Geometry of Dirac structures

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Physical modelling and Port-Hamiltonian systems

Hamiltonian dynamics vs network modelling

- Hamiltonian mechanics: origins in analytical mechanics: principle of least action $\rightarrow$ Euler-Lagrange equations $\rightarrow$ Legendre transform $\rightarrow$ Hamiltonian equations of motion

  analysis of physical systems

- Network modelling: origins in electrical engineering, describes complex networks as interconnection of basic elements, cornerstone of systems theory

  modelling and simulation of physical systems

Port-Hamiltonian systems try to combine both points of view:

- total energy of basic elements $\leftrightarrow$ Hamiltonian

- interconnection structure $\leftrightarrow$ geometric structure, i.e. symplectic, Poisson, or Dirac structure

Modelling

Basic principles of macroscopic physics:

- energy conservation
- positive entropy production
- power continuity

The concept of a power port

Port: Point of interaction of a physical system with its environment

Power port: Port of physical interaction that involves exchange of energy (power)

Mathematically, a power port consists of

a vector space V and its dual V^*, and
two variables $f \in V$ and $e \in V^*$ such that

the dual product $\langle e, f \rangle$ denotes power.

f is called flow, and e is called effort

Examples of physical power ports are

- mechanical: velocities and forces
- electrical: currents and voltages
- thermal: entropy flow and temperature
- hydraulic: volume flow and pressure
- chemical: molar flow and chemical potential

Five types of physical behaviour

- storage (energy conservation)
- supply and demand (boundary conditions)
- irreversible transformations (positive entropy production)
- reversible transformations (power continuity)
- distribution, topology (power continuity)

Elementary energy storing elements

are defined by a power port and an energy function H of the energy variable x:

$$\dot{x} = u$$
$$y = \frac{\mathbf{d}H}{\mathbf{d}x}(x)$$

power port: $(u, y) = (f, e)$ ($\mathbb{C}$-type) or (e, f) ($\mathbb{I}$-type)

u rate of change of energy variable x

y differential of energy function, co-energy variable

Note: $\dot{H} = \langle \frac{\mathbf{d}H}{\mathbf{d}x}, \dot{x} \rangle = \langle u, y \rangle$, i.e.

$$H(x(t)) - H(x(0)) = \int_0^t \langle u, y \rangle d\tau$$

Examples (mechanical)

- Spring: potential energy $H(x) = \frac{x^2}{2\kappa}$, elongation x

$$\dot{x} = u$$
$$y = \frac{x}{\kappa}$$

flow $f = u$ is velocity, effort $e = y$ is force

- Mass: kinetic energy $H(p) = \frac{p^2}{2m}$, momentum p

$$\dot{p} = u$$
$$y = \frac{p}{m}$$

flow $f = y$ is velocity, effort $e = u$ is force

Examples (electrical)

- Capacitor: electrical energy $H(q) = \frac{q^2}{2C}$, charge q

$$\dot{q} = u$$
$$y = \frac{q}{C}$$

flow $f = u$ is current, effort $e = y$ is voltage

- Inductor: magnetic energy $H(\phi) = \frac{\phi^2}{2L}$, magnetic flux ϕ

$$\dot{\phi} = u$$
$$y = \frac{\phi}{L}$$

flow $f = y$ is current, effort $e = u$ is voltage

Examples (thermal)

- Heat capacitor: internal energy $H(S)$ (e.g. of gas), entropy S

$$\begin{aligned} \dot{S} &= u \\ y &= \frac{\mathbf{d}H}{\mathbf{d}S}(S) \end{aligned}$$

flow $f = u$ is entropy flow, effort $e = y$ is temperature

Note: There is only one type of storage element.

Supply and demand: boundaries

A set of power ports

$$(f_b, e_b)$$

through which the system can interact with its environment.

By definition, power towards the system, i.e. into the system's boundaries, is counted positive.

These could be

- flow sources, providing a (fixed) flow, e.g. current source, fluid-flow source

- effort sources, providing a (fixed) effort, e.g. voltage source, pressure source

i.e. fixed "boundary conditions", or

- any open set of ports, connectable to the environment (possibly other (yet) unmodelled systems, e.g. control systems!)

i.e. open boundaries

Irreversible transformations (positive entropy production)

Irreversible transducer:

power-continuous two-port which (irreversibly) transforms energy from one domain (e.g. electrical, mechanical) into the thermal domain

Assume difference in time scales, i.e. temperature is considered constant

- energy $\rightarrow$ *free* energy
- power continuous two-port transducer $\rightarrow$ power discontinuous one-port ("dissipator")

The (non-termal) power port of the one port dissipator is denoted by (f_r, e_r).

By definition, power towards the non-thermal port (i.e. "outside" of the system) is counted positive.

Linear dissipators: $e_r = Rf_r,\ R \geq 0$ such that

$$\int_0^t \langle e_r, f_r \rangle d\tau = \int_0^t \langle Rf_r, f_r \rangle d\tau \geq 0$$

i.e. (free) energy is "dissipated" or lost.

E.g. resistor, damper

Reversible transformations (power continuity)

Reversible transducer: power-continuous two-port which (reversibly) transforms energy from one domain into another domain

- Non-mixing, transformer:

$$\begin{pmatrix} f_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} f_2 \\ e_2 \end{pmatrix}$$

e.g. electric transformer, lever, gear box

- Mixing, gyrator:

$$\begin{pmatrix} f_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 & n \\ 1/n & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ e_2 \end{pmatrix}$$

e.g. electric gyrator

Power continuity: $\langle e_1, f_1 \rangle = \langle e_2, f_2 \rangle$

Distribution, topology (power continuity)

describes how the power ports of all the elements (i.e. storage, boundaries, (ir)reversible transformations) are interconnected

Two types of "junctions"

- Generalized Kirchhoff Current Law & effort identity

$$\sum_{i}^{n} \pm f_i = 0, \quad e_1 = \cdots = e_n$$

- Generalized Kirchhoff Voltage Law & flow identity

$$\sum_{i}^{n} \pm e_i = 0, \quad f_1 = \cdots = f_n$$

Power continuity: $\sum_i^n \pm \langle e_i, f_i \rangle = 0$

The model

The model now consists of the following power ports and their interconnections

- n_s storage elements: (f_s, e_s) (oriented towards the storage elements)
- n_b sources: (f_b, e_b) (oriented outwards of the sources, i.e. towards the system)
- n_r dissipators: (f_r, e_r) with $e_r = Rf_r$ (oriented towards the dissipators)
- power continuous interconnection: transformers, gyrators
- power continuous interconnection: junctions

Power balance

The power ports satisfy

$$\langle e_s, f_s \rangle - \langle e_b, f_b \rangle + \langle e_r, f_r \rangle = 0$$

That is, for a dissipative structure

$$\langle e_s, f_s \rangle + \langle -e_b, f_b \rangle = -\langle R f_r, f_r \rangle \leq 0$$

Or, for a lossless structure (no dissipation)

$$\langle e_s, f_s \rangle + \langle -e_b, f_b \rangle = 0$$

The interconnection structure

Eliminating the dissipative ports, the power continuous interconnections define a relation between the storage and source ports of the form:

$$F \begin{pmatrix} f_s \\ f_b \end{pmatrix} + E \begin{pmatrix} e_s \\ -e_b \end{pmatrix} = 0$$

$F, E \in \mathbb{R}^{(n_s+n_b)\times(n_s+n_b)}$ and rank $[F \quad E] = n_s + n_b$.

This is called the interconnection structure.

Lossless: $FE^T + EF^T = 0$

Dissipative: $FE^T + EF^T \leq 0$

Dirac structure

A constant Dirac structure on an m-dimensional linear space W is an m-dimensional linear subspace $D \subset W \times W^*$ such that

$$\langle w^*, w \rangle = 0, \quad \forall (w, w^*) \in D.$$

Proposition The interconnection structure

$$L = \left\{ (f_s, f_b, e_s, -e_b) \in V_s \times V_b \times V_s^* \times V_b^* \mid F \begin{pmatrix} f_s \\ f_b \end{pmatrix} + E \begin{pmatrix} e_s \\ -e_b \end{pmatrix} = 0 \right\}$$

with $\operatorname{rank} [F \quad E] = n_s + n_b$, is a Dirac structure *if and only if* the interconnection structure is lossless (that is $FE^T + EF^T = 0$).

Port-Hamiltonian systems (with dissipation)

subdividing the storage ports into (u_C, y_C) ($\mathbb{C}$-type) and (u_I, y_I) ($\mathbb{I}$-type) yields the interconnection structure

$$A \begin{pmatrix} u_C \\ u_I \\ f_b \end{pmatrix} + B \begin{pmatrix} y_C \\ y_I \\ -e_b \end{pmatrix} = 0$$

$A, B \in \mathbb{R}^{(n_s+n_b)\times(n_s+n_b)}$ and rank $[A \quad B] = n_s + n_b$.

Again $AB^T + BA^T = 0$ (lossless), or $AB^T + BA^T \leq 0$ (dissipative).

The constitutive relations of the storage elements then yield

$$A \begin{pmatrix} \dot{x}_C \\ \dot{x}_I \\ f_b \end{pmatrix} + B \begin{pmatrix} \frac{\mathrm{d}H_C}{\mathrm{d}x_C}(x_C) \\ \frac{\mathrm{d}H_I}{\mathrm{d}x_I}(x_I) \\ -e_b \end{pmatrix} = 0$$

$\longrightarrow$ a set of ordinary differential equations, or
$\longrightarrow$ a set of differential and algebraic equations (in case of dependent states)

This is called a port-Hamiltonian system

Dissipative Port-Hamiltonian system

In case the interconnection structure is dissipative, $AB^T + BA^T \leq 0$:

$$\langle \frac{\mathbf{d}H_C}{\mathbf{d}x_C}(x_C), \dot{x}_C \rangle + \langle \frac{\mathbf{d}H_I}{\mathbf{d}x_I}(x_I), \dot{x}_I \rangle + \langle -e_b, f_b \rangle \leq 0$$

which yields the energy inequality

$$H_C(x(t)) + H_I(x(t)) - H_C(x(0)) - H_I(x(0)) \leq \int_0^t \langle e_b, f_b \rangle d\tau$$

This is called a Port-Hamiltonian system with dissipation.

Lossless Port-Hamiltonian system

In case the interconnection structure is lossless, $AB^T + BA^T = 0$:

$$\langle \frac{\mathbf{d}H_C}{\mathbf{d}x_C}(x_C), \dot{x}_C \rangle + \langle \frac{\mathbf{d}H_I}{\mathbf{d}x_I}(x_I), \dot{x}_I \rangle + \langle -e_b, f_b \rangle = 0$$

which yields the energy balance

$$H_C(x(t)) + H_I(x(t)) - H_C(x(0)) - H_I(x(0)) = \int_0^t \langle e_b, f_b \rangle d\tau$$

This is called a lossless Port-Hamiltonian system.

Theorem A lossless Port-Hamiltonian system is defined by a total energy function $H(x)$ and a Dirac structure D (i.e. the lossless interconnection structure)

$$\left(\dot{x}, f_b, \frac{\mathbf{d}H}{\mathbf{d}x}(x), -e_b\right) \in D$$

Conservative systems. If there are no sources, then

$$\left(\dot{x}, \frac{\mathbf{d}H}{\mathbf{d}x}(x)\right) \in D$$

and the system is conservative:

$$\dot{H} = \langle \frac{\mathbf{d}H}{\mathbf{d}x}(x), \dot{x} \rangle = 0$$

Examples of Dirac structures and Port-Hamiltonian systems

Mass-spring-damper-force system

Junction: $f_C = f_I = f_r = f_b$ (velocity identity),
$e_C + e_I + e_r - e_b = 0$ (force balance)

Interconnection structure: (recall $(u_C, y_C) = (f_C, e_C)$ and $(u_I, y_I) = (e_I, f_I)$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_C \\ u_I \\ f_b \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & d & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_C \\ y_I \\ -e_b \end{pmatrix} = 0$$

Dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \right] \begin{pmatrix} x/\kappa \\ p/m \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_b$$

$$f_b = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x/\kappa \\ p/m \end{pmatrix}$$

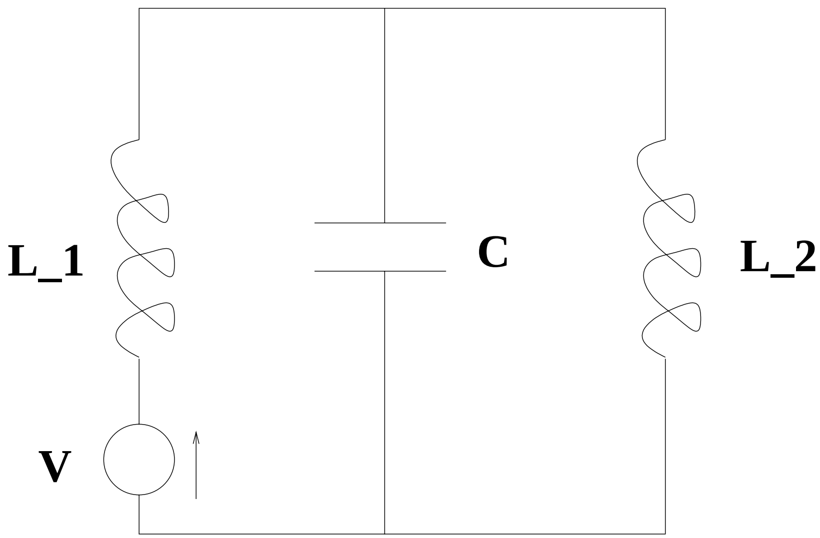
Total energy

$$H(x,p) = \frac{p^2}{2m} + \frac{x^2}{2\kappa}$$

and energy balance

$$\dot{H} = -d\left(\frac{p}{m}\right)^2 + \langle e_b, f_b \rangle \leq \langle e_b, f_b \rangle$$

An LC circuit of order 3



Interconnection structure:

$$\begin{pmatrix} i_C \\ v_1 \\ v_2 \\ i_b \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_1 \\ i_2 \\ -v_b \end{pmatrix}$$

The circuit is lossless (no resistors), hence the interconnection structure is a Dirac structure.

Dynamics:

$$\begin{pmatrix} \dot{q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q/C \\ \phi_1/L_1 \\ \phi_2/L_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_b$$

$$i_b = \phi_1/L_1 \ (= i_1)$$

Note: $(0,1,0)^T \notin \operatorname{Im} \mathbb{J}$, no interaction potential function

If $v_b = 0$ then

- The dynamics is defined w.r.t. a Poisson structure
- $\operatorname{rank} \mathbb{J} = 2$, i.e. $\phi_1 + \phi_2$ is a conserved quantity (inductor loop!)

Total energy

$$H(q, \phi_1, \phi_2) = \frac{q^2}{2C} + \frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2}$$

and energy balance

$$\dot{H} = \langle v_b, i_b \rangle$$

A study of general LC circuits

Note: no resistors, no sources

Consider a simply connected network N and write $N = \Gamma \cup \Sigma$

- Γ: maximal tree
- Σ: set of links, co-tree

Standard network analysis yields:

$$i_\Gamma = P i_\Sigma, \quad v_\Sigma = -P^T v_\Gamma$$

The interconnection structure is lossless (Dirac structure):

$$\langle v_\Gamma, i_\Gamma \rangle + \langle v_\Sigma, i_\Sigma \rangle = \langle v_\Gamma, P i_\Sigma \rangle + \langle -P^T v_\Gamma, i_\Sigma \rangle = 0$$

This is Tellegen's theorem

Divide into capacitor and inductor branches:

$$i_\Gamma = (i_\Gamma^C, i_\Gamma^L),\ i_\Sigma = (i_\Sigma^C, i_\Sigma^L),\ v_\Gamma = (v_\Gamma^C, v_\Gamma^L),\ v_\Sigma = (v_\Sigma^C, v_\Sigma^L)$$

Then

$$i_\Gamma = (\dot{q}_\Gamma, \partial H/\partial \phi_\Gamma), \quad i_\Sigma = (\dot{q}_\Sigma, \partial H/\partial \phi_\Sigma),$$
$$v_\Gamma = (\partial H/\partial q_\Gamma, \dot{\phi}_\Gamma), \quad v_\Sigma = (\partial H/\partial q_\Sigma, \dot{\phi}_\Sigma),$$

where total energy function (Hamiltonian)

$$H = \frac{q_\Gamma^2}{2C_\Gamma} + \frac{q_\Sigma^2}{2C_\Sigma} + \frac{\phi_\Gamma^2}{2L_\Gamma} + \frac{\phi_\Sigma^2}{2L_\Sigma}$$

The interconnection structure becomes

$$\begin{pmatrix} \dot{q}_\Gamma \\ \partial H/\partial\phi_\Gamma \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \dot{q}_\Sigma \\ \partial H/\partial\phi_\Sigma \end{pmatrix},$$

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \dot{\phi}_\Sigma \end{pmatrix} = \begin{pmatrix} -P_{11}^T & -P_{21}^T \\ -P_{12}^T & -P_{22}^T \end{pmatrix} \begin{pmatrix} \partial H/\partial q_\Gamma \\ \dot{\phi}_\Gamma \end{pmatrix}$$

which can be rewritten as

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \partial H/\partial\phi_\Gamma \\ \dot{q}_\Gamma \\ \dot{\phi}_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & -P_{21}^T & -P_{11}^T & 0 \\ P_{21} & 0 & 0 & P_{22} \\ P_{11} & 0 & 0 & P_{12} \\ 0 & -P_{22}^T & -P_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_\Sigma \\ \dot{\phi}_\Gamma \\ \partial H/\partial q_\Gamma \\ \partial H/\partial\phi_\Sigma \end{pmatrix}$$

Define $x_1 = (q_\Sigma, \phi_\Gamma)$ and $x_2 = (q_\Gamma, \phi_\Sigma)$ the system becomes

$$\begin{pmatrix} \partial H/\partial x_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{J}_{11} & \mathbb{J}_{12} \\ \mathbb{J}_{21} & \mathbb{J}_{22} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \partial H/\partial x_2 \end{pmatrix}$$

- Assume x_1 void, i.e. maximal capacitor tree, inductor co-tree:

  $$\dot{x}_2 = \mathbb{J}_{22} \partial H/\partial x_2$$

  is a Poisson dynamical system. Capacitor cutsets or inductor loops correspond to conserved quantities.

- Assume x_2 void, i.e. maximal inductor tree, capacitor co-tree:

  $$\partial H/\partial x_1 = \mathbb{J}_{11} \dot{x}_1$$

  If $\mathbb{J}_{11}$ singular, this is a pre-symplectic dynamical system. Capacitor loops or inductor cutsets correspond to algebraic constraints.

Define $y = x_2 - \mathbb{J}_{21}x_1$ and $z = x_1$ and $\tilde{H}(y,z) = H(x_1,x_2)$:

$$\dot{y} = \mathbb{J}_{22}\partial\tilde{H}/\partial y, \quad \mathbb{J}_{11}\dot{z} = \partial\tilde{H}/\partial z$$

Choose coordinates $y = (y_{11}, y_{12}, y_2)$ and $z = (z_{11}, z_{12}, z_2)$ such that

$$\mathbb{J}_{22} = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{J}_{11} = \begin{pmatrix} 0 & -I_1 & 0 \\ I_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Then, with $\alpha = (y_{11}, z_{11})$ and $\beta = (y_{12}, z_{12})$ and $\tilde{H}(y,z) = \hat{H}(\alpha, \beta, y_2, z_2)$ the dynamical equations become

$$\dot{\alpha} = \frac{\partial\hat{H}}{\partial\beta}, \qquad \dot{y}_2 = 0,$$

$$\dot{\beta} = -\frac{\partial\hat{H}}{\partial\alpha}, \qquad 0 = \frac{\partial\hat{H}}{\partial z_2}$$

Theorem A lossless Port-Hamiltonian system defined by a total energy function H and a constant Dirac structure D can, after a change of coordinates, always be written as

$$\dot{\alpha} = \frac{\partial \widehat{H}}{\partial \beta}, \qquad \dot{y}_2 = 0,$$

$$\dot{\beta} = -\frac{\partial \widehat{H}}{\partial \alpha}, \qquad 0 = \frac{\partial \widehat{H}}{\partial z_2}$$

These are called canonical coordinates.

This is a set of differential and algebraic equations.

Note (1): Port-Hamiltonian systems encompass symplectic, pre-symplectic and Poisson dynamical systems.

Note (2): If D is not constant, integrability conditions are necessary.

Two gases in thermal interaction

through a heat conducting wall, and in thermal interaction with two heat sources.

Total internal energy $H_1(S_1)+H_2(S_2)$, with S_i entropy and $\mathbf{d}H_i/\mathbf{d}S_i = T_i$ temperature. u_i is entropy flow delivered by the heat sources.

Heat flow balances:

$$\begin{aligned} T_1\dot{S}_1 &= \sigma(T_1 - T_2) + T_1 u_1, \\ T_2\dot{S}_2 &= \sigma(T_2 - T_1) + T_2 u_2 \end{aligned}$$

Port-Hamiltonian system

$$\begin{pmatrix} \dot{S}_1 \\ \dot{S}_2 \end{pmatrix} = \sigma\,(1/T_2 - 1/T_1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$$

$$y_1 = T_1, \quad y_2 = T_2$$

Port-Hamiltonian systems as basic building blocks

Example: modelling multibody systems

The rigid body element:

$$\frac{\mathbf{d}}{\mathbf{d}t}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 0 & Q \\ -Q^T & -P\times \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{d}V(Q) \\ M^{-1}\,P \end{pmatrix}}_{\mathbf{d}H(Q,P)} + \begin{pmatrix} 0 \\ I \end{pmatrix} W$$

$$T = (0 \quad I)\begin{pmatrix} \mathbf{d}V(Q) \\ M^{-1}\,P \end{pmatrix}$$

$Q \in SE(3)$: spatial displacement of body
$P \in se^*(3)$: momentum in body frame
$W \in se^*(3)$: external wrench (force) in body frame
$T \in se(3)$: external twist (velocity) in body frame

The total energy of the rigid body element is

$$H(Q,P) = \underbrace{\frac{1}{2}\langle P, M^{-1}P\rangle}_{\text{kinetic energy}} + \underbrace{V(Q)}_{\text{potential energy}}$$

Energy balance:

$$\dot{H}(Q,P) = \langle W, T\rangle$$

i.e.

$$\underbrace{H(Q(t),P(t)) - H(Q(0),P(0))}_{\text{increase in total energy of the rigid body}} = \underbrace{\int_0^t \langle W(s), T(s)\rangle \mathbf{d}s}_{\text{energy supplied trough the port } (W,T)}$$

The rigid body element can be written as the Port-Hamiltonian system

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \\ T \end{pmatrix} = \begin{pmatrix} 0 & Q & 0 \\ -Q^T & -P\times & -I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \mathbf{d}V(Q) \\ M^{-1}P \\ -W \end{pmatrix}$$

The skew-symmetric matrix defines a Dirac structure, depending on the state Q, P of the system (i.e. non-constant).

Links – Spring

The spring element:

$$
\begin{aligned}
\frac{\mathbf{d}}{\mathbf{d}t} Q &= Q\, T \\
W &= Q^T \, \mathbf{d}V(Q)
\end{aligned}
$$

$Q \in SE(3)$: spatial displacement of the spring
$T \in se(3)$: twist in body frame
$W \in se^*(3)$: wrench in body frame

Total energy = potential energy of the spring: $H(Q) = V(Q)$

Energy balance:

$$
\underbrace{H(Q(t)) - H(Q(0))}_{\text{increase in potential energy of the spring}} = \underbrace{\int_0^t \langle W(s), T(s) \rangle \mathbf{d}s}_{\text{energy supplied trough the port } (W,T)}
$$

The spring can be written as the Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} I & -Q \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \dot{Q} \\ T \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ Q^T & I \end{pmatrix}}_{B} \begin{pmatrix} \mathbf{d}V(Q) \\ -W \end{pmatrix}$$

The matrices A and B define a Dirac structure, i.e. $AB^T + BA^T = 0$, depending on the state Q.

Joints – kinematic pairs

A kinematic pair is an *energy conserving* interconnection between:

– two links (e.g. a revolute joint), or

– a link and the environment (e.g. a (non-)holonomic constraint)

(Unactuated) kinematic pairs are described by a multi-port D_{KP}:

$$D_{KP} = \{(T, W) \mid T \in \mathcal{FT}, W \in \mathcal{CW} = \mathcal{FT}^{\perp}\}$$

$\mathcal{FT}$: space of freedom twists (twists allowed by joint)

$\mathcal{CW}$: space of constraint wrenches (constraint forces)

A kinematic pair produces no work: $\langle W, T \rangle = 0$

i.e. energy balance: $\int_0^t \langle W(s), T(s) \rangle \mathbf{d}s = 0$

Examples: $T_{link} = (T^{rot}_{link}, T^{linear}_{link})$, $W_{link} = (W^{rot}_{link}, W^{linear}_{link})$

- revolute joint: $T = \begin{pmatrix} T_{link1} \\ T_{link2} \end{pmatrix}$, $W = \begin{pmatrix} W_{link1} \\ W_{link2} \end{pmatrix}$

$$\mathcal{FT} = \mathrm{Im}\begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & \omega & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \mathcal{CW} = \mathrm{Im}\begin{pmatrix} \zeta_1 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & \zeta_1 & \zeta_2 & 0 \\ 0 & 0 & 0 & 0 & -I_3 \end{pmatrix}$$

where ω is the axis of rotation allowed by the joint, and ω, ζ_1, ζ_2 form an orthonormal basis of $\mathbb{R}^3$.

- sliding surface (holonomic constraint): $T = T_{link}$, $W = W_{link}$

$$\mathcal{FT} = \begin{pmatrix} n & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 \end{pmatrix}, \; \mathcal{CW} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & 0 & n \end{pmatrix}$$

where α_1, α_2 are the tangents of the surface and n the normal.

The interconnected system

The multibody system is defined by:

$$(\dot{Q}^i_{\text{rigid}}, \dot{P}^i_{\text{rigid}}, \mathbf{d}H^i_{\text{rigid}}, T^i_{\text{rigid}}, -W^i_{\text{rigid}}) \in D^i_{\text{rigid}}, \quad i = 1, \ldots, \sharp\text{rigid bodies}$$

$$(\dot{Q}^j_{\text{spring}}, \mathbf{d}H^j_{\text{spring}}, T^j_{\text{spring}}, -W^j_{\text{spring}}) \in D^j_{\text{spring}}, \quad j = 1, \ldots, \sharp\text{springs}$$

$$(T^\ell_{kp}, W^\ell_{kp}) \in D^\ell_{KP}, \quad \ell = 1, \ldots, \sharp\text{kinematic pairs}$$

$$(T_{\text{rigid}}, T_{\text{spring}}, T_{kp}, T_b, -W_{\text{rigid}}, -W_{\text{spring}}, W_{kp}, -W_b) \in D_{\text{topology}}, \ \text{(incl. sources)}$$

The first two equations are dynamic equations. The third is a set of algebraic equations. The last equation defines the topology of the network.

The multibody system is a Port-Hamiltonian system

$$\left(\dot{Q}, \dot{P}, T_b, \mathbf{d}H, -W_b\right) \in D(Q, P)$$

with $Q = (Q^i_{\text{rigid}}, Q^j_{\text{spring}})$ and $P = (P^i_{\text{rigid}})$ and total energy

$$H(Q, P) = \sum_{i,\ell} H^i_{\text{rigid}}(Q^i_{\text{rigid}}, P^i_{\text{rigid}}) + H^\ell_{\text{spring}}(Q^\ell_{\text{spring}})$$

and non-constant Dirac structure D defined by the Dirac structures

$$D^i_{\text{rigid}}, \quad D^j_{\text{spring}}, \quad D^\ell_{KP}, \quad D_{\text{topology}}$$

Interconnected Port-Hamiltonian systems

Theorem The power continuous interconnection of two (or n) Port-Hamiltonian systems is again a Port-Hamiltonian system.

The Hamiltonian is the total energy $H_1 + H_2$.

In case both Port-Hamiltonian systems are lossless, the interconnected system is lossless too, and the Dirac structure is defined only by the two Dirac structures D_1 and D_2.

Interdomain Port-Hamiltonian systems

Example: a magnetically levitated ball

Energy variables: $x = (\phi, z, p) \in \mathbb{R}^3$, i.e. magnetic flux, altitude ball, momentum ball

Total magnetic plus mechanical energy

$$H(\phi, z, p) = \frac{1}{2L(z)}\phi^2 + \frac{1}{2m}p^2 + mgz$$

with $L(z) = \frac{L_0}{z_0 - z}$ for $z < z_0$.

Co-energy variable $\mathbf{d}H/\mathbf{d}x = (i, F, v)$, where

- $i = \phi/L(z)$ current through the inductor
- gravity force minus magnetic force

$$F = mg - \frac{\phi^2}{2L^2(z)}\,\frac{\mathbf{d}L}{\mathbf{d}z}$$

- $v = p/m$ velocity ball

This yields the Port-Hamiltonian system

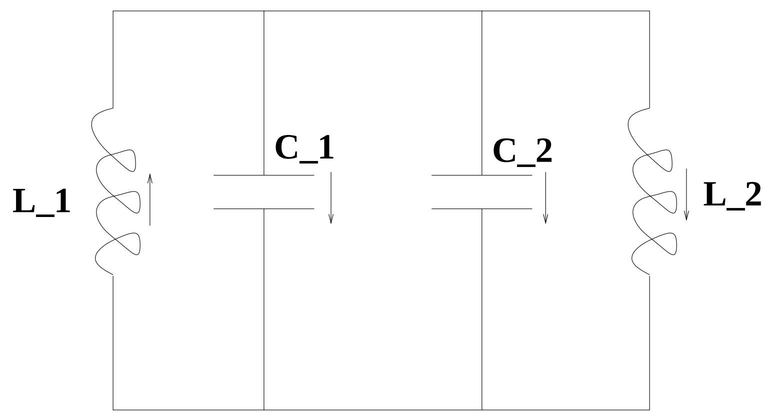
$$\begin{pmatrix}\dot{\phi}\\ \dot{z}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}-R & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\partial H/\partial\phi\\ \partial H/\partial z\\ \partial H/\partial p\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} V$$

$$i = \partial H/\partial\phi = \phi/L(z)$$

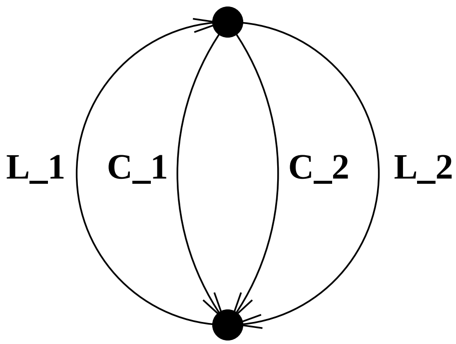
with voltage source V and resistor R

Addendum: LC circuit example

Consider the LC circuit given in the following figure, consisting of two inductors L_1 and L_2 and two capacitors C_1 and C_2.



The graph of the circuit is given in the next figure.



A maximal tree Γ of the graph is given by, for instance, $\Gamma = \{C_1\}$. The corresponding co-tree (i.e., the branches which, when added to the tree, produce a loop) are then given by $\Sigma = \{C_2, L_1, L_2\}$.

