omega'_{\mathcal{O}_{\rho}}$ *on the polar reduced space* $M'_{\mathcal{O}_{\rho}} \simeq G/G_{\rho}$ *that satisfies*

$$i_{\mathcal{O}_{\rho}}^{*}\omega = \pi_{\mathcal{O}_{\rho}}^{*}\omega_{\mathcal{O}_{\rho}} + \mathcal{J}_{\mathcal{O}_{\rho}}^{*}\omega'_{\mathcal{O}_{\rho}}. \tag{5.5}$$

The form $\omega'_{\mathcal{O}_{\rho}}$ *is symplectic if and only if for one point* $z \in \mathcal{J}^{-1}(\mathcal{O}_{\rho})$ *(and, hence, for all) we have*

$$\mathfrak{g}\cdot z\cap(\mathfrak{g}\cdot z)^{\omega}\subset T_zM^z_{G_z}. \tag{5.6}$$

The characterization (5.6) of the symplecticity of $\omega'_{\mathcal{O}_\rho}$ admits a particularly convenient formulation when the G-action on the symplectic manifold (M,ω) admits a standard momentum map $\mathbf{J}$: $M\rightarrow\mathfrak{g}^*$. Indeed, assume that M is connected and let $z\in M$ be such that $\mathbf{J}(z)=\mu\in\mathfrak{g}^*$ and $G_z=H$. Then, if the symbol G_μ denotes the isotropy subgroup of μ with respect to the affine G-action on $\mathfrak{g}^*$ defined with the nonequivariance one-cocycle of $\mathbf{J}$, we have that (5.6) is equivalent to

$$\mathfrak{g}_\mu=\mathrm{Lie}(N_{G_\mu}(H)). \tag{5.7}$$

6. Singular Point Reduction

After this review of some of the main results on optimal momentum maps and reduction we turn our attention to the classical reduction procedure when the freeness hypothesis on the group action as well as the regularity assumption on the momentum value are dropped. In this section we present a summary of the results on point reduction, that is, the generalization to the singular case of the classical Marsden–Weinstein theorem. We shall also connect this reduction procedure to the optimal reduction theorem.

6.1. THE SINGULAR SYMPLECTIC STRATA

Throughout this section the following notations and conventions will be in force. Let (M,ω) be a connected symplectic manifold acted canonically and properly upon by a Lie group G. It is assumed that this action has an associated standard momentum map $\mathbf{J}$: $M\rightarrow\mathfrak{g}^*$ with *nonequivariance one-cocycle* σ: $G\rightarrow\mathfrak{g}^*$, that is, $\sigma(g):=\mathbf{J}(g\cdot m)-\mathrm{Ad}^*_{g^{-1}}\mathbf{J}(m)$ for any $g\in G$ and $m\in M$. Denote by Θ: $G\times\mathfrak{g}^*\rightarrow\mathfrak{g}^*$ the affine action of G on $\mathfrak{g}^*$ defined by σ, that is, $\Theta(g,\nu):=\mathrm{Ad}^*_{g^{-1}}\nu+\sigma(g)$ for any $g\in G$ and $\nu\in\mathfrak{g}^*$. Let $\mu\in\mathfrak{g}^*$ be a value of $\mathbf{J}$, G_μ the isotropy subgroup of μ with respect to the affine action Θ, and $H\subset G$ an isotropy subgroup of the G-action on M. Denote by M^z_H the connected component of the H-isotropy type manifold M_H that contains a given element $z\in M$ such that $\mathbf{J}(z)=\mu$ and let $G_\mu M^z_H$ be its G_μ-saturation, that is, the union of all G_μ-orbits through all points of M^z_H.

THEOREM 6.1 (Singular symplectic point strata). *The following hold:*

(i) *The set $\mathbf{J}^{-1}(\mu)\cap G_\mu M^z_H$ is an embedded submanifold of M.*
(ii) *The set $M^{(H)}_\mu:=[\mathbf{J}^{-1}(\mu)\cap G_\mu M^z_H]/G_\mu$ has a unique quotient differentiable structure such that the canonical projection $\pi^{(H)}_\mu$: $\mathbf{J}^{-1}(\mu)\cap G_\mu M^z_H\longrightarrow M^{(H)}_\mu$ is a surjective submersion.*
(iii) *There is a unique symplectic structure $\omega^{(H)}_\mu$ on $M^{(H)}_\mu$ characterized by $i^{(H)*}_\mu\omega=\pi^{(H)*}_\mu\omega^{(H)}_\mu$, where $i^{(H)}_\mu:\mathbf{J}^{-1}(\mu)\cap G_\mu M^z_H\hookrightarrow M$ is the natural inclusion. The pairs $(M^{(H)}_\mu,\omega^{(H)}_\mu)$ are called* singular symplectic point strata.

(iv) *Let $h \in C^\infty(M)^G$ be a G-invariant Hamiltonian. Then the flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z$ invariant and commutes with the G_μ-action, so it induces a flow F_t^μ on $M_\mu^{(H)}$ that is characterized by $\pi_\mu^{(H)} \circ F_t \circ i_\mu^{(H)} = F_t^\mu \circ \pi_\mu^{(H)}$.*

(v) *The flow F_t^μ is Hamiltonian on $M_\mu^{(H)}$, with* reduced Hamiltonian *function $h_\mu^{(H)}$: $M_\mu^{(H)} \to \mathbb{R}$ defined by $h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ i_\mu^{(H)}$. The vector fields X_h and $X_{h_\mu^{(H)}}$ are $\pi_\mu^{(H)}$-related.*

(vi) *Let k: $M \to \mathbb{R}$ be another G-invariant function. Then $\{h, k\}$ is also G-invariant and $\{h,k\}_\mu^{(H)} = \{h_\mu^{(H)}, k_\mu^{(H)}\}_{M_\mu^{(H)}}$, where $\{\cdot,\cdot\}_{M_\mu^{(H)}}$ denotes the Poisson bracket induced by the symplectic structure on $M_\mu^{(H)}$.*

For the next theorem we need a few preparatory remarks. For any $z \in M$ denote by $N(H)^z$ the set of elements in the normalizer $N(H)$ of H that leaves the submanifold M_H^z invariant. Note that $H \subset N(H)$. The subgroup $N(H)^z$ is open and hence closed in $N(H)$. The Lie group $L^z := N(H)^z/H$ acts freely and canonically on M_H^z with associated momentum map $\mathbf{J}_{L^z}$: $M_H^z \to (\mathrm{Lie}(L^z))^*$ given by

$$\mathbf{J}_{L^z}(z') := \Lambda(\mathbf{J}|_{M_H^z}(z') - \mu), \quad z' \in M_H^z, \tag{6.1}$$

where $\mu := \mathbf{J}(z) \in \mathfrak{g}^*$. In this expression, Λ: $(\mathfrak{g}_z^\circ)^H \to (\mathrm{Lie}(L^z))^*$ denotes the natural L^z-equivariant isomorphism given by

$$\left\langle \Lambda(\beta), \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp t\xi)H \right\rangle = \langle \beta, \xi \rangle, \tag{6.2}$$

for any $\beta \in (\mathfrak{g}_z^\circ)^H$ and $\xi \in \mathrm{Lie}(N(H)^z) = \mathrm{Lie}(N(H))$; $\mathfrak{g}_z^\circ$ denotes the annihilator of $\mathfrak{g}_z$ in $\mathfrak{g}$ and $(\mathfrak{g}_z^\circ)^H$ are the H-fixed points in the vector space $\mathfrak{g}_z^\circ$. The nonequivariance one-cocycle τ: $M_H^z \to (\mathrm{Lie}(L^z))^*$ of the momentum map $\mathbf{J}_{L^z}$ is given by

$$\tau(l) = \Lambda(\sigma(n) + n \cdot \mu - \mu), \quad \text{for any} \quad l = nH \in L^z, \quad n \in N(H)^z. \tag{6.3}$$

Since $N(H)^z$ is open in $N(H)$, it follows that

$$\mathrm{Lie}(N(H)^z/H) = \mathrm{Lie}(N(H)/H) =: \mathfrak{l}.$$

Sjamaar's principle takes the form of a structure theorem for the singular strata.

THEOREM 6.2 (Structure theorem for the singular point strata). *In the setup described above the following statements hold:*

(i) *The canonical projection*

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z \to M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z]/G_\mu$$

defines a smooth fiber bundle with fiber G_μ/H and structure group $N_{G_\mu}(H)^z/H$.

(ii) *Consider the free, proper, and canonical action of $L^z := N(H)^z/H$ on M_H^z and let $\mathbf{J}_{L^z} : M_H^z \to \mathfrak{l}^*$ be the associated momentum map given by* (6.1). *Then the Marsden–Weinstein reduced space $(M_H^z)_0$ at the zero value of this momentum map is given by*

$$(M_H^z)_0 = \mathbf{J}_{L^z}^{-1}(0)/L_0^z = [\mathbf{J}^{-1}(\mu) \cap M_H^z]/(N_{G_\mu}(H)^z/H).$$

Note that L_0^z is, in general, different from L^z because the action is affine and not linear.

(iii) *The projection* π_0: $\mathbf{J}_{L^z}^{-1}(0) \to (M_H^z)_0$ *defines a smooth principal* L_0^z*-bundle. Regarding* G_μ/H *as a right* $(N_{G_\mu}(H)^z/H)$*-space and* $\mathbf{J}^{-1}(\mu) \cap M_H^z$ *as a left* $(N_{G_\mu}(H)^z/H)$*-space, consider the bundle with fiber* G_μ/H *and structure group* G_μ *associated with* π_0*, that is,*

$$G_\mu/H \times_{N_{G_\mu}(H)^z/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^z\right) \longrightarrow [\mathbf{J}^{-1}(\mu) \cap M_H^z]/(N_{G_\mu}(H)^z/H).$$

This bundle is G_μ*-symplectomorphic to* $\pi_\mu^{(H)}$: $\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z \longrightarrow M_\mu^{(H)}$*, that is,* $G_\mu/H \times_{N_{G_\mu}(H)^z/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^z\right)$ *is* G_μ*-diffeomorphic to* $\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z$ *and* $(M_H^z)_0 = \mathbf{J}_{L^z}^{-1}(0)/L_0^z = (\mathbf{J}^{-1}(\mu) \cap M_H^z)/(N_{G_\mu}(H)^z/H)$ *is symplectomorphic to* $M_\mu^{(H)}$*. We will say that* $(M_H^z)_0$ *is a* regularization *of the singular symplectic point stratum* $M_\mu^{(H)}$.

This last part of the theorem and Proposition 5.2 show that, up to connected components, singular symplectic point strata are symplectomorphic to the corresponding optimal reduced spaces. In other words, optimal reduction, which we have already seen that it is always regular, directly yields the strata of the singular reduced spaces.

It turns out that both the level sets and the quotients form a specific kind of stratification that we make precise in the discussion below.

6.2. STRATIFIED SPACES

In this subsection we shall adopt the definitions, notations, and conventions in [45]. For the proofs of the statements reviewed here, we also refer to this work.

Recall that the subset A of a topological space P is said to be *locally closed* if each of its points has an open neighborhood U in P such that $U \cap A$ is closed in U. An injectively immersed submanifold is embedded if and only if it its image is locally closed in the ambient manifold.

