Geometry of Dirac structures

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Physical modelling and Port-Hamiltonian systems

Hamiltonian dynamics vs network modelling

- Hamiltonian mechanics: origins in analytical mechanics: principle of least action $\rightarrow$ Euler-Lagrange equations $\rightarrow$ Legendre transform $\rightarrow$ Hamiltonian equations of motion

  analysis of physical systems

- Network modelling: origins in electrical engineering, describes complex networks as interconnection of basic elements, cornerstone of systems theory

  modelling and simulation of physical systems

Port-Hamiltonian systems try to combine both points of view:

- total energy of basic elements $\leftrightarrow$ Hamiltonian

- interconnection structure $\leftrightarrow$ geometric structure, i.e. symplectic, Poisson, or Dirac structure

Modelling

Basic principles of macroscopic physics:

- energy conservation

- positive entropy production

- power continuity

The concept of a power port

Port: Point of interaction of a physical system with its environment

Power port: Port of physical interaction that involves exchange of energy (power)

Mathematically, a power port consists of

a vector space V and its dual V^*, and
two variables $f \in V$ and $e \in V^*$ such that

the dual product $\langle e, f \rangle$ denotes power.

f is called flow, and e is called effort

Examples of physical power ports are

- mechanical: velocities and forces
- electrical: currents and voltages
- thermal: entropy flow and temperature
- hydraulic: volume flow and pressure
- chemical: molar flow and chemical potential

Five types of physical behaviour

- storage (energy conservation)
- supply and demand (boundary conditions)
- irreversible transformations (positive entropy production)
- reversible transformations (power continuity)
- distribution, topology (power continuity)

Elementary energy storing elements

are defined by a power port and an energy function H of the energy variable x:

$$\dot{x} = u$$
$$y = \frac{dH}{dx}(x)$$

power port: $(u, y) = (f, e)$ ($\mathbb{C}$-type) or (e, f) ($\mathbb{I}$-type)

u rate of change of energy variable x

y differential of energy function, co-energy variable

Note: $\dot{H} = \langle \frac{dH}{dx}, \dot{x} \rangle = \langle u, y \rangle$, i.e.

$$H(x(t)) - H(x(0)) = \int_0^t \langle u, y \rangle d\tau$$

Examples (mechanical)

- Spring: potential energy $H(x) = \frac{x^2}{2\kappa}$, elongation x

$$\dot{x} = u$$
$$y = \frac{x}{\kappa}$$

flow $f = u$ is velocity, effort $e = y$ is force

- Mass: kinetic energy $H(p) = \frac{p^2}{2m}$, momentum p

$$\dot{p} = u$$
$$y = \frac{p}{m}$$

flow $f = y$ is velocity, effort $e = u$ is force

Examples (electrical)

- Capacitor: electrical energy $H(q) = \frac{q^2}{2C}$, charge q

$$\dot{q} = u$$
$$y = \frac{q}{C}$$

flow $f = u$ is current, effort $e = y$ is voltage

- Inductor: magnetic energy $H(\phi) = \frac{\phi^2}{2L}$, magnetic flux ϕ

$$\dot{\phi} = u$$
$$y = \frac{\phi}{L}$$

flow $f = y$ is current, effort $e = u$ is voltage

Examples (thermal)

- Heat capacitor: internal energy $H(S)$ (e.g. of gas), entropy S

$$\dot{S} = u$$
$$y = \frac{dH}{dS}(S)$$

  flow $f = u$ is entropy flow, effort $e = y$ is temperature

Note: There is only one type of storage element.

Supply and demand: boundaries

A set of power ports

$$(f_b, e_b)$$

through which the system can interact with its environment.

By definition, power towards the system, i.e. into the system's boundaries, is counted positive.

These could be

- flow sources, providing a (fixed) flow, e.g. current source, fluid-flow source
- effort sources, providing a (fixed) effort, e.g. voltage source, pressure source

i.e. fixed "boundary conditions", or

- any open set of ports, connectable to the environment (possibly other (yet) unmodelled systems, e.g. control systems!)

i.e. open boundaries

Irreversible transformations (positive entropy production)

Irreversible transducer:

power-continuous two-port which (irreversibly) transforms energy from one domain (e.g. electrical, mechanical) into the thermal domain

Assume difference in time scales, i.e. temperature is considered constant

- energy $\to$ *free* energy
- power continuous two-port transducer $\to$ power discontinuous one-port ("dissipator")

The (non-termal) power port of the one port dissipator is denoted by (f_r, e_r).