Denote the currents and voltages corresponding to the elements by: i_{C_1} and v_{C_1} for C_1; i_{C_2} and v_{C_2} for C_2; i_{L_1} and v_{L_1} for L_1; i_{L_2} and v_{L_2} for L_2. According to standard network theory we can write

$$i_\Gamma = P i_\Sigma, \quad v_\Sigma = -P^T v_\Gamma, \tag{1}$$

for some matrix P. That is, the currents in the tree can be expressed as linear fuctions of the currents in the co-tree and, dually, the voltages in the co-tree can be expressed as linear functions of the voltages in the tree.

Kirchhoff's current law for the circuit yields

$$i_{C_1} + i_{C_2} - i_{L_1} + i_{L_2} = 0. \tag{2}$$

Alternatively, the incoming currents (note the orientation!) at each node of the graph should sum to zero. Kirchhoff's voltage laws yield

$$v_{C_1} - v_{C_2} = 0, \quad v_{C_1} + v_{L_1} = 0, \quad v_{C_1} - v_{L_2} = 0. \tag{3}$$

Alternatively, the voltages over every loop in the graph should sum to zero (again, note the orientation). Now the currents and voltages can be written as in Eq. (**??**):

$$i_{C_1} = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} i_{C_2} \\ i_{L_1} \\ i_{L_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_{C_2} \\ v_{L_1} \\ v_{L_2} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} v_{C_1}. \tag{4}$$

We can write

$$i_{C_1} = \dot{q}_{C_1}, \quad v_{C_1} = \frac{\partial H}{\partial q_{C_1}} \tag{5}$$

and

$$\left(i_{C_2}, i_{L_1}, i_{L_2}\right) = \left(\dot{q}_{C_2}, \frac{\partial H}{\partial \phi_{L_1}}, \frac{\partial H}{\partial \phi_{L_2}}\right), \quad \left(v_{C_2}, v_{L_1}, v_{L_2}\right) = \left(\frac{\partial H}{\partial q_{C_2}}, \dot{\phi}_{L_1}, \dot{\phi}_{L_2}\right), \tag{6}$$

where

$$H(q_{C_1}, q_{C_2}, \phi_{L_1}, \phi_{L_2}) = \frac{q_{C_1}^2}{2C_1} + \frac{q_{C_2}^2}{2C_2} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2} \tag{7}$$

is the total electromagnetic energy in the circuit (the energy variables q_{C_i} denote the charge of the capacitor C_i and ϕ_{L_i} the flux of the inductor L_i, $i = 1, 2$).

The (general) circuit's dynamics can be written as

$$\begin{pmatrix} \partial H/\partial q_\Sigma \\ \partial H/\partial \phi_\Gamma \\ \dot q_\Gamma \\ \dot\phi_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & -P_{21}^T & -P_{11}^T & 0 \\ P_{21} & 0 & 0 & P_{22} \\ P_{11} & 0 & 0 & P_{12} \\ 0 & -P_{22}^T & -P_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \dot q_\Sigma \\ \dot\phi_\Gamma \\ \partial H/\partial q_\Gamma \\ \partial H/\partial \phi_\Sigma \end{pmatrix}. \tag{8}$$

In this example there are no inductors in the tree, hence ϕ_Γ is absent. Therefore, we can eliminate the second row and the second column of the skew-symmetric matrix in (**??**). The matrices P_{11} and P_{12} are given by

$$P_{11} = -1, \quad P_{12} = \begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{9}$$

The dynamics of the circuit can thus be written as

$$\begin{pmatrix} \frac{\partial H}{\partial q_{C_2}} \\ \dot q_{C_1} \\ \dot\phi_{L_1} \\ \dot\phi_{L_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot q_{C_2} \\ \frac{\partial H}{\partial q_{C_1}} \\ \frac{\partial H}{\partial \phi_{L_1}} \\ \frac{\partial H}{\partial \phi_{L_2}} \end{pmatrix}. \tag{10}$$

Hence,

$$\mathbb{J}_{11} = 0, \quad \mathbb{J}_{12} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{11}$$

and

$$\mathbb{J}_{21} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{J}_{22} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

Eq. (**??**) can be written out to obtain

$$\frac{\partial H}{\partial q_{C_1}} - \frac{\partial H}{\partial q_{C_2}} = 0, \tag{13}$$

$$\dot{q}_{C_1} = -\dot{q}_{C_2} + \frac{\partial H}{\partial \phi_{L_1}} - \frac{\partial H}{\partial \phi_{L_2}}, \tag{14}$$

$$\dot{\phi}_{L_1} = -\frac{\partial H}{\partial q_{C_1}}, \tag{15}$$

$$\dot{\phi}_{L_2} = \frac{\partial H}{\partial q_{C_1}}. \tag{16}$$

This is a set of differential and algebraic equations. Eq. (**??**) is an *algebraic constraint*, corresponding to the capacitor loop C_1–C_2 in the circuit, i.e., $v_{C_1} - v_{C_2} = 0$. Eqs. (**??**) and (**??**) imply that $\phi_{L_1} + \phi_{L_2}$ is a *conserved quantity* of the system. This corresponds to the inductor loop L_1–L_2, i.e., $v_{L_1} + v_{L_2} = 0$.

In order to find the canonical coordinates of the system, first define the variables

$$y = x_2 - \mathbb{J}_{21} x_1 \quad \text{and} \quad z = x_1, \tag{17}$$

where $x_1 = (q_\Sigma, \phi_\Gamma)$ and $x_2 = (q_\Gamma, \phi_\Sigma)$. For this example this yields

$$y_1 = q_{C_1} + q_{C_2}, \quad y_2 = \phi_{L_1}, \quad y_3 = \phi_{L_2}, \quad z = q_{C_2}. \tag{18}$$

In these coordinates the Hamiltonian becomes

$$\tilde{H}(y_1, y_2, y_3, z) = \frac{(y_1 - z)^2}{2C_1} + \frac{z^2}{2C_2} + \frac{y_2^2}{2L_1} + \frac{y_3^2}{2L_2} \tag{19}$$

and the system can be written as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{H}}{\partial y_1} \\ \frac{\partial \tilde{H}}{\partial y_2} \\ \frac{\partial \tilde{H}}{\partial y_3} \end{pmatrix}, \qquad 0 = \frac{\partial \tilde{H}}{\partial z}. \tag{20}$$

Canonical coordinates for the skew-symmetric matrix in (**??**) are

$$\xi_1 = y_1, \quad \xi_2 = \frac{1}{2}(y_2 - y_3), \quad \xi_3 = \frac{1}{2}(y_2 + y_3), \tag{21}$$

in which the Hamiltonian becomes

$$\widehat{H}(\xi_1, \xi_2, \xi_3, z) = \frac{(\xi_1 - z)^2}{2C_1} + \frac{z^2}{2C_2} + \frac{(\xi_2 + \xi_3)^2}{2L_1} + \frac{(\xi_3 - \xi_2)^2}{2L_2}. \tag{22}$$

In the *canonical coordinates* (ξ, z) the implicit Hamiltonian system takes the canonical form

$$\dot{\xi}_1 = \frac{\partial \widehat{H}}{\partial \xi_2}, \tag{23}$$

$$\dot{\xi}_2 = -\frac{\partial \widehat{H}}{\partial \xi_1}. \tag{24}$$

$$\dot{\xi}_3 = 0, \tag{25}$$

$$0 = \frac{\partial \widehat{H}}{\partial z}. \tag{26}$$

Note that the canonical coordinates are related to the original energy variables of the circuit by

$$\xi_1 = q_{C_1} + q_{C_2}, \quad \xi_2 = \frac{1}{2}(\phi_{L_1} - \phi_{L_2}), \quad \xi_3 = \frac{1}{2}(\phi_{L_1} + \phi_{L_2}), \quad z = q_{C_2}. \tag{27}$$

The system (**??**)–(**??**) is an implicit Hamiltonian system in canonical form. The underlying geometric structure is that of a *Dirac structure*. One observes that the system has conserved quantities (**??**) as well as algebraic constraints (**??**). As such it *combines* properties of Poisson systems (i.e., (**??**)–(**??**)) and pre-symplectic systems (i.e., (**??**),(**??**),(**??**)). The conserved quantity (**??**) corresponds to the inductor loop L_1–L_2 in the circuit. The algebraic constraint (**??**) corresponds to the capacitor loop C_1–C_2 in the circuit.

References

- Some interesting papers on the modeling of physical systems can be found in the lecture notes by Peter Breedveld on

  `www-lar.deis.unibo.it/euron-geoplex-sumsch/lectures_1.html`.

- Notes on Port-Hamiltonian systems modeling (including LC circuits) can be found in the lecture notes by Arjan van der Schaft and Bernhard Maschke on the website mentioned above.

- The modeling of LC circuits using Dirac structures, and its construction such as used in this example, was first described in:

  A.M. Bloch and P.E. Crouch, Representations of Dirac structures on vector spaces and nonlinear LC circuits, In: H. Hermes, G. Ferraya, R. Gardner and H. Sussmann, editors, *Proc. of Symposia in Pure Mathematics, Differential Geometry and Control Theory*, vol. 64, pp. 103–117, 1999.

Dirac structures

Linear Dirac structures

V vector space, V^* dual, $n = \dim V$

Symmetric non-degenerate bilinear form on $V \oplus V^*$

$$\langle\!\langle (v_1, v_1^*), (v_2, v_2^*) \rangle\!\rangle := \langle v_1^*, v_2 \rangle + \langle v_2^*, v_1 \rangle$$

Linear Dirac structure on V: subspace $D \subset V \oplus V^*$ such that $D = D^\perp$. D is said to be a *Lagrangian subspace* of $V \oplus V^*$.

Proposition A vector subspace $D \subset V \oplus V^*$ is a linear Dirac structure if and only if $\dim D = n$ and $\langle v^*, v \rangle = 0$, $\forall (v^*, v) \in D$.

Proof D Dirac, then taking $v := v_1 = v_2$ and $v^* := v_1^* = v_2^* \Longrightarrow \langle v^*, v \rangle = 0$, $\forall (v^*, v) \in D$. General fact: $\dim D + \dim D^\perp = \dim V \oplus V^*$. So $\dim D = n$.

Conversely, let $(v_1, v_1^*) \in D$. Then for all $(v_2, v_2^*) \in D$ we have

$$\begin{aligned} 0 &= \frac{1}{2} \langle\langle (v_1 + v_2, v_1^* + v_2^*), (v_1 + v_2, v_1^* + v_2^*) \rangle\rangle \\ &= \langle v_1^* + v_2^*, v_1 + v_2 \rangle \\ &= \langle v_1^*, v_1 \rangle + \langle v_1^*, v_2 \rangle + \langle v_2^*, v_1 \rangle + \langle v_2^*, v_2 \rangle \\ &= \langle v_1^*, v_2 \rangle + \langle v_2^*, v_1 \rangle \\ &= \langle\langle (v_1, v_1^*), (v_2, v_2^*) \rangle\rangle . \end{aligned}$$

So $(v_1, v_1^*) \in D^\perp \implies D \subset D^\perp$. But $2n = \dim D + \dim D^\perp = n + \dim D^\perp \implies \dim D^\perp = n \implies D = D^\perp$. ■

Corollary A vector subspace $D \subset V \oplus V^*$ a linear Dirac structure if and only if it is *maximal isotropic* ($D \subset D^\perp$) in $V \oplus V^*$.

Example E subspace of the vector space F. Then $D := E \oplus E^\circ$ is a linear Dirac structure.

Example $\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i$ canonical symplectic structure on $\mathbb{R}^{2k}$:

$$[\omega] = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

Then $D := \left\{(v, \omega(v, \cdot)) \mid v \in \mathbb{R}^{2k}\right\}$ is a linear Dirac structure.

Example $\flat : V \to V^*$ and $\sharp : V^* \to V$ linear skew-symmetric maps ($\flat^* = -\flat$ and $\sharp^* = -\sharp$). Then graph$(\flat)$, graph$(\sharp) \subset V \oplus V^*$ are linear Dirac structures. Conversely, any Dirac structure satisfying $D \cap (\{0\} \oplus V^*) = \{(0,0)\}$ (respectively $D \cap (V \oplus \{0\}) = \{(0,0\})$ defines a skew-symmetric map $\flat : V \to V^*$ (respectively $\sharp : V^* \to V$).

Proof $(v, v_1^*), (v, v_2^*) \in D \Longrightarrow (0, v_1^* - v_2^*) = (v, v_1^*) - (v, v_2^*) \in D \Longrightarrow v_1^* - v_2^* = 0 \Longrightarrow \exists \flat : V \to V^*$ s. t. $D = \{(v, v^\flat) \mid v \in V\}$. $\flat^* = -\flat \Longleftrightarrow \langle v^\flat, u\rangle + \langle u^\flat, v\rangle = \langle\!\langle (v, v^\flat), (u, u^\flat)\rangle\!\rangle = 0$, true since D is Dirac. $\square$

Constant versions of presymplectic and generalized Poisson structures.

Characterization of linear Dirac structures

$\pi_V : V \oplus V^* \to V$ and $\pi_{V^*} : V \oplus V^* \to V^*$ projections. Then

$$\ker \pi_V|_D = D \cap (\{0\} \oplus V^*) \qquad \ker \pi_{V^*}|_D = D \cap (V \oplus \{0\})$$

and we have the *characteristic equations of a Dirac structure*

$$\pi_V(D)^\circ = \pi_{V^*}(\ker \pi_V|_D) = \text{“}D \cap V^*\text{”}$$
$$\pi_{V^*}(D)^\circ = \pi_V(\ker \pi_{V^*}|_D) = \text{“}D \cap V\text{”}$$

Proof

$$v \in \pi_V(D) \Longleftrightarrow \exists v^* \in V^* \text{ such that } (v, v^*) \in D$$
$$u^* \in \pi_{V^*}(\ker \pi_V|_D) \Longleftrightarrow (0, u^*) \in D$$

$u^* \in \pi_V(D)^\circ \Longleftrightarrow 0 = \langle u^*, v\rangle = \langle\langle (0, u^*), (v, v^*)\rangle\rangle \ \forall (v, v^*) \in D \Longleftrightarrow (0, u^*) \in D^\perp = D$ since D is Dirac $\Longleftrightarrow u^* \in \pi_{V^*}(\ker \pi_V|_D)$ $\square$

Proposition The following are equivalent:
(a) a Dirac structure D on V;
(b) a subspace $E \subset V$ and $\flat : E \to E^*$ linear skew-symmetric;
(c) a subspace $F \subset V$ and $\sharp : (V/F)^* \to V/F$ linear skew-symmetric.

Proof (a)$\Longrightarrow$(b): Given D, define $E := \pi_V(D) = [\pi_{V^*}(\ker \pi_V|_D)]^\circ$ and $\flat : E \to E^*$ by $e^\flat := u^*|_E$, where $u^* \in V^*$ satisfies $(e, u^*) \in D$. Well defined since $(e, u_1^*), (e, u_2^*) \in D \Longrightarrow (0, u_1^* - u_2^*) \in D \Longleftrightarrow u_1^* - u_2^* \in \pi_{V^*}(\ker \pi_V|_D) = E^\circ \Longleftrightarrow (u_1^* - u_2^*)|_E \equiv 0$. $\flat$ is clearly linear. $\flat$ is skew-symmetric: for $e_1, e_2 \in E$ we have $e_1^\flat = u_1^*|_E$ and $e_2^\flat = u_2^*|_E$, where $(e_1, u_1^*), (e_2, u_2^*) \in D$, so $\langle e_1^\flat, e_2\rangle + \langle e_2^\flat, e_1\rangle = \langle u_1^*, e_2\rangle + \langle u_2^*, e_1\rangle = \langle\langle (e_1, u_1^*), (e_2, u_2^*)\rangle\rangle = 0$.

(b)$\Longrightarrow$(a): Given subspace $E \subset V$ and a linear skew-symmetric map $\flat : E \to E^*$, define $D = \left\{(u, u^*) \mid u \in E, u^* \in V^* \text{ such that } u^*|_E = u^\flat\right\} \subset V \oplus V^*$. Then, if $(u, u^*) \in D$ we have $u \in E$ and hence $\langle u^*, u\rangle = \langle u^*|_E, u\rangle = \langle u^\flat, u\rangle = 0$, by skew-symmetry of $\flat$.

Show $\dim D = \dim V$: $\iota : E \hookrightarrow V$ inclusion; define the linear maps

$$S : u \in E \mapsto (u, u^\flat) \in E \oplus E^* \subset V \oplus E^* \quad \text{injective}$$
$$T : (v, v^*) \in V \oplus V^* \mapsto (v, \iota^* v^* = v^*|_E) \in V \oplus E^* \quad \text{surjective}$$

and note that $D = T^{-1}(S(E))$: $(v, v^*) \in D \iff v^*|_E = v^\flat \iff S(v) = (v, \iota^* v^*) = T(v, v^*) \iff (v, v^*) \in T^{-1}(S(E))$. Since $\ker T = \{0\} \oplus \ker \iota^* \implies \dim \ker T = \dim \ker \iota^* = \dim V^* - \dim E^* = \dim V - \dim E$. Since S is injective $\implies \dim S(E) = \dim E$. Thus

$$\dim D = \dim S(E) + \dim \ker T = \dim E + \dim V - \dim E = \dim V$$

(a)$\implies$(c) Define $F := [\pi_{V^*}(D)]^\circ = \pi_V(D \cap (V \oplus \{0\}))$ and $\sharp : F^\circ = \pi_{V^*}(D) \to (F^\circ)^* = [\pi_{V^*}(D)]^*$ by $(u^*)^\sharp := u|_{\pi_{V^*}(D)}$, where $u \in V$ satisfies $(u, u^*) \in D$. Well defined since $(u_1, u^*), (u_2, u^*) \in D \implies (u_1 - u_2, 0) \in D \iff u_1 - u_2 \in \pi_V(D \cap (V \oplus \{0\})) = [\pi_{V^*}(D)]^\circ \iff (u_1 - u_2)|_{\pi_{V^*}(D)} \equiv 0$. $\sharp$ is clearly linear. $\sharp$ is skew-symmetric: for $u_1^*, u_2^* \in \pi_{V^*}(D)$ we have $(u_1^*)^\sharp = u_1|_{\pi_{V^*}(D)}$ and $(u_2^*)^\sharp = u_2|_{\pi_{V^*}(D)}$, where $(u_1, u_1^*), (u_2, u_2^*) \in D$, so $\langle (u_1^*)^\sharp, u_2^* \rangle + \langle (u_2^*)^\sharp, u_1^* \rangle = \langle u_1, u_2^* \rangle + \langle u_2, u_1^* \rangle = \langle\!\langle (u_1, u_1^*), (u_2, u_2^*) \rangle\!\rangle = 0$. Composing with $(V/F)^* \cong F^\circ$ get $\sharp : (V/F)^* \cong F^\circ \to (F^\circ)^* \cong V/F$.

(c)$\Longrightarrow$(a) $\sharp : F^\circ \to (F^\circ)^*$, define $D := \{(u, u^*) \mid u^* \in F^\circ, u \in V = V^{**}$ such that $u|_{F^\circ} = (u^*)^\sharp\} \subset V \oplus V^*$. For any $(u, u^*) \in D$, we have $\langle u^*, u\rangle = \langle u^*, (u^*)^\sharp\rangle = 0$ by skew-symmetry of $\sharp$.

Show $\dim D = \dim V$: $\kappa : F^\circ \hookrightarrow V^*$ inclusion; define

$\bar{S} : u^* \in F^\circ \mapsto ((u^*)^\sharp, u^*) \in (F^\circ)^* \oplus F^\circ \subset (F^\circ)^* \oplus V^*$ injective

$\bar{T} : (v, v^*) \in V \oplus V^* \mapsto (\kappa^* v = v|_{F^\circ}, v^*) \in (F^\circ)^* \oplus V^*$ surjective

and note that $D = \bar{T}^{-1}(\bar{S}(F^\circ))$: $(v, v^*) \in D \Longleftrightarrow v|_{F^\circ} = (v^*)^\sharp \Longleftrightarrow \bar{S}(v^*) = (\kappa^* v, v^*) = \bar{T}(v, v^*) \Longleftrightarrow (v, v^*) \in \bar{T}^{-1}(\bar{S}(E))$. Since $\ker \bar{T} = \ker \kappa^* \oplus \{0\} \Longrightarrow \dim \ker \bar{T} = \dim \ker \kappa^* = \dim V - \dim (F^\circ)^* = \dim V - \dim F^\circ$. Since S is injective $\Longrightarrow \dim \bar{S}(F^\circ) = \dim F^\circ$. Thus

$$\dim D = \dim \bar{S}(F^\circ) + \dim \ker \bar{T} = \dim F^\circ + \dim V - \dim F^\circ = \dim V.$$

$\square$

Summary

D Dirac structure on V. Then

- $E := \pi_V(D)$, $F := [\pi_{V^*}(D)]^\circ \subset V$
- $\flat : E \to E^* \,:\, e^\flat := u^*|_E$, for $u^* \in V^*$ such that $(e, u^*) \in D$
- $\sharp : F^\circ \to (F^\circ)^* \,:\, (u^*)^\sharp := u|_{F^\circ}$, for $u \in V$ such that $(u, u^*) \in D$
- $D = \left\{(u, u^*) \mid u \in E, u^* \in V^* \text{ such that } u^*|_E = u^\flat\right\}$
  $= \left\{(u, u^*) \mid u^* \in F^\circ, u \in V = V^{**} \text{ such that } u|_{F^\circ} = (u^*)^\sharp\right\}$
- $\ker \flat = F$, $\ker \sharp = E^\circ$

Proof We show that $\ker \flat = [\pi_{V^*}(D)]^\circ = \pi_V(D \cap (V \oplus \{0\}))$: $e \in E, e^\flat = 0 \iff \forall u^* \in V^*$ such that $(e, u^*) \in D$ we have $0 = e^\flat = u^*|_E \iff \forall u^* \in V^*$ such that $(e, u^*) \in D$ we have $u^* \in E^\circ = \pi_{V^*}(D \cap (\{0\} \oplus V^*)) \iff \forall u^* \in V^*$ such that $(e, u^*) \in D$ we have $(0, u^*) \in D$.

Take an $e \in \pi_V(D \cap (V \oplus \{0\}))$, which means that there is some $v^* \in V$ such that $(e, v^*) \in D \cap (V \oplus \{0\})$, that is, $v^* = 0$ which says that necesarily $(e, 0) \in D$. Need to show that for any $u^* \in V^*$ such that $(e, u^*) \in D$ we necessarily have $(0, u^*) \in D$. But $(e, 0), (e, u^*) \in D \Longrightarrow (0, u^*) = (e, u^*) - (e, 0) \in D$.

Conversely, let $e \in \ker \flat \subset E = \pi_V(D)$. So there is some $u^* \in V^*$ such that $(e, u^*) \in D$. But then necessarily $(0, u^*) \in D$ and hence $(e, 0) = (e, u^*) - (0, u^*) \in D$ which means that $e \in \pi_V(D \cap (V \oplus \{0\}))$.

Conclusion: $\ker \flat = F = [\pi_{V^*}(D)]^\circ = \pi_V(D \cap (V \oplus \{0\}))$

We show that $\ker\sharp = [\pi_V(D)]^\circ = \pi_{V^*}(D \cap (\{0\} \oplus V^*))$, where $\sharp : \pi_{V^*}(D) \to [\pi_{V^*}(D)]^*$: $u^* \in \pi_{V^*}(D), (u^*)^\sharp = 0 \iff \forall u \in V$ such that $(u, u^*) \in D$ we have $0 = (u^*)^\sharp = u|_{\pi_{V^*}(D)} \iff \forall u \in V$ such that $(u, u^*) \in D$ we have $u \in [\pi_{V^*}(D)]^\circ = \pi_V(D \cap (V \cap \oplus \{0\})) \iff \forall u \in V$ such that $(u, u^*) \in D$ we have $(u, 0) \in D$.

Take $u^* \in \pi_{V^*}(D \cap (\{0\} \oplus V^*))$, which means that $(0, u^*) \in D$. Need to show that for any $u \in V$ such that $(u, u^*) \in D$ we necessarily have $(u, 0) \in D$. But $(0, u^*), (u, u^*) \in D \Longrightarrow (u, 0) = (u, u^*) - (0, u^*) \in D$.

Conversely, let $u^* \in \ker\sharp \subset \pi_{V^*}(D)$. So there is some $u \in V$ such that $(u, u^*) \in D$. But then necessarily $(u, 0) \in D$ and hence $(0, u^*) = (u, u^*) - (u, 0) \in D$ which means that $u^* \in \pi_{V^*}(D \cap (\{0\} \oplus V))$.

Conclusion: $\ker\sharp = E^\circ = [\pi_V(D)]^\circ = \pi_{V^*}(D \cap (\{0\} \oplus V^*))$

Example Let $\flat : V \to V^*$ be linear skew-symmetric and $D := \operatorname{graph} \flat$. Then $E = \pi_V(D) = V$, the map $V \to V^*$ coincides with $\flat$, $F = \ker \flat$, and the map $\sharp : (V/F)^* \to V/F$ is the generalized Poisson structure associated to the symplectic vector space V/F.

Example Let $\sharp : V^* \to V$ be linear skew-symmetric and $D := \operatorname{graph} \sharp$. Then $E = \pi_V(D) = (V^*)^\sharp$, $F = [\pi_{V^*}(D)]^\circ = (V^*)^\circ = \{0\} = \ker \flat$ and hence $\flat : E \to E^*$ defines a symplectic structure, the natural one induced by $\sharp$. In addition, the map $V^* \to V$ coincides with $\sharp$.

Dirac bases

Take $\mathbb{R}^n$ with the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Search for linear maps $\mathbf{A} : \mathbb{R}^n \to V$ and $\mathbf{B} : \mathbb{R}^n \to V^*$ such that the image of $\mathbf{A} \oplus \mathbf{B} : \mathbf{e} \in \mathbb{R}^n \mapsto (\mathbf{Ae}, \mathbf{Be}) \in V \oplus V^*$ is a Dirac structure on V. In particular, the range of $\mathbf{A} \oplus \mathbf{B}$ must be n-dimensional, so $\mathbf{A} \oplus \mathbf{B}$ must be injective. Since $\ker \mathbf{A} \oplus \mathbf{B} = \ker \mathbf{A} \cap \ker \mathbf{B}$ this implies

$$\ker \mathbf{A} \cap \ker \mathbf{B} = \{\mathbf{0}\}. \tag{28}$$

$\langle \mathbf{A}^*\mathbf{Be}, \mathbf{e} \rangle = \langle \mathbf{Be}, \mathbf{Ae} \rangle = 0$, $\forall \mathbf{e} \in \mathbb{R}^n \Longrightarrow \mathbf{A}^*\mathbf{B} : \mathbb{R}^n \to (\mathbb{R}^n)^*$ is skew-symmetric, that is,

$$\mathbf{A}^*\mathbf{B} + \mathbf{B}^*\mathbf{A} = \mathbf{0}. \tag{29}$$

Definition $\mathbf{A} : \mathbb{R}^n \to V$ and $\mathbf{B} : \mathbb{R}^n \to V^*$ satisfying (**??**) and (**??**) is called a *basis of a Dirac structure*.

So any basis of D is $\{(\mathbf{Ae}'_1, \mathbf{Be}'_1), \ldots, (\mathbf{Ae}'_n, \mathbf{Be}'_n)\}$, for some basis $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ of $\mathbb{R}^n$.

Assume that $\langle\cdot|\cdot\rangle$ is an inner product on V and identify $V^* \cong V$ via $\mathbf{e} \in V \mapsto \langle \mathbf{e}|\cdot\rangle \in V^*$.

Proposition $\mathbf{A} \pm \mathbf{B} : \mathbb{R}^n \to V$ is invertible.

Proof $\mathbf{e} \in \ker(\mathbf{A} \pm \mathbf{B}) \Longrightarrow \mathbf{Ae} = \mp\mathbf{Be} \Longrightarrow$

$$\begin{aligned}
\|\mathbf{Ae}\|^2 + \|\mathbf{Be}\|^2 &= \langle \mathbf{Ae}|\mathbf{Ae}\rangle + \langle \mathbf{Be}|\mathbf{Be}\rangle = \langle \mathbf{Ae}, \mathbf{Ae}\rangle + \langle \mathbf{Be}, \mathbf{Be}\rangle \\
&= \langle \mathbf{A}^*\mathbf{Ae}, \mathbf{e}\rangle + \langle \mathbf{B}^*\mathbf{Be}, \mathbf{e}\rangle = \mp\langle \mathbf{A}^*\mathbf{Be}, \mathbf{e}\rangle \mp \langle \mathbf{B}^*\mathbf{Ae}, \mathbf{e}\rangle \\
&= \mp\langle (\mathbf{A}^*\mathbf{B} + \mathbf{B}^*\mathbf{A})\mathbf{e}, \mathbf{e}\rangle = 0 \quad \text{by } (\mathbf{??})
\end{aligned}$$

So $\mathbf{e} \in \ker \mathbf{A} \cap \ker \mathbf{B} = \{\mathbf{0}\}$ which shows that $\mathbf{A} \pm \mathbf{B}$ is injective, hence an isomorphism since $n = \dim V$. $\square$

Dirac structures on $\mathbb{R}^n$

Hypothesis: $V \cong \mathbf{R}^n$ and $\mathbf{R}^n \cong (\mathbf{R}^n)^*$ via the usual Euclidean inner product. $\{\mathbf{e}_1 \ldots, \mathbf{e}_n\}$ standard basis of $\mathbb{R}^n$. Then $\mathcal{B}_{\text{standard}} := \{\mathbf{e}_i \oplus \mathbf{0}, \mathbf{0} \oplus \mathbf{e}_j \mid i, j = 1, \ldots, n\}$, is a basis of $\mathbf{R}^n \oplus (\mathbf{R}^n)^*$ in which the non-degenerate form $\langle\, , \rangle$ has the matrix

$$[\langle\, , \rangle]_{\mathcal{B}_{\text{standard}}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

that diagonalizes to

$$[\langle\, , \rangle]_{\mathcal{B}_{\text{diagonal}}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}$$

by choosing the basis $\mathcal{B}_{\text{diagonal}} := \{\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{n}_1, \ldots, \mathbf{n}_n\}$ given by

$$\mathbf{p}_i := \frac{\sqrt{2}}{2}(\mathbf{e}_i \oplus \mathbf{0} + \mathbf{0} \oplus \mathbf{e}_i) \qquad \mathbf{e}_i \oplus \mathbf{0} = \frac{\sqrt{2}}{2}(\mathbf{p}_i + \mathbf{n}_i)$$

$$\mathbf{n}_i := \frac{\sqrt{2}}{2}(\mathbf{e}_i \oplus \mathbf{0} - \mathbf{0} \oplus \mathbf{e}_i) \qquad \mathbf{0} \oplus \mathbf{e}_i = \frac{\sqrt{2}}{2}(\mathbf{p}_i - \mathbf{n}_i)$$

So $\langle , \rangle$ has signature $\{+1, \ldots, +1, -1, \ldots, -1\}$

On $P := \mathrm{span}\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ the form $\langle\, , \rangle$ is positive definite.

On $N := \mathrm{span}\{\mathbf{n}_1, \ldots, \mathbf{n}_n\}$ the form $\langle\, , \rangle$ is negative definite.