Let P be a topological space and $\mathcal{Z}$ a locally finite partition of P into smooth manifolds $S_i \subset P$, $i \in I$, that are locally closed topological subspaces of P (hence, their manifold topology is the relative one induced by P). The pair $(P, \mathcal{Z})$ is called a *decomposition* of P with *pieces* in $\mathcal{Z}$, or a *decomposed space*, if the following *frontier condition* holds:

(DS) If $R, S \in \mathcal{Z}$ are such that $R \cap \bar{S} \neq \emptyset$, then $R \subset \bar{S}$. In this case we write $R \preceq S$. If, in addition, $R \neq S$ we say that R is *incident* to S or that it is a *boundary piece* of S and write $R \prec S$.

The *dimension* of P is defined as $\dim P = \sup\{\dim S_i \mid S_i \in \mathcal{Z}\}$. The *depth* $\mathrm{dp}(z)$ of any point $z \in P$ relative to the decomposition $\mathcal{Z}$ is defined by

$$\mathrm{dp}(z) := \sup\{k \in \mathbb{N} \mid \exists\ S_0, S_1, \ldots, S_k \in \mathcal{Z} \text{ with } z \in S_0 \prec S_1 \prec \ldots \prec S_k\}.$$

Note that $\mathrm{dp}(x) = \mathrm{dp}(y)$ for any $x, y \in S$, $S \in \mathcal{Z}$. Thus the depth $\mathrm{dp}(S)$ of the piece $S \in \mathcal{Z}$ is well defined by $\mathrm{dp}(S) := \mathrm{dp}(x)$, $x \in \mathcal{S}$. The depth $\mathrm{dp}(P)$ of $(P, \mathcal{Z})$ is defined by $\mathrm{dp}(P) := \sup\{\mathrm{dp}(S) \mid S \in \mathcal{Z}\}$.

A continuous mapping f: $P \to Q$ between the decomposed spaces $(P, \mathcal{Z})$ and $(Q, \mathcal{Y})$ is a *morphism of decomposed spaces* if for every piece $S \in \mathcal{Z}$, there is a piece $T \in \mathcal{Y}$ such that $f(S) \subset T$ and the restriction $f|_S$: $S \to T$ is smooth. If $(P, \mathcal{Z})$ and $(P, \mathcal{T})$ are two decompositions of the same topological space we say that $\mathcal{Z}$ is *coarser* than $\mathcal{T}$ or that $\mathcal{T}$ is *finer* than $\mathcal{Z}$ if the identity mapping $(P, \mathcal{T}) \to (P, \mathcal{Z})$ is a morphism of decomposed spaces. A topological subspace $Q \subset P$ is a *decomposed subspace* of $(P, \mathcal{Z})$ if for all pieces $S \in \mathcal{Z}$, the intersection $S \cap Q$ is a submanifold of S and the corresponding partition $\mathcal{Z} \cap Q$ forms a decomposition of Q.

Two subsets A and B of P are said to be equivalent at $z \in P$ if there is an open neighborhood U of z such that $A \cap U = B \cap U$. The equivalence class of $A \subset P$ at z is denoted by $[A]_z$ and called the *set germ* of A at z.

A *stratification* (Definition 1.2.2 in [45]) of the topological space P is a map $\mathcal{S}$ that associates to any $z \in P$ the set germ $\mathcal{S}(z)$ of a closed subset of P such that the following condition is satisfied:

(ST) For every $z \in P$ there is a neighborhood U of z and a decomposition $\mathcal{Z}$ of U such that for all $y \in U$ the germ $\mathcal{S}(y)$ coincides with the set germ of the piece of $\mathcal{Z}$ that contains y.

The pair $(P, \mathcal{S})$ is called a *stratified space* (see Definition 1.2.2 in [45]). Any decomposition of P defines a stratification of P by associating to each of its points the set germ of the piece containing it. The converse is, by definition, locally true.

Two decompositions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of P are said to be equivalent if they induce the same stratification of P. Any stratified space $(P, \mathcal{S})$ has a unique associated decomposition $\mathcal{Z}_{\mathcal{S}}$ with the following maximality property: for any open subset $U \subset P$ and any decomposition $\mathcal{Z}$ of P inducing $\mathcal{S}$ on U, the restriction of $\mathcal{Z}_{\mathcal{S}}$ to U is coarser than the restriction of $\mathcal{Z}$ to U. The decomposition $\mathcal{Z}_{\mathcal{S}}$ is called the *canonical decomposition* associated to the stratification $(P, \mathcal{S})$ and its pieces are called the *strata*. The local finiteness of the decomposition $\mathcal{Z}_{\mathcal{S}}$ implies that for any stratum S of $(P, \mathcal{S})$ there are only finitely many strata R with $S \prec R$. In what follows the symbol $\mathcal{S}$ in the stratification $(P, \mathcal{S})$ denotes both the map that sends each point to a set germ and the set of pieces associated to the canonical decomposition $\mathcal{Z}_{\mathcal{S}}$ induced by the stratification of P.

Let $(P, \mathcal{S})$ be a stratified space. A *singular* or *stratified chart* of P is a homeomorphism ϕ: $U \to \phi(U) \subset \mathbb{R}^n$ from an open set $U \subset P$ to a subset of $\mathbb{R}^n$ such that for every stratum $S \in \mathcal{S}$ the image $\phi(U \cap S)$ is a submanifold of $\mathbb{R}^n$ and the restriction $\phi|_{U \cap S}$: $U \cap S \to \phi(U \cap S)$ is a diffeomorphism. Two singular charts ϕ: $U \to \phi(U) \subset \mathbb{R}^n$ and φ: $V \to \varphi(V) \subset \mathbb{R}^m$ are *C^k-compatible* if for any $z \in U \cap V$ there exist an open neighborhood $W \subset U \cap V$ of z, a natural number $N \geq \max\{n, m\}$, open neighborhoods $O, O' \subset \mathbb{R}^N$ of $\phi(U) \times \{0\}$ and $\varphi(V) \times \{0\}$, respectively, and a C^k-diffeomorphism ψ: $O \to O'$ such that $i_m \circ \varphi|_W = \psi \circ i_n \circ \phi|_W$, where i_n and i_m denote the natural embeddings of $\mathbb{R}^n$ and $\mathbb{R}^m$ into $\mathbb{R}^N$ by using the first n and m coordinates, respectively. A *singular* or *stratified atlas* is defined as for manifolds by using stratified charts. The same is true for compatible and maximal stratified atlases. A maximal atlas on the stratified space $(P, \mathcal{S})$ determines a C^k-*differentiable structure* on P and $(P, \mathcal{S})$ is called a C^k-*stratified space*. If $k = \infty$, $(P, \mathcal{S})$ is called a *smooth stratified space*.

Stratified spaces with smooth structure are naturally presheaf spaces. Let $(P, \mathcal{S})$ be a stratified space with smooth structure. The *presheaf* C_P^∞ *of smooth functions* on P is defined by assigning to any open set $U \subset P$ the algebra $C_P^\infty(U)$ of real-valued functions on U consisting of all continuous functions f: $U \to \mathbb{R}$ with the following property: for all $z \in U$ and any stratified chart ϕ: $V \to \mathbb{R}^n$ such that $z \in V$, there exists an open neighborhood W of z and a smooth function $\bar{f}$: $\mathbb{R}^n \to \mathbb{R}$ such that $W \subset U \cap V$ and $f|_W = \bar{f} \circ \phi|_W$.

Since the stratified space with smooth structure $(P, \mathcal{S})$ can be considered as the presheaf space (P, C_P^∞) the notion of smooth map between stratified spaces with smooth structure can be defined by working in the category of presheaf spaces. Note that smooth maps between stratified spaces are not, in general, stratified maps and, conversely, stratified maps need not be smooth. These remarks allow the introduction of certain particularly well-behaved smooth stratified spaces.

Let P be a smooth stratified space and $R, S \subset M$ two strata. Let ϕ: $U \to \mathbb{R}^n$ be a smooth stratified chart of M around the point z. The *Whitney condition* (B) at the point $z \in R$ with respect to the chart (U, ϕ) is given by the following statement:

(B) Let $\{x_n\}_{n\in\mathbb{N}} \subset R \cap U$ and $\{y_n\}_{n\in\mathbb{N}} \subset S \cap U$ be two sequences with the same limit $z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ and such that $x_n \neq y_n$, for all $n \in \mathbb{N}$. Suppose that the set of connecting lines $\overline{\phi(x_n)\phi(y_n)} \subset \mathbb{R}^n$ converges in projective space to a line L and that the sequence of tangent spaces $\{T_{y_n}S\}_{n\in\mathbb{N}}$ converges in the Grassmann bundle of $\dim S$-dimensional subspaces of TP to $\tau \subset T_zP$. Then, $(T_z\phi)^{-1}(L) \subset \tau$.

This condition does not depend on the chart used to formulate it. If condition (B) is verified for every point $z \in R$, the pair (R, S) is said to satisfy the *Whitney condition* (B) or that S is (B)-*regular* over R. A stratified space with smooth structure such that for every pair of strata Whitney's condition (B) holds, is called a *Whitney* **(B)**-*space*.

There is also a weaker Whitney condition (A). We shall not elaborate on this condition because it is not needed later.

6.3. LOCAL TRIVIALITY AND CONE SPACES

Let P be a topological space. Define the equivalence relation $\sim$ in the product $P \times [0, \infty)$ by $(z, a) \sim (z', a')$ if and only if $a = a' = 0$. The *cone* CP on P is defined as the quotient topological space $P \times [0, \infty)/\sim$. If P is a smooth manifold then the cone CP is a decomposed space with two pieces, namely, $P \times (0, \infty)$ and the *vertex* which is the class corresponding to any element of the form $(z, 0)$, $z \in P$, that is, $P \times \{0\}$. Analogously, if $(P, \mathcal{Z})$ is a decomposed (stratified) space then the associated cone CP is also a decomposed (stratified) space whose pieces (strata) are the vertex and the sets of the form $S \times (0, \infty)$, with $S \in \mathcal{Z}$.

A stratified space $(P, \mathcal{S})$ is said to be *locally trivial* if for any $z \in P$ there exist a neighborhood U of z, a stratified space $(F, \mathcal{S}^F)$, a distinguished point $\mathbf{0} \in F$, and an isomorphism of stratified spaces ψ: $U \to (S \cap U) \times F$, where S is the stratum that contains z and ψ satisfies $\psi^{-1}(y, \mathbf{0}) = y$, for all $y \in S \cap U$. If F is a cone CL over a compact stratified space L, then L is called the *link* of z.

An important corollary of Thom's first isotopy lemma guarantees that every Whitney (B) stratified space is locally trivial (see [31, 50]). A converse to this implication needs the introduction of the so called cone spaces which will be discussed next.

Let $m \in \mathbb{N} \cup \{\infty, \omega\}$. A *cone space* of class C^m and depth 0 is the union of countably many C^m manifolds together with the stratification whose strata are the unions of the connected components of equal dimension. A cone space of class C^m and depth $d+1$, $d \in \mathbb{N}$, is a stratified space $(P, \mathcal{S})$ with a C^m differentiable structure such that for any $z \in P$ there exists a connected neighborhood U of z, a compact cone space L of class C^m and depth d called the *link*, and a stratified isomorphism ψ: $U \to (S \cap U) \times CL$, where S is the stratum that contains the point z, the map ψ satisfies that $\psi^{-1}(y, \mathbf{0}) = y$, for all $y \in S \cap U$, and $\mathbf{0}$ is the vertex of the cone CL.