By definition, power towards the non-thermal port (i.e. ''outside'' of the system) is counted positive.

Linear dissipators: $e_r = R f_r,\ R \geq 0$ such that

$$\int_0^t \langle e_r, f_r \rangle d\tau = \int_0^t \langle R f_r, f_r \rangle d\tau \geq 0$$

i.e. (free) energy is ''dissipated'' or lost.

E.g. resistor, damper

Reversible transformations (power continuity)

Reversible transducer: power-continuous two-port which (reversibly) transforms energy from one domain into another domain

- Non-mixing, transformer:

$$\begin{pmatrix} f_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} f_2 \\ e_2 \end{pmatrix}$$

e.g. electric transformer, lever, gear box

- Mixing, gyrator:

$$\begin{pmatrix} f_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 & n \\ 1/n & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ e_2 \end{pmatrix}$$

e.g. electric gyrator

Power continuity: $\langle e_1, f_1 \rangle = \langle e_2, f_2 \rangle$

Distribution, topology (power continuity)

describes how the power ports of all the elements (i.e. storage, boundaries, (ir)reversible transformations) are interconnected

Two types of "junctions"

- Generalized Kirchhoff Current Law & effort identity

$$\sum_i^n \pm f_i = 0, \quad e_1 = \cdots = e_n$$

- Generalized Kirchhoff Voltage Law & flow identity

$$\sum_i^n \pm e_i = 0, \quad f_1 = \cdots = f_n$$

Power continuity: $\sum_i^n \pm \langle e_i, f_i \rangle = 0$

The model

The model now consists of the following power ports and their interconnections

- n_s storage elements: (f_s, e_s) (oriented towards the storage elements)
- n_b sources: (f_b, e_b) (oriented outwards of the sources, i.e. towards the system)
- n_r dissipators: (f_r, e_r) with $e_r = Rf_r$ (oriented towards the dissipators)
- power continuous interconnection: transformers, gyrators
- power continuous interconnection: junctions

Power balance

The power ports satisfy

$$\langle e_s, f_s \rangle - \langle e_b, f_b \rangle + \langle e_r, f_r \rangle = 0$$

That is, for a dissipative structure

$$\langle e_s, f_s \rangle + \langle -e_b, f_b \rangle = -\langle R f_r, f_r \rangle \leq 0$$

Or, for a lossless structure (no dissipation)

$$\langle e_s, f_s \rangle + \langle -e_b, f_b \rangle = 0$$

The interconnection structure

Eliminating the dissipative ports, the power continuous interconnections define a relation between the storage and source ports of the form:

$$F\begin{pmatrix} f_s \\ f_b \end{pmatrix} + E\begin{pmatrix} e_s \\ -e_b \end{pmatrix} = 0$$

$F, E \in \mathbb{R}^{(n_s+n_b)\times(n_s+n_b)}$ and rank $[F \quad E] = n_s + n_b$.

This is called the interconnection structure.

Lossless: $FE^T + EF^T = 0$

Dissipative: $FE^T + EF^T \leq 0$

Dirac structure

A constant Dirac structure on an m-dimensional linear space W is an m-dimensional linear subspace $D \subset W \times W^*$ such that

$$\langle w^*, w \rangle = 0, \quad \forall (w, w^*) \in D.$$

Proposition The interconnection structure

$$L = \left\{ (f_s, f_b, e_s, -e_b) \in V_s \times V_b \times V_s^* \times V_b^* \mid F \begin{pmatrix} f_s \\ f_b \end{pmatrix} + E \begin{pmatrix} e_s \\ -e_b \end{pmatrix} = 0 \right\}$$

with rank $[F \quad E] = n_s + n_b$, is a Dirac structure *if and only if* the interconnection structure is lossless (that is $FE^T + EF^T = 0$).