For any Dirac structure D: $D \cap P = D \cap N = \{\mathbf{0}\}$ because $D = D^{\perp}$. Since $n = \dim D \Longrightarrow D = \mathrm{graph}\, L$, where $L : N \to P$ is a linear map. If $(\mathbf{n}, \mathbf{p}) \in D \subset N \oplus P$, then $\mathbf{p} = L\mathbf{n}$ and $0 = \langle (\mathbf{n}, \mathbf{p}), (\mathbf{n}, \mathbf{p}) \rangle = \|\mathbf{n}\|^2 - \|\mathbf{p}\|^2 = \|\mathbf{n}\|^2 - \|L\mathbf{n}\|^2 \Longrightarrow \|L\mathbf{n}\| = \|\mathbf{n}\|, \forall \mathbf{n} \in N$. Conversely, the same computation shows that if $L : N \to P$ is norm preserving, then $\mathrm{graph}\, L$ is a Dirac structure. Therefore,

$$\mathfrak{Dir}(\mathbb{R}^n) \longleftrightarrow \{L : N \to P \mid L \text{ linear and norm preserving}\},$$

where $\mathfrak{Dir}(\mathbb{R}^n)$ is the set of Dirac structures on $\mathbb{R}^n$. By polarization, L is norm preserving if and only if $L \in \mathrm{O}(n)$.

Conclusion: $\mathfrak{Dir}(\mathbb{R}^n) \longleftrightarrow \mathrm{O}(n)$.

Can make this more precise.

Given two vectors $\mathbf{a} = a^i \mathbf{e}_i, \mathbf{b} = b^i \mathbf{e}_i \in \mathbb{R}^n$ form the vectors

$$\frac{\sqrt{2}}{2}(\mathbf{a} - \mathbf{b}) := \frac{\sqrt{2}}{2}(a^i - b^i)\mathbf{n}_i \in N, \quad \frac{\sqrt{2}}{2}(\mathbf{a} + \mathbf{b}) := \frac{\sqrt{2}}{2}(a^i + b^i)\mathbf{p}_i \in P$$

and note that $(\mathbf{a}, \mathbf{b}) = \frac{\sqrt{2}}{2}(\mathbf{a} - \mathbf{b}) + \frac{\sqrt{2}}{2}(\mathbf{a} + \mathbf{b}) \in N \oplus P$.

Let $\mathbf{A}, \mathbf{B} : \mathbb{R}^n \to \mathbb{R}^n$ be a basis of a given Dirac structure $D \iff D = \{(\mathbf{Ae}, \mathbf{Be}) \mid \mathbf{e} \in \mathbb{R}^n\}$. Then in the splitting $N \oplus P$ we have

$$\mathbf{n} = \frac{\sqrt{2}}{2}(\mathbf{A} - \mathbf{B})\mathbf{e}, \qquad \mathbf{p} = \frac{\sqrt{2}}{2}(\mathbf{A} + \mathbf{B})\mathbf{e}.$$

But $\mathbf{A} - \mathbf{B}$ is invertible so we get

$$\mathbf{p} = (\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})^{-1}\mathbf{n}$$

Note that $(\mathbf{??}) \Longrightarrow (\mathbf{A}^* + \mathbf{B}^*)(\mathbf{A} + \mathbf{B}) = (\mathbf{A}^* - \mathbf{B}^*)(\mathbf{A} - \mathbf{B})$. Therefore,

$$\begin{aligned}
&(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})^{-1}\left[(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})^{-1}\right]^* \\
&\quad = (\mathbf{A} + \mathbf{B})\left[(\mathbf{A}^* - \mathbf{B}^*)(\mathbf{A} - \mathbf{B})\right]^{-1}(\mathbf{A}^* + \mathbf{B}^*) \\
&\quad = (\mathbf{A} + \mathbf{B})\left[(\mathbf{A}^* + \mathbf{B}^*)(\mathbf{A} + \mathbf{B})\right]^{-1}(\mathbf{A}^* + \mathbf{B}^*) = \mathbf{I}
\end{aligned}$$

So $\mathbf{R} := (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1} \in \mathrm{O}(n)$, *generalized Cayley transform*. $\mathbf{A}$ (or $\mathbf{B}$) invertible, since $\mathbf{B}\mathbf{A}^{-1}$ is skew-symmetric (by (**??**)), implies

$$(\mathbf{A},\mathbf{B}) \mapsto (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1} = (\mathbf{I}+\mathbf{B}\mathbf{A}^{-1})(\mathbf{I}-\mathbf{B}\mathbf{A}^{-1})^{-1}$$

the usual Cayley transform. So, by $\mathbf{p} = (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1}\mathbf{n} \Longrightarrow (\mathfrak{Dir}(\mathbb{R}^n) \overset{Cayley}{\longleftrightarrow} \mathrm{O}(n))$.

$(\mathbf{A},\mathbf{B}) \sim (\underline{\mathbf{A}},\underline{\mathbf{B}}) \overset{def}{\Longleftrightarrow} D = \{(\mathbf{A}\mathbf{e},\mathbf{B}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\} = \{(\underline{\mathbf{A}}\mathbf{e},\underline{\mathbf{B}}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\}$, for pairs of maps $(\mathbf{A},\mathbf{B}),(\underline{\mathbf{A}},\underline{\mathbf{B}})$ satisfying (**??**), (**??**).

Proposition The following are equivalent:

(1) $(\mathbf{A},\mathbf{B}) \sim (\underline{\mathbf{A}},\underline{\mathbf{B}})$.

(2) $(\underline{\mathbf{A}},\underline{\mathbf{B}}) = (\mathbf{A}\mathbf{M},\mathbf{B}\mathbf{M})$, for some $\mathbf{M} \in \mathrm{GL}(n)$

(3) $\mathbf{A}^*\underline{\mathbf{B}} + \mathbf{B}^*\underline{\mathbf{A}} = \mathbf{0}$.

(4) $\mathbf{R} := (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1} = (\underline{\mathbf{A}}+\underline{\mathbf{B}})(\underline{\mathbf{A}}-\underline{\mathbf{B}})^{-1} =: \underline{\mathbf{R}}$.

Proof (1)$\Longleftrightarrow$(2) Let $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Since any basis of $D := \{(\mathbf{A}\mathbf{e}, \mathbf{B}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\}$ is $\{(\mathbf{A}\mathbf{e}'_1, \mathbf{B}\mathbf{e}'_1), \ldots, (\mathbf{A}\mathbf{e}'_n, \mathbf{B}\mathbf{e}'_n)\}$, for some basis $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ of $\mathbb{R}^n$ and $\{(\underline{\mathbf{A}}\mathrm{e}_1, \underline{\mathbf{B}}\mathrm{e}_1), \ldots, (\underline{\mathbf{A}}\mathrm{e}_n, \underline{\mathbf{B}}\mathrm{e}_n)\}$ is another basis of D, it must be of the same form. So there is some $\mathbf{M} \in \mathrm{GL}(n)$ such that the basis $\{(\mathbf{A}\mathbf{M}\mathbf{e}_1, \mathbf{B}\mathbf{M}\mathbf{e}_1), \ldots, (\mathbf{A}\mathbf{M}\mathbf{e}_n, \mathbf{B}\mathbf{M}\mathbf{e}_n)\}$ of D coincides with$\{(\underline{\mathbf{A}}\mathrm{e}_1, \underline{\mathbf{B}}\mathrm{e}_1), \ldots, (\underline{\mathbf{A}}\mathrm{e}_n, \underline{\mathbf{B}}\mathrm{e}_n)\}$, that is, $(\underline{\mathbf{A}}, \underline{\mathbf{B}}) = (\mathbf{A}\mathbf{M}, \mathbf{B}\mathbf{M})$.

Conversely, if $(\underline{\mathbf{A}}, \underline{\mathbf{B}}) = (\mathbf{A}\mathbf{M}, \mathbf{B}\mathbf{M})$ for some $\mathbf{M} \in \mathrm{GL}(n)$, then $\{(\underline{\mathbf{A}}\mathrm{e}, \underline{\mathbf{B}}\mathrm{e}) \mid \mathbf{e} \in \mathbb{R}^n\} = \{(\mathbf{A}\mathbf{M}\mathbf{e}, \mathbf{B}\mathbf{M}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\} = \{(\mathbf{A}\mathbf{e}, \mathbf{B}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\}$.

(2)$\Longrightarrow$(3) $\mathbf{A}^*\underline{\mathbf{B}} + \mathbf{B}^*\underline{\mathbf{A}} = (\mathbf{A}^*\mathbf{B} + \mathbf{B}^*\mathbf{A})\mathbf{M} = \mathbf{0}$.

(3)$\Longleftrightarrow$(4) $\mathbf{A}^*\underline{\mathbf{B}} + \mathbf{B}^*\underline{\mathbf{A}} = \mathbf{0} \Longleftrightarrow (\mathbf{A}^* + \mathbf{B}^*)(\underline{\mathbf{A}} + \underline{\mathbf{B}}) = (\mathbf{A}^* - \mathbf{B}^*)(\underline{\mathbf{A}} - \underline{\mathbf{B}})$ $\Longleftrightarrow (\underline{\mathbf{A}} + \underline{\mathbf{B}})(\underline{\mathbf{A}} - \underline{\mathbf{B}})^{-1} = (\mathbf{A}^* + \mathbf{B}^*)^{-1}(\mathbf{A}^* - \mathbf{B}^*) \Longleftrightarrow \underline{\mathbf{R}} = (\mathbf{R}^*)^{-1} = \mathbf{R}$.

(4)$\Longleftrightarrow$(1) $\mathrm{graph}\,\mathbf{R} = \{(\mathbf{A}\mathbf{e}, \mathbf{B}\mathbf{e}) \mid \mathbf{e} \in \mathbb{R}^n\}$ and $\mathrm{graph}\,\underline{\mathbf{R}} = \{(\underline{\mathbf{A}}\mathrm{e}, \underline{\mathbf{B}}\mathrm{e}) \mid \mathbf{e} \in \mathbb{R}^n\}$, so $\mathrm{graph}\,\mathbf{R} = \mathrm{graph}\,\underline{\mathbf{R}} \Longleftrightarrow (\mathbf{A}, \mathbf{B}) \sim (\underline{\mathbf{A}}, \underline{\mathbf{B}})$ $\square$

Induced Dirac structures

Let the Dirac structure on D be given by a subspace $E_V \subset V$ and a linear skew-symmetric map $\flat_V : E \to E^*$. Let $W \subset V$ be a subspace. Then $E_W := W \cap E$ is a subspace of W and $\flat_W := \flat|_{W\cap E} : W \cap E \to (E \cap W)^*$ is skew-symmetric so it defines a Dirac structure on W.

Alternatively, let D be given by a subspace $F_V \subset V$ and a linear skew-symmetric map $\sharp_V : (V/F_V)^* \to V/F_V$. Let $W \subset V$ be a subspace. Then $F_{V/W} := \Pi(F) \subset V/W$ is a subspace, where $\Pi : V \to V/W$ is the projection which also induces the map $[\Pi] : V/F_V \to (V/W)/F_{V/W}$. Define $\sharp_{V/W} : \left[(V/W)/F_{V/W}\right]^* \to (V/W)/F_{V/W}$ by

$$\left[(V/W)/F_{V/W}\right]^* \xrightarrow{[\Pi]^*} (V/F_V)^* \xrightarrow{\sharp_V} V/F_V \xrightarrow{[\Pi]} (V/W)/F_{V/W}$$

$$\sharp_{V/W} := [\Pi] \circ \sharp_V \circ [\Pi]^*$$

$\sharp_{V/W}$ is skew: $\alpha_1, \alpha_2 \in \left[(V/W)/F_{V/W}\right]^* \Longrightarrow \left\langle \alpha_1^{\sharp_{V/W}}, \alpha_2 \right\rangle + \left\langle \alpha_2^{\sharp_{V/W}}, \alpha_1 \right\rangle$ $= \left\langle (\alpha_1 \circ [\Pi])^{\sharp_V}, (\alpha_2 \circ [\Pi]) \right\rangle + \left\langle (\alpha_2 \circ [\Pi])^{\sharp_V}, (\alpha_1 \circ [\Pi]) \right\rangle = 0.$

Conclusion: D Dirac structure on V, $W \subset V$ subspace induces Dirac structures on both W and on V/W.

Dirac maps

Backward Dirac Maps. $L : W \to V$ linear map. D_V a Dirac structure given by $E_V \subset V$, $\flat_V : E_V \to E_V^*$ skew-symmetric. Define $E_W := L^{-1}(E_V) \subset W$, $\flat_W := L^* \circ \flat_V \circ L : E_W \to E_W^*$ skew. Let $\mathcal{B}L(D_V) = \{(w, w^*) \in W \oplus W^* \mid w \in E_W, w^*|_{E_W} = w^{\flat_W}\}$ be the Dirac structure thus defined.

Special case: $L = \iota : W \hookrightarrow V$, injection of a subspace.

Get the *backward Dirac map* $\mathcal{B}L : \mathfrak{Dir}(V) \to \mathfrak{Dir}(W)$.

Proposition $\mathcal{B}L(D_V) = \{(w, L^*v^*) \mid w \in W, v^* \in V^*, (Lw, v^*) \in D_V\}$

Proof $\supseteq$: (w, L^*v^*) satisfies $(Lw, v^*) \in D_V \iff Lw \in E_V, v^*|_{E_V} = (Lw)^{\flat_V}$. Then $w \in L^{-1}(E_V) = E_W$. If $e \in E_W \implies \langle (L^*v^*)|_{E_W}, e\rangle = \langle v^*, Le\rangle = \langle (Lw)^{\flat_V}, Le\rangle = \langle L^*((Lw)^{\flat_V}), e\rangle = \langle w^{\flat_W}, e\rangle \implies (w, L^*v^*) \in \mathcal{B}L(D_V)$.

$\subseteq$: $(w, w^*) \in \mathcal{B}L(D_V) \iff w \in E_W, w^*|_{E_W} = w^{\flat_W} = L^*((Lw)^{\flat_V})$. Need to find $v^* \in V^*$ such that $w^* = L^*v^*$ and $(Lw, v^*) \in D_V \iff w^* = L^*v^*, Lw \in E_V, v^*|_{E_V} = (Lw)^{\flat_V}$. First note that $Lw \in L(E_W) = E_V$. Second, L induces an injective linear map $[L] : W/E_W \to V/E_V \iff [L]^* : (V/E_V)^* \to (W/E_W)^*$ surjective $\iff L^* : E_V^\circ \to E_W^\circ$ surjective.

Since $Lw \in E_V = \pi_V(D_V), \exists u^* \in V^*$ such that $(Lw, u^*) \in D_V \implies u^*|_{E_V} = (Lw)^{\flat_V} \implies (L^*u^*)|_{E_W} = L^*((Lw)^{\flat_V}) = w^{\flat_W} \implies (w^* - L^*u^*)|_{E_W} = 0 \iff w^* - L^*u^* \in E_W^\circ \implies \exists v_1^* \in E_V^\circ$ such that $L^*v_1^* = w^* - L^*u^* \implies w^* = L^*(u^* + v_1^*)$ and $(u^* + v_1^*)|_{E_V} = u^*|_{E_V} + v_1^*|_{E_V} = (Lw)^{\flat_V} + 0$, so $v^* := u^* + v_1^* \in V^*$ is the desired element. $\square$

Forward Dirac Maps. $L : V \to W$ linear map. D_V a Dirac structure given by $F_V \subset V$, $\sharp_V : F_V^\circ \to (F_V^\circ)^*$ skew-symmetric. Define $F_W := L(F_V) \subset W$. Then $L^*(F_W^\circ) \subset F_V^\circ$: for any $w^* \in F_W^\circ$ and $v \in F_V$, we have $\langle L^*(w^*), v\rangle = \langle w^*, Lv\rangle = 0$ since $Lv \in L(F_V) = F_W$.

But note that $(L^*)^{-1}(F_V^\circ) = F_W^\circ$.

L induces $\bar{L} : (F_V^\circ)^* \to (F_W^\circ)^*$ by $\langle \bar{L}(\alpha), w^* \rangle = \langle \alpha, L^* w^* \rangle, \forall w^* \in F_W^\circ$.

Define $\sharp_W := \bar{L} \circ \sharp_V \circ L^* : F_W^\circ \to (F_W^\circ)^*$ skew-symmetric and $\mathcal{F}L(D_V) = \{(w, w^*) \in W \oplus W^* \mid w^* \in F_W^\circ, w|_{F_W^\circ} = (w^*)^{\sharp_W}\}$.

Special case: $L = \Pi : V \to V/W$, projection onto the quotient.

Get the *forward Dirac map* $\mathcal{F}L : \mathfrak{Dir}(V) \to \mathfrak{Dir}(W)$.

Proposition $\mathcal{F}L(D_V) = \{(Lv, w^*) \mid v \in V, w^* \in W^*, (v, L^* w^*) \in D_V\}$

Proof $\supseteq$: (Lv, w^*) satisfies $(v, L^* w^*) \in D_V \iff L^* w^* \in F_V^\circ, v|_{F_V^\circ} = (L^* w^*)^{\sharp_V}$. Then $w^* \in (L^*)^{-1}(F_V^\circ) = F_W^\circ$. If $\beta \in F_W^\circ \Longrightarrow \langle (Lv)|_{F_W^\circ}, \beta \rangle = \langle v, L^* \beta \rangle = \langle (L^* w^*)^{\sharp_V}, L^* \beta \rangle = \langle \bar{L}((L^* w^*)^{\sharp_V}), \beta \rangle = \langle (w^*)^{\sharp_W}, \beta \rangle \Longrightarrow (Lv)|_{F_W^\circ} = (w^*)^{\sharp_W} \Longrightarrow (Lv, w^*) \in \mathcal{F}L(D_V)$.

$\subseteq$: $(w, w^*) \in \mathcal{F}L(D_V) \Leftrightarrow w^* \in F_W^\circ, w|_{F_W^\circ} = (w^*)^{\sharp_W} = \bar{L}((L^* w^*)^{\sharp_V})$. Need to find $v \in V$ such that $w = Lv$ and $(v, L^* w^*) \in D_V \iff w = Lv, L^* w^* \in F_V^\circ, v|_{F_V^\circ} = (L^* w^*)^{\sharp_V}$.

Since $L^*w^* \in F_V^\circ = \pi_{V^*}(D_V), \exists u \in V$ such that $(u, L^*w^*) \in D_V \Longrightarrow u|_{F_V^\circ} = (L^*w^*)^{\sharp_V} \Longrightarrow \forall z^* \in F_W^\circ$ we have $\langle Lu, z^*\rangle = \langle u, L^*z^*\rangle = \langle \bar{L}(u|_{F_V^\circ}), z^*\rangle = \langle \bar{L}((L^*w^*)^{\sharp_V}), z^*\rangle = \langle (w^*)^{\sharp_W}, z^*\rangle = \langle w, z^*\rangle$. Therefore $w - Lu \in (F_W^\circ)^\circ = F_W = L(F_V) \Longrightarrow \exists v_1 \in F_V$ such that $Lv_1 = w - Lu \Longrightarrow w = L(v_1 + u)$. Also $(v_1 + u)|_{F_V^\circ} = v_1|_{F_V^\circ} + u|_{F_V^\circ} = 0 + (L^*w^*)^{\sharp_V}$ so the desired element is $v := v_1 + u$. $\square$

Summary $L : V \to W$ linear map induces *forward* and *backward Dirac maps*

$$\mathcal{F}L : \mathfrak{Dir}(V) \to \mathfrak{Dir}(W) \qquad \text{and} \qquad \mathcal{B}L : \mathfrak{Dir}(W) \to \mathfrak{Dir}(V)$$

$$\mathcal{F}L(D_V) := \{(Lv, w^*) \mid v \in V, w^* \in W^*, (v, L^*w^*) \in D_V\}$$
$$\mathcal{B}L(D_W) := \{(v, L^*w^*) \mid v \in V, w^* \in W^*, (Lv, w^*) \in D_W\}$$

Proposition If $L : V \to W$ is surjective, then $\mathcal{F}L \circ \mathcal{B}L = I_{\mathfrak{Dir}(W)}$. If $L : V \to W$ is injective, then $\mathcal{B}L \circ \mathcal{F}L = I_{\mathfrak{Dir}(V)}$.

Proof Let $D_W \in \mathfrak{Dir}(W)$. Then $(w, w^*) \in \mathcal{F}L(\mathcal{B}L(D_W)) \iff \exists v_1 \in V, w_1^* \in W^*$ such that $(w, w^*) = (Lv_1, w_1^*)$ and $(v_1, L^*w_1^*) \in \mathcal{B}L(D_W)$ $\iff \exists v_1 \in V$ such that $w = Lv_1$ and $(v_1, L^*w^*) \in \mathcal{B}L(D_W) \iff$ $\exists v_1 \in V$ such that $w = Lv_1$ and $(Lv_1, w^*) \in D_W$. So, if L is surjective then there is always a $v_1 \in V$ such that $w = Lv_1$ and then the condition is equivalent to $(w, w^*) \in D_W$.

Let $D_V \in \mathfrak{Dir}(V)$. Then $(v, v^*) \in \mathcal{B}L(\mathcal{F}L(D_V)) \iff \exists v_1 \in V, w_1^* \in W^*$ such that $(v, v^*) = (v_1, L^*w_1^*)$ and $(Lv_1, w_1^*) \in \mathcal{F}L(D_V) \iff \exists w_1^* \in W^*$ such that $v^* = L^*w_1^*$ and $(Lv, w_1^*) \in \mathcal{F}L(D_V) \iff \exists w_1^* \in W^*$ such that $v^* = L^*w_1^*$ and $(v, L^*w_1^*) \in D_V$. So, if L^* is surjective then there is always a $w_1^* \in W^*$ such that $v^* = L^*w_1^*$ and then the condition is equivalent to $(v, v^*) \in D_V$. But surjectivity of L^* is equivalent to injectivity of L. $\square$

Given a Dirac structure D_V on V, recall that $\flat_V : \pi_V(D_V) \to [V(D_V)]^*$ is defined by $e^\flat := u^*|_E$, for $u^* \in V^*$ such that $(e, u^*) \in D_V$. $\sharp : \pi_{V^*}(D_V) \to [\pi_{V^*}(D_V)]^*$ defined by: for $\alpha, \beta \in F^\circ \subset V^*$ set $\langle \alpha^\sharp, \beta \rangle := \langle \alpha, e \rangle$ where $e \in E$ satisfies $\beta|_{E^\flat} = e^\flat$.

Definition $D_V \in \mathfrak{Dir}(V), D_W \in \mathfrak{Dir}(W)$. $L : V \to W$ is called *forward (backward) Dirac* if $\mathcal{F}L(D_V) = D_W$ $(\mathcal{B}L(D_W) = D_V)$.

Example: (V, D_V), $\Pi : V \to V/F_V$ forward Dirac

Proposition (i) $(V, \sharp_V), (W, \sharp_W)$ generalized Poisson vector spaces. A linear map $L : V \to W$ is generalized Poisson $\iff$ L is forward Dirac.

(ii) $(V, \flat_V), (W, \flat_W)$ presymplectic vector spaces. A linear map $L : V \to W$ is presymplectic $\iff$ L is backward Dirac.

Proposition $(V, D_V), (W, D_W), L : V \to W$ forward Dirac. Then $[L] : V/F_V \to W/F_W$ is generalized Poisson.

Proof $\mathcal{F}[L](\mathrm{graph}\, \sharp_{V/F_V}) = \mathcal{F}[L](\mathcal{F}\Pi_V(D_V)) = \mathcal{F}\Pi_W(\mathcal{F}(D_V)) = \mathcal{F}\Pi_W(D_W) = \mathrm{graph}\, \sharp_{W/F_W}$. $\square$

(Co)distributions

M n-manifold, $U \subseteq M$ open, $\mathfrak{X}(U)$ vector fields, $\Omega^k(U)$ k-forms

• *Distribution* Δ on M is an assignment of a vector subspace $\Delta(x) \subset T_xM$ to each $x \in M$.

• Δ is *smooth* if $\forall x_0 \in M, \exists U \ni x_0, \exists X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that $\Delta(x) = \operatorname{span}\{X_1(x), \ldots, X_k(x)\}, \forall x \in U$.

• Δ is *constant dimensional* if the dimension of the linear subspace $\Delta(x) \subset T_xM$ does not depend on the point $x \in M$.

Δ smooth constant dimensional $\Longrightarrow$ Δ vector subbundle of TM

• *Codistribution* Γ on M is an assignment of a vector subspace $\Gamma(x) \subset T_x^*M$ to each $x \in M$.

Smoothness and constant dimensionality are defined similarly. Γ smooth constant dimensional $\Longrightarrow$ Γ vector subbundle of T^*M.

Dirac structures on manifolds

A smooth vector subbundle $D \subset TM \oplus T^*M, x \in M$, is a *Dirac structure* if $D(x) = D^{\perp}(x), \forall x \in M$, where

$$D^{\perp}(x) = \{(w, w^*) \in T_xM \times T_x^*M \mid \langle v^*, w\rangle + \langle w^*, v\rangle = 0, \forall (v, v^*) \in D(x)\}.$$

The bilinear form $\langle\langle (v, v^*), (w, w^*)\rangle\rangle := \langle v^*, w\rangle + \langle w^*, v\rangle$ on $TM \oplus T^*M$ is nondegenerate.

So D defines for each $x \in M$ a linear Dirac structure on T_xM. Converse not true. Discuss later regularity conditions when true.

Proposition A Dirac structure is a smooth vector subbundle $D \subset TM \oplus T^*M$ such that D is *isotropic*: $\forall (X, \alpha), (Y, \beta) \in \mathfrak{D}_{\mathrm{loc}}$

$$\langle\langle (X, \alpha), (Y, \beta)\rangle\rangle := \langle \alpha, Y\rangle + \langle \beta, X\rangle = 0, \tag{30}$$

and D is *maximal*: if (Y, β) is such that (**??**) holds for all $(X, \alpha) \in \mathfrak{D}_{\mathrm{loc}}$, then $(Y, \beta) \in \mathfrak{D}_{\mathrm{loc}}$.

Proof D subbundle $\Longrightarrow \forall (v, v^*) \in D(x), \exists (X, \alpha) \in \mathfrak{D}_{loc}$ such that $(v, v^*) = (X(x), \alpha(x))$. D subbundle and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ nondegenerate $\Longrightarrow$ $D^\perp \subset TM \oplus T^*M$ is a subbundle whose fibers are $D^\perp(x) \Longrightarrow$ every $(w, w^*) \in D^\perp(x)$ can be extended to a local section (Y, β) of $D^\perp$.

D isotropic $\Longleftrightarrow D \subset D^\perp$: if D is isotropic and $(v, v^*) \in D(x)$, let $(X, \alpha) \in \mathfrak{D}_{\text{loc}}$ extension to a local section. Then $\langle\!\langle (X, \alpha), (Y, \beta) \rangle\!\rangle = 0, \forall (Y, \beta) \in \mathfrak{D}_{\text{loc}}$. Evaluating at $x \in M$ gives $\langle\!\langle (v, v^*), (w, w^*) \rangle\!\rangle = 0, \forall (w, w^*) \in D(x) \Longrightarrow (v, v^*) \in D^\perp(x)$. So $D \subset D^\perp$. Conversely, if $D \subset D^\perp$ and $(X, \alpha), (Y, \beta) \in \mathfrak{D}_{\text{loc}}$, then $\langle\!\langle (X, \alpha), (Y, \beta) \rangle\!\rangle(x) = \langle\!\langle (X(x), \alpha(x)), (Y(x), \beta(x)) \rangle\!\rangle = 0, \forall x$ in the domain of the local sections. So $\langle\!\langle (X, \alpha), (Y, \beta) \rangle\!\rangle = 0$, i.e., D is isotropic.

So, for the notion of isotropy of a subbundle $D \subset TM \oplus T^*M$, we can use the standard definition, either pointwise or in terms of local sections and get the same answer.

If D Dirac on M $\Longrightarrow D(x)$ linear Dirac on $T_xM, \forall x \in M \Longleftrightarrow D(x)$ maximal isotropic in $T_xM, \forall x \in M$. So D is isotropic from the above. Now assume (Y, β) is such that (**??**) holds for all $(X, \alpha) \in \mathfrak{D}_{\mathrm{loc}}$. Then $0 = \langle\langle (X, \alpha), (Y, \beta) \rangle\rangle(x) = \langle\langle (X(x), \alpha(x)), (Y(x), \beta(x)) \rangle\rangle, \forall x$ in the domain of definition of the local sections $\Longrightarrow (Y(x), \beta(x)) \in D^{\perp}(x) = D(x), \forall x$ in the domain of definition of the local sections $\Longrightarrow (Y, \beta) \in \mathfrak{D}_{\mathrm{loc}}$.

Conversely, assume that D is maximal isotropic. Then, from the considerations above, $D(x)$ is isotropic ($D(x) \subset D^{\perp}(x)$) in $T_xM, \forall x \in M$. If $(w, w^*) \in D^{\perp}(x)$ extend it to a local section (Y, β) of $D^{\perp}$. But then (Y, β) is such that (**??**) holds for all $(X, \alpha) \in \mathfrak{D}_{\mathrm{loc}}$, so by maximality, $(Y, \beta) \in \mathfrak{D}_{\mathrm{loc}} \Longrightarrow (w, w^*) = (Y(x), \beta(x)) \in D(x)$. So $D^{\perp} \subset D \Longrightarrow D^{\perp} = D \Longleftrightarrow D$ Dirac structure on M. $\square$

Associated smooth (co)distributions

$D \subset TM \oplus T^*M$ Dirac structure. Then $G_0, G_1 \subset TM$ defined by

$$G_0(x) := \{X(x) \in T_xM \mid X \in \mathfrak{X}_{\mathrm{loc}}(M), (X, 0) \in \mathfrak{D}_{\mathrm{loc}}\}$$
$$G_1(x) := \{X(x) \in T_xM \mid X \in \mathfrak{X}_{\mathrm{loc}}(M), \exists\alpha \in \Omega^1_{\mathrm{loc}}(M), (X, \alpha) \in \mathfrak{D}_{\mathrm{loc}}\}$$

are smooth distributions on M and $P_0, P_1 \subset T^*M$ defined by

$$P_0(x) := \{\alpha(x) \in T^*_xM \mid \alpha \in \Omega^1_{\mathrm{loc}}(M), (0, \alpha) \in \mathfrak{D}_{\mathrm{loc}}\}$$
$$P_1(x) := \{\alpha(x) \in T^*_xM \mid \alpha \in \Omega^1_{\mathrm{loc}}(M), \exists X \in \mathfrak{X}_{\mathrm{loc}}(M), (X, \alpha) \in \mathfrak{D}_{\mathrm{loc}}\}$$

are smooth codistributions on M.

The annihilators are always taken pointwise in each fiber.

Proposition (i) $G_0 \subset P_1^\circ$, $P_0 \subset G_1^\circ$, $P_1 \subset G_0^\circ$, $G_1 \subset P_0^\circ$

(ii) If P_1 has constant rank $\Longrightarrow P_1 = G_0^\circ, G_0 = P_1^\circ$. If G_1, has constant rank $\Longrightarrow P_0 = G_1^\circ, G_1 = P_0^\circ$.

Proof (i) $v \in G_0(x) \Longleftrightarrow \exists Y \in \mathfrak{X}_{\mathrm{loc}}(M), v = Y(x), (Y,0) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow \exists Y \in \mathfrak{X}_{\mathrm{loc}}(M), v = Y(x), 0 = \langle \alpha, Y\rangle + \langle 0, X\rangle = \langle \alpha, Y\rangle, \forall (X,\alpha) \in \mathfrak{D}_{\mathrm{loc}} \Longrightarrow \langle \alpha(x), v\rangle = 0, \forall \alpha(x) \in P_1 \Longleftrightarrow v \in P_1(x)^\circ$.