If $m \neq 0$ then L is required to be embedded into a sphere via a fixed smooth global singular chart φ: $L \to S^l$ that determines the smooth structure of CL. More specifically, the smooth structure of CL is generated by the global chart τ: $[z, t] \in CL \mapsto t\varphi(z) \in \mathbb{R}^{l+1}$. The maps ψ: $U \to (S \cap U) \times CL$ and φ: $L \to S^l$ are referred to as a *cone chart* and a *link chart* respectively. Moreover, if $m \neq 0$ then ψ and ψ^{-1} are required to be differentiable of class C^m as maps between stratified spaces with a smooth structure.

The cone charts and the link charts in the definition of a cone space imply that it is a stratified space with smooth structure. It is proved in [45] that any cone space of class C^m with $m \geq 2$ is a Whitney (B) stratified space.

Whitney stratified spaces are, in general, *not* cone spaces. A counterexample is given by Neil's parabola (see [45]). However, Mather's theory of control data (see [31] and page 410 of [46] for an outline of the construction of the link) implies that Whitney (B) stratified subsets of Euclidean space are cone spaces. We caution that the terminology in this area is not uniformly accepted; some authors (for instance [46]) use cone spaces as the definition of stratified spaces.

6.4. THE STRATIFICATION THEOREMS

With this quick review of stratified and cone spaces the structure of the level sets of the momentum map and that of the reduced spaces can be rigorously stated.

THEOREM 6.3. *Consider the closed subset $\mathbf{J}^{-1}(\mu) \subset M$ as a topological subspace of M. Then the submanifolds of the type $\mathbf{J}^{-1}(\mu) \cap G_\mu M_H^z$, with M_H^z the connected component of the H-isotropy type submanifold that contains a point z such that $\mathbf{J}(z) = \mu$, form a Whitney (B) stratification of $\mathbf{J}^{-1}(\mu)$.*

THEOREM 6.4. (Stratification Theorem). *The symplectic strata $M_\mu^{(H)}$ introduced in Theorem 6.1 form a symplectic Whitney (B) stratification of the quotient topological space $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$. In addition, the quotient M_μ is a cone space when considered as a stratified space with strata $M_\mu^{(H)}$.*

Unlike the orbit type stratification of any orbit space of a proper Lie group action on a manifold, the symplectic stratification described in Theorem 6.4 is, in general, *not* minimal among all the Whitney stratifications of the quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ when the value $\mu \in \mathfrak{g}^*$ is not zero. As a corollary of M_μ being a cone space one obtains the following result (see Theorem 5.9 in [46]).

THEOREM 6.5 (Maximal Stratum Theorem). *Each connected component of M_μ contains a unique open stratum that is connected, open, and dense in the connected component of M_μ that contains it.*

7. Singular Orbit Reduction

With the same notations and conventions employed till now, consider the orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ of the affine action Θ through μ. It is important to remark that $\mathcal{O}_\mu$ is only an initial submanifold of $\mathfrak{g}^*$, in general. If the group G is algebraic, semisimple, or compact then it is an embedded submanifold. It is straightforward to verify that the natural inclusion $\mathbf{J}^{-1}(\mu) \hookrightarrow \mathbf{J}^{-1}(\mathcal{O}_\mu)$ induces a bijective map between $\mathbf{J}^{-1}(\mu)/G_\mu$ and $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$. Even if μ is a regular value of $\mathbf{J}$ and G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$ it is not clear what the manifold structure on the quotient $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ should be. If, moreover, the orbit $\mathcal{O}_\mu$ is an *embedded* submanifold, then it is easy to show that $\mathbf{J}$ is transverse to it and hence $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is also an embedded submanifold of M. So if the G-action on M is free and proper and μ is a regular value of $\mathbf{J}$, both quotients $\mathbf{J}^{-1}(\mu)/G_\mu$ and $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ are smooth manifolds with their respective projections surjective submersions and are, in addition, diffeomorphic. It turns out that they are symplectomorphic if we endow $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ with a symplectic structure intimately connected to the symplectic structure on the orbit $\mathcal{O}_\mu$ that we study next.

Let $\mathfrak{g}$ be a Lie algebra acting canonically on the connected symplectic manifold (M, ω) with momentum map $\mathbf{J}$: $M \to \mathfrak{g}^*$ having nonequivariance one-cocycle σ: $G \to \mathfrak{g}^*$. Define the *infinitesimal nonequivariance two-cocycle* of $\mathbf{J}$ as the element $\Sigma \in \Lambda^2(\mathfrak{g})$ given by

$$\Sigma(\xi, \eta) := \mathbf{J}^{[\xi,\eta]}(z) - \{\mathbf{J}^\xi, \mathbf{J}^\eta\}(z), \quad z \in M, \quad \xi, \eta \in \mathfrak{g}, \tag{7.1}$$

where $\mathbf{J}^{\xi}(z) := \langle \mathbf{J}(z), \xi \rangle$, for any $z \in M$. As the definition implies, the left-hand side of this equation does not depend on $z \in M$. As was the case for the nonequivariance one-cocycle, this independence on z follows from the connectedness of M. The relationship between σ: $G \to \mathfrak{g}^*$ and Σ: $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is given by $\Sigma(\xi, \eta) = \mathrm{d}\widehat{\sigma}_\eta(e) \cdot \xi$, where $\widehat{\sigma}_\eta$: $G \to \mathbb{R}$ is defined by $\widehat{\sigma}_\eta(g) := \langle \sigma(g), \eta \rangle$, for any $\xi, \eta \in \mathfrak{g}$.

The *affine Lie–Poisson space* determined by the two-cocycle $\Sigma \in Z^2(\mathfrak{g}; \mathbb{R})$ is defined as the vector space $\mathfrak{g}^*$ endowed with the Poisson bracket

$$\{f, g\}_{\pm}^{\Sigma}(\mu) := \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle \mp \Sigma \left(\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right), \tag{7.2}$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$. The brackets (7.2) are also called the $\pm\Sigma$-*Lie–Poisson structures*. In this formula, the functional derivative $\delta f / \delta \mu$ is defined as the unique element of $\mathfrak{g}$ satisfying

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle = Df(\mu)(\nu)$$

for any $\mu, \nu \in \mathfrak{g}^*$, where $Df(\mu) \in \mathfrak{g}^{**}$ denotes the Fréchet derivative of f at μ. The leaves of the Poisson structure 7.2 are the orbits $\mathcal{O}_\mu$ of the affine action Θ endowed with the G-invariant *orbit* (or Kirillov–Kostant–Souriau) *symplectic form*

$$\omega_{\mathcal{O}_\mu}^{\pm}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle \mp \Sigma(\xi, \eta), \tag{7.3}$$

for arbitrary $\nu \in \mathcal{O}_\mu$, and $\xi, \eta \in \mathfrak{g}$. In this formula $\xi_{\mathfrak{g}^*}$ denotes the infinitesimal generator vector field relative to the action Θ given by $\xi \in \mathfrak{g}$, that is, $\xi_{\mathfrak{g}^*}(\nu) := -\mathrm{ad}_\xi^* \nu + \Sigma(\xi, \cdot)$.

7.1. REGULAR ORBIT REDUCTION

With these preparatory remarks, if $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ and if the action is free, proper, and Hamiltonian, we can state the following result [16, 24, 25]. The set $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i_{\mathcal{O}_\mu}^* \omega = \pi_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu} + \mathbf{J}_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu}^+, \tag{7.4}$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ is the restriction of $\mathbf{J}$ to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ and $\omega_{\mathcal{O}_\mu}^+$ is the $+$-symplectic structure on the affine orbit $\mathcal{O}_\mu$ (see (7.3)). The maps $i_{\mathcal{O}_\mu}$: $\mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow M$ and $\pi_{\mathcal{O}_\mu}$: $\mathbf{J}^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ are the natural injection and the projection, respectively. The pair $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the *symplectic orbit reduced space*. This result can be used to reduce Hamiltonian G-equivariant dynamics. We will not discuss this here because that result will be stated below in total generality for the singular case. We emphasize the similarity between the orbit reduction formula (7.4) and its counterpart (5.5) in the optimal context.

What if the orbit $\mathcal{O}_\mu$ is not embedded or, equivalently, not locally closed in $\mathfrak{g}^*$? One proceeds in the following way ([44]). The freeness of the G-action guarantees that $\mathbf{J}$ is a submersion onto some open subset of $\mathfrak{g}^*$. Since $\mathcal{O}_\mu$ is an initial submanifold, this implies that $\mathbf{J}$ is transverse to $\mathcal{O}_\mu$ and hence, by the transversality theorem for initial manifolds, $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is an initial submanifold of M whose tangent space at z is $T_z(\mathbf{J}^{-1}(\mathcal{O}_\mu)) = \mathfrak{g} \cdot z + A'_G(z)$, where $\mathfrak{g} \cdot z$ denotes the tangent space at z to the orbit $G \cdot z \subset M$. The free and proper G-action on M restricts to a free proper smooth G-action on the G-invariant initial submanifold $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ and, consequently, the quotient $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient manifold with $\pi_{\mathcal{O}_\mu}: \mathbf{J}^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ a surjective submersion. The proof of these statements uses various properties of initial submanifolds. From this point, the proof of the statement proceeds as in the case when $\mathcal{O}_\mu$ was an embedded submanifold. In other words, in the orbit reduction theorem quoted above, one can drop the assumption that the orbit $\mathcal{O}_\mu$ is embedded.

The final result is that if G acts freely and properly on M and $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$, the point reduced space (M_μ, ω_μ) and the orbit reduced space $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ are symplectomorphic.

7.2. THE SINGULAR ORBIT REDUCTION THEOREMS

Based on the model of the manifold structure on the orbit reduced space discussed previously, we turn now to the singular case. A very important technical point is the choice of the topology for the set $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$. In the point reduction approach $\mathbf{J}^{-1}(\mu)$ was thought of as a topological subspace of M and of $\mathbf{J}^{-1}(\mu)/G_\mu$ was the resulting topological quotient. This is *not* the right way to proceed when dealing with orbit reduction; in this situation $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ needs to be endowed not with the relative topology but with the initial topology induced by the map $\mathbf{J}_{\mathcal{O}_\mu} := \mathbf{J}|_{\mathbf{J}^{-1}(\mathcal{O}_\mu)}$: $\mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$, where the orbit $\mathcal{O}_\mu$ comes with its own smooth structure diffeomorphic to G/G_μ. This topology on $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is called the *initial topology*. Recall that the initial topology induced by the map $\mathbf{J}_{\mathcal{O}_\mu}: \mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$ is characterized by the fact that for any topological space Z and any map $\varphi: Z \to \mathbf{J}^{-1}(\mathcal{O}_\mu)$ we have that φ is continuous if and only if $\mathbf{J}_{\mathcal{O}_\mu} \circ \varphi$ is continuous. Additionally, the set $\mathcal{B} = \{\mathbf{J}_{\mathcal{O}_\mu}^{-1}(U) \mid U \text{ open in } \mathcal{O}_\mu\}$ is a subbase of this topology. In particular, this implies that $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is first countable.

The following proposition shows that the initial topology of $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ generalizes to the singular case the smooth structure for this set considered in the regular situation discussed above.

PROPOSITION 7.1. *Endowing $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ with its initial topology, the map f: $G \times_{G_\mu} \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mathcal{O}_\mu)$ given by $[g, z] \mapsto g \cdot z$ is a homeomorphism.*

At this point all the necessary background for orbit reduction has been explained and we can state the following result. We are using all notations in force till now.