Port-Hamiltonian systems (with dissipation)

subdividing the storage ports into (u_C, y_C) ($\mathbb{C}$-type) and (u_I, y_I) ($\mathbb{I}$-type) yields the interconnection structure

$$A \begin{pmatrix} u_C \\ u_I \\ f_b \end{pmatrix} + B \begin{pmatrix} y_C \\ y_I \\ -e_b \end{pmatrix} = 0$$

$A, B \in \mathbb{R}^{(n_s+n_b)\times(n_s+n_b)}$ and rank $[A \quad B] = n_s + n_b$.

Again $AB^T + BA^T = 0$ (lossless), or $AB^T + BA^T \leq 0$ (dissipative).

The constitutive relations of the storage elements then yield

$$A \begin{pmatrix} \dot{x}_C \\ \dot{x}_I \\ f_b \end{pmatrix} + B \begin{pmatrix} \frac{dH_C}{dx_C}(x_C) \\ \frac{dH_I}{dx_I}(x_I) \\ -e_b \end{pmatrix} = 0$$

→ a set of ordinary differential equations, or
→ a set of differential and algebraic equations (in case of dependent states)

This is called a port-Hamiltonian system

Dissipative Port-Hamiltonian system

In case the interconnection structure is dissipative, $AB^T + BA^T \leq 0$:

$$\langle \frac{dH_C}{dx_C}(x_C), \dot{x}_C \rangle + \langle \frac{dH_I}{dx_I}(x_I), \dot{x}_I \rangle + \langle -e_b, f_b \rangle \leq 0$$

which yields the energy inequality

$$H_C(x(t)) + H_I(x(t)) - H_C(x(0)) - H_I(x(0)) \leq \int_0^t \langle e_b, f_b \rangle d\tau$$

This is called a Port-Hamiltonian system with dissipation.

Lossless Port-Hamiltonian system

In case the interconnection structure is lossless, $AB^T + BA^T = 0$:

$$\langle \frac{dH_C}{dx_C}(x_C), \dot{x}_C \rangle + \langle \frac{dH_I}{dx_I}(x_I), \dot{x}_I \rangle + \langle -e_b, f_b \rangle = 0$$

which yields the energy balance

$$H_C(x(t)) + H_I(x(t)) - H_C(x(0)) - H_I(x(0)) = \int_0^t \langle e_b, f_b \rangle d\tau$$

This is called a lossless Port-Hamiltonian system.

Theorem A lossless Port-Hamiltonian system is defined by a total energy function $H(x)$ and a Dirac structure D (i.e. the lossless interconnection structure)

$$\left(\dot{x}, f_b, \frac{dH}{dx}(x), -e_b\right) \in D$$

Conservative systems. If there are no sources, then

$$\left(\dot{x}, \frac{dH}{dx}(x)\right) \in D$$

and the system is conservative:

$$\dot{H} = \langle \frac{dH}{dx}(x), \dot{x} \rangle = 0$$

Examples of Dirac structures and Port-Hamiltonian systems

Mass-spring-damper-force system

Junction: $f_C = f_I = f_r = f_b$ (velocity identity),
$e_C + e_I + e_r - e_b = 0$ (force balance)

Interconnection structure: (recall $(u_C, y_C) = (f_C, e_C)$ and $(u_I, y_I) = (e_I, f_I)$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_C \\ u_I \\ f_b \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & d & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_C \\ y_I \\ -e_b \end{pmatrix} = 0$$

Dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \right] \begin{pmatrix} x/\kappa \\ p/m \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_b$$

$$f_b = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x/\kappa \\ p/m \end{pmatrix}$$