$v^* \in P_0(x) \Longleftrightarrow \exists \beta \in \Omega^1_{\mathrm{loc}}(M), v^* = \beta(x), (0,\beta) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow \exists \beta \in \Omega^1_{\mathrm{loc}}(M), v^* = \beta(x), \langle \beta, X\rangle + \langle \alpha, 0\rangle = \langle \beta, X\rangle = 0, \forall (X,\alpha) \in \mathfrak{D}_{\mathrm{loc}} \Longrightarrow \langle v^*, X(x)\rangle = 0, \forall X(x) \in G_1 \Longleftrightarrow v^* \in G_1^\circ$.

$P_1 = P_1^{\circ\circ} \subset G_0^\circ$ and $G_1 = G_1^{\circ\circ} \subset P_0^\circ$.

(ii) If $G_1 \subset TM$ has constant rank, then $G_1^\circ \subset T^*M$ has constant rank. Let $w^* \in G_1^\circ$ and let $\beta \in \Omega^1_{\mathrm{loc}}(M)$ be an extension of w^* to a local section of G_1°. Then $0 = \langle \beta, X\rangle, \forall X \in \mathfrak{X}_{\mathrm{loc}}(M)$, local section of $G_1 \Longleftrightarrow 0 = \langle \beta, X\rangle, \forall (X,\alpha) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow 0 = \langle \beta, X\rangle + \langle \alpha, 0\rangle = \langle\!\langle (\alpha, X), (\beta, 0)\rangle\!\rangle, \forall (X,\alpha) \in \mathfrak{D}_{\mathrm{loc}} \Longrightarrow (0,\beta) \in \mathfrak{D}_{\mathrm{loc}}$ since D is maximal isotropic. So $w^* = \beta(x)$ where $(0,\beta) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow w^* \in P_0$. Thus $G_1^\circ \subset P_0$.

Same proof if P_1 has constant rank. $\square$

Important remarks. **(1)** To obtain a smooth distribution, it is important to define G_0 in terms of local sections. E.g., it is *not* true that $v \in G_0(x)$ if and only if $(v, 0) \in D(x)$. Same for P_0.

Counterexample: $M = \mathbb{R}^2 = \{x = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. Consider the closed two-form $\omega = \|x\|^2 \mathbf{d}x_1 \wedge \mathbf{d}x_2$, and define D by

$$D(x) = \{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid v^* = \omega(x)(v, \cdot)\}.$$

D is a Dirac structure on M: D is a vector subbundle of $TM \oplus T^*M$ generated by the global basis $\left\{\frac{\partial}{\partial x_1} + \|x\|^2 \mathbf{d}x_2, \frac{\partial}{\partial x_2} - \|x\|^2 \mathbf{d}x_1\right\}$. This also shows that $D^\perp(x) = D(x)$. Since ω is nondegenerate outside $x = (0, 0)$ the smooth distribution G_0 is given by $G_0 = \{0\}$ (the zero section of TM). However, $(v, 0) \in D(0)$ for *every* $v \in \mathbb{R}^2$.

This example illustrates also something else. Notice that P_1 is the smooth codistribution whose basis is given by the one-forms $-\|x\|^2 \mathbf{d}x_1$ and $\|x\|^2 \mathbf{d}x_2$. In particular, $P_1^\circ(0) = (\{0\})^\circ = T_0M = \mathbb{R}^2$, which does not equal $G_0(0) = \{0\}$. Hence $G_0 \subsetneq P_1^\circ(x)$.

The problem stems from the fact that P_1 does not have constant rank in this example. In general, *if* P_1 has constant rank, *then* $P_1^\circ(x) = G_0(x)$, as we saw before.

(2) On the other hand, the codistribution P_1 can be equivalently defined pointwise by:

$$P_1(x) = \{v^* \in T_x^*M \mid \exists v \in T_xM \text{ such that } (v, v^*) \in D(x)\}.$$

To see this, recall that, by definition, D is a smooth vector sub-bundle of $TM \oplus T^*M$. Hence there exists a smooth local basis for its fibers. The canonical projection of this basis to T^*M yields a smooth local basis for P_1 (around the point x). Therefore, the two definitions are equivalent.

Representation Theorem D Dirac structure on M.

(i) Locally, around every $x \in M$ there exist linear operators $E(x) : T_x^*M \to \mathbb{R}^n$ and $F(x) : T_xM \to \mathbb{R}^n$ depending smoothly on x s.t.

$$\mathrm{im}(F(x) \oplus E(x)) = \mathbb{R}^n \quad \text{and} \quad E(x)F(x)^* + F(x)E(x)^* = 0$$

such that D can be locally expressed as

$$D(x) = \{(v, v^*) \in T_xM \oplus T_x^*M \mid F(x)v + E(x)v^* = 0\}.$$

Concretely, writing $E(x)$ and $F(x)$ as $n \times n$ matrices, this means $\mathrm{rank}[F(x) \mid E(x)] = n$ and $E(x)F(x)^T + F(x)E(x)^T = 0$.

(ii) Let P be a constant rank codistribution of M and $\flat : P^\circ \to (P^\circ)^*$ a skew-symmetric vector bundle map (in every fiber). Then

$$D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v^*|_{P^\circ} = \flat(x)v, v \in P^\circ(x), x \in M\} \quad (31)$$

is a Dirac structure on M.

Converserly, if D is a Dirac structure on M having the property that G_1 is a constant rank distribution on M, then there exists a vector bundle map $\flat : G_1 \to G_1^*$ such that D is given by (**??**) with $P := P_0 = G_1^\circ$. Also, $\ker(\flat : G_1 \to G_1^*) = G_0$.

(iii) Let G be a constant rank distribution on M and $\sharp : G^\circ \to (G^\circ)^*$ a skew-symmetric vector bundle map (in every fiber). Then

$$D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v|_{G^\circ} = \sharp(x)v^*, v^* \in G^\circ(x), x \in M\} \quad (32)$$

is a Dirac structure on M.

Converserly, if D is a Dirac structure on M having the property that P_1 is a constant rank codistribution on M, then there exists a vector bundle map $\sharp : P_1 \to P_1^*$ such that D is given by (**??**) with $G := G_0 = P_1^\circ$. Also, $\ker(\sharp : P_1 \to P_1^*) = P_0$.

E.g. non-degenerate two-form ($G_0 = \{0\}, G_1 = TM$), generalized Poisson ($P_1 = T^*M$), $B \oplus B^\circ$ for $B \subset TM$ subbundle ($G_0 = G_1 = B$).

Admissible functions

$\mathcal{A}_D := \{H \in C^\infty(M) \mid \mathbf{d}H(x) \in P_1(x), \forall x \in M\}$ *admissible functions*

Define $\{\cdot,\cdot\}_D : \mathcal{A}_D \times \mathcal{A}_D \to C^\infty(M)$ by $\{H_1, H_2\}_D := \langle \mathbf{d}H_1, X_2\rangle = -\langle \mathbf{d}H_2, X_1\rangle$, where $(X_1, \mathbf{d}H_1), (X_2, \mathbf{d}H_2) \in \mathfrak{D}$.

Well defined: if $(Y_1, \mathbf{d}H_1)$, $(Y_2, \mathbf{d}H_2) \in \mathfrak{D} \Longrightarrow (X_2 - Y_2, 0) \in \mathfrak{D} \Longrightarrow$ $\langle \mathbf{d}H_1, Y_2 - X_2\rangle = \langle\!\langle (Y_2 - X_2, 0), (X_1, \mathbf{d}H_1)\rangle\!\rangle = 0$ since D is isotropic. Thus, $\langle \mathbf{d}H_1, Y_2\rangle = \langle \mathbf{d}H_1, Y_2 - X_2\rangle + \langle \mathbf{d}H_1, X_2\rangle = \langle \mathbf{d}H_1, X_2\rangle$.

Second equality: Since D is isotropic $\Longrightarrow 0 = \langle\!\langle (X_1, \mathbf{d}H_1), (X_2, \mathbf{d}H_2)\rangle\!\rangle$ $= \langle \mathbf{d}H_1, X_2\rangle + \langle \mathbf{d}H_2, X_1\rangle$. $\{\cdot,\cdot\}_D$ depends only on $\mathbf{d}H_1, \mathbf{d}H_2$.

$\mathcal{A}_D$ is not closed under the bracket $\{\cdot,\cdot\}_D$.

$\{\cdot,\cdot\}_D$ is $\mathbb{R}$-bilinear and skew-symmetric.

$\{\cdot,\cdot\}_D$ satisfies the Leibniz identity: $(X_i, \mathbf{d}H_i) \in \mathfrak{D}, i = 1, 2, 3 \Longrightarrow$ $\{H_1H_2, H_3\}_D = \langle \mathbf{d}(H_1H_2), X_3\rangle = \langle H_1\mathbf{d}H_2 + H_2\mathbf{d}H_1, X_3\rangle =$ $H_1\{H_2, H_3\}_D + H_2\{H_1, H_3\}_D$.

Assume that D is given by (**??**), that is, by a subbundle $G \subset TM$ and $\sharp : T^*M \to TM$. If $H_1, H_2 \in \mathcal{A}_D \Longrightarrow \exists X_2 \in \mathfrak{X}_{\mathrm{loc}}(M)$ such that $(X_2, \mathbf{d}H_2) \in \mathfrak{D}_{\mathrm{loc}}$ and $\mathbf{d}H_1(x) \in P_1(x) = G_0^\circ(x), \forall x$. Therefore $X_2(x) - \sharp\mathbf{d}H_2(x) =: Y(x) \in G_0(x), \forall x$ and hence $\{H_1, H_2\}_D = \langle \mathbf{d}H_1, X_2 \rangle = \langle \mathbf{d}H_1, \sharp\mathbf{d}H_2 \rangle + \langle \mathbf{d}H_1, Y \rangle = \langle \mathbf{d}H_1, \sharp\mathbf{d}H_2 \rangle$ since $\langle \mathbf{d}H_1, Y \rangle = 0$ because $\mathbf{d}H_1(x) \in P_1(x) = G_0^\circ(x)$ and $Y(x) \in G_0(x), \forall x$.

If D is given by a constant rank distribution $G \subset TM$ and a skew symmetric linear vector bundle map $\sharp : T^*M \to TM$, then $P_1 = G^\circ$ and the D-bracket on $\mathcal{A}_D$ is given by the familiar formula

$$\{H_1, H_2\}_D = \langle \mathbf{d}H_1, \sharp\mathbf{d}H_2 \rangle = -\langle \mathbf{d}H_2, \sharp\mathbf{d}H_1 \rangle = \{H_1, H_2\},$$

where $\{\cdot, \cdot\}$ is the generalized Poisson bracket defined by $\sharp$.

In particular, if D is given by a generalized Poisson structure on M then $P_1 = T^*M$, $\mathcal{A}_D = C^\infty(M)$, and the D-bracket coincides on $C^\infty(M)$ with the generalized Poisson bracket defined by $\sharp$.

Closed (or integrable) Dirac structures

D is closed (or integrable) if for all $(X_i, \alpha_i) \in \mathfrak{D}_{\mathrm{loc}}, i = 1, 2, 3$,

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0.$$

Examples: symplectic form ($\mathbf{d}\omega = 0$), Poisson bracket (Jacobi identity), differential inclusion (involutivity).

Theorem D is closed $\iff \forall (X_1, \alpha_1), (X_2, \alpha_2) \in \mathfrak{D}_{\mathrm{loc}}$

$[(X_1, \alpha_1), (X_2, \alpha_2)] := ([X_1, X_2], \mathbf{i}_{X_1} \mathbf{d}\alpha_2 - \mathbf{i}_{X_2} \mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1 \rangle) \in \mathfrak{D}_{\mathrm{loc}}$.

Remark Since $0 = \langle\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle\rangle = \mathbf{i}_{X_1} \alpha_2 - \mathbf{i}_{X_2} \alpha_1, \forall (X_1, \alpha_1)$, $(X_2, \alpha_2) \in \mathfrak{D}_{\mathrm{loc}}$, we have $[(X_1, \alpha_1), (X_2, \alpha_2)] = ([X_1, X_2], \pounds_{X_1} \alpha_2 - \pounds_{X_2} \alpha_1 + \mathbf{d}\langle \alpha_2, X_1 \rangle)$. This is NOT a Lie bracket if D is not closed!

Corollary D closed $\iff (D, \pi_{TM}, [\cdot, \cdot])$ is a Lie algebroid. D closed $\Longrightarrow \pi_{TM}(D)$ induces a singular foliation on M.

Proof For $(X_i, \alpha_i) \in \mathfrak{D}_{\mathrm{loc}}, i = 1, 2, 3$ we have

$$
\begin{aligned}
&\Big\langle \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1\rangle,\ X_3\Big\rangle + \langle \alpha_3, [X_1, X_2]\rangle \\
&\quad = \mathbf{d}\alpha_2(X_1, X_3) - \mathbf{d}\alpha_1(X_2, X_3) + X_3[\langle \alpha_2, X_1\rangle] \\
&\qquad\quad + \langle \boldsymbol{\mathcal{L}}_{X_2}\alpha_3, X_1\rangle - \boldsymbol{\mathcal{L}}_{X_2}\langle \alpha_3, X_1\rangle \\
&\quad = X_1[\langle \alpha_2, X_3\rangle] - \langle \alpha_2, [X_1, X_3]\rangle + \mathbf{d}\alpha_1(X_3, X_2) + \langle \boldsymbol{\mathcal{L}}_{X_2}\alpha_3, X_1\rangle \\
&\qquad\quad - X_2[\langle \alpha_3, X_1\rangle] \\
&\quad = \langle \boldsymbol{\mathcal{L}}_{X_1}\alpha_2, X_3\rangle + \langle \boldsymbol{\mathcal{L}}_{X_2}\alpha_3, X_1\rangle + \mathbf{d}\alpha_1(X_3, X_2) + X_2[\langle \alpha_1, X_3\rangle]
\end{aligned}
$$

since $\langle \alpha_3, X_1\rangle + \langle \alpha_1, X_3\rangle = \langle\!\langle (X_1, \alpha_1), (X_3, \alpha_3)\rangle\!\rangle = 0$. Hence get

$$
\begin{aligned}
&\langle \boldsymbol{\mathcal{L}}_{X_1}\alpha_2, X_3\rangle + \langle \boldsymbol{\mathcal{L}}_{X_2}\alpha_3, X_1\rangle + X_3[\langle \alpha_1, X_2\rangle] - \langle \alpha_1, [X_3, X_2]\rangle \\
&\quad = \langle \boldsymbol{\mathcal{L}}_{X_1}\alpha_2, X_3\rangle + \langle \boldsymbol{\mathcal{L}}_{X_2}\alpha_3, X_1\rangle + \langle \boldsymbol{\mathcal{L}}_{X_3}\alpha_1, X_2\rangle.
\end{aligned}
$$

Therefore, D is closed $\Longleftrightarrow$ $\Big\langle \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1\rangle,\ X_3\Big\rangle +$ $\langle \alpha_3, [X_1, X_2]\rangle = \langle\!\langle ([X_1, X_2], \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1\rangle), (X_3, \alpha_3)\rangle\!\rangle =$ $0, \forall (X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow ([X_1, X_2], \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 +$ $\mathbf{d}\langle \alpha_2, X_1\rangle) \in \mathfrak{D}_{\mathrm{loc}}, \forall (X_1, \alpha_1), (X_2, \alpha_2), \in \mathfrak{D}_{\mathrm{loc}}$. $\square$

Lemma If $(X_i, \alpha_i) \in \mathfrak{D}_{\mathsf{loc}}, i = 1, \ldots, n$, satisfy $([X_i, X_j], \mathbf{i}_{X_i}\mathbf{d}\alpha_j - \mathbf{i}_{X_j}\mathbf{d}\alpha_i + \mathbf{d}\langle \alpha_j, X_i\rangle) \in \mathfrak{D}_{\mathsf{loc}}, \forall i, j = 1, \ldots n$, then also $([X, Y], \mathbf{i}_X\mathbf{d}\beta - \mathbf{i}_Y\mathbf{d}\alpha + \mathbf{d}\langle \beta, X\rangle) \in \mathfrak{D}_{\mathsf{loc}}$ where $(X, \alpha) = \sum_{i=1}^n f_i(X_i, \alpha_i)$ and $(Y, \beta) = \sum_{j=1}^n g_i(X_i, \alpha_i)$ for arbitrary $f_i, g_j \in C^\infty(M)$.

Proof $[X, Y] = \sum_{i,j=1}^n \left(f_i X_i[g_j] X_j - g_j X_j[f_i] X_i + f_i g_j [X_i, X_j] \right)$ and $\mathbf{i}_X\mathbf{d}\beta - \mathbf{i}_Y\mathbf{d}\alpha + \mathbf{d}\langle \beta, X\rangle =$

$$\sum_{i,j=1}^n \left(f_i X_i[g_j]\alpha_j - g_j X_j[f_i]\alpha_i + f_i g_j (\mathbf{i}_{X_i}\mathbf{d}\alpha_j - \mathbf{i}_{X_j}\mathbf{d}\alpha_i + \mathbf{d}\langle \alpha_j, X_i\rangle)\right)$$

so that

$$\begin{aligned}
&([X, Y], \mathbf{i}_X\mathbf{d}\beta - \mathbf{i}_Y\mathbf{d}\alpha + \mathbf{d}\langle \beta, X\rangle) \\
&\quad = \sum_{i,j=1}^n f_i X_i[g_j](X_j, \alpha_j) - \sum_{i,j=1}^n g_j X_j[f_i](X_i, \alpha_i) \\
&\qquad + \sum_{i,j=1}^n f_i g_j ([X_i, X_j], \mathbf{i}_{X_i}\mathbf{d}\alpha_j - \mathbf{i}_{X_j}\mathbf{d}\alpha_i + \mathbf{d}\langle \alpha_j, X_i\rangle) \in \mathfrak{D}_{\mathsf{loc}}. \qquad \square
\end{aligned}$$

Corollary D closed Dirac structure on M. Then

(i) G_0 and G_1 are algebraically involutive distributions;

(ii) $H_1, H_2 \in \mathcal{A}_D \Longrightarrow \{H_1, H_2\}_D \in \mathcal{A}_D$;

(iii) $H_1, H_2, H_3 \in \mathcal{A}_D \Longrightarrow \{H_1, \{H_2, H_3\}_D\}_D + \{H_2, \{H_3, H_1\}_D\}_D + \{H_3, \{H_1, H_2\}_D\}_D = 0$.

Proof (i) X_1, X_2 local sections of $G_0 \iff (X_1, 0), (X_2, 0) \in \mathfrak{D}_{\text{loc}}$. By the theorem $\Longrightarrow ([X_1, X_2], 0) \in \mathfrak{D}_{\text{loc}} \iff [X_1, X_2]$ local section of G_0.

X_1, X_2 local sections of $G_1 \iff \exists \alpha_1, \alpha_2 \in \Omega^1_{\text{loc}}(M)$ such that (X_1, α_1), $(X_2, \alpha_2) \in \mathfrak{D}_{\text{loc}} \Longrightarrow ([X_1, X_2], \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1\rangle) \in \mathfrak{D}_{\text{loc}} \iff [X_1, X_2]$ local section of G_1.

(ii) Let $H_1, H_2 \in \mathcal{A}_D \iff \exists X_1, X_2 \in \mathfrak{X}_{\text{loc}}(M), (X_1, \mathbf{d}H_1), (X_2, \mathbf{d}H_2) \in \mathfrak{D}_{\text{loc}} \Longrightarrow ([X_1, X_2], \mathbf{d}\langle \mathbf{d}H_2, X_1\rangle) \in \mathfrak{D}_{\text{loc}} \iff \mathbf{d}\{H_1, H_2\} = \mathbf{d}\langle \mathbf{d}H_2, X_1\rangle \in P_1 \iff \{H_1, H_2\} \in \mathcal{A}_D$.

Let $H_1, H_2, H_3 \in \mathcal{A}_D \Longleftrightarrow \exists X_1, X_2, X_3 \in \mathfrak{X}_{\mathrm{loc}}(M)$ such that $(X_1, \mathbf{d}H_1)$, $(X_2, \mathbf{d}H_2), (X_3, \mathbf{d}H_3) \in \mathfrak{D}_{\mathrm{loc}}$. Then, using $\boldsymbol{\pounds}_Z = \mathbf{i}_Z\mathbf{d} + \mathbf{d}\mathbf{i}_Z$

$$
\begin{aligned}
0 &= \langle \boldsymbol{\pounds}_{X_1}\mathbf{d}H_2, X_3\rangle + \langle \boldsymbol{\pounds}_{X_2}\mathbf{d}H_3, X_1\rangle + \langle \boldsymbol{\pounds}_{X_3}\mathbf{d}H_1, X_2\rangle \\
&= \langle \mathbf{d}\langle \mathbf{d}H_2, X_1\rangle, X_3\rangle + \langle \mathbf{d}\langle \mathbf{d}H_3, X_2\rangle, X_1\rangle + \langle \mathbf{d}\langle \mathbf{d}H_1, X_3\rangle, X_2\rangle \\
&= \langle \mathbf{d}\{H_2, H_1\}_D, X_3\rangle + \langle \mathbf{d}\{H_3, H_2\}_D, X_1\rangle + \langle \mathbf{d}\{H_1, H_3\}_D, X_2\rangle \\
&= \{\{H_2, H_1\}_D, H_3\}_D + \{\{H_3, H_2\}_D, H_1\}_D + \{\{H_1, H_3\}_D, H_2\}_D \\
&= -\{\{H_1, H_2\}_D, H_3\}_D - \{\{H_2, H_3\}_D, H_1\}_D - \{\{H_3, H_1\}_D, H_2\}_D. \qquad \square
\end{aligned}
$$

Proposition D closed Dirac structure on M with P_1 a constant rank codistribution on M. Then D is closed $\Longleftrightarrow$

(i) $G_0 = P_1^\circ$ (algebraically) involutive subbundle of TM;

(ii) $H_1, H_2 \in \mathcal{A}_D \Longrightarrow \{H_1, H_2\}_D \in \mathcal{A}_D$;

(iii) $H_1, H_2, H_3 \in \mathcal{A}_D \Longrightarrow \{H_1, \{H_2, H_3\}_D\}_D + \{H_2, \{H_3, H_1\}_D\}_D + \{H_3, \{H_1, H_2\}_D\}_D = 0$.

How is integrability of the Dirac structure expressed in the three representations?

Representation I Locally, around every point $x \in M$ there exist $n \times n$ matrices $E(x), F(x)$ depending smoothly on x satisfying

$$\mathrm{rank}[F(x) \mid E(x)] = n \quad \text{and} \quad E(x)F(x)^T + F(x)E(x)^T = 0$$

such that D can be locally expressed as

$$D(x) = \{(v, v^*) \in T_xM \oplus T_x^*M \mid F(x)v + E(x)v^* = 0\}.$$

Define $X_i = -E_i^T$, $\alpha_i = -F_i^T$, where E_i^T and F_i^T are the ith columns of E^T and F^T, respectively. Then D is closed $\Longleftrightarrow \forall i, j = 1, \ldots, n$

$$([X_i, X_j], \mathbf{i}_{X_i}\mathbf{d}\alpha_j - \mathbf{i}_{X_j}\mathbf{d}\alpha_i + \mathbf{d}\langle \alpha_j, X_i \rangle) \in \mathfrak{D}_{\mathrm{loc}}$$

Proof $\ker[F(x) \mid E(x)] = \mathrm{im} \begin{bmatrix} -E(x)^T \\ -F(x)^T \end{bmatrix}$. Since $\mathrm{rank}[F(x) \mid E(x)] = n \Longrightarrow (X_i, \alpha_i)$ locally span $\mathfrak{D}_{\mathrm{loc}}$. Lemma and Theorem prove the statement. $\square$

Representation II $P \subset T^*M$ vector subbundle and $\flat : TM \to T^*M$ a skew-symmetric vector bundle map (in every fiber). Then

$$D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v^* - \flat(x)v \in P(x), v \in P^\circ, x \in M\}$$

is a Dirac structure on M. Define $\omega \in \Omega^2(M)$ by $\omega(X, Y) = \langle X^\flat, Y \rangle$.

Then D is closed $\iff$ P° is involutive and $\mathbf{d}\omega(X_1, X_2, X_3) = 0, \forall X_1, X_2, X_3 \in \Gamma_{\text{loc}}(P^\circ)$.

Proof $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathfrak{D}_{\text{loc}} \Longrightarrow X_1, X_2$ are local sections of P° and $\exists \gamma_1, \gamma_2$ local sections of P such that $\alpha_i = \mathbf{i}_{X_i}\omega + \gamma_i, i = 1, 2$.

$$\begin{aligned}
&\mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1 \rangle \\
&\quad = \mathbf{i}_{X_1}\mathbf{d}(\mathbf{i}_{X_2}\omega + \gamma_2) - \mathbf{i}_{X_2}\mathbf{d}(\mathbf{i}_{X_1}\omega + \gamma_1) + \mathbf{d}\mathbf{i}_{X_1}(\mathbf{i}_{X_2}\omega + \gamma_2) \\
&\quad = \mathbf{i}_{X_1}\mathbf{d}\mathbf{i}_{X_2}\omega + \mathbf{i}_{X_1}\mathbf{d}\gamma_2 - \mathbf{i}_{X_2}\mathbf{d}\mathbf{i}_{X_1}\omega - \mathbf{i}_{X_2}\mathbf{d}\gamma_1 + \mathbf{d}\mathbf{i}_{X_1}\mathbf{i}_{X_2}\omega + \mathbf{d}\mathbf{i}_{X_1}\gamma_2
\end{aligned}$$

But $\mathbf{d}\mathbf{i}_{X_1}\omega = \boldsymbol{\pounds}_{X_1}\omega - \mathbf{i}_{X_1}\mathbf{d}\omega$ and $\mathbf{d}\mathbf{i}_{X_1}\mathbf{i}_{X_2}\omega = \boldsymbol{\pounds}_{X_1}\mathbf{i}_{X_2}\omega - \mathbf{i}_{X_1}\mathbf{d}\mathbf{i}_{X_2}\omega = \boldsymbol{\pounds}_{X_1}\mathbf{i}_{X_2}\omega - \mathbf{i}_{X_1}\boldsymbol{\pounds}_{X_2}\omega + \mathbf{i}_{X_1}\mathbf{i}_{X_2}\mathbf{d}\omega$. Also $\mathbf{i}_{X_1}\gamma_2 = 0$. Hence

$$\begin{aligned}
&\mathbf{i}_{X_1}\mathbf{d}\mathbf{i}_{X_2}\omega - \mathbf{i}_{X_2}\mathbf{d}\mathbf{i}_{X_1}\omega + \mathbf{d}\mathbf{i}_{X_1}\mathbf{i}_{X_2}\omega \\
&= \mathbf{i}_{X_1}\mathbf{d}\mathbf{i}_{X_2}\omega - \mathbf{i}_{X_2}\boldsymbol{\pounds}_{X_1}\omega + \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \boldsymbol{\pounds}_{X_1}\mathbf{i}_{X_2}\omega - \mathbf{i}_{X_1}\boldsymbol{\pounds}_{X_2}\omega + \mathbf{i}_{X_1}\mathbf{i}_{X_2}\mathbf{d}\omega \\
&= -\mathbf{i}_{X_2}\boldsymbol{\pounds}_{X_1}\omega + \boldsymbol{\pounds}_{X_1}\mathbf{i}_{X_2}\omega + \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \mathbf{i}_{X_1}\left(\mathbf{d}\mathbf{i}_{X_2} + \mathbf{i}_{X_2}\mathbf{d} - \boldsymbol{\pounds}_{X_2}\right)\omega \Longrightarrow
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle\alpha_2, X_1\rangle \\
&= -\mathbf{i}_{X_2}\boldsymbol{\pounds}_{X_1}\omega + \boldsymbol{\pounds}_{X_1}\mathbf{i}_{X_2}\omega + \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \mathbf{i}_{X_1}\mathbf{d}\gamma_2 - \mathbf{i}_{X_2}\mathbf{d}\gamma_1 \\
&= \mathbf{i}_{[X_1,X_2]}\omega + \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \mathbf{i}_{X_1}\mathbf{d}\gamma_2 - \mathbf{i}_{X_2}\mathbf{d}\gamma_1
\end{aligned}
$$

since $\mathbf{i}_{[X_1,X_2]} = \boldsymbol{\pounds}_{X_1} \circ \mathbf{i}_{X_2} - \mathbf{i}_{X_2} \circ \boldsymbol{\pounds}_{X_1}$. So D is closed $\Longleftrightarrow$

$$
\begin{gathered}
([X_1, X_2], \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle\alpha_2, X_1\rangle) \in \mathfrak{D}_{\mathrm{loc}}, \\
\forall (X_1, \alpha_1), (X_2, \alpha_2) \in \mathfrak{D}_{\mathrm{loc}} \Longleftrightarrow \\
([X_1, X_2], \mathbf{i}_{[X_1,X_2]}\omega + \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \mathbf{i}_{X_1}\mathbf{d}\gamma_2 - \mathbf{i}_{X_2}\mathbf{d}\gamma_1) \in \mathfrak{D}_{\mathrm{loc}}, \\
\forall X_1, X_2 \in \Gamma_{\mathrm{loc}}(P^\circ), \gamma_1, \gamma_2 \in \Gamma_{\mathrm{loc}}(P) \Longleftrightarrow \\
[X_1, X_2] \in \Gamma_{\mathrm{loc}}(P^\circ),\ \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega + \mathbf{i}_{X_1}\mathbf{d}\gamma_2 - \mathbf{i}_{X_2}\mathbf{d}\gamma_1 \in \Gamma_{\mathrm{loc}}(P) \\
\forall X_1, X_2 \in \Gamma_{\mathrm{loc}}(P^\circ), \gamma_1, \gamma_2 \in \Gamma_{\mathrm{loc}}(P)
\end{gathered}
$$

D closed $\Longrightarrow P^\circ$ is involutive. Since P° involutive *subbundle*, if $\gamma \in \Gamma_{\mathrm{loc}}(P) \overset{\text{Frobenius}}{\Longleftrightarrow} \exists \bar{\gamma}_i \in \Gamma_{\mathrm{loc}}(P)$ such that $\mathbf{d}\gamma = \zeta^i \wedge \bar{\gamma}_i$ for some locally defined one-forms $\zeta^i \Longrightarrow \mathbf{i}_X\mathbf{d}\gamma = \zeta^i(X)\bar{\gamma} \in \Gamma_{\mathrm{loc}}(P), \forall X \in \Gamma_{\mathrm{loc}}(P^\circ)$. So $\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega \in \Gamma_{\mathrm{loc}}(P), X_1, X_2 \in \Gamma_{\mathrm{loc}}(P^\circ) \Longleftrightarrow \mathbf{d}\omega(X_1, X_2, X_3) = (\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\omega)(X_3) = 0, \forall X_1, X_2, X_3 \in \Gamma_{\mathrm{loc}}(P^\circ)$. $\square$

Representation II $G \subset TM$ vector subbundle and $\sharp : T^*M \to TM$ a skew-symmetric vector bundle map (in every fiber). Then

$$D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v - \sharp(x)v^* \in G(x), v \in G^\circ, x \in M\}$$

is a Dirac structure on M. Define for $H_1, H_2 \in \mathcal{A}_D$ the *generalized Poisson bracket* $\{H_1, H_2\} := \langle \mathbf{d}H_1, \sharp\mathbf{d}H_2\rangle$. Then D is closed $\iff G$ is involutive and $\forall H_1, H_2, H_3 \in \mathcal{A}_D$ we have $\{H_1, H_2\} \in \mathcal{A}_D$, and $\{\{H_1, H_2\}, H_3\} + \{\{H_2, H_3\}, H_1\} + \{\{H_3, H_1\}, H_2\} = 0$.