THEOREM 7.2 (Singular symplectic orbit strata). *Let $\mu = \mathbf{J}(z)$. The following hold*:

(i) *The set* $G(\mathbf{J}^{-1}(\mu) \cap M_H^z)$ *is an initial submanifold of* M *whose tangent space is given by*

$$\begin{aligned} &T_m\big(G(\mathbf{J}^{-1}(\mu) \cap M_H^z)\big) \\ &\quad = \operatorname{span}\{\xi_M(m) + X_f(m) \mid \xi \in \mathfrak{g}, f \in C^\infty(M)^G\} \\ &\quad = \mathfrak{g} \cdot m + A'_G(m), \end{aligned} \tag{7.5}$$

with A'_G *the polar distribution associated to the* G*-action on* M.

(ii) *The set* $M_{\mathcal{O}_\mu}^{(H)} := [G(\mathbf{J}^{-1}(\mu) \cap M_H^z)]/G$ *has a unique quotient differentiable structure such that the canonical projection*

$$\pi_{\mathcal{O}_\mu}^{(H)} : G(\mathbf{J}^{-1}(\mu) \cap M_H^z) \longrightarrow M_{\mathcal{O}_\mu}^{(H)}$$

is a surjective submersion.

(iii) *There is a unique symplectic structure* $\omega_{\mathcal{O}_\mu}^{(H)}$ *on* $M_{\mathcal{O}_\mu}^{(H)}$ *characterized by*

$$i_{\mathcal{O}_\mu}^{(H)*}\omega = \pi_{\mathcal{O}_\mu}^{(H)*}\omega_{\mathcal{O}_\mu}^{(H)} + \mathbf{J}_{\mathcal{O}_\mu}^{(H)*}\omega_{\mathcal{O}_\mu}^{+}, \tag{7.6}$$

where $i_{\mathcal{O}_\mu}^{(H)} : G(\mathbf{J}^{-1}(\mu) \cap M_H^z) \hookrightarrow M$ *is the inclusion,* $\mathbf{J}_{\mathcal{O}_\mu}^{(H)} : G(\mathbf{J}^{-1}(\mu) \cap M_H^z) \to \mathcal{O}_\mu$ *is obtained by restriction of the momentum map* $\mathbf{J}$, *and* $\omega_{\mathcal{O}_\mu}^{+}$ *is the* $+$*-symplectic form on* $\mathcal{O}_\mu$ *defined in (7.3). The pairs* $(M_{\mathcal{O}_\mu}^{(H)}, \omega_{\mathcal{O}_\mu}^{(H)})$ *are called the* singular symplectic orbit strata.

(iv) *Let* $h \in C^\infty(M)^G$ *be a* G*-invariant Hamiltonian. Then the flow* F_t *of* X_h *leaves the connected components of* $G(\mathbf{J}^{-1}(\mu) \cap M_H^z)$ *invariant and commutes with the* G*-action, so it induces a flow* $F_t^{\mathcal{O}_\mu}$ *on* $M_{\mathcal{O}_\mu}^{(H)}$ *that is characterized by*

$$\pi_{\mathcal{O}_\mu}^{(H)} \circ F_t \circ i_{\mathcal{O}_\mu}^{(H)} = F_t^{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu}^{(H)}.$$

(v) *The flow* $F_t^{\mathcal{O}_\mu}$ *is Hamiltonian on* $(M_{\mathcal{O}_\mu}^{(H)}, \omega_{\mathcal{O}_\mu}^{(H)})$ *relative to the* reduced Hamiltonian $h_{\mathcal{O}_\mu}^{(H)} : M_{\mathcal{O}_\mu}^{(H)} \to \mathbb{R}$ *defined by*

$$h_{\mathcal{O}_\mu}^{(H)} \circ \pi_{\mathcal{O}_\mu}^{(H)} = h \circ i_{\mathcal{O}_\mu}^{(H)}.$$

The vector fields X_h *and* $X_{h_{\mathcal{O}_\mu}^{(H)}}$ *are* $\pi_{\mathcal{O}_\mu}^{(H)}$*-related.*

(vi) *Let* $k : M \to \mathbb{R}$ *be another* G*-invariant function. Then* $\{h, k\}$ *is also* G*-invariant and*

$$\{h, k\}_{\mathcal{O}_\mu}^{(H)} = \{h_{\mathcal{O}_\mu}^{(H)}, k_{\mathcal{O}_\mu}^{(H)}\}_{M_{\mathcal{O}_\mu}^{(H)}},$$

where $\{\cdot, \cdot\}_{M_{\mathcal{O}_\mu}^{(H)}}$ *denotes the Poisson bracket induced by the symplectic structure on* $M_{\mathcal{O}_\mu}^{(H)}$.

As for singular point reduced strata, there is a structure theorem for the singular orbit strata or, equivalently, an orbit form of Sjamaar's principle.

THEOREM 7.3 (Structure theorem for the singular orbit strata). *The following hold:*

(i) *The canonical projection* $\pi_{\mathcal{O}_\mu}^{(H)} : G(\mathbf{J}^{-1}(\mu) \cap M_H^z) \longrightarrow M_{\mathcal{O}_\mu}^{(H)} = [G(\mathbf{J}^{-1}(\mu) \cap M_H^z)]/G$ *defines a smooth fiber bundle with fiber* G/H *and structure group* $N(H)^z/H$. *We recall that* $N(H)^z$ *is the open and hence closed subgroup of* $N(H)$ *that leaves* M_H^z *invariant.*

(ii) *Consider the free, proper, and canonical action of* $L^z := N(H)^z/H$ *on* M_H^z *and let* $\mathbf{J}_{L^z}\colon M_H^z \to \mathfrak{l}^*$ *be the associated momentum map given by* $\mathbf{J}_{L^z}(m) = \Lambda(\mathbf{J}|_{M_H^z}(m) - \mu)$, *for any* $m \in M_H^z$. *Then the regular orbit reduced space* $(M_H^z)_{\mathcal{O}_0}$ *at the affine orbit corresponding to* $0 \in \mathfrak{l}^*$ *is given by*

$$(M_H^z)_{\mathcal{O}_0} = \mathbf{J}_{L^z}^{-1}(\mathcal{O}_0)/L^z = \left[\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z\right]/(N(H)^z/H) \tag{7.7}$$

(iii) *The projection* $\pi_{\mathcal{O}_0}\colon \mathbf{J}_{L^z}^{-1}(\mathcal{O}_0) \to (M_H^z)_{\mathcal{O}_0}$ *defines a smooth principal* L^z*-bundle. Regarding* G/H *as a right* $(N(H)^z/H)$*-space and* $\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z$ *as a left* $(N(H)^z/H)$*-space, consider the bundle with fiber* G/H *and structure group* G *associated to* $\pi_{\mathcal{O}_0}$, *that is,*

$$G/H \times_{N(H)^z/H} \left(\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z\right) \longrightarrow \left[\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z\right]/(N(H)^z/H).$$

This bundle is G*-symplectomorphic to* $\pi_{\mathcal{O}_\mu}^{(H)}\colon G(\mathbf{J}^{-1}(\mu) \cap M_H^z) \longrightarrow M_{\mathcal{O}_\mu}^{(H)}$, that is, $G/H \times_{N(H)^z/H} \left(\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z\right)$ is *G-diffeomorphic to* $G(\mathbf{J}^{-1}(\mu) \cap M_H^z)$ *and the orbit reduced space*

$$(M_H^z)_{\mathcal{O}_0} = \mathbf{J}_{L^z}^{-1}(\mathcal{O}_0)/L^z = \left[\mathbf{J}^{-1}(N(H)^z \cdot \mu) \cap M_H^z\right]/(N(H)^z/H)$$

is symplectomorphic to $M_{\mathcal{O}_\mu}^{(H)}$. *We say that* $(M_H^z)_{\mathcal{O}_0}$ *is a* regularization *of the singular symplectic orbit stratum* $M_{\mathcal{O}_\mu}^{(H)}$.

The singular symplectic orbit strata form a stratification in the same sense as the singular point strata.

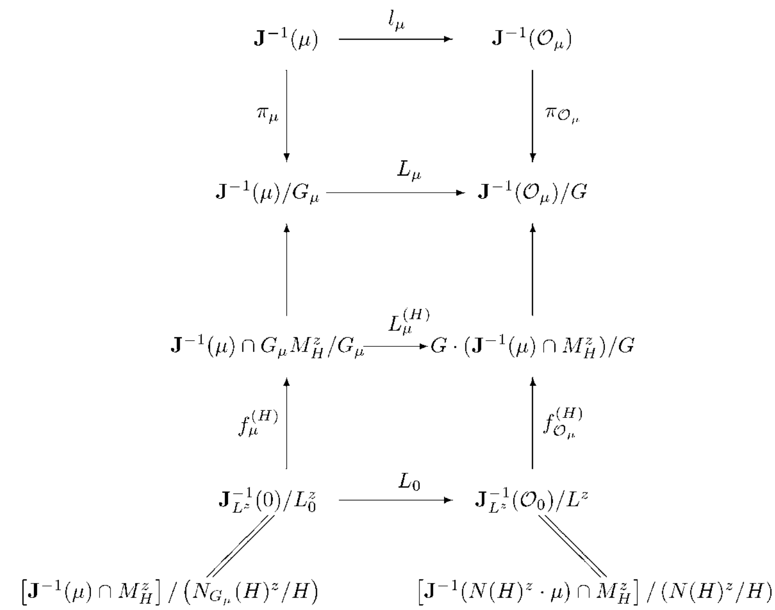
THEOREM 7.4 (Orbit reduction stratification theorem and the singular reduction diagram). *Let* $l_\mu\colon \mathbf{J}^{-1}(\mu) \hookrightarrow \mathbf{J}^{-1}(\mathcal{O}_\mu)$ *be the inclusion and* $L_\mu\colon \mathbf{J}^{-1}(\mu)/G_\mu \to \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ *the map defined by the commutative diagram*

$$\begin{array}{ccc}
\mathbf{J}^{-1}(\mu) & \xrightarrow{\;l_\mu\;} & \mathbf{J}^{-1}(\mathcal{O}_\mu) \\
{\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{O}_\mu}} \\
\mathbf{J}^{-1}(\mu)/G_\mu & \xrightarrow{\;L_\mu\;} & \mathbf{J}^{-1}(\mathcal{O}_\mu)/G.
\end{array}$$

Consider $\mathbf{J}^{-1}(\mu)/G_\mu$ *as a smooth symplectically stratified topological space with the stratification introduced in Theorem 6.4. Then*

(i) *The submanifolds in Theorem 7.2 induce a smooth symplectic stratification of* $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ *that makes it into a cone (and, hence, Whitney (B)) space.*
(ii) *The map* L_μ *is a homeomorphism of cone spaces.*

The Structure Theorem, Sjamaar's Principle, and the singular reduction diagram are illustrated in the following commutative diagram:



In this diagram, the symplectomorphisms $f_\mu^{(H)}$ and $f_{\mathcal{O}_\mu}^{(H)}$ are the explicit implementation of Sjamaar's Principle (see Theorems 6.2 and 7.3). We recall that L_0 and $L_\mu^{(H)}$ are also symplectomorphisms and that L_μ is a homeomorphism of smooth symplectic Whitney (B) stratified spaces.

8. Poisson Reduction

This section reviews the main theorems in the theory of Poisson reduction. The hypotheses of the first theorem are strong and are rarely verified in physical applications. Nevertheless, this theorem serves as a model for the type of results that one would like to have. The subsequent theorems will weaken and eliminate various assumptions.

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and G a Lie group acting canonically on M. If the G-action Φ: $G \times M \to M$ is free and proper, the orbit space M/G is a smooth manifold and the canonical projection π: $M \to M/G$ is a smooth surjective submersion.