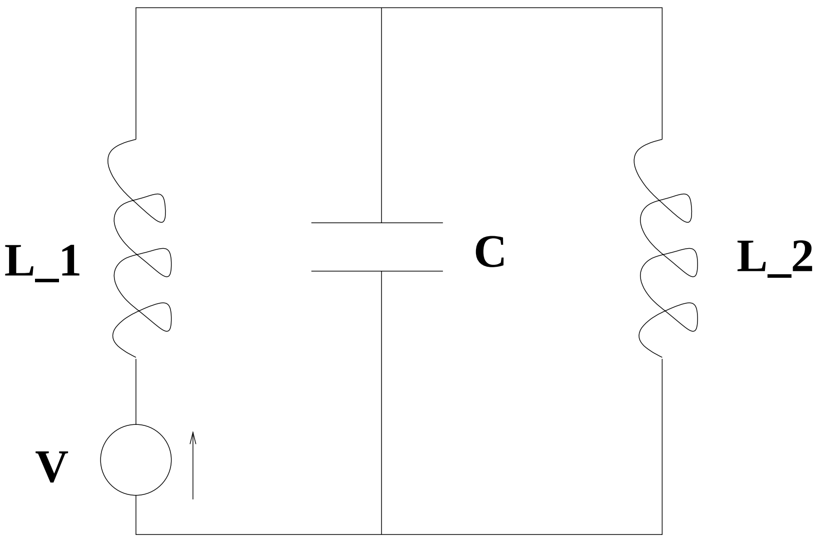
Total energy

$$H(x,p) = \frac{p^2}{2m} + \frac{x^2}{2\kappa}$$

and energy balance

$$\dot{H} = -d\left(\frac{p}{m}\right)^2 + \langle e_b, f_b \rangle \leq \langle e_b, f_b \rangle$$

An LC circuit of order 3



Interconnection structure:

$$\begin{pmatrix} i_C \\ v_1 \\ v_2 \\ i_b \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_1 \\ i_2 \\ -v_b \end{pmatrix}$$

The circuit is lossless (no resistors), hence the interconnection structure is a Dirac structure.

Dynamics:

$$\begin{pmatrix} \dot{q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q/C \\ \phi_1/L_1 \\ \phi_2/L_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_b$$

$$i_b = \phi_1/L_1 \ (= i_1)$$

Note: $(0,1,0)^T \notin \operatorname{Im} \mathbb{J}$, no interaction potential function

If $v_b = 0$ then

- The dynamics is defined w.r.t. a Poisson structure
- $\operatorname{rank} \mathbb{J} = 2$, i.e. $\phi_1 + \phi_2$ is a conserved quantity (inductor loop!)

Total energy

$$H(q, \phi_1, \phi_2) = \frac{q^2}{2C} + \frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2}$$

and energy balance

$$\dot{H} = \langle v_b, i_b \rangle$$

A study of general LC circuits

Note: no resistors, no sources

Consider a simply connected network N and write $N = \Gamma \cup \Sigma$

- Γ: maximal tree
- Σ: set of links, co-tree

Standard network analysis yields:

$$i_\Gamma = P i_\Sigma, \quad v_\Sigma = -P^T v_\Gamma$$

The interconnection structure is lossless (Dirac structure):

$$\langle v_\Gamma, i_\Gamma \rangle + \langle v_\Sigma, i_\Sigma \rangle = \langle v_\Gamma, P i_\Sigma \rangle + \langle -P^T v_\Gamma, i_\Sigma \rangle = 0$$

This is Tellegen's theorem

Divide into capacitor and inductor branches:

$$i_\Gamma = (i_\Gamma^C, i_\Gamma^L),\ i_\Sigma = (i_\Sigma^C, i_\Sigma^L),\ v_\Gamma = (v_\Gamma^C, v_\Gamma^L),\ v_\Sigma = (v_\Sigma^C, v_\Sigma^L)$$

Then

$$i_\Gamma = (\dot{q}_\Gamma, \partial H/\partial \phi_\Gamma), \quad i_\Sigma = (\dot{q}_\Sigma, \partial H/\partial \phi_\Sigma),$$
$$v_\Gamma = (\partial H/\partial q_\Gamma, \dot{\phi}_\Gamma), \quad v_\Sigma = (\partial H/\partial q_\Sigma, \dot{\phi}_\Sigma),$$

where total energy function (Hamiltonian)

$$H = \frac{q_\Gamma^2}{2C_\Gamma} + \frac{q_\Sigma^2}{2C_\Sigma} + \frac{\phi_\Gamma^2}{2L_\Gamma} + \frac{\phi_\Sigma^2}{2L_\Sigma}$$

The interconnection structure becomes

$$\begin{pmatrix} \dot{q}_\Gamma \\ \partial H/\partial \phi_\Gamma \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \dot{q}_\Sigma \\ \partial H/\partial \phi_\Sigma \end{pmatrix},$$