Proof $G_0 = G$ and $\{\cdot, \cdot\}_D = \{\cdot, \cdot\}$. Apply last Proposition. $\square$

Definition D Dirac on M. A point $x \in M$ is *regular* if the dimension of G_1 and P_1 (and hence also of G_0 and P_0) is constant in a neighborhood of x.

Normal Form Theorem D Dirac structure on n-dimensional manifold M. Assume $P_1 \subset T^*M$ subbundle, so $D = \{(v, v^*) \in T_xM \oplus T_x^*M \mid v - \sharp(x)v^* \in P_1^\circ(x), v^* \in P_1(x), x \in M\}$. If D is closed, then $\forall x \in M$ regular point, $\exists U \subset M$ open, $x \in U$, and local canonical coordinates $(q^1, \ldots, q^k, p_1, \ldots, p_k, r_1, \ldots, r_l, s_1, \ldots, s_m)$ on U, $2k + l + m = n$, such that in these coordinates

$$\sharp|_U = \begin{bmatrix} 0 & I_k & 0 & * \\ -I_k & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix} \text{ and } \begin{cases} G_0 = P_1^\circ = \operatorname{span}_{C^\infty(U)} \left\{ \frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_m} \right\} \\ m = n - \dim P_1(x), l = n - \dim G_1(x) \end{cases}$$

Conversely, if D is given as above for some subbundle $P_1 \subset T^*M$ and the expressions above hold in a neighborhood U of a given point $x \in M$, then the Dirac structure D is closed in U.

Structure Theorem A closed Dirac structure decomposes as the disjoint union of leaves of a generalized foliation, all of whose leaves are presymplectic, and the closed Dirac structure on each leaf defined by this presymplectic form coincides with the restriction of the given Dirac structure to this leaf.

Proof As in the linear case define $\flat(x) : \pi_{TM}(D) \to [\pi_{TM}(D)]^*$ by $v^\flat := \alpha|_{\pi_{T_xM}(D(x))}$, for $\alpha \in T_x^*M$ such that $(v, \alpha) \in D$. We have $\ker \flat(x) = [\pi_{T_x^*M}(D(x))]^\circ \subset T_xM$. Then define $\Omega(x)(u, v) := \langle u^\flat, v\rangle$, where $x \in M$ and $u, v \in \pi_{T_xM}(D(x)) \iff \Omega(x)(u, v) = \langle \alpha, v\rangle, \forall (u, \alpha), (v, \beta) \in D(x)$. Since $0 = \langle\langle (u, \alpha), (v, \beta)\rangle\rangle = \langle \alpha, v\rangle + \langle \beta, u\rangle \implies \Omega(x)(u, v) = \langle\langle (u, \alpha), (v, \beta)\rangle\rangle_- := \frac{1}{2}(\langle \alpha, u\rangle - \langle \beta, v\rangle)$. This shows that Ω is the pull-back by the inclusion $D \hookrightarrow TM \oplus T^*M$ of the smooth skew-symmetric bilinear form $\langle\langle \cdot, \cdot\rangle\rangle_-$ to D. So Ω is a skew-symmetric two-tensor on the vector bundle D.

$v \in T_xM$ is tangent to the leaf $\implies \exists \alpha \in T_x^*M$ s. t. $(v, \alpha) \in D(x)$.

$(X_i, \alpha_i) \in \mathfrak{D}_{\text{loc}}$. Then $X_1[\Omega(X_2, X_3)] = X_1[\langle \alpha_2, X_3\rangle] = \langle \boldsymbol{\pounds}_{X_1}\alpha_2, X_3\rangle + \langle \alpha_2, [X_1, X_3]\rangle$ and $\Omega(X_2, [X_3, X_1]) = \langle \alpha_2, [X_3, X_1]\rangle$. Therefore

$$\begin{aligned}\mathbf{d}\Omega(X_1, X_2, X_3) &= X_1[\Omega(X_2, X_3)] + X_2[\Omega(X_3, X_1)] + X_3[\Omega(X_1, X_2)]\\ &\quad + \Omega(X_1, [X_2, X_3]) + \Omega(X_2, [X_3, X_1]) + \Omega(X_3, [X_1, X_2])\\ &= \langle \boldsymbol{\pounds}_{X_1}\alpha_2, X_3\rangle + \langle \boldsymbol{\pounds}_{X_2}\alpha_3, X_1\rangle + \langle \boldsymbol{\pounds}_{X_3}\alpha_1, X_2\rangle = 0\end{aligned}$$

since D is closed. So each leaf is presmplectic. $\square$

Implicit Hamiltonian systems

Consider a Dirac structure D, and smooth function H on M. The *implicit Hamiltonian system* (M, D, H) is defined as the set of C^∞ solutions $x(t)$ satisfying the condition

$$(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)), \forall t.$$

- Energy conservation: $\dot{H}(t) = \langle \mathbf{d}H(x(t)), \dot{x}(t) \rangle = 0, \forall t.$

- Algebraic constraints: $\mathbf{d}H(x(t)) \in P_1(x(t)), \forall t.$

- $\dot{x}(t) \in G_1$, set of admissible flows. If G_1 is an involutive subbundle of $TM, \exists n - \dim G_1$ independent conserved quantities.

Hence, in general, an implicit Hamiltonian system defines a set of *differential and algebraic equations*.

Standard existence and uniqueness theorems do not apply!

Examples

1. D given by (M,ω), ω non-degenerate

$D(x) := \{(u,u^*) \in T_xM \oplus T_x^*M \mid u^* = \omega(x)(u,\cdot)\}$

$$\begin{aligned}(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) &\Longleftrightarrow \mathbf{d}H(x(t)) = \omega(x(t))(\dot{x}(t),\cdot)\\ &\Longleftrightarrow \dot{x}(t) = X_H(x(t))\end{aligned}$$

So get standard Hamilton equations. $\pi_{TM}(D) = TM$. So D is closed $\Longleftrightarrow \mathbf{d}\omega = 0 \Longleftrightarrow \omega$ is symplectic. Then Normal Form Theorem = Darboux Theorem, so $\exists(q^1,\ldots,q^n,p_1,\ldots,p_n), \omega = \mathbf{d}q^i \wedge \mathbf{d}p_i$ and Hamilton's equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \forall i = 1,\ldots,k$$

Here H is a function of (q^i, p_i).

2. D given by generalized Poisson bracket $\sharp : T^*M \to TM$

$D(x) := \{(u, u^*) \in T_xM \oplus T_x^*M \mid u = (u^*)^\sharp\}$

$$(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow \dot{x}(t) = (\mathbf{d}H(x(t)))^\sharp = X_H(x(t))$$

which are Hamilton's equations on a generalized Poisson manifold. These are equivalent to $\dot{F} = \{F, H\}, \forall F$. Here $\{F, H\} := \langle \mathbf{d}F, (\mathbf{d}H)^\sharp \rangle$. D is closed $\Longleftrightarrow \{\cdot, \cdot\}$ satisfies Jacobi identity. In this case, the Normal Form Theorem states $\exists (q^1, \ldots, q^n, p_1, \ldots, p_n, r_1, \ldots, r_l)$ around each regular point such that Hamilton's equations take the form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \forall i = 1, \ldots, k \qquad \dot{r}_j = 0, \quad \forall j = 1, \ldots, l$$

Here H is a function of (q^i, p_i, r_j).

Problem: These equations are only around a regular point! We know that in the case of Poisson manifolds, in general, the r_j's define the transverse Poisson structure which is very important in unuderstanding the structure of Poisson manifolds. Such a theorem is still missing for Dirac manifolds!

3. D given by a subbundle $B \subset TM$

$D := B \oplus B^\circ = \{(u, u^*) \in T_xM \oplus T_x^*M \mid u \in B(x), u^* \in B^\circ(x)\}$

$(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow \dot{x}(t) \in B(x(t))$ and $\mathbf{d}H(x(t)) \in B^\circ(x(t))$

One usually writes: $\dot{x} = X_H(x)$ with the conditions $X_H(x) \in B(x)$, $\mathbf{d}H(x) \in B^\circ(x)$. The system is given by *differential inclusions*. In this case $G_0 = G_1 = B = P_0^\circ = P_1^\circ$.

If D is closed $\overset{Theorem}{\Longrightarrow}$ B integrable. Also, around any regular point $\exists(q^i, p_i, r_j, s_b) =: (x^a, s_b)$ such that $B = G_0 = \operatorname{span}\{\partial/\partial s_1, \ldots, \partial/\partial s_m\}$. Therefore, $\mathbf{d}H(x^a, s_b) \in B^\circ \Longrightarrow \partial H/\partial s_b = 0$. Also $(\dot{x}^a, \dot{s}_b) \in B \Longrightarrow \dot{x}^a = 0$. So the equations are:

$$\dot{x}^a = 0, \qquad \frac{\partial H}{\partial s_b}(x^a, s_b) = 0.$$

Electrical network consisting of only two capacitors Silly example with no dynamics. Constitutive equations of a capacitor C

$$\dot{q} = i_C, \qquad v_C = \frac{\partial H_C}{\partial q}, \qquad H_C(q) = \frac{1}{2C}q^2, \qquad C > 0,$$

wherer q is the charge, i_C is the current, v_C is the tension (voltage), H is the electrical energy of the capacitor.

Interconnection between two capacitors is given by Kirchhoff's Laws:

$$\dot{q}_1 - \dot{q}_2 = 0, \qquad \frac{\partial H_{C_1}}{\partial q_1} + \frac{\partial H_{C_2}}{\partial q_2} = 0.$$

$H(q_1, q_2) := H_{C_1}(q_1) + H_{C_2}(q_2)$ is the total energy of the system. Change variables: $x := q_1 - q_2, s := q_1 + q_2 \Longrightarrow \dot{x} = 0, \partial H/\partial s = 0$. Since the Hessian of H relative to s is positive definite, by IFT $\Longrightarrow s$ is a function of $x \Longrightarrow \dot{s} = 0 \Longrightarrow$ no dynamics. Of course, if there are only capacitors, there is no energy exchange between the elements, so no dynamics.

4. D **given locally by** E, F : $D(x) = \{(v, v^*) \in T_xM \oplus T_x^*M \mid F(x)v + E(x)v^* = 0\}$, where rank$[F(x) \mid E(x)] = n$ and $E(x)F(x)^T + F(x)E(x)^T = 0$. So

$$(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow F(x)\dot{x} + E(x)\mathbf{d}H(x) = 0.$$

5. D **given by** $G_0 \subset TM$ **and** $\sharp : T^*M \to TM$ $D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v - \sharp(x)v^* \in G_0(x), v^* \in G_0^\circ(x), x \in M\}$.

$$\begin{aligned}&(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow \dot{x} - X_H(x) \in G_0(x), \quad \mathbf{d}H(x) \in G_0^\circ(x)\\ &\Longleftrightarrow \dot{x} = X_H(x) + g(x)\lambda, \quad g^T(x)\mathbf{d}H(x) = 0,\end{aligned}$$

where $X_H := (\mathbf{d}H)^\sharp$ and $g(x)$ is a full rank matrix such that im $g(x) = G_0(x)$. λ is a Lagrange multiplier needed to insure that the constraint equations $g^T(x)\mathbf{d}H(x) = 0$ hold for all time.

6. D **given by** $P_0 \subset T^*M$ **and** $\flat : TM \to T^*M$ $D := \{(v, v^*) \in T_xM \oplus T_x^*M \mid v^* - \flat(x)v \in P_0(x), v \in P_0^\circ(x), x \in M\}$

$$(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow \mathbf{d}H(x) - (\dot{x})^\flat \in P_0(x), \quad \dot{x} \in P_0^\circ(x)$$
$$\Longleftrightarrow \mathbf{d}H(x) = (\dot{x})^\flat + p^T(x)\lambda, \quad p(x)\dot{x} = 0,$$

where $p(x)$ is a full rank matrix such that $\operatorname{im} p(x) = P_0(x)$ and λ is a Lagrange multiplier needed to insure that $p(x)\dot{x} = 0$ for all time.

Note the difference between this and the last representation: Here the flow constraints $p(x)\dot{x} = 0$ are explicit whereas in the previous situation it was the algebraic constraints $g^T(x)\mathbf{d}H(x) = 0$ that were explicit.

7. D **in normal form around a regular point** Here one uses the same expression as in Example 5, except that the map $\sharp$ and G_0 are known explicitly.

So, $(\dot{x}, \mathbf{d}H(x(t))) \in D(x(t)) \Longleftrightarrow$

$$\begin{bmatrix} \dot{q}^i \\ \dot{p}_j \\ \dot{r}_a \\ \dot{s}_b \end{bmatrix} - \begin{bmatrix} 0 & I_k & 0 & * \\ -I_k & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} H_{q^i} \\ H_{p_j} \\ H_{r_a} \\ H_{s_b} \end{bmatrix} \in \text{span}\left\{ \frac{\partial}{\partial s_1}, \dots \frac{\partial}{\partial s_m} \right\} \text{ and}$$

$$\left\langle \mathbf{d}H(q^i, p_j, r_a, s_b), \frac{\partial}{\partial s_b} \right\rangle = 0, \ \forall b = 1, \dots, m.$$

The second relation says that $\frac{\partial H}{\partial s_c}(q^i, p_j, r_a, s_b) = 0, \forall c = 1, \dots, m$, whereas the first one gives information only about the dynamics in the variables (q^i, p_j, r_a). There is no information about the dynamics in the variables s_b. So the equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{r}_a = 0, \quad 0 = \frac{\partial H}{\partial s_c}(q^i, p_j, r_a, s_b),$$

$\forall i = 1, \dots, k, a = 1, \dots, l$

8. Implicit generalized Hamiltonian systems of index one

Assume $P_1 \subset T^*M$ is a vector subbundle, $G_0(x) = \operatorname{im} g(x) = \operatorname{span}\{g_1(x), \ldots, g_m(x)\}$, with $g_1, \ldots, g_m \in \mathfrak{X}(M)$ linearly independent vector fields over $C^\infty(M)$, and that the matrix $[\pounds_{g_i}\pounds_{g_j}H(x)]_{i,j=1,\ldots,m}$ is non-singular $\forall x \in M_c := \{y \in M \mid \mathbf{d}H(y) \in P_1\} = \{y \in M \mid \pounds_{g_i}H(y) = 0, j = 1, \ldots, m\}$.

Then the implicit Hamiltonian system $(\dot{x}, \mathbf{d}H(x)) \in D(x)$ reduces to the explicit system $\dot{x}_c = X_{H_c}(x_c)$ on the constraint manifold M_c, where $x_c(t) \in M_c, \forall t$, H_c is the restriction of H to M_c, and $\sharp_c : T^*M_c \to TM_c$.

In coordinates around a regular point this is easy to see. The assumption is equivalent to the Hessian of H relative to the s_b variables to be non-degenerate. So by IFT, s_b is a function of the other variables $\Longrightarrow H_c(q^i, p_j, r_a) := H(q^i, p_j, r_a, s_b(q^i, p_j, r_a))$ and the previous equations become

$$\dot{q}^i = \frac{\partial H_c}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_c}{\partial q^i}, \quad \dot{r}_a = 0.$$

Constrained mechanical systems

• $M := T^*Q$, $\pi : T^*Q \to Q$, $\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i$, $\flat : TM \to T^*M$, $\{\cdot,\cdot\}$, $\sharp : T^*M \to TM$;

• Assume that the kinematic constraints are linear in the velocities and that they are independent everywhere, i.e. $\exists \alpha^1, \ldots, \alpha^k \in \Omega^1(M)$ independent such that $\langle \alpha^i(q), \dot{q} \rangle = \alpha^i_j(q)\dot{q}^j = 0, \forall i = 1, \ldots, k$.

• $\Delta := [\mathrm{span}\{\alpha^1, \ldots, \alpha^k\}]^\circ \subset TQ$ is called the *constraint distribution*. It describes the allowed infinitesimal motions of the system.

• The constraints are called *holonomic* if they can be integrated to a set of configuration constraints $\{f^1(q) = 0, \ldots, f^k(q) = 0\}$. If this is not possible, then the constraints are called *nonholonomic*.

• In general, constraints given by a smooth distribution $\Delta \subset TQ$ are called *holonomic* if Δ is integrable, in which case its integral manifolds (in Q) completetely describe the constraints, i.e. the constraints on the velocities can be obtained by taking the time derivative of the point constraints. If Δ is not integrable, the constraints are called *nonholonomic*.

• Lift α^i to $\pi_Q^*\alpha^i \in \Omega^1(T^*Q)$, $P_0 = G_1^\circ := \mathrm{span}\{\pi_Q^*\alpha^1, \ldots, \pi_Q^*\alpha^k\} \subset T^*(T^*Q)$, subbundle. Define

$$D(q,p) = \{(v, v^*) \in T_{(q,p)}M \times T^*_{(q,p)}M \mid v^* - v^\flat = (\pi_Q^*\alpha^i)\lambda_i \text{ and } \langle \pi_Q^*\alpha^i, v\rangle = 0\}$$

• The vector fields $X_i = -(\pi_Q^*\alpha^i)^\sharp \in \mathfrak{X}(T^*Q)$, $i = 1, \ldots, k$, satisfy $0 - X_i^\flat = \pi_Q^*\alpha^i \in P_0 \Longleftrightarrow X_i \in \Gamma(G_0)$. So $G_0 = \mathrm{span}\{X_1, \ldots, X_k\}$. Since the vector fields are independent, the distribution has constant rank k and defines hence a vector subbundle of $T(T^*Q)$.

• Recall D closed $\iff P_1^\circ$ is involutive and $\mathbf{d}\omega$ vanishes on local sections of P_1°. Second condition holds, since in this case $\mathbf{d}\omega = 0$. But P_1° involutive $\iff$ integrable $\overset{Frobenius}{\iff} 0 = \mathbf{d}(\pi_Q^*\alpha^i) \wedge (\pi_Q^*\alpha^1) \wedge \cdots \wedge (\pi_Q^*\alpha^k) = \pi_Q^*\left(\mathbf{d}\alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^k\right), \forall i = 1, \ldots, k \iff \mathbf{d}\alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^k = 0, \forall i = 1, \ldots, k \overset{Frobenius}{\iff}$ vector subbundle of TQ who is the annihilator of span$\{\alpha^1, \ldots, \alpha^k\}$ is integrable. The leaves of the induced foliation on Q are *holonomic* constraints.

Conclusion: D closed $\iff P_1^\circ$ is involutive $\iff$ constraint distribution Δ is involutive $\iff$ constraints defined by $\{\alpha_1, \ldots, \alpha_k\}$ are holonomic.

Moral: In nonholonomic mechanics one needs to have the Dirac structure non-integrable!

• Hamiltonian $H(q^i, p_i)$: setting $v = (\dot{q}^i, \dot{p}_i)$ and $v^* = \mathbf{d}H(q^i, p_i)$ leads to equations of motion (in implicit form).

• Define the matrix A^T, whose ith row are the components of α^i. Kinematic constraints can be equivalently written (classical)

$$A^T(q)\dot{q} = 0. \tag{33}$$

The matrix $A^T(q)$ is a $k \times n$ matrix, $n = \dim Q$, with full row rank k at every point $q \in Q$. The constraint distribution is $\Delta = \ker A^T(q)$

$$G_0(q,p) = \mathrm{im} \begin{bmatrix} 0_{n\times k} \\ A(q) \end{bmatrix}, \quad (q,p) \in T^*Q. \tag{34}$$

• Equations of motion: $(\dot{x}, \mathbf{d}H(x)) \in D \Longleftrightarrow \mathbf{d}H(x) - (\dot{x})^\flat = (\pi_Q^* \alpha^i)\lambda_i$, $\langle \pi_Q^* \alpha^i, \dot{x} \rangle = 0 \Longleftrightarrow \dot{x} = X_H(x) + \sum_{i=1}^k X_i \lambda_i, \langle \pi_Q^* \alpha^i, \dot{x} \rangle = 0 \Longleftrightarrow$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix} + \begin{pmatrix} 0_{n\times k} \\ A(q) \end{pmatrix} \lambda, \tag{35a}$$

$$0 = \begin{pmatrix} 0_{k\times n} & A^T(q) \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix}. \tag{35b}$$

• These equations can be obtained from classical mechanics, using the Lagrange-d'Alembert's principle: the constraints (**??**) generate constraint forces of the form $F_c = A(q)\lambda$, where $\lambda \in \mathbb{R}^k$ are the Lagrange multipliers. The general equations of motion are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda_j \alpha_i^j, \quad \alpha_i^j(q)\dot{q}^i = 0, \quad i = 1, \ldots, n,\ j = 1, \ldots, k.$$

Legendre transform this to (q^i, p_i) variables and get the implicit Hamiltonian system (**??**), (**??**).

• If the kinetic energy is defined by a positive definite metric on Q, then the constraints are of index 1. In that case, the Lagrange multipliers λ can be solved uniquely. Hence the constrained mechanical system on T^*Q can be written as an *unconstrained* generalized Hamiltonian system on M_c. Van der Schaft and Maschke (1994) have shown that the corresponding generalized Poisson bracket on M_c satisfies the Jacobi identity if and only if the kinematic constraints are *holonomic*.

Symmetries of Dirac structures

A vector field Y on M is an (infinitesimal) symmetry of a Dirac structure D on M if $(L_Y X, L_Y \alpha) \in \mathfrak{D}_{loc}$ for all $(X, \alpha) \in \mathfrak{D}_{loc}$.

Analogously, a diffeomorphism $\phi : M \to M$ is called a symmetry of D if $(\phi_* X, (\phi^*)^{-1}\alpha) \in \mathfrak{D}_{loc}$ for all $(X, \alpha) \in \mathfrak{D}_{loc}$.

Examples

1. Let $\omega \in \Omega^2(M)$ a nondegenerate two-form on M and
$$D(x) = \{(v, v^*) \in T_x M \times T_x^* M \mid v^* = \omega(x)v\}, \quad x \in M$$
Then Y is a symmetry of D if and only if $L_Y \omega = 0$.

Examples (cont'd)

2. Let $J(x) : T_x^*M \to T_xM$, $x \in M$, be a skew-symmetric vector bundle map and

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v = J(x)v^*\}, \quad x \in M$$

Then Then Y is a symmetry of D if and only if $L_Y \circ J = J \circ L_Y$ (with $J : \Omega^1_{loc} \to \mathfrak{X}_{loc}(M)$), or

$$L_Y\{H_1, H_2\} = \{L_Y H_1, H_2\} + \{H_1, L_Y H_2\}, \quad \forall H_1, H_2 \in C^\infty(M)$$

3. Let $\Delta \subset TM$ be a smooth constant dimensional distribution on M and

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v \in \Delta(x),\ v^* \in \Delta(x)^\circ\}, \quad x \in M$$

Then Y is a symmetry of D if and only if $[Y, \Delta] \subset \Delta$.

Examples (cont'd)

5. Let $J(x) : T_x^*M \to T_xM$, $x \in M$, be a skew-symmetric vector bundle map, $\Delta \subset TM$ be a smooth constant dimensional distribution on M and

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v - J(x)v^* \in \Delta(x),\ v^* \in \Delta(x)^\circ\}$$

Then Y is a symmetry of D if $L_Y \circ J = J \circ L_Y$ and $[Y, \Delta] \subset \Delta$.

E.g. mechanical systems with kinematic constraints

Proposition If the vector field Y is symmetry of a generalized Dirac structure D, then

- Y is canonical with respect to $\{\cdot, \cdot\}_D$, i.e.

$$L_Y\{H_1, H_2\}_D = \{L_Y H_1, H_2\}_D + \{H_1, L_Y H_2\}_D, \quad \forall H_1, H_2 \in \mathcal{A}_D,$$

- $L_Y \mathsf{G}_i \subset \mathsf{G}_i$, $L_Y \mathsf{P}_i \subset \mathsf{P}_i$, $i = 0, 1$.

If P_1 is constant dimensional and involutive then the converse is also true.

Some basic properties

- Let $\{(X_1, \alpha_1), \ldots, (X_n, \alpha_n)\}$ be a basis of D. Then Y is a symmetry of D if and only if

$$(L_Y X_i, L_Y \alpha_i) \in \mathfrak{D}_{loc}, \quad i = 1, \ldots, n$$

- If Y_1 and Y_2 are symmetries of D, then also $[Y_1, Y_2]$ is a symmetry of D. I.e. the set of symmetries of a Dirac structure D forms a Lie algebra.

Symmetries for closed Dirac structures

Consider a closed Dirac structure D.

- Let $(Y, \alpha) \in \mathfrak{D}$. Then Y is a symmetry of D if and only if $\mathbf{d}\alpha|_{\mathsf{G}_1} = 0$.

- Let $(Y, \mathbf{d}P) \in \mathfrak{D}$. Then Y is a symmetry of D.

- In particular, every $X \in \mathsf{G}_0$ is a symmetry of D.

In general, let $(Y, \mathbf{d}P) \in \mathfrak{D}$ and assume Y is a symmetry of D. Then Y is called a Hamiltonian symmetry of D.

Symmetries of implicit Hamiltonian systems

A vector field Y on M is a weak symmetry of the implicit Hamiltonian system (M, D, H) if

1. Y is an infinitesimal symmetry of D

2. $L_Y H(x(t)) = 0$ for all solutions $x(t)$ of (M, D, H)

Note A sufficient condition for 2. is that

$$L_Y H(x) = 0, \quad \forall x \in M_c$$

where

$$M_c = \{x \in M \mid \mathbf{d}H(x) \in \mathsf{P}_1(x)\}$$

First integrals or conserved quantities

A (non-constant) function $P \in C^\infty(M)$ is a first integral of (M, D, H) if $\dot{P} = 0$ along all solutions $x(t)$ of (M, D, H). I.e.

$$\frac{dP}{dt}(x(t)) = \langle \mathbf{d}P(x(t)), \dot{x}(t)) \rangle = 0$$

A sufficient condition for P to be a first integral is that

$$\langle \mathbf{d}P(x), X_H(x) + \mathsf{G}_0(x) \rangle = 0, \quad \forall x \in M_c,$$

where $X_H(x)$ is such that $(X_H(x), \mathbf{d}H(x)) \in D(x)$, for every $x \in M_c$.

Consider (M, D, H) and let D be closed. Then the set of first integrals in $\mathcal{A}_D$ is a Lie algebra under the Poisson bracket $\{\cdot, \cdot\}_D$

$$P_1, P_2 \in \mathcal{A}_D \text{ first integrals } \Rightarrow \{P_1, P_2\}_D \in \mathcal{A}_D \text{ first integral}$$

A function $C \in C^{\infty}(M)$ is a Casimir function of D if $\mathbf{d}C \in \mathsf{P}_0$.

A Casimir function C is a first integral of (M, D, H) for every $H \in C^{\infty}(M)$.

Example Recall that G_1 describes the set of admissible flows, i.e.

$$\dot{x}(t) \in \mathsf{G}_1(x(t)).$$

Assume that G_1 is constant dimensional and involutive. Then there exist local coordinates $(x_1, \ldots, x_n)$ such that

$$\mathsf{G}_1 = \mathrm{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right\}.$$

This implies $\mathsf{P}_0 = \mathrm{span}\ \{\mathbf{d}x_{m+1}, \ldots, \mathbf{d}x_n\}$. Hence $x_{m+1}, \ldots, x_n$ are Casimir functions.

Symmetries vs. first integrals

Proposition 1 Let Y be a weak Hamiltonian symmetry of (M, D, H), i.e. $(Y, \mathbf{d}P) \in \mathfrak{D}$. Then P is a first integral.

Proof: $D = D^{\perp}$ implies

$$0 = \langle \mathbf{d}H(x(t)), Y(x(t)) \rangle + \langle dP(x(t)), \dot{x}(t)) \rangle = L_Y H(x(t)) + \dot{P}(x(t))$$

Proposition 2 (Noether's Theorem) Consider (M, D, H) and assume D be closed. Let $(Y, \mathbf{d}P) \in \mathfrak{D}$. Then Y is a weak Hamiltonian symmetry if and only if P is a first integral.

$\tilde{P}$ is a second function such that $(Y, \mathbf{d}\tilde{P}) \in \mathfrak{D}$ if and only if $C = P - \tilde{P}$ is a Casimir function.

Restriction of implicit Hamiltonian systems to submanifolds

Restriction of Dirac structures

Let D be a Dirac structure on M and $N \subset M$ a submanifold of M.

Define the map $\sigma(x) : T_xN \times T_x^*M \to T_xN \times T_x^*N$, $x \in N$, by

$$\sigma(x)(v, v^*) = (v, v^*|_{T_xN}).$$

Assume that the dimension of $D(x) \cap (T_xN \times T_x^*M)$ is independent of $x \in N$, and define the vector subbundle $D_N \subset TN \oplus T^*N$

$$D_N(x) = \sigma(x)\left(D(x) \cap (T_xN \times T_x^*M)\right), \quad x \in N$$

Then $D_N = D_N^\perp$, and hence D_N is a Dirac structure on N.

Local sections of D_N.

Let $\iota : N \hookrightarrow M$ denote the inclusion map.

Then $(\bar{X}, \bar{\alpha})$ is a local section of D_N if and only if there exists a local section (X, α) of D such that $\bar{X} \sim_\iota X$ and $\bar{\alpha} = \iota^* \alpha$.

Otherwise stated

$$(\mathfrak{D}_N)_{loc} = \{(\bar{X}, \bar{\alpha}) \in \mathfrak{X}_{loc}(N) \oplus \Omega^1_{loc}(N) \mid \exists (X, \alpha) \in \mathfrak{D}_{loc} \text{ such that } \bar{X} \sim_\iota X \text{ and } \bar{\alpha} = \iota^* \alpha\}$$

Furthermore, if D is closed, then also D_N is closed.

Proof:

$$\langle L_{\bar{X}_1} \bar{\alpha}_2, \bar{X}_3 \rangle + \langle L_{\bar{X}_2} \bar{\alpha}_3, \bar{X}_1 \rangle + \langle L_{\bar{X}_3} \bar{\alpha}_1, \bar{X}_2 \rangle = \big(\langle L_{X_1} \alpha_2, X_3 \rangle + \langle L_{X_2} \alpha_3, X_1 \rangle + \langle L_{X_3} \alpha_1, X_2 \rangle \big) \circ \iota$$

Restriction of implicit Hamiltonian systems

Assume $N \subset M$ is invariant under the flow of (M, D, H).

E.g. $N = P^{-1}(\mu)$ is the level set of a first integral.

Define H_N to be the restriction of H to N, i.e.

$$H_N = H \circ \iota$$

Then every solution $x(t)$ of (M, D, H) which leaves N invariant (i.e. $x(t) \subset N$) is a solution of (N, D_N, H_N).

Note The reverse is generally not true. E.g. presympletic structures.

Restriction by Casimirs

Let $N \subset M$ be such that every $X \in \mathsf{G}_1$ is tangent to N (i.e. $X(\bar{x}) \in T_{\bar{x}}N$, $\forall \bar{x} \in N$).

E.g. $N = C^{-1}(\mu)$ is the level set of a Casimir function $\mathbf{d}C \in \mathsf{P}_0$.

Then the solutions of (M, D, H) contained in N are exactly the solutions of the implicit generalized Hamiltonian system (N, D_N, H_N).