THEOREM 8.1 (Regular Poisson reduction). *Assume the hypotheses above. Let $\mathcal{J}$: $M \to M/A'_G$ be the corresponding optimal momentum map. Then*

(i) *The orbit space M/G is a Poisson manifold with the Poisson bracket $\{\cdot,\cdot\}^{M/G}$, uniquely characterized by the relation*

$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m), \tag{8.1}$$

for any $m \in M$ and $f, g \in C^\infty(M/G)$.

(ii) *The Poisson structure induced by the bracket* $\{\cdot,\cdot\}^{M/G}$ *on* M/G *is the only one for which the projection* π: $(M,\{\cdot,\cdot\}) \to (M/G,\{\cdot,\cdot\}^{M/G})$ *is a Poisson map.*

(iii) *Let* $h \in C^\infty(M)^G$ *be a G-invariant smooth function on M. The Hamiltonian flow* F_t *of* X_h *commutes with the G-action, so it induces a flow* $F_t^{M/G}$ *on* M/G *characterized by* $\pi \circ F_t = F_t^{M/G} \circ \pi$. *The flow* $F_t^{M/G}$ *is Hamiltonian on* $(M/G, \{\cdot,\cdot\}^{M/G})$, *for the* reduced Hamiltonian *function* $[h] \in C^\infty(M/G)$ *defined by* $[h]\circ\pi = h$. *The vector fields* X_h *and* $X_{[h]}$ *are* π*-related.*

(iv) *The symplectic leaves of* $(M/G,\{\cdot,\cdot\}^{M/G})$ *are given by the optimal orbit reduced spaces* $\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/G, \omega_{\mathcal{O}_\rho}\right)$, $\rho \in M/A'_G$, *introduced in Theorem 5.3.*

(v) *If the Poisson manifold* $(M, \{\cdot,\cdot\})$ *is symplectic with form* ω *and the G-action has an associated standard momentum map* $\mathbf{J}$: $M \to \mathfrak{g}^*$, *then the symplectic leaves of* $(M/G,\{\cdot,\cdot\}^{M/G})$ *are given by the spaces* $\left(M^c_{\mathcal{O}_\mu} := G\cdot\mathbf{J}^{-1}(\mu)^c/G, \omega^c_{\mathcal{O}_\mu}\right)$, *where* $\mathbf{J}^{-1}(\mu)^c$ *is a connected component of the fiber* $\mathbf{J}^{-1}(\mu)$ *and* $\omega^c_{\mathcal{O}_\mu}$ *the restriction to* $M^c_{\mathcal{O}_\mu}$ *of the symplectic form* $\omega_{\mathcal{O}_\mu}$ *of the orbit reduced space* $M_{\mathcal{O}_\mu}$ *defined in (7.4). If, additionally, G is compact, M is connected, and the momentum map* $\mathbf{J}$ *is proper, then* $M^c_{\mathcal{O}_\mu} = M_{\mathcal{O}_\mu}$.

8.1. POISSON REDUCTION BY PSEUDOGROUPS

The reduction theorem just presented is valid under very strong regularity hypotheses that insure the smoothness of the orbit space onto which the Poisson bracket and the corresponding equivariant dynamics can be projected. When these hypotheses are not present, the orbit space is not smooth anymore and one needs to work with presheaves of Poisson algebras.

Let M be a topological space with a presheaf $\mathcal{F}$ of smooth functions. A *presheaf of Poisson algebras* on $(M,\mathcal{F})$ is a map $\{\cdot,\cdot\}$ that assigns to each open set $U\subset M$ a bilinear operation $\{\cdot,\cdot\}_U$: $\mathcal{F}(U)\times\mathcal{F}(U)\to\mathcal{F}(U)$ such that the pair $(\mathcal{F}(U),\{\cdot,\cdot\}_U)$ is a Poisson algebra. A presheaf of Poisson algebras will be usually denoted as a triple $(M,\mathcal{F},\{\cdot,\cdot\})$. The presheaf of Poisson algebras $(M,\mathcal{F},\{\cdot,\cdot\})$ is *nondegenerate* if the following condition holds: if $f\in\mathcal{F}(U)$ is such that $\{f,g\}_{U\cap V}=0$, for any $g\in\mathcal{F}(V)$ and any open set of V, then f is constant on the connected components of U.

Any Poisson manifold $(M,\{\cdot,\cdot\})$ has a natural presheaf of Poisson algebras on its presheaf of smooth functions C^∞_M that associates to any open subset U of M the restriction $\{\cdot,\cdot\}|_U$ of the bracket $\{\cdot,\cdot\}$ to $C^\infty(U)\times C^\infty(U)$. We shall formulate below a result that fully characterizes the situations in which the presheaf C^∞_M of Poisson algebras on $(M,\{\cdot,\cdot\})$ behaves properly under restrictions to subsets and projections to the orbit spaces of pseudogroups of local Poisson diffeomorphisms of $(M,\{\cdot,\cdot\})$. To do this, we return to the discussion on pseudogroups in Section 3.

Let M be a smooth manifold and A a pseudogroup of local diffeomorphisms of M. Let $S\subset M$ be a subset of M endowed with a topology $\mathcal{T}$ that, in general, does not coincide with the relative or subspace topology. The presheaf C^∞_M of smooth functions on M induces a quotient presheaf $C^\infty_{M/A}$ on the orbit space M/A. Consider now the subset

$$A_S := \{a \in A \mid a(s) \in S \text{ for any } s \in S,\ s \text{ in the domain of } a\}.$$

Throughout this section we will assume that A_S is a subpseudogroup of A. This hypothesis allows the formation of the quotients S/A_S and M/A_S. Since the quotient S/A_S can be seen as a subset of M/A_S, there is a well defined presheaf of Whitney smooth functions $C^\infty_{S/A_S,M/A_S}$ on S/A_S induced by C^∞_{M/A_S}. The openness of the projection $M \to M/A_S$ guarantees, by Proposition 3.2, that

$$C^\infty_{S/A_S,M/A_S} = C^\infty_{S/A_S}(\cdot)^{A_S},$$

where $C^\infty_{S/A_S}(\cdot)^{A_S}$ is the quotient presheaf on S/A_S associated to the presheaf $C^\infty_{S,M}(\cdot)^{A_S}$ of Whitney A_S-invariant functions on S induced by $C^\infty_M(\cdot)^{A_S}$. In order to simplify notation, define

$$W^\infty_{S/A_S} := C^\infty_{S/A_S,M/A_S} = C^\infty_{S/A_S}(\cdot)^{A_S}.$$

We recall that for any open set $V \subset S/A_S$, the elements $f \in W^\infty_{S/A_S}(V)$ are characterized by the fact that if π_S: $S \to S/A_S$ is the projection onto orbit space then for any $m \in \pi_S^{-1}(V)$ there exists an open A_S-invariant neighborhood of m in M and $F \in C^\infty_M(U_m)^{A_S}$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V)\cap U_m} = F|_{\pi_S^{-1}(V)\cap U_m}. \tag{8.2}$$

The function F is called a *local extension* of $f \circ \pi_S$ at the point m.

Now assume that the given topology $\mathcal{T}$ on S is stronger than or equal to the relative topology on S. The presheaf W^∞_{S/A_S} is said to have the *(A, A_S)-local extension property* if A_S is a subpseudogroup of A and for any $f \in W^\infty_{S/A_S}(V)$ and $m \in \pi_S^{-1}(V)$ there exist an open A-invariant neighborhood U_m of m in M and $F \in C^\infty_M(U_m)^A$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V)\cap U_m} = F|_{\pi_S^{-1}(V)\cap U_m}.$$

The function F is called an *A-invariant local extension* of $f \circ \pi_S$ at m.

Finally, let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold, $A \subset \mathcal{P}_L(M)$ a pseudogroup of local Poisson diffeomorphisms of M, and $S \subset M$ a subset of M such that W^∞_{S/A_S} has the (A, A_S)-local extension property. Then $(M, \{\cdot,\cdot\}, A, S)$ is said to be *Poisson reducible* if $\left(S/A_S, W^\infty_{S/A_S}, \{\cdot,\cdot\}^{S/A_S}\right)$ is a well defined presheaf of Poisson algebras where, for any open set $V \subset S/A_S$, the bracket $\{\cdot,\cdot\}_V^{S/A_S}$: $W^\infty_{S/A_S}(V) \times W^\infty_{S/A_S}(V) \to W^\infty_{S/A_S}(V)$ is given by

$$\{f, g\}_V^{S/A_S}(\pi_S(m)) = \{F, G\}(m) \tag{8.3}$$

for any $m \in \pi_S^{-1}(V)$ and where F, G are A-invariant local extensions at m of $f \circ \pi_S$ and $g \circ \pi_S$, respectively.

Using the concepts just introduced we formulate now a Poisson reduction theorem for actions of pseudogroups of local Poisson diffeomorphisms. The following notations are used below. If $B \in \Lambda^2(M)$ is the Poisson tensor associated to $(M, \{\cdot,\cdot\})$, that is, $B(m)(\mathrm{d}f(m), \mathrm{d}g(m)) := \{f, g\}(m)$, for any smooth locally defined functions f, g in a neighborhood of $m \in M$, then $B^\sharp : T^*M \to TM$ denotes the bundle map given by $B^\sharp(\mathrm{d}g) := \{\cdot, g\}$. If $V \subset T_mM$ is a subspace, then its *annihilator* $V^\circ \subset T^*_mM$ is defined by $V^\circ := \{\alpha \in T^*_mM \mid \langle \alpha, v\rangle = 0 \text{ for all } v \in V\}$.

THEOREM 8.2 ([42]). *Let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold, $A \subset \mathcal{P}_L(M)$ a pseudogroup of local Poisson diffeomorphisms of M, and $S \subset M$ a subset of M such that W^∞_{S/A_S} has the (A, A_S)-local extension property. Let $B^\sharp\colon T^*M \to TM$ be the bundle map associated to the Poisson tensor B of $(M, \{\cdot,\cdot\})$. Then $(M, \{\cdot,\cdot\}, A, S)$ is Poisson reducible if and only if for any $m \in S$ we have*

$$B^\sharp(\Delta_m) \subset \left[\Delta^S_m\right]^\circ, \tag{8.4}$$

where $\Delta_m := \{\mathrm{d}F(m) \mid F \in C^\infty_M(U_m)^A$, for any open A-invariant neighborhood U_m of m in $M\}$, and where $\Delta^S_m := \{\mathrm{d}F(m) \in \Delta_m \mid F|_{U_m \cap V_m}$ is constant, *for an open A-invariant neighborhood U_m of m in M and an open A_S-invariant neighborhood V_m of m in $S\}$.*

Even though in this theorem only the subpseudogroup A_S is needed in the construction of the quotient space S/A_S, the full pseudogroup A is used in the definition of the Poisson bracket on this quotient when $(M, \{\cdot,\cdot\}, A, S)$ is Poisson reducible. Actually, in spite of the fact that the reduction of $(M, \{\cdot,\cdot\}, A, S)$ and $(M, \{\cdot,\cdot\}, A_S, S)$ gives the same quotient manifold S/A_S it does *not* yield the same Poisson brackets on this quotient since different sets of functions are involved. There are even instances in which $(M, \{\cdot,\cdot\}, A, S)$ is Poisson reducible whereas $(M, \{\cdot,\cdot\}, A_S, S)$ is not, as will be shown explicitly later on.

Theorem 8.2 has several useful corollaries which we now state.