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \dot{\phi}_\Sigma \end{pmatrix} = \begin{pmatrix} -P_{11}^T & -P_{21}^T \\ -P_{12}^T & -P_{22}^T \end{pmatrix} \begin{pmatrix} \partial H/\partial q_\Gamma \\ \dot{\phi}_\Gamma \end{pmatrix}$$

which can be rewritten as

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \partial H/\partial \phi_\Gamma \\ \dot{q}_\Gamma \\ \dot{\phi}_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & -P_{21}^T & -P_{11}^T & 0 \\ P_{21} & 0 & 0 & P_{22} \\ P_{11} & 0 & 0 & P_{12} \\ 0 & -P_{22}^T & -P_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_\Sigma \\ \dot{\phi}_\Gamma \\ \partial H/\partial q_\Gamma \\ \partial H/\partial \phi_\Sigma \end{pmatrix}$$

Define $x_1 = (q_\Sigma, \phi_\Gamma)$ and $x_2 = (q_\Gamma, \phi_\Sigma)$ the system becomes

$$\begin{pmatrix} \partial H/\partial x_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{J}_{11} & \mathbb{J}_{12} \\ \mathbb{J}_{21} & \mathbb{J}_{22} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \partial H/\partial x_2 \end{pmatrix}$$

- Assume x_1 void, i.e. maximal capacitor tree, inductor co-tree:

  $$\dot{x}_2 = \mathbb{J}_{22} \partial H/\partial x_2$$

  is a Poisson dynamical system. Capacitor cutsets or inductor loops correspond to conserved quantities.

- Assume x_2 void, i.e. maximal inductor tree, capacitor co-tree:

  $$\partial H/\partial x_1 = \mathbb{J}_{11} \dot{x}_1$$

  If $\mathbb{J}_{11}$ singular, this is a pre-symplectic dynamical system. Capacitor loops or inductor cutsets correspond to algebraic constraints.

Define $y = x_2 - \mathbb{J}_{21} x_1$ and $z = x_1$ and $\tilde{H}(y,z) = H(x_1, x_2)$:

$$\dot{y} = \mathbb{J}_{22} \partial \tilde{H} / \partial y, \quad \mathbb{J}_{11} \dot{z} = \partial \tilde{H} / \partial z$$

Choose coordinates $y = (y_{11}, y_{12}, y_2)$ and $z = (z_{11}, z_{12}, z_2)$ such that

$$\mathbb{J}_{22} = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{J}_{11} = \begin{pmatrix} 0 & -I_1 & 0 \\ I_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Then, with $\alpha = (y_{11}, z_{11})$ and $\beta = (y_{12}, z_{12})$ and $\tilde{H}(y,z) = \hat{H}(\alpha, \beta, y_2, z_2)$ the dynamical equations become

$$\dot{\alpha} = \frac{\partial \hat{H}}{\partial \beta}, \qquad \dot{y}_2 = 0,$$

$$\dot{\beta} = -\frac{\partial \hat{H}}{\partial \alpha}, \qquad 0 = \frac{\partial \hat{H}}{\partial z_2}$$

Theorem A lossless Port-Hamiltonian system defined by a total energy function H and a constant Dirac structure D can, after a change of coordinates, always be written as

$$\dot{\alpha} = \frac{\partial \hat{H}}{\partial \beta}, \qquad \dot{y}_2 = 0,$$

$$\dot{\beta} = -\frac{\partial \hat{H}}{\partial \alpha}, \qquad 0 = \frac{\partial \hat{H}}{\partial z_2}$$

These are called canonical coordinates.

This is a set of differential and algebraic equations.

Note (1): Port-Hamiltonian systems encompass symplectic, pre-symplectic and Poisson dynamical systems.

Note (2): If D is not constant, integrability conditions are necessary.

Two gases in thermal interaction

through a heat conducting wall, and in thermal interaction with two heat sources.

Total internal energy $H_1(S_1)+H_2(S_2)$, with S_i entropy and $dH_i/dS_i = T_i$ temperature. u_i is entropy flow delivered by the heat sources.