Example

Consider (M, D, H) and assume D closed and G_1 constant dimensional.

Then there exists a skew-symmetric linear map $\omega(x) : \mathsf{G}_1(x) \to (\mathsf{G}_1(x))^*$, $x \in M$, with kernel G_0, such that

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v^* - \omega(x)v \in \mathsf{G}_1(x)^\circ,\ v \in \mathsf{G}_1(x)\}, \quad x \in M.$$

G_1 is involutive and defines a regular foliation partitioning M into integral submanifolds of G_1.

Restricting D to an integral submanifold N yields

$$D_N(x) = \{(\bar{v}, \bar{v}^*) \in T_xN \times T_x^*N \mid \bar{v}^* = \bar{\omega}(x)\bar{v},\ \forall x \in N\}$$

with $\bar{\omega}$ the restriction of ω to N.

Example (cont'd)

$\bar{\omega}$ is a closed two-form on N, with kernel G_0. Hence D_N is a presymplectic structure on N.

The restriction (N, D_N, H_N) represents a presymplectic Hamiltonian system on N.

Theorem A closed Dirac structure with G_1 constant dimensional has a regular foliation by presymplectic leaves.

Note In case D is a Poisson structure on M, then N is a symplectic submanifold of M (and D_N is a symplectic structure).

Projection of implicit Hamiltonian systems to quotient manifolds

Symmetry Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and smooth left action $\phi : G \times M \to M$.

G is a symmetry Lie group of D if for every $g \in G$

$$(X, \alpha) \in \mathfrak{D}_{loc} \Rightarrow \left(\phi_g^* X, \phi_g^* \alpha\right) \in \mathfrak{D}_{loc}$$

Equivalently, for every $\xi \in \mathfrak{g}$

$$(X, \alpha) \in \mathfrak{D}_{loc} \Rightarrow \left(L_{\xi_M} X, L_{\xi_M} \alpha\right) \in \mathfrak{D}_{loc}$$

G is a symmetry group of (M, D, H) if, in addition, H is G-invariant, i.e. $H \circ \phi_g = H$ for all $g \in G$. Equivalently, $L_{\xi_M} H = 0$ for all $\xi \in \mathfrak{g}$.

Throughout we assume that the action of G on M is free and proper.

Then the orbit space M/G is a smooth manifold and the canonical projection $\pi : M \to M/G$ is a surjective submersion.

Denote the following spaces:

1. vertical subbundle $V = \ker(T\pi)$, with fiber

$$V(x) = \mathrm{span}\,\{\xi_M(x) \mid \xi \in \mathfrak{g}\}, \quad x \in M$$

2. bundle of projectable one-forms $E \subset TM \oplus T^*M$, with sections

$$\Gamma_{loc}(E) = \{(X, \alpha) \in \mathfrak{X}_{loc}(M) \oplus \Omega^1_{loc}(M) \mid \alpha = \pi^*\widehat{\alpha} \text{ for some } \widehat{\alpha} \in \Omega^1_{loc}(M/G)\}$$

Projection of Dirac structures

Assume that

- $V + \mathsf{G}_0$ is a smooth vector subbundle of TM
- $D \cap E$ is a smooth vector subbundle of $TM \oplus T^*M$

Then D projects to a Dirac structure $\hat{D}$ on $\hat{M} := M/G$. In local sections:

$$\widehat{\mathfrak{D}}_{loc} = \{(\hat{X}, \widehat{\alpha}) \in \mathfrak{X}_{loc}(\widehat{M}) \oplus \Omega^1_{loc}(\widehat{M}) \mid \exists (X, \alpha) \in \mathfrak{D}_{loc} \text{ such that } X \sim_\pi \hat{X} \text{ and } \alpha = \pi^* \widehat{\alpha}\}$$

Furthermore, if D is closed, then also $\hat{D}$ is closed.

Projection of implicit Hamiltonian systems

(M, D, H) projects to an implicit Hamiltonian system $(\widehat{M}, \widehat{D}, \widehat{H})$, with $H = \widehat{H} \circ \pi$.

G-projectable solution $x(t)$ of (M, D, H): $\exists X \sim_\pi \hat{X} \in \mathfrak{X}_{loc}(M/G)$

$$\text{such that } (\dot{x}(t), \mathbf{d}H(x(t)) = (X(x(t), \mathbf{d}H(x(t)) \in D(x(t))$$

- If $x(t)$ is a G-projectable solution of (M, D, H) then $\widehat{x}(t) := \pi(x(t))$ is a solution of $(\widehat{M}, \widehat{D}, \widehat{H})$.

- Every solution $\widehat{x}(t)$ of $(\widehat{M}, \widehat{D}, \widehat{H})$ is locally the projection under π of a G-projectable solution $x(t)$ of (M, D, H).

Not every solution is G-projectable!

$$D(x) = \{(v, v^*) \in \mathbb{R}^3 \times (\mathbb{R}^3)^* \mid v \in \mathsf{P}_1^\circ(x),\ v^* \in \mathsf{P}_1(x)\}, \qquad x \in M = \mathbb{R}^3$$

with

$$\mathsf{P}_1 = \mathsf{P}_0 = \operatorname*{span}_{C^\infty(M)} \{\mathbf{d}x_3\}, \text{ i.e., } \mathsf{G}_1 = \mathsf{G}_0 = \mathsf{P}_1^\circ = \operatorname*{span}_{C^\infty(M)} \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$$

Since D is linear (constant), $\frac{\partial}{\partial x_1}$ is a symmetry of D. Assume $H(x_1, x_2, x_3) = \widehat{H}(x_3)$, then $\frac{\partial}{\partial x_1}$ is a symmetry of (M, D, H).

Any solution $x(t)$ of (M, D, H) satisfies $\dot{x}(t) = X(x(t))$, where

$$X(x_1, x_2, x_3) = h_1(x_1, x_2, x_3)\frac{\partial}{\partial x_1} + h_2(x_1, x_2, x_3)\frac{\partial}{\partial x_2},\ h_1, h_2 \in C^\infty(M)$$

Only the solutions for which $L_{\frac{\partial}{\partial x_1}} h_2 = 0$ are G-projectable.

In that case

$$\widehat{X}(x_2, x_3) = h_2(x_2, x_3)\frac{\partial}{\partial x_2} \in \mathfrak{X}(\widehat{M})$$

with $\widehat{M} = \mathbb{R}^3/\mathbb{R} = \mathbb{R}^2$

The reduced generalized Dirac structure is

$$D(x) = \{(v, v^*) \in \mathbb{R}^2 \times (\mathbb{R}^2)^* \mid v \in \widehat{\mathsf{P}}_1^\circ(x),\ v^* \in \widehat{\mathsf{P}}_1(x)\}, \quad x \in \mathbb{R}^2$$

with

$$\widehat{\mathsf{P}}_1 = \widehat{\mathsf{P}}_0 = \underset{C^\infty(\mathbb{R}^2)}{\mathsf{span}} \{\mathbf{d}x_3\}, \text{ i.e., } \mathsf{G}_1 = \mathsf{G}_0 = \mathsf{P}_1^\circ = \underset{C^\infty(\mathbb{R}^2)}{\mathsf{span}} \left\{\frac{\partial}{\partial x_2}\right\}$$

Induced generalized Poisson bracket

Let $\widehat{H}_1, \widehat{H}_2 \in C^\infty(\widehat{M})$ such that $\widehat{H}_1, \widehat{H}_2 \in \mathcal{A}_{\widehat{D}}$, i.e.

$$\exists \widehat{X}_j \text{ such that } (\widehat{X}_j, \mathbf{d}\widehat{H}_j) \in \widehat{\mathfrak{D}}_{loc}, \quad j = 1, 2$$

There exist $(X_j, \mathbf{d}H_j) \in \mathfrak{D}_{loc}$ such that $X_j \sim_\pi \widehat{X}_j$ and $H_j = \widehat{H}_j \circ \pi$.

Then

$$\{\widehat{H}_1, \widehat{H}_2\}_{\widehat{D}}(\widehat{x}) = \langle \mathbf{d}\widehat{H}_2, \widehat{X}_1 \rangle(\widehat{x}) = \langle \mathbf{d}H_2, X_1 \rangle(x) = \{H_1, H_2\}_D(x)$$

with $\pi(x) = \widehat{x}$.

Hence one obtains the induced generalized Poisson bracket on $\mathcal{A}_{\widehat{D}}$

$$\{\widehat{H}_1, \widehat{H}_2\}_{\widehat{D}} \circ \pi = \{\widehat{H}_1 \circ \pi, \widehat{H}_2 \circ \pi\}_D,$$

Example

Assume D closed and P_1 constant dimensional.

Then there exists a skew-symmetric linear map $J : \mathsf{P}_1 \to (\mathsf{P}_1)^*$, such that

$$\mathfrak{D}_{loc} = \{(X, \alpha) \in \mathfrak{X}_{loc}(M) \oplus \Omega^1_{loc}(M) \mid X - J(\alpha) \in \mathsf{G}_0,\ \alpha \in \mathsf{P}_1 = \mathsf{G}_0^\circ\}$$

$\mathsf{G}_0 = \mathsf{P}_1^\circ$ is constant dimensional and involutive, and defines a regular foliation Φ partitioning M into integral submanifolds of G_0.

Note that every $X \in \mathsf{G}_0$ is a symmetry of D.

The set of leafs $\widehat{M} = M/\Phi$ is a smooth manifold.

Example (cont'd)

Note that $\widehat{\mathsf{P}}_1 = \Omega^1_{loc}(\widehat{M})$. Indeed, $\alpha = \pi^* \widehat{\alpha} \in \mathsf{G}_0^\circ = \mathsf{P}_1$.

Then D restricts to a Dirac structure $\hat{D}$ on $\hat{M}$ given by

$$\widehat{\mathfrak{D}}_{loc} = \{(\hat{X}, \widehat{\alpha}) \in \mathfrak{X}_{loc}(\widehat{M}) \oplus \Omega^1_{loc}(\widehat{M}) \mid \hat{X} = \hat{J}(\widehat{\alpha})\}$$

for some skew-symmetric linear map $\hat{J} : \Omega^1_{loc}(\widehat{M}) \to \mathfrak{X}_{loc}(\widehat{M})$

Hence $\mathcal{A}_{\widehat{D}} = C^\infty(\widehat{M})$ and $\hat{J}$ is exactly the structure matrix of the Poisson bracket $\{\cdot, \cdot\}_{\widehat{D}}$.

Theorem A closed Dirac structure with P_1 constant dimensional projects to a Poisson structure of the leaf space M/Φ.

Regular reduction of implicit Hamiltonian systems

Hamiltonian symmetry Lie group

Let G be a symmetry Lie group of (M, D, H).

Assume that there exists an Ad^*-equivariant momentum map $P : M \to \mathfrak{g}^*$ such that

$$(\xi_M, \mathbf{d}P_\xi) \in \mathfrak{D}, \quad \forall \xi \in \mathfrak{g},$$

where $P_\xi(x) = \langle P(x), \xi \rangle$, $x \in M$.

I.e. every ξ_M is a Hamiltonian symmetry.

Then P_ξ, $\xi \in \mathfrak{g}$ is a first integral of (M, D, H).

Projection after restriction (1/2)

Assume $\mu \in \mathfrak{g}^*$ is a regular value of P, then $P^{-1}(\mu)$ is a smooth submanifold of M.

Assuming the constant dimensionality conditions are satisfied, (M, D, H) restricts to an implicit Hamiltonian system (N, D_N, H_N) on $N = P^{-1}(\mu)$.

Consider the isotropy group

$$\begin{aligned} G_\mu &= \{g \in G \mid \phi_g(P^{-1}(\mu)) \subset P^{-1}(\mu)\} \\ &= \{g \in G \mid Ad_g^*(\mu) = \mu\} \end{aligned}$$

Then G_μ is a symmetry Lie group of (N, D_N, H_N).

Projection after restriction (2/2)

Assume the action of G_μ on N is free and proper, then $M_\mu = N/G_\mu = P^{-1}(\mu)/G_\mu$ is a smooth manifold.

Assuming the constant dimensionality conditions are satisfied, (N, D_N, H_N) projects to an implicit Hamiltonian system (M_μ, D_μ, H_μ).

Note If D is closed, then also D_μ is closed.

Examples

- If D is the graph of a symplectic structure ω, then D_μ is the graph of the Marsden-Weinstein reduced symplectic structure ω_μ.

- If D is the graph of a Poisson structure $\{\cdot,\cdot\}$, then D_μ is the graph of the reduced Poisson structure $\{\cdot,\cdot\}_\mu$.

Restriction after projection (1/2)

Assume the action of G on M is free and proper.

Assuming the constant dimensionality conditions are satisfied, (M, D, H) projects to an implicit Hamiltonian system $(M/G, \hat{D}, \hat{H})$, with $H = \hat{H} \circ \pi$.

Define the coadjoint orbits, forming a regular foliation of $\mathfrak{g}^*$,

$$\mathcal{O}_\mu = \{Ad_g^*(\mu) \mid g \in G\}, \quad \mu \in \mathfrak{g}^*$$

The leaf space is the quotient space $\hat{\mathfrak{g}}^* = \mathfrak{g}^*/G$, with projection map $\varpi : \mathfrak{g}^* \to \hat{\mathfrak{g}}^*$.

The momentum map P defines a map $\hat{P} : M/G \to \hat{\mathfrak{g}}^*$ by

$$\hat{P} \circ \pi = \varpi \circ P$$

Restriction after projection (2/2)

Assume $\widehat{\mu}$ is a regular value of $\widehat{P}$.

$\widehat{P}$ is a "Casimir" function of $(M/G, \widehat{D}, \widehat{H})$, in the sense that every $\hat{X} \in \widehat{\mathsf{G}}_1$ is tangent to the level set $\widehat{P}^{-1}(\widehat{\mu})$.

Then $(M/G, \widehat{D}, \widehat{H})$ restricts to an implicit Hamiltonian system $(\widehat{P}^{-1}(\widehat{\mu}), D_{\widehat{\mu}}, H_{\widehat{\mu}})$ (the constant dimensionality conditions are satisfied because $\widehat{P}$ is a "Casimir"!).

Note If D is closed, then also $D_{\widehat{\mu}}$ is closed.

Remark

- Take $\widehat{\mu} = \varpi(\mu)$, then $\widehat{P}^{-1}(\widehat{\mu})$ is equal to the quotient space $P^{-1}(\mathcal{O}_\mu)/G$.

Diagram Let ψ be such that the diagram commutes ($\widehat{\mu} = \varpi(\mu)$)

$$\begin{tikzcd}
 & M \arrow[dr, "\pi"] & \\
P^{-1}(\mu) \arrow[ur, "\iota_1"] \arrow[d, "\pi_\mu"'] & & M/G \\
P^{-1}(\mu)/G_\mu \arrow[rr, "\psi"] & & \widehat{P}^{-1}(\widehat{\mu}) \arrow[u, "\iota_2"']
\end{tikzcd}$$

Theorem D_μ and $D_{\widehat{\mu}}$ are isomorphic via ψ, i.e.

$$(X, \alpha) \in (\mathfrak{D}_\mu)_{loc} \Longleftrightarrow (\psi_* X, (\psi^*)^{-1}\alpha) \in (\mathfrak{D}_{\widehat{\mu}})_{loc}$$

Theorem $H_\mu = H_{\widehat{\mu}} \circ \psi$. Hence the implicit Hamiltonian systems (M_μ, D_μ, H_μ) and $(\widehat{P}^{-1}(\widehat{\mu}), D_{\widehat{\mu}}, H_{\widehat{\mu}})$ are equivalent up to isomorphism.

In particular, $x(t)$ is a solution of (M_μ, D_μ, H_μ) if and only if $\psi(x(t))$ is a solution of $(\widehat{P}^{-1}(\widehat{\mu}), D_{\widehat{\mu}}, H_{\widehat{\mu}})$.

Horizontal symmetries

Recall the assumption that G is a Hamiltonian symmetry group of (M, D, H), i.e.

$$(\xi_M, \mathbf{d}P_\xi) \in \mathfrak{D}, \ \forall \xi \in \mathfrak{g}$$

Thus $\xi_M \in \mathsf{G}_1$, $\forall \xi \in \mathfrak{g}$.

A vector field Y is called a horizontal symmetry of D if Y is a symmetry of D and $Y \in \mathsf{G}_1$.

In particular, every Hamiltonian symmetry is horizontal.

In general Let G be a symmetry Lie group of D. Then the set of horizontal symmetries is generated by a normal Lie subgroup $K \subset G$.

Reduction using horizontal symmetries

Consider (M, D, H) and assume G_1 constant dimensional.

Let $\omega(x) : T_xM \to T_x^*M$, $x \in M$, be a skew-symmetric linear map such that

$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v^* - \omega(x)v \in \mathsf{G}_1(x)^\circ, \ \forall x \in M, \ v \in \mathsf{G}_1(x)\}.$

Assume there exists an Ad^*-equivariant momentum map $P : M \to \mathfrak{g}^*$ such that

$$\mathbf{d}\langle P, \xi\rangle = \omega(\xi_M), \quad \forall\, \xi \in \mathfrak{g}$$

I.e. G is a Hamiltonian symmetry group with respect to ω.

Let $K \subset G$ be the horizontal symmetry Lie group of (M, D, H), with Lie algebra $\mathfrak{k}$.

Since $\xi_M \in \mathsf{G}_1$ for all $\xi \in \mathfrak{k}$:

$$(\xi_M, \mathbf{d}\langle P, \xi\rangle) \in \mathfrak{D}, \quad \xi \in \mathfrak{k}$$

Define the horizontal momentum map $P_h : M \to \mathfrak{k}^*$ to be the restriction of P to $\mathfrak{k}$. Note that P_h is Ad^*-equivariant.

Then P_h is a first integral.

(M, D, H) can be reduced to an implicit Hamiltonian system (M_μ, D_μ, H_μ), where $M_\mu = P_h^{-1}(\mu)/K_\mu$.

The reduced system has a symmetry Lie group $L = G/K$ and can be reduced further to the quotient space M_μ/L. This gives a total reduction of dimension $G + K_\mu$.

Regular reduction of constrained mechanical systems

Consider the configuration manifold Q and let $\{\alpha_1, \ldots, \alpha_k\}$ be a set of independent one-forms on Q, defining the kinematic constraints

$$\alpha_i(q)\dot{q} = 0, \quad q \in Q$$

Define the codistribution $\mathsf{P}_Q = \text{span}\,\{\alpha_1, \ldots, \alpha_k\} \subset T^*Q$.

The cotangent bundle projection $\pi_Q : T^*Q \to Q$ defines a codistribution $\mathsf{P}_0 \subset T^*(T^*Q)$ by vertical lift

$$\mathsf{P}_0 = \text{span}\,\{\pi_Q^*\alpha_1, \ldots, \pi_Q^*\alpha_k\}$$

The constrained mechanical system is defined by (T^*Q, D, H), where

$$D(q,p) = \{(v, v^*) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid v^* - \omega v \in \mathsf{P}_0,\ v \in \mathsf{P}_0^\circ\}$$

with $\omega = \mathbf{d}q \wedge \mathbf{d}p$ the canonical symplectic form on T^*Q.

Let G be a Lie group acting on Q such that $L_{\xi_Q}\mathsf{P}_Q \subset \mathsf{P}_Q$, $\forall \xi \in \mathfrak{g}$.

Moreover, assume G acts horizontally, i.e. $\xi_Q \in \mathsf{P}_Q^{\circ}$, $\forall \xi \in \mathfrak{g}$.

The action of G lifts to an action on T^*Q satisfying

$$L_{\xi_{T^*Q}}\omega = 0 \quad \text{and} \quad L_{\xi_{T^*Q}}\mathsf{P}_0 \subset \mathsf{P}_0$$

as well as

$$L_{\xi_{T^*Q}} \subset \mathsf{G}_1 = \mathsf{P}_0^{\circ}$$

Hence, assuming G-invariance of H, G is a horizontal symetry Lie group of (M, D, H).

The action admits an Ad^*-equivariant momentum map $P : T^*Q \to \mathfrak{g}^*$ defined by

$$\langle P(q,p), \xi \rangle = p^T \xi_Q(q), \quad \xi \in \mathfrak{g}$$

such that

$$\mathbf{d}\langle P, \xi \rangle = \omega(\xi_{T^*Q})$$

Since the symmetries are horizontal, it follows that

$$(\xi_{T^*Q}, \mathbf{d}P_\xi) \in \mathfrak{D}, \quad \forall \xi \in \mathfrak{g}$$

and P is a first integral of (M, D, H).

Reduction yields the implicit Hamiltonian system (M_μ, D_μ, H_μ) on $M_\mu = P^{-1}(\mu)/G_\mu$ with

$$D(x) = \{(w, w^*) \in T_x M_\mu \times T_x^* M_\mu \mid w^* - \omega_\mu(x) w \in \mathsf{P}_0^\mu,\ w \in (\mathsf{P}_0^\mu)^\circ\}$$

where

- ω_μ is a symplectic form on M_μ, defined by $\iota^*\omega = \pi_\mu^*\omega_\mu$

$$\Big(\iota : P^{-1}(\mu) \to T^*Q \quad \text{and} \quad \pi_\mu : P^{-1}(\mu) \to M_\mu\Big)$$

- $\mathsf{P}_0^\mu \subset T^*M_\mu$ is defined by

$$\mathsf{P}_0^\mu = \text{span}\,\{\beta \in \Omega^1(M_\mu) \mid \pi_\mu^*\beta \in \text{ span}\,\{\iota^*(\pi_Q^*\alpha_1), \ldots, \iota^*(\pi_Q^*\alpha_k)\}\}$$

The system is again of a "constrained mechanical format".

References

G. Blankenstein and A.J. van der Schaft, Symmetry and reduction in implicit gemeralized Hamiltonian systems, *Reports on Mathematical Physics*, vol. 47, pp. 57–100, 2001.

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Singular reduction of implicit Hamiltonian systems

The singular reduction problem

Given is a vector subbundle $G_0 \subset TM$ and a generalized Poisson structure $\sharp : G_0^\circ \to (G_0^\circ)^*$. Define the Dirac structure by

$$D(x) = \{(v, v^*) \in T_xM \oplus T_x^*M \mid v|_{G^\circ} = \sharp(x)v^*, \ v^* \in G_0^\circ(x)\}.$$

The vector field $Y \in \mathfrak{X}_{\mathrm{loc}}(M)$ is a *symmetry* of D if $\boldsymbol{\pounds}_Y \circ \sharp = \sharp \circ \boldsymbol{\pounds}_Y$ and $Z \in \Gamma_{\mathrm{loc}}(G_0) \Longrightarrow \boldsymbol{\pounds}_Y Z \in \Gamma_{\mathrm{loc}}(G_0)$.

A *symmetry Lie group* of a Dirac structure of this type is a smooth left action $\phi : G \times M \to M$ satisfying $\phi_g^* \circ \sharp = \sharp \circ \phi_g^*$ and $Z \in \Gamma_{\mathrm{loc}}(G_0) \Longrightarrow \phi_g^* Z \in \Gamma_{\mathrm{loc}}(G_0), \forall g \in G$.

So $\forall \xi \in \mathfrak{g} \Longrightarrow \xi_M$ is a symmetry of D.

It is not assumed that G acts regularly on M.

Assume ϕ admits a momentum map $\mathbf{J} : M \to \mathfrak{g}^* \overset{def}{\Longleftrightarrow} (\xi_M, \mathbf{dJ}^\xi) \in \mathfrak{D}, \forall \xi \in \mathfrak{g}$, where $\mathbf{J}^\xi(x) := \langle \mathbf{J}(x), \xi \rangle, \forall x \in M$. Suppose $\mathbf{J}$ is Ad^*-equivariant.

If $\mu = 0$ is a singular value of $\mathbf{J}$ form the quotient topological space $M_0 := \mathbf{J}^{-1}(0)/G$. $\pi : \mathbf{J}^{-1}(0) \to M_0$ canonical projection.

$f_0 \in C^0(M_0)$ is called *smooth*, denoted $f_0 \in C^\infty(M_0)$, if $\exists f \in C^\infty(M)^G$ such that $f_0 \circ \pi = f|_{\mathbf{J}^{-1}(0)}$.

Given the singular reduced space M_0, together with its topology and the set of smooth functions, the goal is to define a reduced Dirac structure on M_0.

Reduced generalized Poisson bracket

Let $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ denote the generalized Poisson bracket corresponding to $\sharp$.

Define a generalized Poisson bracket $\{\cdot,\cdot\}_0 : C^\infty(M_0) \times C^\infty(M_0) \to C^\infty(M_0)$ as follows:

If $f_0, h_0 \in C^\infty(M_0)$ let $f, h \in C^\infty(M)^G$ be such that $f_0 \circ \pi = f|_{\mathbf{J}^{-1}(0)}$ and $h_0 \circ \pi = h|_{\mathbf{J}^{-1}(0)}$. Then define the bracket $\{\cdot,\cdot\}_0$ by

$$\{f_0, h_0\}_0 \circ \pi = \{f, h\}|_{\mathbf{J}^{-1}(0)}$$

Does not depend on the choice of the G-invariant extensions f, h.

Reduction of the 'characteristic distribution' G_0

M_0 is not a manifold, so it is hopeless to search for a subbundle of the inexistent tangent bundle of M_0. Seek a vector space of derivations $\widehat{G}_0$ on $C^\infty(M_0)$ naturally induced by G_0. We shall show that every vector field $X \in \Gamma_{\mathrm{loc}}(G_0)$ is "tangent" to $N = \mathbf{J}^{-1}(0)$.

• μ regular value of $\mathbf{J} \Longrightarrow N = \mathbf{J}^{-1}(\mu)$ is a submanifold of M, this means that X restricts to a well defined vector field $\bar{X}$ on N.

• $\mu = 0$ singular value of the momentum map $\Longrightarrow N$ is not a smooth manifold. What is "tangent"? Recall $\mathfrak{X}(M) \longleftrightarrow$ {derivations on $C^\infty(M)$}: $X \longleftrightarrow \pounds_X \equiv X$

A derivation X on $C^\infty(M)$ is said to be *tangent* to the subset $N \subset M$ if it restricts to a well defined derivation $\bar{X}$ on the set of Whitney smooth functions on N.

A continuous function $\bar{f}$ on N is said to be a *Whitney smooth function* if there exists a smooth function f on M such that $\bar{f} = f|_N$; the set of Whitney smooth functions on N is denoted by $W^\infty(N)$.

1.) *X is tangent to N if there exists a derivation $\bar{X}$ on $W^\infty(N)$ such that $X[f](x) = \bar{X}[f|_N](x)$ for all $f \in C^\infty(M)$ and all $x \in N$.*

2.) *A necessary and sufficient condition for X to be tangent to N is that $X[f](x) = X[h](x), \forall x \in N, \forall f, h \in C^\infty(M)$ such that $f|_N = h|_N$.*

3.) *If N is a smooth* closed submanifold *of M and M is* paracompact, *then $W^\infty(N) = C^\infty(N)$ (relative to the differential structure on N).* In this case, the previous definition has the usual meaning of a vector field X being tangent to the submanifold N. Consequently, its restriction $\bar{X}$ to N yields a vector field on N.

$\gamma(t)$ is an *integral curve* of $X \in \mathfrak{X}(M)$ (as a derivation) through $x_0 \in M$ if

$$\frac{d}{dt} f(\gamma(t)) = X[f](\gamma(t)), \quad \forall t, \; \forall f \in C^\infty(M), \; \gamma(0) = x_0.$$

$X \in \Gamma_{\mathrm{loc}}(G_0)$, $\gamma(t)$ integral curve of X through $x_0 \in \mathbf{J}^{-1}(0)$, so

$$\frac{d}{dt}\mathbf{J}^{\xi}(\gamma(t)) = X[\mathbf{J}^{\xi}](\gamma(t)) = 0, \ \forall t, \ \forall \xi \in \mathfrak{g},$$

since $(\xi_M, \mathbf{dJ}^{\xi}) \in \mathfrak{D}$ which implies that $\mathbf{dJ}^{\xi}(x) \in (G_0)^{\circ}(x)$, $\forall x \in M$. Thus the integral curve of $X \in \Gamma_{\mathrm{loc}}(G_0)$ through every $x_0 \in \mathbf{J}^{-1}(0)$ is contained in $\mathbf{J}^{-1}(0)$. By the equivalence of derivations and velocity vectors it then follows that

$$X[f](x_0) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} h(\gamma(t)) = X[h](x_0), \tag{36}$$

for all $f, h \in C^{\infty}(M)$ satisfying $f|_N = h|_N$. This shows that every vector field $X \in \Gamma_{\mathrm{loc}}(G_0)$ is tangent to $N = \mathbf{J}^{-1}(0)$. Consequently, every vector field $X \in \Gamma_{\mathrm{loc}}(G_0)$ restricts to a well defined derivation $\bar{X}$ on $W^{\infty}(N)$. So, the constant dimensional distribution G_0 on M restricts to a vector space $\bar{G}_0$ of derivations on $W^{\infty}(N)$. If G_0 is locally spanned by the independent vector fields $X_1, \ldots, X_m$, then $\bar{G}_0$ is locally spanned by the independent derivations $\bar{X}_1, \ldots, \bar{X}_m$.

Show that the distribution G_0 on M projects to a well defined vector space $\widehat{G}_0$ of derivations on the smooth functions $C^\infty(M_0)$.

$X \in \mathfrak{X}(M)$ *projects* to M_0 if $\exists \widehat{X}$, derivation on $C^\infty(M_0)$, such that $X[f](x) = \widehat{X}[f_0](\pi(x))$, $\forall x \in N, \forall f \in C^\infty(M)^G$, where f_0 is defined by $f_0 \circ \pi = f|_N$. It is clear that X restricts to a well defined derivation $\hat{X}$ on $C^\infty(M_0)$ if and only if

1. $X[f](x)$ does not depend on the extension of $f_0 \circ \pi$ off N to M,

2. $X[f](x) = X[f](y)$ for all $x, y \in N$ such that $\pi(x) = \pi(y)$.

Recall $X \in \Gamma_{\mathrm{loc}}(G_0) \Longrightarrow X$ tangent to $N \Longrightarrow X[f](x) = \bar{X}[f|_N](x) = \bar{X}[f_0 \circ \pi](x)$, $\forall x \in N$. Therefore its value does not depend on the extension of $f_0 \circ \pi$ off N to M. It remains to show that

$$X[f](x) = X[f](y), \quad \forall x, y \in N \text{ such that } \pi(x) = \pi(y). \qquad (37)$$

In general, this condition will not be satisfied by every local section X of G_0. To see this, assume that Y is a local section of G_0 for which condition (**??**) is satisfied. Clearly, $X = hY$ is also a local section of G_0, for any $h \in C^\infty(M)$. However, X will satisfy condition (**??**) if *and only if* h is G-invariant, i.e. $h \in C^\infty(M)^G$.

However, we show: *There exists a basis of local sections $X_1, \ldots, X_m \in \Gamma_{\mathrm{loc}}(G_0)$, spanning G_0, all of whose elements satisfy* (**??**).

Proposition $\forall x_0 \in M, \exists U$, open neighborhood of x_0, and a finite set $X_j \in \Gamma(G_0)|_U$ with the property that span$\{X_j(x)\} = G_0(x), \forall x \in U$, such that $[\xi_M, X_j](x) \in \mathfrak{g} \cdot x, \forall \xi \in \mathfrak{g}$.