COROLLARY 8.3. *Let S be an embedded submanifold of the Poisson manifold $(M, \{\cdot,\cdot\})$. The triple $(M, \{\cdot,\cdot\}, S)$ is Poisson reducible if and only if*

$$B^\sharp(T^*_mM) \subset T_mS, \quad \text{for any } m \in S, \tag{8.5}$$

or, equivalently, whenever

$$T_m\mathcal{L}_m \subset T_mS, \quad \text{for any } m \in S, \tag{8.6}$$

where $\mathcal{L}_m$ is the symplectic leaf of $(M, \{\cdot,\cdot\})$ containing the point $m \in S$. If S is only an immersed submanifold of M then the conditions (8.5) *or* (8.6) *are sufficient but, in general, not necessary conditions for the Poisson reducibility of $(M, \{\cdot,\cdot\}, S)$. In both cases, the Poisson reducibility of $(M, \{\cdot,\cdot\}, S)$ implies that $(S, \{\cdot,\cdot\}|_S)$ is a Poisson manifold.*

COROLLARY 8.4. *Let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold, $B \in \Lambda^2(M)$ the associated Poisson tensor, and D a smooth, integrable, and regular distribution on M generated by a family of local infinitesimal Poisson automorphisms of M. Then there is a unique Poisson bracket $\{\cdot,\cdot\}^{M/D}$ on the quotient manifold M/D for which the projection $\pi_D\colon M \to M/D$ is a Poisson map.*

If *M is symplectic with form ω then the rank of the Poisson structure $\{\cdot,\cdot\}^{M/D}$ at the point $\pi_D(m)$ is*

$$\operatorname{rank}\left(B^{\sharp}_{M/D}(\pi_D(m))\right) = \dim M - \dim D(m) - \dim[(D(m))^{\omega} \cap D(m)], \tag{8.7}$$

where $B_{M/D} \in \Lambda^2(T^*(M/D))$ *is the Poisson tensor associated to the bracket* $\{\cdot,\cdot\}^{M/D}$ *on* M/D.

COROLLARY 8.5. *Let G be a Lie group acting freely, properly, and canonically on the Poisson manifold* $(M, \{\cdot,\cdot\})$ *via the map* $\Phi\colon G \times M \to M$. *Let* $A := A_G = \{\Phi_g \mid g \in G\} \subset \mathcal{P}(M)$ *and let S be an embedded G-invariant submanifold of M. Then* $(M, \{\cdot,\cdot\}, A, S)$ *is Poisson reducible if and only if*

$$B^{\sharp}((\mathfrak{g}\cdot m)^{\circ}) \subset T_m S, \quad \text{for any} \quad m \in S. \tag{8.8}$$

If the G-action on M is not free, the inclusion

$$B^{\sharp}\left(((\mathfrak{g}\cdot m)^{\circ})^{G_m}\right) \subset T_m S, \quad \text{for any} \quad m \in S, \tag{8.9}$$

implies that $(M, \{\cdot,\cdot\}, A, S)$ *is Poisson reducible.*

8.2. POISSON REDUCTION BY DISTRIBUTIONS

Next, we want to analyze the Poisson reduction procedure by generalized distributions. The existence of a pseudogroup of global Poisson diffeomorphisms will not be required anymore in the following theorems. We begin by extending the notion of integrability of generalized distributions to decomposed subsets.

Let M be a differentiable manifold and $S \subset M$ a decomposed subset of M. Let $\{S_i\}_{i\in I}$ be the pieces of this decomposition. The topology of S is not necessarily the relative topology as a subset of M. We say that $D \subset TM|_S$ is a *smooth distribution on S adapted to the decomposition* $\{S_i\}_{i\in I}$, if $D \cap TS_i$ is a smooth distribution on each S_i for all $i \in I$. The distribution D is said to be *integrable* if $D \cap TS_i$ is integrable for each $i \in I$.

The integrability of the distributions $D_{S_i} := D \cap TS_i$ on S_i allows the partitioning of each S_i into the corresponding maximal integral manifolds. Thus, there is an equivalence relation on S_i whose equivalence classes are precisely these maximal integral manifolds. Doing this on each S_i, gives an equivalence relation D_S on the whole set S by taking the union of the different equivalence classes corresponding to all the D_{S_i}. The quotient space S/D_S is defined by $S/D_S := \bigcup_{i\in I} S_i/D_{S_i}$ and π_{D_S}: $S \to S/D_S$ denotes the natural projection.

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $D \subset TM$ a smooth distribution on M. The distribution D is called *Poisson* or *canonical*, if the condition $\mathrm{d}f|_D = \mathrm{d}g|_D = 0$, for any $f, g \in C^{\infty}_M(U)$ and any open subset $U \subset P$, implies that $\mathrm{d}\{f, g\}|_D = 0$. Note that if D is spanned by a family of infinitesimal Poisson automorphisms then D is a Poisson distribution. The converse is not necessarily true.

We shall define now a presheaf of smooth functions on S/D_S that requires less invariance properties than those that appeared in the context of quotients by pseudogroups of transformations. Define the presheaf of smooth functions C^∞_{S/D_S} on S/D_S by associating to any open subset V of S/D_S the set of functions $C^\infty_{S/D_S}(V)$ characterized by the following property: $f \in C^\infty_{S/D_S}(V)$ if and only if for any $z \in V$ there exists $m \in \pi^{-1}_{D_S}(V)$, an open neighborhood U_m of m in M, and $F \in C^\infty_M(U_m)$ such that

$$f \circ \pi_{D_S}|_{\pi^{-1}_{D_S}(V)\cap U_m} = F|_{\pi^{-1}_{D_S}(V)\cap U_m}. \tag{8.10}$$

The function F is called, as before, a *local extension* of $f \circ \pi_{D_S}$ at the point $m \in \pi^{-1}_{D_S}(V)$.

The presheaf C^∞_{S/D_S} is said to have the (D, D_S)*-local extension property* if the topology of S is stronger than the relative topology and the local extensions of $f \circ \pi_{D_S}$ defined in (8.10) can always be chosen to satisfy

$$\mathrm{d}F(n)|_{D(n)} = 0, \quad \text{for any } n \in \pi^{-1}_{D_S}(V) \cap U_m.$$

The function F is called a *local D-invariant extension* of $f \circ \pi_{D_S}$ at the point $m \in \pi^{-1}_{D_S}(V)$.

PROPOSITION 8.6. *Suppose that S is a smooth embedded submanifold of M and that D_S is a smooth, integrable, and regular distribution on S. Then the presheaf C^∞_{S/D_S} coincides with the presheaf of smooth functions on S/D_S when considered as a regular quotient manifold.*

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, S a decomposed subset of M, and $D \subset TM|_S$ a Poisson integrable generalized distribution adapted to the decomposition of S. Assume that C^∞_{S/D_S} has the (D, D_S)-local extension property. We say that $(M, \{\cdot,\cdot\}, D, S)$ is *Poisson reducible* if $(S/D_S, C^\infty_{S/D_S}, \{\cdot,\cdot\}^{S/D_S})$ is a well defined presheaf of Poisson algebras where, for any open set $V \subset S/D_S$, the bracket $\{\cdot,\cdot\}^{S/D_S}_V \colon C^\infty_{S/D_S}(V) \times C^\infty_{S/D_S}(V) \to C^\infty_{S/D_S}(V)$ is given by

$$\{f, g\}^{S/D_S}_V(\pi_{D_S}(m)) := \{F, G\}(m),$$

for any $m \in \pi^{-1}_{D_S}(V)$. In this formula, the maps F, G are local D-invariant extensions at m of $f \circ \pi_{D_S}$ and $g \circ \pi_{D_S}$, respectively.

THEOREM 8.7. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$, S a decomposed space, and $D \subset TM|_S$ a Poisson integrable generalized distribution adapted to the decomposition of S. Assume that C^∞_{S/D_S} has the (D, D_S)-local extension property. Then $(M, \{\cdot,\cdot\}, D, S)$ is Poisson reducible if for any $m \in S$*

$$B^\sharp(\Delta_m) \subset \left[\Delta^S_m\right]^\circ, \tag{8.11}$$

where

$$\Delta_m := \Big\{\mathrm{d}F(m) \mid F \in C^\infty_M(U_m), \mathrm{d}F(z)|_{D(z)} = 0, \textit{ for all } z \in U_m \cap S, \textit{ and for any open neighborhood } U_m \textit{ of } m \textit{ in } M\Big\}$$

and

$$\Delta_m^S := \{\mathrm{d}F(m) \in \Delta_m \mid F|_{U_m \cap V_m} \textit{ is constant for an open neighborhood } U_m \textit{ of } m \textit{ in } M \textit{ and an open neighborhood } V_m \textit{ of } m \textit{ in } S\}.$$

Note that if S is endowed with the relative topology then

$$\Delta_m^S := \{\mathrm{d}F(m) \in \Delta_m \mid F|_{U_m \cap V_m} \text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } M\}.$$

As opposed to the situation in Theorem 8.2, condition (8.11) is sufficient for Poisson reducibility but, in general, is not necessary. The reason behind this is that the functions that define the spaces Δ_m and Δ_m^S are not defined on saturated open sets. As we will see in Theorem 8.8, an alternative hypothesis that makes this condition necessary and sufficient is, roughly speaking, the regularity of the distribution $D_S := D \cap TS$.

8.3. THE REGULAR CASE

Next, we investigate the consequences of this theorem if the distribution D is regular. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and S an embedded submanifold of M. Let $D \subset TM|_S$ be a subbundle of the tangent bundle of M restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, regular distribution on S and D is canonical.

THEOREM 8.8 ([28]). *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$ and S an embedded smooth submanifold of M. Let $D \subset TM|_S$ be a canonical subbundle of the tangent bundle of M restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, and regular distribution on S. Then $(M, \{\cdot,\cdot\}, D, S)$ is Poisson reducible if and only if*

$$B^\sharp(D^\circ) \subset TS + D. \tag{8.12}$$

One of the key technical difficulties in proving this theorem is given by the following statement that is useful also in other contexts when carrying out Poisson reduction.

LEMMA 8.9. *Let M be a smooth manifold and S an embedded submanifold of M. Let $D \subset TM|_S$ be a subbundle of the tangent bundle of M restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, regular distribution on S. Then the presheaf C^∞_{S/D_S} has the (D, D_S)-local extension property.*

Remark 8.10. Even though in the previous theorem only the distribution D_S intervenes in the construction of the quotient manifold S/D_S, the full distribution D is used in the definition of the Poisson bracket on this quotient when $(M, \{\cdot,\cdot\}, D, S)$ is Poisson reducible. Actually, in spite of the fact that the reduction of $(M, \{\cdot,\cdot\}, D, S)$ and $(M, \{\cdot,\cdot\}, D_S, S)$ gives the same quotient manifold S/D_S it does *not* yield the same Poisson brackets on this quotient since different sets of functions are involved. This is particularly evident in the following example in which we show, using Theorem 8.8, that $(M, \{\cdot,\cdot\}, D, S)$ is reducible whereas $(M, \{\cdot,\cdot\}, D_S, S)$ is not.

Let (M, ω) be a connected symplectic manifold acted freely and canonically upon by a connected and compact Lie group G. Let $\mathbf{J}$: $M \to \mathfrak{g}^*$ be a coadjoint equivariant standard momentum map associated to this action, $\mu \in \mathfrak{g}^*$ one of its values, and $G_\mu \subset G$ its coadjoint isotropy subgroup. Let D be the distribution on M given by the tangent spaces to the orbits of the G-action and $S := \mathbf{J}^{-1}(\mu)$, which is a smooth closed submanifold of M because of the freeness of the action. In this case D_S is the distribution given by the tangent spaces to the orbits of the G_μ-action. The compactness and connectedness of G implies that G_μ is connected (see Theorem 3.3.1 in [11]) and, hence, $S/D_S = \mathbf{J}^{-1}(\mu)/G_\mu$.