Heat flow balances:

$$\begin{aligned} T_1\dot{S}_1 &= \sigma(T_1 - T_2) + T_1 u_1, \\ T_2\dot{S}_2 &= \sigma(T_2 - T_1) + T_2 u_2 \end{aligned}$$

Port-Hamiltonian system

$$\begin{pmatrix} \dot{S}_1 \\ \dot{S}_2 \end{pmatrix} = \sigma\,(1/T_2 - 1/T_1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$$

$$y_1 = T_1, \quad y_2 = T_2$$

Port-Hamiltonian systems as basic building blocks

Example: modelling multibody systems

The rigid body element:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 0 & Q \\ -Q^T & -P\times \end{pmatrix} \underbrace{\begin{pmatrix} \mathrm{d}V(Q) \\ M^{-1}P \end{pmatrix}}_{\mathrm{d}H(Q,P)} + \begin{pmatrix} 0 \\ I \end{pmatrix} W$$

$$T = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \mathrm{d}V(Q) \\ M^{-1}P \end{pmatrix}$$

$Q \in SE(3)$: spatial displacement of body

$P \in se^*(3)$: momentum in body frame

$W \in se^*(3)$: external wrench (force) in body frame

$T \in se(3)$: external twist (velocity) in body frame

The total energy of the rigid body element is

$$H(Q,P) = \underbrace{\frac{1}{2}\langle P, M^{-1}P\rangle}_{\text{kinetic energy}} + \underbrace{V(Q)}_{\text{potential energy}}$$

Energy balance:

$$\dot{H}(Q,P) = \langle W, T\rangle$$

i.e.

$$\underbrace{H(Q(t),P(t)) - H(Q(0),P(0))}_{\text{increase in total energy of the rigid body}} = \underbrace{\int_0^t \langle W(s), T(s)\rangle \mathrm{d}s}_{\text{energy supplied trough the port } (W,T)}$$

The rigid body element can be written as the Port-Hamiltonian system

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \\ T \end{pmatrix} = \begin{pmatrix} 0 & Q & 0 \\ -Q^T & -P\times & -I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}V(Q) \\ M^{-1}P \\ -W \end{pmatrix}$$

The skew-symmetric matrix defines a Dirac structure, depending on the state Q, P of the system (i.e. non-constant).

Links – Spring

The spring element:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Q &= Q\,T \\ W &= Q^T\,\mathrm{d}V(Q) \end{aligned}$$

$Q \in SE(3)$: spatial displacement of the spring
$T \in se(3)$: twist in body frame
$W \in se^*(3)$: wrench in body frame

Total energy = potential energy of the spring: $H(Q) = V(Q)$

Energy balance:

$$\underbrace{H(Q(t)) - H(Q(0))}_{\text{increase in potential energy of the spring}} = \underbrace{\int_0^t \langle W(s), T(s)\rangle \mathrm{d}s}_{\text{energy supplied trough the port } (W,T)}$$

The spring can be written as the Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} I & -Q \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \dot{Q} \\ T \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ Q^T & I \end{pmatrix}}_{B} \begin{pmatrix} \mathrm{d}V(Q) \\ -W \end{pmatrix}$$

The matrices A and B define a Dirac structure, i.e. $AB^T + BA^T = 0$, depending on the state Q.

Joints – kinematic pairs

A kinematic pair is an *energy conserving* interconnection between:
– two links (e.g. a revolute joint), or
– a link and the environment (e.g. a (non-)holonomic constraint)

(Unactuated) kinematic pairs are described by a multi-port D_{KP}:

$$D_{KP} = \{(T, W) \mid T \in \mathcal{FT}, W \in \mathcal{CW} = \mathcal{FT}^{\perp}\}$$

$\mathcal{FT}$: space of freedom twists (twists allowed by joint)
$\mathcal{CW}$: space of constraint wrenches (constraint forces)

A kinematic pair produces no work: $\langle W, T \rangle = 0$

i.e. energy balance: $\int_0^t \langle W(s), T(s) \rangle \mathrm{d}s = 0$

Examples: $T_{link} = (T^{rot}_{link}, T^{linear}_{link})$, $W_{link} = (W^{rot}_{link}, W^{linear}_{link})$

• revolute joint: $T = \begin{pmatrix} T_{link1} \\ T_{link2} \end{pmatrix}$, $W = \begin{pmatrix} W_{link1} \\ W_{link2} \end{pmatrix}$

$$\mathcal{FT} = \mathrm{Im}\begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & \omega & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \mathcal{CW} = \mathrm{Im}\begin{pmatrix} \zeta_1 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & \zeta_1 & \zeta_2 & 0 \\ 0 & 0 & 0 & 0 & -I_3 \end{pmatrix}$$

where ω is the axis of rotation allowed by the joint, and ω, ζ_1, ζ_2 form an orthonormal basis of $\mathbb{R}^3$.