So $\forall f \in C^\infty(M)^G \Longrightarrow 0 = [X_i, \xi_M]\,[f] = X_i\left[\boldsymbol{\mathcal{L}}_{\xi_M} f\right] - \boldsymbol{\mathcal{L}}_{\xi_M}(X_i[f]) = -\boldsymbol{\mathcal{L}}_{\xi_M}(X_i[f]),\ \forall \xi \in \mathfrak{g} \Longrightarrow X_i[f]$ is G-invariant, so satisfies (**??**). Thus, there exists a basis $X_1, \ldots, X_m \in \Gamma_{\mathrm{loc}}(G_0)$, spanning G_0, such that each X_i projects to a well defined derivation $\widehat{X}_i$ on $C^\infty(M_0)$. The derivations $\widehat{X}_1, \ldots, \widehat{X}_m$ locally span a vector space of derivations on $C^\infty(M_0)$, which we will denote by $\widehat{G}_0$.

Singular reduced Dirac structure on M_0

The *singular reduced Dirac structure* D_0 is defined by the pair $(\{\cdot,\cdot\}_0, \widehat{G}_0)$. D_0 is a *topological Dirac structure*.

Note: In the case of regular reduction, D_0 as defined above equals the regular reduced Dirac structure.

Integral curves

A continuous function γ on M_0 is said to be *smooth* if $f_0 \circ \gamma$ is smooth, for all $f_0 \in C^\infty(M_0)$.

Let $\hat{X}$ be a derivation on $C^\infty(M_0)$. An integral curve of $\hat{X}$, through some point $x_0 \in M_0$, is a smooth curve γ for which

$$\frac{\mathrm{d}}{\mathrm{d}t} f_0(\gamma(t)) = \hat{X}[f_0](\gamma(t)), \quad \forall t, \forall f_0 \in C^\infty(M_0), \gamma(0) = x_0.$$

Singular dynamics

A smooth curve γ if called a *solution* of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) if there exists a derivation $\hat{X}$ on $C^\infty(M_0)$ such that γ is an integral curve of $\hat{X}$, and

$$\hat{X}(\gamma(t)) - \{\cdot, H_0\}_0(\gamma(t)) \in \hat{G}_0(\gamma(t)), \quad \forall t \tag{38}$$

$$\hat{Z}[H_0](\gamma(t)) = 0, \quad \forall t, \forall \hat{Z} \in \hat{G}_0.$$

Notice: No DAEs!

Example: Assume (M, D, H) does not include algebraic constraints, i.e., $G_0 = 0$. Then also $\hat{G}_0 = 0$ and (**??**) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} f_0(\gamma(t)) = \{f_0, H_0\}_0(\gamma(t)), \quad \forall t, \forall f_0 \in C^\infty(M_0)$$

Solutions on M_0

A G-projectable solution is a solution $x(t)$ of (M, D, H), which is the integral curve of a projectable vector field X on M, such that X projects to a well defined derivation $\hat{X}$ on $C^\infty(M_0)$.

Proposition Every G-projectable solution $x(t)$ of (M, D, H), with $x(0) \in \mathbf{J}^{-1}(0)$, projects to a solution $\gamma(t) = \pi(x(t))$ of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0).

Orbit type decomposition

Specific assumptions

Recall the Dirac structure: $G \subset TM$ vector subbundle, $\sharp : G^\circ \to (G^\circ)^*$ skew,

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v|_{G^\circ} = \sharp(x)v^*,\ v^* \in G_0^\circ(x)\}$$

and the momentum map $(\xi_M, \mathbf{dJ}^\xi) \in \mathfrak{D}, \forall \xi \in \mathfrak{g}$. Hypotheses:

- $\sharp$ is nondegenerate
- $\xi_M = (\mathbf{dJ}^\xi)^\sharp$ and $\mathbf{dJ}^\xi \in G_0^\circ$, for all $\xi \in \mathfrak{g}$
- G acts properly, M is paracompact

Orbit type decomposition of M_0

Under these conditions it follows that:

$$\operatorname{im} T_x\mathbf{J} = \mathfrak{g}_x^\circ, \quad \forall x \in M$$

Hence, $\mathbf{J}^{-1}(0) \cap M_{(K)}$ is a smooth submanifold of M, for every compact subgroup $K \subset G$.

(As usual $M_{(K)} = \{x \in M \mid G_x \sim K\}$.)

Moreover, the quotient $(M_0)_{(K)} := (\mathbf{J}^{-1}(0) \cap M_{(K)})/G = \pi(\mathbf{J}^{-1}(0) \cap M_{(K)})$ is a smooth manifold.

This yields the orbit type decomposition

$$M_0 = \coprod_{(K)} (M_0)_{(K)}$$

Reduction to the pieces

- The generalized Poisson bracket $\{\cdot,\cdot\}$ on M induces a generalized Poisson bracket $\{\cdot,\cdot\}_{(K)}$ on $C^\infty((M_0)_{(K)})$.

Assume there exists a set of local sections X_j of G_0, spanning G_0, such that the flow of X_j commutes with the G-action. So flow of X_j preserves the submanifold $M_{(K)}$.) (E.g., assume $G_0 \cap V = 0$.)

- The distribution G_0 on M reduces to a distribution $G_{(K)}$ on each piece $(M_0)_{(K)}$.

- Define the Hamiltonian $H_{(K)} \circ \pi_{(K)} = H|_{\mathbf{J}^{-1}(0)\cap M_{(K)}}$.

The triple $((M_0)_{(K)}, D_{(K)}, H_{(K)})$ defines an implicit Hamiltonian system on the piece $(M_0)_{(K)}$.

Notice: DAEs!

The reduced implicit Hamiltonian system

- The system $((M_0)_{(K)}, D_{(K)}, H_{(K)})$ is the regular reduction of (M, D, H) to the piece $(M_0)_{(K)}$.

- The system $((M_0)_{(K)}, D_{(K)}, H_{(K)})$ is exactly the restriction of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) to the piece $(M_0)_{(K)}$.

- A solution $\gamma(t)$ of (M_0, D_0, H_0), with $\gamma(0) \in (M_0)_{(K)}$, preserves the piece $(M_0)_{(K)}$ and restricts to a sulotion $\bar{\gamma}(t)$ of $((M_0)_{(K)}, D_{(K)}, H_{(K)})$.

Example: the spherical pendulum

cf. [Arms, Cushman & Gotay, 1991], [Cushman & Bates, 1997]

Configuration space and constraints

Configuration space $Q = \mathbb{R}^3 \setminus 0$.

Holonomic constraint:

$$\alpha(q)\dot{q} = 0, \text{ with } \alpha(q) = q_1 \mathbf{d}q_1 + q_2 \mathbf{d}q_2 + q_3 \mathbf{d}q_3$$

which integrates to $q_1^2 + q_2^2 + q_3^2 = 1$.

Symmetry Lie group S^1 acting on Q by rotations about the vertical q_3-axis.

$$\xi_Q(q) = -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2}$$

The constraint is invariant under the S^1-action. Moreover, the action is horizontal.

The cotangent bundle

Cotangent bundle $M := T^*Q = (\mathbb{R}^3 \setminus 0) \times \mathbb{R}^3$, with canonical symplectic form $\omega = \mathbf{d}q^1 \wedge \mathbf{d}p_1 + \mathbf{d}q^2 \wedge \mathbf{d}p_2 + \mathbf{d}q^3 \wedge \mathbf{d}p_3$.

Characteristic distribution:

$$G_0 = \text{span}\,\{X := q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2} + q_3 \frac{\partial}{\partial p_3}\}$$

Hamiltonian energy function:

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_3$$

Lifted action on M:

$$\xi_M(q,p) = -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}$$

with conserved momentum map $\mathbf{J}(q,p) = q_1 p_2 - q_2 p_1$.

Singular reduced space

Algebra of S^1-invariant polynomials on M:

$$\sigma_1 = q_3, \qquad \sigma_3 = p_1^2 + p_2^2 + p_3^2, \qquad \sigma_5 = q_1^2 + q_2^2,$$
$$\sigma_2 = p_3, \qquad \sigma_4 = q_1 p_1 + q_2 p_2, \qquad \sigma_6 = q_1 p_2 - q_2 p_1 = P(q,p)$$

with (in-)equalities

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \qquad \sigma_3 \geq 0, \qquad \sigma_5 \geq 0$$

The *Hilbert map* for the S^1-action is defined by

$$\sigma : \mathbb{R}^6 \to \mathbb{R}^6, \quad (q,p) \mapsto (\sigma_1(q,p), \ldots, \sigma_6(q,p))$$

The singular reduced space is given by the semialgebraic variety $M_0 = \sigma(\mathbf{J}^{-1}(0))$:

$$M_0 = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 \geq 0,\ \sigma_5 \geq 0\}$$

Reduced Poisson bracket & derivation

A function f_0 on M_0 is smooth if and only if $\exists$ a smooth S^1-invariant function $f(\sigma_1, \ldots, \sigma_6)$ such that $f(\sigma_1, \ldots, \sigma_5, 0) = f_0(\sigma_1, \ldots, \sigma_5)$.

The reduced Poisson bracket is given by

$\{\sigma_i, \sigma_j\}_0$	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	0	1	$2\sigma_2$	0	0
σ_2	-1	0	0	0	0
σ_3	$-2\sigma_2$	0	0	$-2(\sigma_3 - \sigma_2^2)$	$-4\sigma_4$
σ_4	0	0	$2(\sigma_3 - \sigma_2^2)$	0	$-2\sigma_5$
σ_5	0	0	$4\sigma_4$	$2\sigma_5$	0

The vector field X reduces to a derivation $\hat{X}$ on $C^\infty(M_0)$ by

$$\widehat{X}[f_0] = \frac{\partial f_0}{\partial \sigma_2}\sigma_1 + \frac{\partial f_0}{\partial \sigma_3} 2(\sigma_4 + \sigma_1\sigma_2) + \frac{\partial f_0}{\partial \sigma_4}\sigma_5$$

Reduced dynamics

The reduced Hamiltonian is given by $H_0 = \frac{1}{2}\sigma_3 + \sigma_1$.

The singular reduced Hamiltonian dynamics is defined by the triple

$$\left(M_0, D_0 = (\{\cdot,\cdot\}_0, \widehat{G}_0), H_0\right)$$

"Integration" of the holonomic constraint: the function $C(q^1, q^2, q^3) = (q^1)^2 + (q^2)^2 + (q^3)^2 - 1$ projects to the function

$$C_0 = \sigma_5 + \sigma_1^2 - 1$$

which is a Casimir function of the system.

(I.e. $\hat{X}[C_0] = 0$ and $\{\cdot, C_0\}_0 = -2\hat{X} \in \widehat{G}_0$.)

Orbit type decomposition

The singular reduced space M_0 is decomposed into two smooth manifolds:

- The first piece corresponds to the fixed points of the S^1-action (i.e., $q^1 = q^2 = p_1 = p_2 = 0$).

$$(M_0)_{(S^1)} = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_3 - \sigma_2^2 = 0,\ \sigma_4 = 0,\ \sigma_5 = 0\}$$

- The second piece corresponds to the complement, on which the S^1-action is free and proper.

$$\begin{aligned}(M_0)_{(e)} = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\\ \sigma_3 - \sigma_2^2 \geq 0,\ \sigma_5 \geq 0,\ \sigma_5 + \sigma_3 - \sigma_2^2 > 0\}\end{aligned}$$

Dynamics on the pieces

- The dynamics on $(M_0)_{(S^1)}$ consists of the stable and unstable equilibrium points $(0, 0, \pm 1, 0, 0, 0)$.

- The dynamics on $(M_0)_{(e)}$ is given by the ordinary differential equations:

  $$\dot{\sigma}_1 = \sigma_2, \qquad \dot{\sigma}_2 = -1 + \sigma_1(\sigma_1 - \sigma_3), \qquad \dot{\sigma}_3 = -2\sigma_2,$$
  $$\dot{\sigma}_4 = \sigma_1 - \sigma_2^2 - \sigma_1^2(\sigma_1 - \sigma_3), \qquad \dot{\sigma}_5 = -2\sigma_1\sigma_2,$$

  satisfying the constraint $\sigma_4 + \sigma_1\sigma_2 = 0$.

The planar pendulum equation is obtained by taking $q^1 = p_1 = 0$ and $q^2(t) = \sin\psi(t)$ and $q^3(t) = -\cos\psi(t)$.

Then the above equations lead to the usual equation of motion:

$$\ddot{\psi} = -\sin\psi$$

Optimal control and implicit Hamiltonian systems

Nonlinear control systems

Let

- Q be a smooth n-dimensional manifold

- U be a smooth m-dimensional manifold

- $F : Q \times U \to TQ$ be a smooth function, such that

  $$\pi_Q \circ F = \rho, \quad \text{i.e. } F(q,u) = (q, f(q,u)) \text{ with } f(q,u) \in T_q Q,$$

  with $\pi_Q : TQ \to Q$ and $\rho : Q \times U \to Q$

Note that F defines a set of smooth vector fields $\{f(\cdot, u)\}_{u \in U}$ on Q.

Then a nonlinear control system is defined by

$$\dot{q} = f(q,u), \quad (q, \dot{q}, u) \in TQ \times U$$

Optimal control problems

Let

- $L : Q \times U \to \mathbb{R}$ be a smooth function (called "Lagrangian")

- $K : Q \to \mathbb{R}$ be a smooth function ("end cost")

Define the cost functional $J : Q^{\mathbb{R}} \times U^{\mathbb{R}} \to \mathbb{R}$

$$J(q(\cdot), u(\cdot)) = \int_0^T L(q(t), u(t))dt + K(q(T))$$

The (fixed time, free terminal point) optimal control problem is defined by:

minimize J *under the constraints* $\dot{q} = f(q, u)$ *and* $q(0) = q_0 \in Q$.

The Maximum Principle

Let (q, p) be local coordinates for T^*Q and define the Hamiltonian

$$H(q,p,u) = p^T f(q,u) - L(q,u), \quad (q,p,u) \in T^*Q \times U$$

Theorem (M.P.) A necessary condition for $(q(t), u(t))$ to be a solution of the optimal control problem is the following: there exists a smooth curve $(q(t), p(t), u(t)) \in T^*Q \times U$ such that

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), u(t))$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), u(t))$$

$$0 = \frac{\partial H}{\partial u}(q(t), p(t), u(t))$$

under the boundary conditions $q(0) = q_0$ and $p(T) = -\frac{\partial K}{\partial q}(q(T))$.

Remarks

- The boundary conditions

$$p(T) = -\frac{\partial K}{\partial q}(q(T))$$

are called the transversality conditions

- The general condition is

$$H(q(t), p(t), u(t)) = \max_{\widehat{u} \in U} H(q(t), p(t), \widehat{u})$$

hence the name maximum principle.

- Throughout we will assume that the optimal control problem has a solution, and only discuss the geometry of the problem.

Example – Euler-Lagrange equations

A typical variational problem is given by

minimize $\int_0^T L(q, \dot{q})dt$ *over all smooth curves* $q(t) \in \mathbb{R}^n$ *satisfying* $q(0) = q_0$.

Setting the first order variation $\delta \int_0^T L(q, \dot{q})dt$ to zero leads to the Euler-Lagrange equations.

Alternatively, define the optimal control problem

minimize $\int_0^T L(q, u)dt$ *under the constraints* $\dot{q} = u$ *and* $q(0) = q_0 \in Q$.

Define the Hamiltonian (Legendre transform)

$$H(q, p, u) = p^T u - L(q, u)$$

Then the Maximum Principle yields the necessary conditions

$$\begin{aligned}
\dot{q}(t) &= \frac{\partial H}{\partial p}(q(t), p(t), u(t)) && = u(t), \\
\dot{p}(t) &= -\frac{\partial H}{\partial q}(q(t), p(t), u(t)) && = \frac{\partial L}{\partial q}(q(t), u(t)), \\
0 &= \frac{\partial H}{\partial u}(q(t), p(t), u(t)) && = p(t) - \frac{\partial L}{\partial u}(q(t), u(t)).
\end{aligned}$$

These are equivalent to the Euler-Lagrange equations

$$\frac{\mathbf{d}}{\mathbf{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))\right) - \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = 0$$

When the regularity condition $\det\left(\frac{\partial^2 L}{\partial u_i \partial u_j}\right)_{i,j=1,\ldots,n} \neq 0$ is satisfied, then the Hamiltonian equations follow:

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t))$$

Implicit Hamiltonian system

Let $\omega = \mathbf{d}q \wedge \mathbf{d}p$ be the canonical symplectic form on T^*Q, and let ω^e be its trivial extension to $T^*Q \times U$. In local coordinates

$$\omega^e(q,p,u) = \mathbf{d}q \wedge \mathbf{d}p, \quad (q,p,u) \in T^*Q \times U$$

Define the closed Dirac structure D on $T^*Q \times U$ by $(x = (q,p,u))$

$$D(x) = \{(v, v^*) \in T_x(T^*Q \times U) \times T_x^*(T^*Q \times U) \mid \; v^* = \omega^e v\}$$

(pre-symplectic structure). I.e.,

$$D(x) = \left\{ (v, v^*) \in T_x(T^*Q \times U) \times T_x^*(T^*Q \times U) \middle| \begin{bmatrix} v_q^* \\ v_p^* \\ v_u^* \end{bmatrix} \right.$$
$$\left. = \begin{bmatrix} 0 & -\mathrm{I}_n & 0 \\ \mathrm{I}_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_q \\ v_p \\ v_u \end{bmatrix} \right\}$$

Then the optimal control problem defines an implicit Hamiltonian system $(T^*Q \times U, D, H)$.

Symmetries of optimal control problems

Consider a Lie group G, with smooth left action $\phi : G \times Q \to Q$.

G is a symmetry Lie group of the optimal control problem if

$$[f(\cdot,u),\xi_Q] = 0,\ L_{\xi_Q}L(\cdot,u) = 0,\ \forall u \in U \text{ and } L_{\xi_Q}K = 0, \quad \forall \xi \in \mathfrak{g}$$

Assume the action of G is free and proper so that Q/G is a smooth manifold, with canonical projection $\pi : Q \to Q/G$ a surjective submersion.

Then

- $f(\cdot,u)$ projects to a vector field $\hat{f}(\cdot,u) = \pi_* f(\cdot,u)$ on Q/G, for all $u \in U$

- $L(\cdot,u)$ projects to a function $\hat{L}(\cdot,u) : Q/G \to \mathbb{R}$, by $L(\cdot,u) = \hat{L}(\cdot,u) \circ \pi$, for all $u \in U$

- K projects to a smooth function $\hat{K}$ on Q/G, by $K = \hat{K} \circ \pi$

The cost functional $\hat{J} : (Q/G)^{\mathbb{R}} \times U^{\mathbb{R}} \to \mathbb{R}$ becomes

$$\hat{J}(\hat{q}(\cdot), u(\cdot)) = \int_0^T \hat{L}(\hat{q}(t), u(t))dt + \hat{K}(\hat{q}(T))$$

The reduced optimal control problem is given by

minimize $\hat{J}$ *under the constraints* $\dot{\hat{q}} = \hat{f}(\hat{q}, u)$ *and* $\hat{q}(0) = \pi(q_0) \in Q$.

If $(q(t), u(t))$ is a solution of the optimal control problem, then $(\hat{q}(t), u(t)) = (\pi(q(t)), u(t))$ is a solution of the reduced optimal control problem, and

$$J(q(\cdot), u(\cdot)) = \hat{J}(\hat{q}(\cdot), u(\cdot))$$

Define the reduced Hamiltonian

$$\widehat{H}(\widehat{q}, \widehat{p}, u) = \widehat{p}^T \widehat{f}(\widehat{q}, u) - \widehat{L}(\widehat{q}, u), \quad (\widehat{q}, \widehat{p}, u) \in T^*(Q/G) \times U$$

Note that $\widehat{H}(\widehat{q}, \widehat{p}, u) = H(q, \pi_q^*(\widehat{p}), u)$, where $\pi_q^* : T^*_{\widehat{q}}(Q/G) \to T^*_q Q$ is the dual of $T_q\pi : T_qQ \to T_{\widehat{q}}(Q/G)$.

The Maximum principle yields

$$\dot{\widehat{q}}(t) = \frac{\partial \widehat{H}}{\partial \widehat{p}}(\widehat{q}(t), \widehat{p}(t), u(t)),$$

$$\dot{\widehat{p}}(t) = -\frac{\partial \widehat{H}}{\partial \widehat{q}}(\widehat{q}(t), \widehat{p}(t), u(t)),$$

$$0 = \frac{\partial \widehat{H}}{\partial u}(\widehat{q}(t), \widehat{p}(t), u(t))$$

with the boundary conditions $\widehat{q}(0) = \pi(q_0)$ and $\widehat{p}(T) = -\frac{\partial \widehat{K}}{\partial \widehat{q}}(\widehat{q}(T))$.

This defines a reduced implicit Hamiltonian system $(T^*(Q/G) \times U, \widehat{D}, \widehat{H})$

Alternatively, consider the implicit Hamiltonian system $(T^*Q \times U, D, H)$.

The action ϕ of G on Q lifts to an action ψ on T^*Q by

$$\psi_g = (\phi_{g^{-1}})^*, \quad \forall g \in G$$

This action leaves the sympletic form $\omega = \mathbf{d}q \wedge \mathbf{d}p$ invariant, and hence is Hamiltonian, i.e.:

$$\omega(\xi_{T^*Q}) = \mathbf{d}P_\xi, \quad \forall \xi \in \mathfrak{g}$$

with Ad^*-equivariant momentum map $P : T^*Q \to \mathfrak{g}^*$ defined by

$$P(q,p)(\xi) = p^T \xi_Q(q), \quad \forall\, \xi \in \mathfrak{g},\ (q,p) \in T^*Q.$$

Consider the trivial fiber bundle $M = T^*Q \times U$ and the projections $\pi_{T^*Q} : M \to T^*Q$ and $\pi_U : M \to U$.

Define the trivial extensions

- $\xi^e_{T^*Q} \in TM$ such that $T\pi_{T^*Q}(\xi^e_{T^*Q}) = \xi_{T^*Q}$ and $T\pi_U(\xi^e_{T^*Q}) = 0$

- $P^e : M \to \mathfrak{g}^*$ such that $P^e = P \circ \pi_{T^*Q}$

Then $\omega^e(\xi^e_{T^*Q}) = \mathbf{d}P^e_\xi$, $\forall \xi \in \mathfrak{g}$, and hence

$$(\xi^e_{T^*Q}, \mathbf{d}P^e_\xi) \in \mathfrak{D}$$

Since D is closed this implies that $\xi^e_{T^*Q}$ is a symmetry of D.

In local coordinates, if $\xi_Q(q) = h(q)\partial_q$, then

$$\xi^e_{T^*Q}(q,p,u) = h(q)\partial q - p^T \frac{\partial h}{\partial q}(q)\partial p$$

and hence

$$\begin{aligned} L_{\xi^e_{T^*Q}} H(q,p,u) &= p^T \frac{\partial f}{\partial q}(q,u)h(q) - \frac{\partial L}{\partial q}(q,u)h(q) - p^T \frac{\partial h}{\partial q}(q) f(q,u) \\ &= p^T [f(\cdot,u), \xi_Q(\cdot)](q) - L_{\xi_Q} L(q,u) \\ &= 0, \quad \forall\, (q,p,u) \in T^*Q \times U \end{aligned}$$

This implies that G is a symmetry Lie group of $(T^*Q \times U, D, H)$, and hence P^e is a first integral.

In fact, the transversality conditions imply

$$P(q(T),p(T))(\xi) = p(T)^T \xi_Q(q(T)) = -(L_{\xi_Q} K)(q(T)) = 0, \quad \forall \xi \in \mathfrak{g}$$

hence the momentum map P^e has constant value zero.

The implicit Hamiltonian system $(T^*Q \times U, D, H)$ can be reduced to an implicit Hamiltonian system (M_0, D_0, H_0) on $M_0 = (P^e)^{-1}(0)/G$.

The reduced Dirac structure is given in terms of local section by

$$(\mathfrak{D}_0)_{loc} = \{(\tilde{X}, \tilde{\alpha}) \in \mathfrak{X}(M_0) \oplus \Omega^1(M_0) \mid \tilde{\alpha} = \omega_0^e(\tilde{X})\}$$

where ω_0^e is the trivial extension of the symplectic form ω_0 on $P^{-1}(0)/G$ defined by

$$\pi_0^* \omega_0 = \iota^* \omega$$

with $\iota : P^{-1}(0) \to T^*Q$ and $\pi_0 : P^{-1}(0) \to P^{-1}(0)/G$.

Standard classical references show that $(P^{-1}(0)/G, \omega_0)$ is symplectomorphic to $T^*(Q/G)$ with its canonical symplectic form $\widehat{\omega}$.

Let $\tau : P^{-1}(0)/G \to T^*(Q/G)$ denote the symplectomorphism between $(P^{-1}(0)/G, \omega_0)$ and $(T^*(Q/G), \widehat{\omega})$.

Let $\tau^e : M_0 \to T^*(Q/G) \times U$ be its trivial extension. Then

- D_0 and $\widehat{D}$ are isomorphic via τ^e
- $H_0 = \widehat{H} \circ \tau^e$

Theorem (M_0, D_0, H_0) is isomorphic to $(T^*(Q/G) \times U, \widehat{D}, \widehat{H})$.

In other words: the Maximum Principle commutes with reduction.

Generalized symmetries

Recall the implicit Hamiltonian system $(T^*Q \times U, D, H)$ with Dirac structure

$$D(x) = \{(v, v^*) \in T_x(T^*Q \times U) \times T_x^*(T^*Q \times U) \mid \ v^* = \omega^e v\}$$

Recall the bundle $M = T^*Q \times U$ and its projections $\pi_{T^*Q} : M \to T^*Q$ and $\pi_U : M \to U$.

A vector field $X \in \mathfrak{X}(M)$ is horizontal if $(\pi_U)_* X = 0$, and vertical if $(\pi_{T^*Q})_* X = 0$.

The (co-)distributions of D are given by

$$\begin{aligned} \mathsf{G}_0 &= T_{\text{vert}}\,(T^*Q \times U), & \mathsf{G}_1 &= T(T^*Q \times U), \\ \mathsf{P}_0 &= 0, & \mathsf{P}_1 &= T^*_{\text{hor}}\,(T^*Q \times U). \end{aligned}$$

The set of admissible functions is given by

$$\mathcal{A}_D = \{H \in C^\infty(T^*Q \times U) \mid H(q,p,u) = H(q,p)\}.$$

Let $Y \in \mathfrak{X}(M)$ be a symmetry of D, in local coordinates

$$Y(q,p,u) = Y_q(q,p,u)\frac{\partial}{\partial q} + Y_p(q,p,u)\frac{\partial}{\partial p} + Y_u(q,p,u)\frac{\partial}{\partial u}$$

Then

- $L_Y \mathsf{G}_0 \subset \mathsf{G}_0$ (so $[Y, \frac{\partial}{\partial u}] \in$ span $\{\frac{\partial}{\partial u}\}$) implies $Y_q(q,p,u) = Y_q(q,p)$ and $Y_p(q,p,u) = Y_p(q,p)$

- $L_Y\{H_1, H_2\}_D = \{L_Y H_1, H_2\}_D + \{H_1, L_Y H_2\}_D$, for all $H_1, H_2 \in \mathcal{A}_D$ implies

$$L_{\tilde{Y}}\omega = 0, \text{ where } \tilde{Y} = (\pi_{T^*Q})_* Y$$

Hence $\tilde{Y}$ is locally Hamiltonian, i.e. $\exists \tilde{H} \in C^\infty(T^*Q)$ such that

$$\mathbf{d}\tilde{H} = i_{\tilde{Y}}\omega$$

Define $\tilde{H}^e = \tilde{H} \circ \pi_{T^*Q}$, then

$$(Y, \mathbf{d}\tilde{H}^e) \in \mathfrak{D}$$

Noether theorem for optimal control problems

Let $Y \in \mathfrak{X}(T^*Q \times U)$ be a weak symmetry of $(T^*Q \times U, D, H)$. Then there exists (locally) a function $\tilde{H} \in C^\infty(T^*Q)$ such that $\tilde{H}^e$ is a first integral of $(T^*Q \times U, D, H)$.

Conversely, let $\tilde{H} \in C^\infty(T^*Q)$ be such that $\tilde{H}^e$ is a first integral of $(T^*Q \times U, D, H)$. Then the horizontal vector field $\tilde{Y} \in \mathfrak{X}(T^*Q \times U)$ defined by $\mathbf{d}\tilde{H} = i_{\tilde{Y}}\omega$ is a weak symmetry of $(T^*Q \times U, D, H)$.

Reduction for configuration-control symmetries

Consider a Lie group G with smooth left action on both the configurations and the control, defined by

$$\theta : G \times Q \times U \to Q \times U$$

such that $\phi_g \circ \pi_Q = \pi_Q \circ \theta_g$ for some smooth left action $\phi : G \times Q \to Q$.

That is

$$\theta_g(q,u) = \Big(\phi_g(q),\, \theta_g^u(q,u)\Big)$$

This defines a symmetry of the optimal control problem if

$$T\phi_g \cdot f(\cdot,\cdot) = f \circ \theta_g(\cdot,\cdot),\; L \circ \theta_g = L,\; K \circ \phi_g = K, \quad \forall g \in G.$$

The action θ of G on $Q \times U$ lifts to an action $\widehat{\theta}$ on $T^*Q \times U$ by

$$\widehat{\theta}(q,p,u) = \Big((\phi_{g^{-1}})^*(q,p),\ \theta^u_g(q,u)\Big)$$

It follows that $H \circ \widehat{\theta} = H$, hence G is a symmetry Lie group of the implicit Hamiltonian system $(T^*Q \times U, D, H)$.

Regular state feedback

Recall the nonlinear control system $\dot{q} = f(q, u)$.

Consider a vector bundle isomorphism $\gamma : Q \times U \to Q \times U$, i.e. $\pi_Q \circ \gamma = \pi_Q$ and $\gamma_q : U \to U$ is a diffeomorphism for every $q \in Q$.

A regular state feedback is defined by: $(q, u) = \gamma(q, v) = (q, \gamma_q(v))$

It defines a new control system: $\dot{q} = f \circ \gamma(q, v) = f(q, \gamma_q(v))$

The optimal control problem is invariant under regular state feedback.

There exists a regular state feedback $\gamma : Q \times U \to Q \times U$ such that the optimal control problem defined by

$$f' = f \circ \gamma,\ L' = L \circ \gamma,\ K' = K$$

has state space symmetry $\phi : G \times Q \to Q$, i.e.

$$[f'(\cdot, v), \xi_Q] = 0,\ L_{\xi_Q} L'(\cdot, v) = 0,\ \forall v \in U \text{ and } L_{\xi_Q} K' = 0, \quad \forall \xi \in \mathfrak{g}$$

and we can reduce the optimal control problem to $(Q/G) \times U$.

Alternatively, define a bundle isomorphism $\gamma^e : T^*Q \times U \to T^*Q \times U$:

$$\gamma^e(q, p, v) = (q, p, \gamma_q(v))$$

Then $\gamma^*\omega^e = \omega^e$ and $H' = H \circ \gamma^e$.

Hence the two implicit Hamiltonian systems $(T^*Q \times U, D, H)$ and $(T^*Q \times U, D', H')$ are isomorphic.

$(T^*Q \times U, D', H')$ has state space symmetry Lie group G and can be reduced as described previously.