The quadruple $(M, \omega, D, \mathbf{J}^{-1}(\mu))$ satisfies (8.12) and is hence Poisson reducible. Indeed, in this case the expression (8.12) is $(\mathfrak{g}\cdot m)^\omega \subset \ker T_m\mathbf{J} + \mathfrak{g}\cdot m$, for any $m \in \mathbf{J}^{-1}(\mu)$, which amounts to $\ker T_m\mathbf{J} \subset \ker T_m\mathbf{J} + \mathfrak{g}\cdot m$ (since $(\mathfrak{g}\cdot m)^\omega = \ker T_m\mathbf{J}$), which is an obvious inclusion.

On the other hand, the quadruple $(M, \omega, D_S, \mathbf{J}^{-1}(\mu))$ is *not* Poisson reducible even though the corresponding quotient manifold is the same as for $(M, \omega, D, \mathbf{J}^{-1}(\mu))$. Indeed, condition (8.12) reads in this case

$$(\mathfrak{g}_\mu\cdot m)^\omega \subset \ker T_m\mathbf{J} + \mathfrak{g}_\mu\cdot m = \ker T_m\mathbf{J}, \quad \text{for any } m \in \mathbf{J}^{-1}(\mu)$$

However,

$$(\mathfrak{g}_\mu\cdot m)^\omega = (\ker T_m\mathbf{J} \cap \mathfrak{g}\cdot m)^\omega = (\ker T_m\mathbf{J})^\omega + (\mathfrak{g}\cdot m)^\omega = \mathfrak{g}\cdot m + \ker T_m\mathbf{J}$$

which is, in general, not a subset of $\ker T_m\mathbf{J}$.

A useful consequence of Theorems 8.7 and 8.8 is given by the following statement:

PROPOSITION 8.11. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$. Let S be an embedded submanifold of M and $D := B^\sharp((TS)^\circ) \subset TM|_S$. Assume that the* characteristic distribution *$D_S := D \cap TS$ of S relative to the Poisson bracket $\{\cdot,\cdot\}$ is a smooth and integrable generalized distribution on S such that C^∞_{S/D_S} has the (D, D_S)-local extension property. Then $(M, \{\cdot,\cdot\}, D, S)$ is Poisson reducible.*

The next topic in this section is the reduction of coisotropic submanifolds. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$ and S an immersed smooth submanifold of M. Denote by $(TS)^\circ := \{\alpha_s \in T^*_sM \mid \langle\alpha_s, v_s\rangle = 0$, for all $s \in S, v_s \in T_sS\} \subset T^*M$ the *conormal bundle* of the manifold S; it is a vector subbundle of $T^*M|_S$. The manifold S is called *coisotropic* if $B^\sharp((TS)^\circ) \subset TS$. Note that this is the straightforward generalization of the definition of a coisotropic submanifold of a symplectic manifold. Indeed, if the Poisson bracket on M is defined by a symplectic form $\omega \in \Omega^2(M)$, then $B^\sharp((TS)^\circ) = (TS)^\omega$ and the condition given above becomes $(TS)^\omega \subset TS$, that is, S is coisotropic in (M, ω). In the symplectic case, coisotropic submanifolds appear sometimes in the physics literature under the name of *first class constraints*. The main properties of coisotropic submanifolds are summarized in the following proposition.

PROPOSITION 8.12. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$ and S an embedded smooth submanifold of M. The following are equivalent:*

(i) *S is coisotropic;*
(ii) *if $f \in C^\infty(M)$ satisfies $f|_S \equiv 0$ then $X_f|_S \in \mathfrak{X}(\mathfrak{S})$;*
(iii) *for any $s \in S$, any open neighborhood U_s of s in M, and any function $g \in C^\infty(U_s)$ such that $X_g(s) \in T_sS$, if $f \in C^\infty(U_s)$ satisfies $\{f, g\}(s) = 0$, it follows that $X_f(s) \in T_sS$;*
(iv) *the subalgebra $\{f \in C^\infty(M) \mid f|_S \equiv 0\}$ is a Poisson subalgebra of $(C^\infty(M), \{\cdot,\cdot\})$.*

Coisotropic submanifolds naturally induce distributions with good properties relative to reduction.

PROPOSITION 8.13. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with associated Poisson tensor $B \in \Lambda^2(M)$. Let S be an embedded coisotropic submanifold of M and $D := B^\sharp((TS)^\circ)$. Then:*

(i) *$D = D \cap TS = D_S$ is a smooth generalized distribution on S.*
(ii) *D is integrable.*
(iii) *If C^∞_{S/D_S} has the (D, D_S)-local extension property then $(M, \{\cdot,\cdot\}, D, S)$ is Poisson reducible.*

Remark 8.14. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $B \in \Lambda^2(M)$ the corresponding Poisson tensor. Let S be an embedded submanifold such that the characteristic distribution $D_S := B^\sharp((TS)^\circ) \cap TS$ is a smooth, integrable, Poisson, and regular distribution on S. Even though the quotient manifolds associated to the quadruples $(M, \{\cdot,\cdot\}, D, S)$ and $(M, \{\cdot,\cdot\}, D_S, S)$ are the same and $(M, \{\cdot,\cdot\}, D, S)$ is reducible by Proposition (8.11), the quadruplet $(M, \{\cdot,\cdot\}, D_S, S)$ is, in general, *not* reducible. Actually, its reducibility is, by Theorem 8.8, equivalent to S being a coisotropic submanifold of M. Indeed, by Proposition 8.13 and Lemma 8.9 if S is coisotropic then $(M, \{\cdot,\cdot\}, D_S, S)$ is reducible. Conversely, if $(M, \{\cdot,\cdot\}, D_S, S)$ is reducible then by Theorem 8.8 we have $B^\sharp(D_S^\circ) \subset TS + D_S = TS$ since $D_S \subset TS$. Additionally,

$$
\begin{aligned}
B^\sharp((TS)^\circ) \subset B^\sharp((TS)^\circ) + B^\sharp([B^\sharp((TS)^\circ)]^\circ) &= B^\sharp((TS)^\circ + [B^\sharp((TS)^\circ)]^\circ) \\
&= B^\sharp([B^\sharp((TS)^\circ) \cap TS]^\circ) = B^\sharp(D_S^\circ)
\end{aligned}
$$

Thus $(M, \{\cdot,\cdot\}, D_S, S)$ *is Poisson reducible relative to the characteristic distribution if and only if S is coisotropic.*

The difference in terms of reducibility between $(M, \{\cdot,\cdot\}, D, S)$ and $(M, \{\cdot,\cdot\}, D_S, S)$ is the same as that between the systems $(M, \omega, D, \mathbf{J}^{-1}(\mu))$ and $(M, \omega, D_S, \mathbf{J}^{-1}(\mu))$ that we considered in Remark 8.10.

EXAMPLE. Coisotropic submanifolds appear naturally when one has integrals in involution. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with Poisson tensor B and let $f_1, \ldots, f_k \in C^\infty(M)$ be k smooth functions in *involution*, that is,

$$\{f_i, f_j\} = 0, \quad \text{for any} \quad i, j \in \{1, \ldots, k\}.$$

Assume that $\mathbf{0} \in \mathbb{R}^k$ is a regular value of the function $F := (f_1, \ldots, f_k) : M \to \mathbb{R}^k$ and let $S := F^{-1}(\mathbf{0})$. Since for any $s \in S$, $\operatorname{span}\{df_1(s), \ldots, df_k(s)\} \subset (T_sS)^\circ$ and the dimensions of both sides of this inclusion are equal we get

$$\operatorname{span}\{df_1(s), \ldots, df_k(s)\} = (T_sS)^\circ.$$

Hence,

$$B^\sharp(s)((T_sS)^\circ) = \operatorname{span}\{X_{f_1}(s), \ldots, X_{f_k}(s)\},$$

and $B^\sharp(s)((T_sS)^\circ) \subset T_sS$ by the involutivity of the components of F. Consequently, S is a coisotropic submanifold of $(M, \{\cdot,\cdot\})$ and Proposition 8.13 can be applied to it.

9. Cosymplectic Submanifolds and Dirac's Formula

The main goal of this section is to study certain submanifolds of a Poisson submanifold that are not Poisson themselves but to which the Poisson reduction method in Theorem 8.8 can be applied. As we shall see, these manifolds are intimately related to constraints and, in particular, to Dirac's formula for constrained Poisson brackets.

9.1. COSYMPLECTIC SUBMANIFOLDS

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $B \in \Lambda^2(M)$ its associated Poisson tensor. An embedded submanifold $S \subset M$ is called *cosymplectic* if

(i) $B^\sharp((TS)^\circ) \cap TS = \{0\}$.
(ii) $T_sS + T_s\mathcal{L}_s = T_sM$,

for any $s \in S$ and $\mathcal{L}_s$ the symplectic leaf of $(M, \{\cdot,\cdot\})$ containing $s \in S$.

The cosymplectic submanifolds of a symplectic manifold (M, ω) are its symplectic submanifolds. In the physics literature, if the phase space is given by a symplectic (as opposed to a Poisson) manifold, coisotropic submanifolds appear often under the name of *second-class constraints*. The main properties of cosymplectic submanifolds are summarized in the following proposition.

PROPOSITION 9.1. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, $B \in \Lambda^2(M)$ the corresponding Poisson tensor, and S a cosymplectic submanifold of M. Then for any $s \in S$,*

(i) $T_s\mathcal{L}_s = (T_sS \cap T_s\mathcal{L}_s) \oplus B^\sharp(s)((T_sS)^\circ)$, *where* $\mathcal{L}_s$ *is the symplectic leaf of* $(M, \{\cdot,\cdot\})$ *that contains* $s \in S$.

(ii) $(T_sS)^\circ \cap \ker B^\sharp(s) = \{0\}$.

(iii) $T_sM = B^\sharp(s)((T_sS)^\circ) \oplus T_sS$.

(iv) $B^\sharp((TS)^\circ)$ *is a subbundle of* $TM|_S$ *and, hence,* $TM|_S = B^\sharp((TS)^\circ) \oplus TS$.

(v) *The symplectic leaves of* $(M, \{\cdot,\cdot\})$ *intersect* S *transversely and hence* $S \cap \mathcal{L}$ *is an initial submanifold of* S, *for any symplectic leaf* $\mathcal{L}$ *of* $(M, \{\cdot,\cdot\})$.

The following theorem is due to Weinstein [51].