• sliding surface (holonomic constraint): $T = T_{link}$, $W = W_{link}$

$$\mathcal{FT} = \begin{pmatrix} n & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 \end{pmatrix}, \mathcal{CW} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & n \end{pmatrix}$$

where α_1, α_2 are the tangents of the surface and n the normal.

The interconnected system

The multibody system is defined by:

$$(\dot{Q}^i_{\text{rigid}}, \dot{P}^i_{\text{rigid}}, dH^i_{\text{rigid}}, T^i_{\text{rigid}}, -W^i_{\text{rigid}}) \in D^i_{\text{rigid}}, \quad i = 1, \ldots, \sharp\text{rigid bodies}$$

$$(\dot{Q}^j_{\text{spring}}, dH^j_{\text{spring}}, T^j_{\text{spring}}, -W^j_{\text{spring}}) \in D^j_{\text{spring}}, \quad j = 1, \ldots, \sharp\text{springs}$$

$$(T^\ell_{kp}, W^\ell_{kp}) \in D^\ell_{KP}, \quad \ell = 1, \ldots, \sharp\text{kinematic pairs}$$

$$(T_{\text{rigid}}, T_{\text{spring}}, T_{kp}, T_b, -W_{\text{rigid}}, -W_{\text{spring}}, W_{kp}, -W_b) \in D_{\text{topology}}, \text{ (incl. sources)}$$

The first two equations are dynamic equations. The third is a set of algebraic equations. The last equation defines the topology of the network.

The multibody system is a Port-Hamiltonian system

$$\left(\dot{Q}, \dot{P}, T_b, \mathrm{d}H, -W_b\right) \in D(Q, P)$$

with $Q = (Q^i_{\text{rigid}}, Q^j_{\text{spring}})$ and $P = (P^i_{\text{rigid}})$ and total energy

$$H(Q, P) = \sum_{i,\ell} H^i_{\text{rigid}}(Q^i_{\text{rigid}}, P^i_{\text{rigid}}) + H^\ell_{\text{spring}}(Q^\ell_{\text{spring}})$$

and non-constant Dirac structure D defined by the Dirac structures

$$D^i_{\text{rigid}}, \quad D^j_{\text{spring}}, \quad D^\ell_{KP}, \quad D_{\text{topology}}$$

Interconnected Port-Hamiltonian systems

Theorem The power continuous interconnection of two (or n) Port-Hamiltonian systems is again a Port-Hamiltonian system.

The Hamiltonian is the total energy $H_1 + H_2$.

In case both Port-Hamiltonian systems are lossless, the interconnected system is lossless too, and the Dirac structure is defined only by the two Dirac structures D_1 and D_2.

Interdomain Port-Hamiltonian systems

Example: a magnetically levitated ball

Energy variables: $x = (\phi, z, p) \in \mathbb{R}^3$, i.e. magnetic flux, altitude ball, momentum ball

Total magnetic plus mechanical energy

$$H(\phi, z, p) = \frac{1}{2L(z)}\phi^2 + \frac{1}{2m}p^2 + mgz$$

with $L(z) = \frac{L_0}{z_0 - z}$ for $z < z_0$.

Co-energy variable $\mathrm{d}H/\mathrm{d}x = (i, F, v)$, where

- $i = \phi/L(z)$ current through the inductor
- gravity force minus magnetic force

$$F = mg - \frac{\phi^2}{2L^2(z)} \frac{\mathrm{d}L}{\mathrm{d}z}$$

- $v = p/m$ velocity ball

This yields the Port-Hamiltonian system

$$\begin{pmatrix} \dot{\phi} \\ \dot{z} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial \phi \\ \partial H/\partial z \\ \partial H/\partial p \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} V$$

$$i = \partial H/\partial \phi = \phi/L(z)$$

with voltage source V and resistor R