Constrained optimal control problems

Consider the nonlinear control system

$$\dot{q} = f(q, u)$$

subject to the constraints

$$b(q, u) = 0$$

Assume the following regularity condition is satisfied:

$$\frac{\partial b}{\partial u}(q, u) \ \text{ has full row rank } \ \forall (q, u) \in Q \times U$$

The constraints can be holonomic

$$h(q) = 0 \quad \Rightarrow \quad b(q, u) = \frac{\partial h}{\partial q}^T (q) f(q, u) = 0$$

or nonholonomic

$$h(q, \dot{q}) = 0 \quad \Rightarrow \quad b(q, u) = h(q, f(q, u)) = 0$$

Let the cost functional be given by

$$J(q(\cdot), u(\cdot)) = \int_0^T L(q(t), u(t))dt + K(q(T))$$

Then the constrained optimal control problem is defined by

minimize J under the constraints $\dot{q} = f(q, u)$ and $b(q, u) = 0$ and $q(0) = q_0 \in Q$.

Again, define the Hamiltonian

$$H(q, p, u) = p^T f(q, u) - L(q, u), \quad (q, p, u) \in T^*Q \times U$$

The Maximum principle

A necessary condition for $(q(t), u(t))$ to be a solution of the optimal control problem is the following: there exists a smooth curve $(q(t), p(t), u(t)) \in T^*Q \times U$ such that

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), u(t)),$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), u(t)) + \frac{\partial b}{\partial q}(q(t), u(t))\lambda(t),$$

$$0 = \frac{\partial H}{\partial u}(q(t), p(t), u(t)) - \frac{\partial b}{\partial u}(q(t), u(t))\lambda(t),$$

along with the constraints

$$b(q(t), u(t)) = 0$$

and the boundary conditions $q(0) = q_0$ and $p(T) = -\frac{\partial K}{\partial q}(q(T))$.

Differentiate the constraints to get

$$\frac{\partial b}{\partial q}(q(t), u(t))\dot{q}(t) + \frac{\partial b}{\partial u}(q(t), u(t))\dot{u}(t) = 0$$

The constrained optimal control problem defines an implicit Hamiltonian system $(T^*Q \times U, D, H)$ with Dirac structure

$$D(x) = \{(v, v^*) \in T_x(T^*Q \times U) \times T_x^*(T^*Q \times U) \mid$$

$$\begin{bmatrix} v_q^* \\ v_p^* \\ v_u^* \end{bmatrix} - \begin{bmatrix} 0 & -\mathrm{I}_n & 0 \\ \mathrm{I}_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_q \\ v_p \\ v_u \end{bmatrix} \in \text{span} \begin{bmatrix} \frac{\partial b}{\partial q}^T \\ 0 \\ \frac{\partial b}{\partial u}^T \end{bmatrix},$$

$$0 = [\ \frac{\partial b}{\partial q} \quad 0 \quad \frac{\partial b}{\partial u}\ ] \begin{bmatrix} v_q \\ v_p \\ v_u \end{bmatrix} \},$$

where $x = (q, p, u) \in T^*Q \times U$.

Define the distribution

$$\mathsf{G}_1 = (\mathbf{d}b)^{\circ}$$

Then the Dirac structure D can be written as

$$D(x) = \{(v, v^*) \in T_x(T^*Q \times U) \times T_x^*(T^*Q \times U) \mid v^* - \omega^e v \in \mathsf{G}_1^{\circ}(x),\ v \in \mathsf{G}_1(x)\}$$

The regularity condition implies that G_1 is constant dimensional. As the kernel of an exact one-form it is also involutive.

Hence, D is closed.

(Note that D is *not* a pre-symplectic structure.)

Consider a Lie group G, with smooth left action $\phi : G \times Q \to Q$.

G is a symmetry Lie group of the constrained optimal control problem if

$$[f(\cdot, u), \xi_Q] = 0,\ L_{\xi_Q} L(\cdot, u) = 0,\ \forall u \in U \text{ and } L_{\xi_Q} K = 0, \quad \forall \xi \in \mathfrak{g}$$

and

$$L_{\xi_Q} b(\cdot, u) = 0, \quad \forall\, \xi \in \mathfrak{g}.$$

The reduced constrained optimal control problem is given by

$$\textit{minimize } \widehat{J} \textit{ under the constraints } \dot{\hat{q}} = \widehat{f}(\widehat{q}, u) \textit{ and } \widehat{b}(\widehat{q}(t), u(t)) = 0$$
$$\textit{and } \widehat{q}(0) = \pi(q_0) \in Q.$$

This defines a reduced implicit Hamiltonian system $(T^*(Q/G) \times U, \widehat{D}, \widehat{H})$.

Alternatively, the action ϕ of G on Q lifts to an action ψ on T^*Q:

$$\psi_g = (\phi_{g^{-1}})^*, \quad \forall g \in G,$$

which is Hamiltonian

$$\omega(\xi_{T^*Q}) = \mathbf{d}P_\xi, \quad \forall \xi \in \mathfrak{g}.$$

Now, $L_{\xi_Q} b(\cdot, u) = 0$ implies $\xi^e_{T^*Q} \in \mathsf{G}_1$. Hence

$$(\xi^e_{T^*Q}, \mathbf{d}P^e_\xi) \in \mathfrak{D}$$

and it follows that G is a symmetry Lie group of $(T^*Q \times U, D, H)$, with first integral P^e.

The system can be reduced to an implicit Hamiltonian system (M_0, D_0, H_0) on $M_0 = (P^e)^{-1}(0)/G$.

Again, the Maximum Principle commutes with reduction.

Generalized symmetries

The (co-)distributions of D are given by

$$\mathsf{G}_0 = T_{\text{vert}}(T^*Q \times U) \cap (\mathbf{d}b)^\circ, \qquad \mathsf{G}_1 = (\mathbf{d}b)^\circ,$$
$$\mathsf{P}_0 = \text{span}\,\{\mathbf{d}b\}, \qquad \mathsf{P}_1 = T^*_{\text{hor}}(T^*Q \times U) + \text{span}\,\{\mathbf{d}b\}.$$

The admissible functions include

$$\mathcal{A}_D \supset \{H \in C^\infty(T^*Q \times U) \mid H(q,p,u) = H(q,p)\}.$$

Let $Y \in \mathfrak{X}(M)$ be a symmetry of D, in local coordinates

$$Y(q,p,u) = Y_q(q,p,u)\frac{\partial}{\partial q} + Y_p(q,p,u)\frac{\partial}{\partial p} + Y_u(q,p,u)\frac{\partial}{\partial u}$$

Assume $[Y, T_{\text{vert}}(T^*Q \times U)] \subset T_{\text{vert}}(T^*Q \times U)$, then $Y_q(q,p,u) = Y_q(q,p)$ and $Y_p(q,p,u) = Y_p(q,p)$.

Again, $L_Y\{H_1, H_2\}_D = \{L_Y H_1, H_2\}_D + \{H_1, L_Y H_2\}_D$, for all $H_1, H_2 \in \mathcal{A}_D$ implies

$$L_{\tilde{Y}}\omega = 0, \text{ where } \tilde{Y} = (\pi_{T^*Q})_* Y,$$

hence $\tilde{Y}$ is locally Hamiltonian, i.e. $\exists \tilde{H} \in C^\infty(T^*Q)$ such that

$$\mathbf{d}\tilde{H} = i_{\tilde{Y}}\omega$$

Define $\tilde{H}^e = \tilde{H} \circ \pi_{T^*Q}$ and assume $Y \in \mathsf{G}_1$, then

$$(Y, \mathbf{d}\tilde{H}^e) \in \mathfrak{D}$$

Noether theorem for constrained optimal control problems

Let $Y \in \mathfrak{X}(T^*Q \times U)$ be a weak symmetry of $(T^*Q \times U, D, H)$ such that $[Y, T_{\text{vert}}(T^*Q \times U)] \subset T_{\text{vert}}(T^*Q \times U)$ and $Y \in \mathsf{G}_1$.

Then there exists (locally) a function $\tilde{H} \in C^\infty(T^*Q)$ such that $\tilde{H}^e$ is a first integral of $(T^*Q \times U, D, H)$.

References

G. Blankenstein and A.J. van der Schaft, Optimal control and implicit Hamiltonian systems, In A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek, editors, *Nonlinear Control in the Year 2000*, pp. 185–206, Springer-Verlag, 2000

(See also Chapter 6 in G. Blankenstein, *Implicit Hamiltonian systems: symmetry and interconnection*, PhD Thesis, University of Twente, The Netherlands, November 2000.)

Port-Hamiltonian formulation of distributed parameter systems

The Hamiltonian formulation of distributed parameter systems is usually based on the definition of a Poisson bracket with the use of the differential operator $\frac{\mathbf{d}}{\mathbf{d}z}$.

Assuming

- an infinite spatial domain where variables go to zero as $|z| \to \infty$
- or, bounded domain with boundary conditions such that the energy exchange through the boundary is zero

Then $\frac{\mathbf{d}}{\mathbf{d}z}$ is a skew-symmetric operator (by integration by parts) and defines a Poisson structure.

Example: inviscid Burger's equation

One-dimensional spatial domain $M = [a, b] \subset \mathbb{R}$, state variable $\alpha(z, t)$, $z \in \mathbb{R}$, is the Eulerian velocity

$$\frac{\partial \alpha}{\partial t} + \alpha \frac{\partial \alpha}{\partial z} = 0$$

Define the Hamiltonian

$$H(\alpha) = \int_a^b \frac{\alpha^3}{6} \mathbf{d}z$$

The variational derivative is given by $\frac{\delta H}{\delta \alpha} = \frac{\alpha^2}{2}$ and hence the system can be written as the infinite-dimensional Hamiltonian system

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha}$$

Recall the definition of the **variational derivative**

$$H(\alpha + \epsilon\eta) = H(\alpha) + \epsilon \int_a^b \frac{\delta H}{\delta \alpha} \eta \mathbf{d}z + O(\epsilon^2)$$

Then

$$\begin{aligned} H(\alpha + \epsilon\eta) &= \int_a^b \frac{(\alpha + \epsilon\eta)^3}{6} \mathbf{d}z \\ &= \int_a^b \frac{\left(\alpha^3 + 3\epsilon\alpha^2\eta + O(\epsilon^2)\right)}{6} \mathbf{d}z \\ &= H(\alpha) + \epsilon \int_a^b \frac{\alpha^2}{2} \eta \mathbf{d}z + O(\epsilon^2) \end{aligned}$$

and hence

$$\frac{\delta H}{\delta \alpha} = \frac{\alpha^2}{2}$$

The differential operator $-\frac{\partial}{\partial z}$ defines a bracket

$$\{H_1, H_2\} = \int_a^b \frac{\delta H_1}{\delta \alpha} \cdot \left(-\frac{\partial}{\partial z}\right) \frac{\delta H_2}{\delta \alpha} \mathbf{d} z$$

which is a Poisson bracket under zero boundary conditions

$$\begin{aligned} \{H_1, H_2\} &= \int_a^b \frac{\delta H_1}{\delta \alpha} \cdot \left(-\frac{\partial}{\partial z}\right) \frac{\delta H_2}{\delta \alpha} \mathbf{d} z \\ &= \int_a^b \frac{\partial}{\partial z} \frac{\delta H_1}{\delta \alpha} \cdot \frac{\delta H_2}{\delta \alpha} \mathbf{d} z - \left. \frac{\delta H_1}{\delta \alpha} \cdot \frac{\delta H_2}{\delta \alpha} \right|_a^b \\ &= -\{H_2, H_1\} \end{aligned}$$

iff

$$\left. \frac{\delta H_1}{\delta \alpha} \cdot \frac{\delta H_2}{\delta \alpha} \right|_a^b = 0$$

Under zero boundary conditions (e.g., $\alpha(a,t) = \alpha(b,t) = 0$) the Hamiltonian is conserved

$$\dot{H} = \{H, H\} = -\{H, H\} = 0$$

However, in general

$$\begin{aligned}
\frac{\mathbf{d}}{\mathbf{d}t} H(\alpha) &= \int_a^b \frac{\delta H}{\delta \alpha} \cdot \frac{\partial \alpha}{\partial t} \mathbf{d}z \\
&= \int_a^b \frac{\delta H}{\delta \alpha} \cdot -\frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha} \mathbf{d}z \\
&= \frac{1}{2} \left(\beta^2(a) - \beta^2(b) \right)
\end{aligned}$$

where $\beta = \frac{\delta H}{\delta \alpha} = \frac{\alpha^2}{2}$.

Hence the Hamiltonian is not constant anymore, but depends on the boundary conditions.

Example: the vibrating string

One dimensional spatial domain $M = [a, b] \in \mathbb{R}$. Let

- $x(z, t)$ denote the displacement of the string, and
- $v(z, t)$ its velocity

Then the vibrating string is described by

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{1}{\mu}\frac{\partial}{\partial z}\left(T\frac{\partial x}{\partial z}\right) \end{aligned}$$

with μ the mass density and T the elasticity modulus of the string.

This is equivalent to the wave equation

$$\mu\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial z}\left(T\frac{\partial x}{\partial z}\right)$$

Hamiltonian formulation of the vibrating string

Define the momentum $\alpha_1 = \mu v$ and the strain $\alpha_2 = \frac{\partial x}{\partial z}$.

The total (kinetic + potential) energy is given by

$$H = \int_a^b \frac{1}{2}\left(\frac{1}{\mu}\alpha_1^2 + T\alpha_2^2\right)\mathbf{d}z$$

The variational derivative is defined as

$$H(\alpha_1+\epsilon\eta_1, \alpha_2+\epsilon\eta_2) = H(\alpha_1, \alpha_2)+\epsilon\int_a^b\left(\frac{\delta H}{\delta\alpha_1}\eta_1 + \frac{\delta H}{\delta\alpha_2}\eta_2\right)\mathbf{d}z + O(\epsilon^2)$$

which simply yields $\frac{\delta H}{\delta\alpha_1} = \frac{\alpha_1}{\mu}$ and $\frac{\delta H}{\delta\alpha_2} = T\alpha_2$

The dynamics of the vibrating string can be written as the Hamiltonian system

$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}}_{J} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}$$

Again, J defines a Poisson bracket under zero boundary conditions.

E.g.

$$\alpha_1(a,t) = \alpha_1(b,t) = \mu \frac{\partial x}{\partial t}(z,t)\Big|_{z=a,b} = 0,$$

in which case the total energy is conserved: $\dot{H} = 0$.

However, in case of free boundary conditions we get

$$
\begin{aligned}
\frac{\mathbf{d}}{\mathbf{d}t}H &= \int_a^b \left(\frac{\delta H}{\delta \alpha_1} \cdot \frac{\partial \alpha_1}{\partial t} + \frac{\delta H}{\delta \alpha_2} \cdot \frac{\partial \alpha_1}{\partial t} \right) \mathbf{d}z \\
&= \int_a^b \left(\frac{\delta H}{\delta \alpha_1} \cdot \frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha_2} + \frac{\delta H}{\delta \alpha_2} \cdot \frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha_1} \right) \mathbf{d}z \\
&= \frac{\delta H}{\delta \alpha_1} \cdot \frac{\delta H}{\delta \alpha_2} \Bigg|_a^b \\
&= f(b) \cdot e(b) - f(a) \cdot e(a)
\end{aligned}
$$

where we defined the boundary variables

- $f = \frac{\delta H}{\delta \alpha_1}\Big|_{z=a,b}$, the velocity, and
- $e = \frac{\delta H}{\delta \alpha_2}\Big|_{z=a,b}$, the stress

The energy balance

$$\frac{\mathbf{d}}{\mathbf{d}t}H = f(b) \cdot e(b) - f(a) \cdot e(a)$$

shows that the time-derivative of the total energy is the balance of the mechanical work done at the boundary points $z = a$ and $z = b$.

The system can be written as

$$\left(\frac{\partial \alpha_1}{\partial t}, \frac{\partial \alpha_2}{\partial t}, f, \frac{\delta H}{\delta \alpha_1}, \frac{\delta H}{\delta \alpha_2}, -e \right) \in D$$

where

$$D = \left\{ (u_1, u_2, f, y_1, y_2, -e) \mid \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} f \\ -e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Bigg|_{z=a,b} \right\}$$

is a Dirac structure

Note that the system has two conservation laws:

- The total momentum $P = \int_a^b \alpha_1(z,t)\mathbf{d}z$, with

$$\frac{\mathbf{d}}{\mathbf{d}t}P(t) = e_b(1,t) - e_b(0,t)$$

- The total strain $S = \int_a^b \alpha_2(z,t)\mathbf{d}z$, with

$$\frac{\mathbf{d}}{\mathbf{d}t}S = f_b(1,t) - f_b(0,t)$$

I.e., their time-derivatives only depend on the boundary variables.

The Stokes-Dirac structure

Let (everything assumed to be smooth)

- M be a n-dimensional manifold,
- with $(n-1)$-dimensional boundary ∂M.
- $\Omega^k(M)$, $k = 0, 1, \dots n$, be the space of k-forms on M,
- $\Omega^k(\partial M)$, $k = 0, 1, \dots, n-1$ be the space of k-forms on ∂M

There is a nondegenerate pairing between $\Omega^k(M)$ and $\Omega^{n-k}(M)$ defined by

$$\langle \beta, \alpha \rangle = \int_M \beta \wedge \alpha, \quad \alpha \in \Omega^k(M),\ \beta \in \Omega^{n-k}(M)$$

Similarly,

$$\langle \beta, \alpha \rangle = \int_{\partial M} \beta \wedge \alpha, \quad \alpha \in \Omega^k(\partial M),\ \beta \in \Omega^{n-1-k}(\partial M)$$

Hence $\Omega^k(M)$ and $\Omega^{n-k}(M)$, respectively $\Omega^k(\partial M)$ and $\Omega^{n-1-k}(\partial M)$, can be regarded as dual spaces.

Define the linear space

$$\mathcal{F} = \Omega^p(M) \times \Omega^q(M) \times \Omega^{n-p}(\partial M)$$

and its dual

$$\mathcal{F}^* = \Omega^{n-p}(M) \times \Omega^{n-q}(M) \times \Omega^{n-q}(\partial M)$$

for any pair p, q of positive integers satisfying

$$p + q = n + 1$$

(Note that $(n-p) + (n-q) = n - 1$.)

There is a pairing between $\mathcal{F}$ and $\mathcal{F}^*$

$$\langle (y_p, y_q, -e_b), (u_p, u_q, f_b) \rangle = \int_M (y_p \wedge u_p + y_q \wedge u_q) - \int_{\partial M} e_b \wedge f_b$$

with

$$u_p \in \Omega^p(M), \quad u_q \in \Omega^q(M), \quad f_b \in \Omega^{n-p}(\partial M),$$
$$y_p \in \Omega^{n-p}(M), \quad y_q \in \Omega^{n-q}(M), \quad -e_b \in \Omega^{n-q}(\partial M)$$

Symmetrize the pairing to get a nondegenerate bilinear form on $\mathcal{F} \times \mathcal{E}$:

$$\langle\langle (u_p^1, u_q^1, f_b^1, y_p^1, y_q^1, -e_b^1)\ ,\ (u_p^2, u_q^2, f_b^2, y_p^2, y_q^2, -e_b^2) \rangle\rangle =$$

$$\int_M \left(y_p^1 \wedge u_p^2 + y_q^1 \wedge u_q^2 + y_p^2 \wedge u_p^1 + y_q^2 \wedge u_q^1 \right) - \int_{\partial M} \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1 \right)$$

Theorem The subspace $D \subset \mathcal{F} \times \mathcal{F}^*$ defined by

$$D = \left\{ (u_p, u_q, f_b, y_p, y_q, -e_b) \mid \begin{pmatrix} u_p \\ u_q \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{pq}\, \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} \begin{pmatrix} y_q \\ y_p \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} f_b \\ -e_b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{n-q} \end{pmatrix} \begin{pmatrix} y_p \\ y_q \end{pmatrix} \Bigg|_{\partial M} \right\}$$

is a Dirac structure, i.e., $D = D^{\perp}$.

The proof is based on Stokes' theorem, hence D is also called a Stokes-Dirac structure.

Distributed parameter Port-Hamiltonian systems

Define a Hamiltonian density function

$$\mathcal{H} : \Omega^p(M) \times \Omega^q(M) \times M \to \Omega^n(M)$$

and the Hamiltonian

$$H = \int_M \mathcal{H}$$

Let $\alpha_p, \eta_p \in \Omega^p(M)$ and $\alpha_q, \eta_q \in \Omega^q(M)$. The variational derivative of H is defined as

$$\left(\frac{\delta H}{\delta \alpha_p}, \frac{\delta H}{\delta \alpha_q}\right) \in \Omega^{n-p}(M) \times \Omega^{n-q}(M)$$

such that

$$H(\alpha_p + \epsilon \eta_p, \alpha_q + \epsilon \eta_q) = H(\alpha_p, \alpha_q) + \epsilon \int_M \left(\frac{\delta H}{\delta \alpha_p} \wedge \eta_p + \frac{\delta H}{\delta \alpha_q} \wedge \eta_q\right) + O(\epsilon^2)$$

Consider the state variables $(\alpha_p, \alpha_q) \in \Omega^p(M) \times \Omega^q(M)$. Then the distributed Port-Hamiltonian system (M, D, H) is defined by

$$\left(\frac{\partial \alpha_p}{\partial t}, \frac{\partial \alpha_q}{\partial t}, f_b, \frac{\delta H}{\delta \alpha_p}, \frac{\delta H}{\delta \alpha_q}, -e_b \right) \in D$$

I.e.

$$\begin{pmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{pq}\,\mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix}, \quad \begin{pmatrix} f_b \\ -e_b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{n-q} \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \Bigg|_{\partial M}$$

$D = D^{\perp}$ yields the energy balance

$$\int_M \left(\frac{\delta H}{\delta \alpha_p} \wedge \frac{\partial \alpha_p}{\partial t} + \frac{\delta H}{\delta \alpha_q} \wedge \frac{\partial \alpha_q}{\partial t} \right) - \int_{\partial M} e_b \wedge f_b = 0$$

I.e.

$$\frac{\mathbf{d}}{\mathbf{d}t} H = \int_{\partial M} e_b \wedge f_b$$

In case of zero energy flow through the boundary, $\dot{H} = 0$ and H is conserved.

Example: A lossless transmission line

$M = [0, 1] \subset \mathbb{R}$.

Let

- $q(z,t)$ denote the charge density,
- $\phi(z,t)$ the flux density,
- $i(z,t)$ the current,
- $v(z,t)$ the voltage of the line

The telegrapher's equations are

$$\frac{\partial q}{\partial t} = -\frac{\partial i}{\partial z}$$
$$\frac{\partial \phi}{\partial t} = -\frac{\partial v}{\partial z}$$

Port-Hamiltonian formulation $p = q = n = 1$ hence

$$\mathcal{F} = \Omega^1(M) \times \Omega^1(M) \times \Omega^0(M)$$

and

$$\mathcal{F}^* = \Omega^0(M) \times \Omega^0(M) \times \Omega^0(M)$$

The energy variables are

- $\alpha_1 = \alpha_p = q(z,t)\mathbf{d}z \in \Omega^1(M)$, the charge density one-form,
- $\alpha_2 = \alpha_q = \phi(z,t)\mathbf{d}z \in \Omega^1(M)$, the flux density one-form

The total energy stored in the line is given by

$$H(\alpha_1, \alpha_2) = \int_0^1 \frac{1}{2}\left(\frac{\star\alpha_1 \wedge \alpha_1}{2C(z)} + \frac{\star\alpha_2 \wedge \alpha_2}{2L(z)}\right)$$

where $C(z)$ and $L(z)$ are the distributed capacitance and inductance of the line. ($\star$ denotes the Hodge star)

The dynamics can be written as

$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix}, \qquad \begin{pmatrix} f_b \\ -e_b \end{pmatrix} = \left. \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix} \right|_{z=0,1}$$

where $\frac{\delta H}{\delta \alpha_1} = \frac{\star \alpha_1}{C(z)} = \frac{q(z,t)}{C(z)} = v(z,t)$ and $\frac{\delta H}{\delta \alpha_2} = \frac{\star \alpha_2}{L(z)} = \frac{\phi(z,t)}{L(z)} = i(z,t)$

The energy balance is given by

$$\frac{\mathbf{d}H}{\mathbf{d}t} = \int_{\partial M} e_b f_b = i(t,0)v(t,0) - i(t,1)v(t,1)$$

i.e., the time derivative of the energy equals the power going into the line at $z = 0$ minus the power going out of the line at $z = 1$.

Note that the system has two conservation laws:

- The total charge $C = \int_a^b \alpha_1(z,t)\mathbf{d}z$, with

$$\frac{\mathbf{d}}{\mathbf{d}t}C(t) = e_b(1,t) - e_b(0,t)$$

- The total flux $F = \int_a^b \alpha_2(z,t)\mathbf{d}z$, with

$$\frac{\mathbf{d}}{\mathbf{d}t}F = f_b(0,t) - f_b(1,t)$$

I.e., their time-derivatives only depend on the boundary variables.

Example: Maxwell's equations Spatial domain is submanifold $M \subset \mathbb{R}^3$ with boundary ∂M. $n = 3, p = q = 2$, hence

$$\mathcal{F} = \Omega^2(M) \times \Omega^2(M) \times \Omega^1(M)$$

and

$$\mathcal{F}^* = \Omega^1(M) \times \Omega^1(M) \times \Omega^1(M)$$

The energy variables are

- $\alpha_1 = \mathcal{D} = \frac{1}{2} D_{ij}(z,t) \mathbf{d}z^i \wedge \mathbf{d}z^j$, electric field induction two-form
- $\alpha_2 = \mathcal{B} = \frac{1}{2} B_{ij}(z,t) \mathbf{d}z^i \wedge \mathbf{d}z^j$, magnetic field induction two-form

Define the (co-energy) variables

- $\mathcal{E} = E_i(z,t) \mathbf{d}z^i$, electric field intensity
- $\mathcal{M} = M_i(z,t) \mathbf{d}z^i$, magnetic field intensity

Maxwell's equations are given by

$$\frac{\partial \mathcal{D}}{\partial t} = \mathbf{d}\mathcal{M}, \quad \frac{\partial \mathcal{B}}{\partial t} = -\mathbf{d}\mathcal{E}$$

The constitutive relations of the medium are

$$\star\mathcal{D} = \epsilon\mathcal{E}, \quad \star\mathcal{B} = \mu\mathcal{M}$$

with $\epsilon(z,t)$ the electric permittivity and $\mu(z,t)$ the magnetic permeability

Define the total energy

$$H = \int_M \frac{1}{2}(\mathcal{E} \wedge \mathcal{D} + \mathcal{M} \wedge \mathcal{B}) = \int_M \frac{1}{2}\left(\epsilon^{-1} \star \mathcal{D} \wedge \mathcal{D} + \mu^{-1} \star \mathcal{B} \wedge \mathcal{B}\right)$$

so that $\frac{\delta H}{\delta \mathcal{D}} = \epsilon^{-1} \star \mathcal{D}$ and $\frac{\delta H}{\delta \mathcal{B}} = \mu^{-1} \star \mathcal{B}$

The system can be written in port-Hamiltonian form

$$\begin{pmatrix} \frac{\partial \mathcal{D}}{\partial t} \\ \frac{\partial \mathcal{B}}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \mathcal{D}} \\ \frac{\delta H}{\delta \mathcal{B}} \end{pmatrix}, \quad \begin{pmatrix} f_b \\ -e_b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \mathcal{D}} \\ \frac{\delta H}{\delta \mathcal{B}} \end{pmatrix} \Bigg|_{\partial M}$$

The energy balance is given by

$$\frac{\mathbf{d}H}{\mathbf{d}t} = \int_{\partial M} e_b \wedge f_b = \int_{\partial M} \mathcal{E} \wedge \mathcal{M}$$

where $\mathcal{E} \wedge \mathcal{M}$ is known as (minus) the Poynting vector

I.e., the time derivative of the total electromagnetic energy in M is equal to the electromagnetic power radiated through the boundary.

Ideal isentropic fluid

Euler equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v), \quad \frac{\partial v}{\partial t} = -v \cdot \nabla v - \frac{1}{\rho} \nabla p$$

where $\rho(z,t) \in \mathbb{R}$ is the mass density, $v(z,t) \in \mathbb{R}^3$ is the Eulerian velocity, and $p(z,t) \in \mathbb{R}$ the pressure function given by

$$p(z,t) = \rho^2(z,t) \frac{\partial U}{\partial \rho}(\rho(z,t))$$

for some internal energy function $U(\rho)$.

Spatial domain $\mathcal{D} \subset \mathbb{R}^3$ filled with fluid. Assume there exists a Riemannian metric $\langle , \rangle$ on $\mathcal{D}$ (e.g., Euclidean on $\mathbb{R}^3$). Let $M \subset \mathcal{D}$ 3-dimensional smooth submanifold with smooth boundary ∂M.

$n = 3, p = 3, q = 1$ hence

$$\mathcal{F} = \Omega^3(M) \times \Omega^1(M) \times \Omega^0(M)$$

and

$$\mathcal{F}^* = \Omega^0(M) \times \Omega^2(M) \times \Omega^2(M)$$

The energy variables are

- $\alpha_1 = \rho \in \Omega^3(M)$ mass-density three form
- $\alpha_2 = \bar{v} = v^\flat \in \Omega^1(M)$ Eulerian velocity one-form

The total energy is given by

$$H(\rho, \bar{v}) = \int_M \left(\frac{1}{2} \langle \bar{v}^\sharp, \bar{v}^\sharp \rangle \rho + U(\star\rho)\rho \right)$$

Hence

$$\frac{\delta H}{\delta \rho} = \frac{1}{2} \langle \bar{v}^\sharp, \bar{v}^\sharp \rangle + \frac{\partial}{\partial \tilde{\rho}} (\tilde{\rho} U(\tilde{\rho}))$$

where $\tilde{\rho} = \star\rho$, and

$$\frac{\delta H}{\delta \bar{v}} = i_{\bar{v}^\sharp} \rho$$

Euler's equations can be written in port-Hamiltonian form (see the references)

$$\begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial \bar{v}}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{d} \\ -\mathbf{d} & \frac{1}{\star\rho}\star\left((\star\mathbf{d}\bar{v}) \wedge (\star\bullet)\right) \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \rho} \\ \frac{\delta H}{\delta \bar{v}} \end{pmatrix}$$

where

$$\frac{1}{\star\rho}\star\left((\star\mathbf{d}\bar{v}) \wedge (\star\bullet)\right)\frac{\delta H}{\delta \bar{v}} := \frac{1}{\star\rho}\star\left((\star\mathbf{d}\bar{v}) \wedge (\star\frac{\delta H}{\delta \bar{v}})\right)$$

The boundary variables are defined by

$$\begin{pmatrix} f_b \\ -e_b \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \rho} \\ \frac{\delta H}{\delta \bar{v}} \end{pmatrix}\Bigg|_{\partial M}$$

Note that the skew-symmetric matrix in the first equation depends on the energy variables ρ and $\bar{v}$, hence this defines a non-constant Dirac structure.

The energy balance is given by (see references)

$$\frac{\mathbf{d}H}{\mathbf{d}t} = \int_{\partial M} e_b \wedge f_b = -\int_{\partial M} i_{\bar{v}} \left(\frac{1}{2} \langle \bar{v}, \bar{v} \rangle \rho + U(\star\rho)\rho \right) - \int_{\partial M} i_{\bar{v}} (\star p)$$

The first terms is the convected energy through the boundary, whereas the second term is (minus) the external work (static pressure times velocity).

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