THEOREM 9.2 (The Poisson structure of a cosymplectic submanifold). *Let* $(M, \{\cdot,\cdot\})$ *be a Poisson manifold,* $B \in \Lambda^2(M)$ *the corresponding Poisson tensor, and* S *a cosymplectic submanifold of* M. *Let* $D := B^\sharp((TS)^\circ) \subset TM|_S$. *Then*

(i) $(M, \{\cdot,\cdot\}, D, S)$ *is Poisson reducible.*

(ii) *The corresponding quotient manifold equals* S *and the reduced bracket* $\{\cdot,\cdot\}^S$ *is given by*

$$\{f, g\}^S(s) = \{F, G\}(s), \tag{9.1}$$

where $f, g \in C^\infty_{S,M}(V)$ *are arbitrary and* $F, G \in C^\infty_M(U)$ *are local* D*-invariant extensions of* f *and* g *around* $s \in S$, *respectively.*

(iii) *The Hamiltonian vector field* X_f *of an arbitrary function* $f \in C^\infty_{S,M}(V)$ *is given by*

$$Ti \circ X_f = X_F \circ i, \tag{9.2}$$

where $F \in C^\infty_M(U)$ *is a local* D*-invariant extension of* f *and* $i\colon S \hookrightarrow M$ *is the inclusion.*

(iv) *The Hamiltonian vector field* X_f *of an arbitrary function* $f \in C^\infty_{S,M}(V)$ *can be written as*

$$Ti \circ X_f = \pi_S \circ X_F \circ i, \tag{9.3}$$

where $F \in C^\infty_M(U)$ *is an arbitrary local extension of* f *and* $\pi_S\colon TM|_S \to TS$ *is the projection induced by the Whitney sum decomposition* $TM|_S = B^\sharp((TS)^\circ) \oplus TS$ of $TM|_S$.

(v) *The symplectic leaves of* $(S, \{\cdot,\cdot\}^S)$ *are the connected components of the intersections* $S \cap \mathcal{L}$, *with* $\mathcal{L}$ *a symplectic leaf of* $(M, \{\cdot,\cdot\})$. *Any symplectic leaf of* $(S, \{\cdot,\cdot\}^S)$ *is a symplectic submanifold of the symplectic leaf of* $(M, \{\cdot,\cdot\})$ *that contains it.*

(vi) *Let* $\mathcal{L}_s$ *and* $\mathcal{L}^S_s$ *be the symplectic leaves of* $(M, \{\cdot,\cdot\})$ *and* $(S, \{\cdot,\cdot\}^S)$, *respectively, that contain the point* $s \in S$. *Let* $\omega_{\mathcal{L}_s}$ *and* $\omega_{\mathcal{L}^S_s}$ *be the corresponding symplectic forms. Then* $B^\sharp(s)((T_sS)^\circ)$ *is a symplectic subspace of* $T_s\mathcal{L}_s$ *and*

$$B^\sharp(s)((T_sS)^\circ) = \left(T_s\mathcal{L}^S_s\right)^{\omega_{\mathcal{L}_s}(s)}. \tag{9.4}$$

(vii) *Let* $B_S \in \Lambda^2(S)$ *be the Poisson tensor associated to* $(S, \{\cdot,\cdot\}^S)$. *Then*

$$B^\sharp_S = \pi_S \circ B^\sharp|_S \circ \pi^*_S, \tag{9.5}$$

where $\pi^*_S\colon T^*S \to T^*M|_S$ *is the dual of* $\pi_S\colon TM|_S \to TS$.

Note that this theorem provides presheaves of Poisson algebras on $(S, C^{\infty}_{S,M})$ and on (S, C^{∞}_{S}). When S is paracompact both presheaves coincide.

COROLLARY 9.3. *Let* $(M, \{\cdot,\cdot\})$ *be a Poisson manifold and* $S \subset M$ *an embedded submanifold. Then* S *is a cosymplectic submanifold of* $(M, \{\cdot,\cdot\})$ *if and only if it satisfies the following two properties:*

(i) $T_sS \cap T_s\mathcal{L}_s$ *is a symplectic subspace of* $(T_s\mathcal{L}_s, \omega_{\mathcal{L}_s}(s))$*, for any* $s \in S$*, where* $\mathcal{L}_s$ *is the symplectic leaf of* $(M, \{\cdot,\cdot\})$ *that contains* $s \in S$*;*
(ii) $T_sS + T_s\mathcal{L}_s = T_sM$*, for any* $s \in S$*.*

9.2. THE DIRAC CONSTRAINTS FORMULA

Next, we show that the classical formula of Dirac [10] for constrained brackets generalizes to the Poisson context if the constraint is a cosymplectic submanifold. Let $(M, \{\cdot,\cdot\})$ be a n-dimensional Poisson manifold and let S be a k-dimensional cosymplectic submanifold of M. Let z_0 be an arbitrary point in S and $(U, \overline{\kappa})$ a submanifold chart around z_0 such that $\overline{\kappa} = (\overline{\varphi}, \overline{\psi})\colon U \to V_1 \times V_2$. V_1 and V_2 are two open neighborhoods of the origin in two Euclidean spaces such that $\overline{\kappa}(z_0) = \big(\overline{\varphi}(z_0), \overline{\psi}(z_0)\big) = (0,0)$ and

$$\overline{\kappa}(U \cap S) = V_1 \times \{0\}. \tag{9.6}$$

Let $\overline{\varphi} =: (\overline{\varphi}^1, \ldots, \overline{\varphi}^k)$ be the components of $\overline{\varphi}$ and define $\widehat{\varphi}^1 := \overline{\varphi}^1|_{U\cap S}$, $\ldots, \widehat{\varphi}^k := \overline{\varphi}^k|_{U\cap S}$. Use now Lemma 8.9 to extend $\widehat{\varphi}^1, \ldots, \widehat{\varphi}^k$ to D-invariant functions $\varphi^1, \ldots, \varphi^k$ on U. Since the differentials $\mathrm{d}\widehat{\varphi}^1(s), \ldots, \mathrm{d}\widehat{\varphi}^k(s)$ are linearly independent for any $s \in U \cap S$, we can assume (by shrinking U if necessary) that $\mathrm{d}\varphi^1(z), \ldots, \mathrm{d}\varphi^k(z)$ are also linearly independent for any $z \in U$. Consequently, (U, κ) with $\kappa := (\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$, is a submanifold chart for M around z_0 with respect to S such that, by construction,

$$\mathrm{d}\varphi^1(s)|_{B^\sharp(s)((T_sS)^\circ)} = \cdots = \mathrm{d}\varphi^k(s)|_{B^\sharp(s)((T_sS)^\circ)} = 0,$$

for any $s \in U \cap S$. This implies that for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n-k\}$, and $s \in S$

$$\{\varphi^i, \psi^j\}(s) = \mathrm{d}\varphi^i(s) \cdot X_{\psi^j}(s) = 0$$

since $\mathrm{d}\psi^j(s) \in (T_sS)^\circ$ (by (9.6)) and, hence,

$$X_{\psi^j}(s) \in B^\sharp(s)((T_sS)^\circ). \tag{9.7}$$

Additionally, since the functions $\varphi^1, \ldots, \varphi^k$ are D-invariant we have, by (9.2), that

$$X_{\varphi^1}(s) = X_{\widehat{\varphi}^1}(s) \in T_sS, \ldots, X_{\varphi^k}(s) = X_{\widehat{\varphi}^k}(s) \in T_sS,$$

for any $s \in S$. Consequently, $\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s), X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\}$ spans $T_s\mathcal{L}_s$ with

$$\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s)\} \subset T_sS \cap T_s\mathcal{L}_s$$

and

$$\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\} \subset B^\sharp(s)((T_sS)^\circ).$$

By Proposition 9.1(i),

$$\operatorname{span}\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s)\} = T_sS \cap T_s\mathcal{L}_s$$

and

$$\operatorname{span}\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\} = B^\sharp(s)((T_sS)^\circ).$$

Since $\dim\big(B^\sharp(s)((T_sS)^\circ)\big) = n - k$ by Proposition (9.1)(iii), it follows that $\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\}$ is a basis of $B^\sharp(s)((T_sS)^\circ)$.

By Theorem (9.2)(vi), $B^\sharp(s)((T_sS)^\circ)$ is a symplectic subspace of $T_s\mathcal{L}_s$, so there exists some $r \in \mathbb{N}$ such that $n - k = 2r$ and, additionally, the matrix $C(s)$ with entries

$$C^{ij}(s) := \{\psi^i, \psi^j\}(s), \qquad i, j \in \{1, \ldots, n-k\}$$

is invertible. Therefore, in the coordinates $(\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$ the matrix associated to the Poisson tensor $B(s)$ is

$$\begin{pmatrix} B_S & 0 \\ 0 & C \end{pmatrix}.$$

Let $C_{ij}(s)$ be the entries of the matrix $C^{-1}(s)$.

PROPOSITION 9.4 (Dirac formulas). *In the coordinate neighborhood* $(\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$ *constructed above and for* $s \in S$ *we have, for any* $f, g \in C^\infty_{S,M}(V)$*:*

$$X_f(s) = X_F(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s) C_{ij}(s) X_{\psi^j}(s) \tag{9.8}$$

and

$$\{f, g\}^S(s) = \{F, G\}(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s) C_{ij}(s) \{\psi^j, G\}(s), \tag{9.9}$$

where $F, G \in C^\infty_M(U)$ *are arbitrary local extensions of* f *and* g*, respectively, around* $s \in S$.

The proof proceeds along the same lines as in the symplectic case (see, for example, [29]). Here is a sketch. By Theorem 9.2(iv), we have $X_f(s) = \pi_S(X_F(s))$. Therefore, the equality (9.8) is equivalent to

$$(\mathrm{Id} - \pi_S) X_F(s) = \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s) C_{ij}(s) X_{\psi^j}(s). \tag{9.10}$$

By Proposition 9.1(ii) this amounts to showing that the right-hand side of (9.10) is the projection of $X_F(s)$ onto $B^\sharp(s)((T_sS)^\circ)$. This is achieved by proving that this term

(i) is an element of $B^\sharp(s)((T_sS)^\circ)$;
(ii) equals $X_F(s)$ if $X_F(s) \in B^\sharp(s)((T_sS)^\circ)$;
(iii) equals 0 if $X_F(s) \in T_sS$.

Part (i) follows from (9.7). To prove (ii) assume that $X_F(s) \in B^\sharp(s)((T_sS)^\circ)$. Since the set $\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\}$ is a basis of $B^\sharp(s)((T_sS)^\circ)$, there exist constants $\{a_1, \ldots, a_k\}$ such that

$$X_F(s) = \sum_{l=1}^{n-k} a_l X_{\psi^l}(s).$$

A direct verification shows that

$$\sum_{i,j=1}^{n-k} \{F, \psi^i\}(s) C_{ij}(s) X_{\psi^j}(s) = X_F(s).$$

Finally, to show (iii) let $X_F(s) \in T_sS$. Since, by construction, $\mathrm{d}\psi^i(s) \in (T_sS)^\circ$, for any $i \in \{1, \ldots, n-k\}$, we get $\{F, \psi^i\}(s) = -\mathrm{d}\psi^i(s) \cdot X_F(s) = 0$. This proves (9.10) and hence the proposition.

Dirac's formula (9.9) provides an explicit local expression for the transverse Poisson structure of a Poisson manifold $(M, \{\cdot,\cdot\})$ at any of its points since the local transverse slice given by the points of the form $(\mathbf{0}, \mathbf{0}, \mathbf{z})$ is a local cosymplectic submanifold of M. In particular, applying this formula to the Lie–Poisson structure on $\mathfrak{g}^*$ at a point μ satisfying the condition $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{k}$, with $\mathfrak{k}$ a linear subspace such that $[\mathfrak{g}_\mu, \mathfrak{k}] \subset \mathfrak{k}$, it follows that the transverse Poisson structure is the Lie–Poisson structure of $\mathfrak{g}_\mu^*$, a result due to Weinstein [51], Molino [35], and Givental. If $\mathfrak{g}_\mu$ has a complement that is a Lie subalgebra, then the transverse structure as expressed by the Dirac formula, is at most quadratic, a result due to Oh [36].

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