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Poisson Structures and their Normal Forms

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These notes are based on some sections of our book,"Poisson structures and their normal form", Progress in Mathematics, Vol. 242, Birkhäuser, to appear this year. In turn, our lectures will be based on some parts of these notes.

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Contents

CHAPTER 1

Generalities on Poisson structures

1.1. Poisson brackets

Definition 1.1.1. A C^∞-smooth ***Poisson structure*** on a C^∞-smooth finite-dimensional manifold M is an $\mathbb{R}$-bilinear antisymmetric operation

$$\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \ (f,g) \longmapsto \{f,g\} \tag{1.1}$$

on the space $\mathcal{C}^\infty(M)$ of real-valued C^∞-smooth functions on M, which verifies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \tag{1.2}$$

and the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}, \quad \forall f,g,h \in \mathcal{C}^\infty(M). \tag{1.3}$$

In other words, $\mathcal{C}^\infty(M)$, equipped with $\{,\}$, is a Lie algebra whose Lie bracket satisfies the Leibniz identity. This bracket $\{,\}$ is called a ***Poisson bracket***. A manifold equipped with such a bracket is called a ***Poisson manifold***.

Similarly, one can define real analytic, holomorphic, and formal Poisson manifolds, if one replaces $\mathcal{C}^\infty(M)$ by the corresponding sheaf of local analytic (respectively, holomorphic, formal) functions. In order to define C^k-smooth Poisson structures ($k \in \mathbb{N}$), we will have to express them in terms of 2-vector fields. This will be done in the next section.

Remark 1.1.2. In this book, when we say that something is smooth without making precise its smoothness class, we usually mean that it is C^∞-smooth. However, most of the time, being C^1-smooth or C^2-smooth will also be good enough, though we don't want to go into these details. Analytic means either real analytic or holomorphic. Though we will consider only finite-dimensional Poisson structures in this book, let us mention that infinite-dimensional Poisson structures also appear naturally (especially in problems of mathematical physics), see, e.g., [**163, 164**] and references therein.

Example 1.1.3. One can define a trivial Poisson structure on any manifold by putting $\{f,g\} = 0$ for all functions f and g.

Example 1.1.4. Take $M = \mathbb{R}^2$ with coordinates (x,y) and let $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an arbitrary smooth function. One can define a smooth Poisson structure on $\mathbb{R}^2$ by putting

$$\{f,g\} = (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})p \ . \tag{1.4}$$

EXERCISE 1.1.5. Verify the Jacobi identity and the Leibniz identity for the above bracket. Show that any smooth Poisson structure of $\mathbb{R}^2$ has the above form.

DEFINITION 1.1.6. A ***symplectic manifold*** (M, ω) is a manifold M equipped with a nondegenerate closed differential 2-form ω, called the ***symplectic form***.

The nondegeneracy of a differential 2-form ω means that the corresponding homomorphism $\omega^\flat : TM \to T^*M$ from the tangent space of M to its cotangent space, which associates to each vector X the covector $i_X\omega$, is an isomorphism. Here $i_X\omega = X \lrcorner \omega$ is the contraction of ω by X and is defined by $i_X\omega(Y) = \omega(X, Y)$.

If $f : M \to \mathbb{R}$ is a function on a symplectic manifold (M, ω), then we can define its ***Hamiltonian vector field***, denoted by X_f, as follows:

$$i_{X_f}\omega = -\mathrm{d}f \ . \tag{1.5}$$

We can also define on (M, ω) a natural bracket, called the Poisson bracket of ω, as follows:

$$\{f, g\} = \omega(X_f, X_g) = -\langle \mathrm{d}f, X_g\rangle = -X_g(f) = X_f(g). \tag{1.6}$$

PROPOSITION 1.1.7. *If (M, ω) is a smooth symplectic manifold then the bracket $\{f, g\} = \omega(X_f, X_g)$ is a smooth Poisson structure on M.*

Proof. The Leibniz identity is obvious. Let us show the Jacobi identity. Recall the following ***Cartan's formula*** for the differential of a k-form η (see, e.g., [**27**]):

$$\begin{aligned} \mathrm{d}\eta(X_1, \ldots, X_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i-1} X_i \left(\eta(X_1, \ldots \widehat{X_i} \ldots, X_{k+1})\right) \\ & + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta\left([X_i, X_j], X_1, \ldots \widehat{X_i} \ldots \widehat{X_j} \ldots, X_{k+1}\right), \end{aligned} \tag{1.7}$$

where $X_1, \ldots, X_{k+1}$ are vector fields, and the hat means that the corresponding entry is omitted. Applying Cartan's formula to ω and X_f, X_g, X_h, we get:

$$\begin{aligned} 0 &= \mathrm{d}\omega(X_f, X_g, X_h) \\ &= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) \\ &\quad -\omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g) \\ &= X_f\{g, h\} + X_g\{h, f\} + X_h\{f, g\} \\ &\quad +[X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g) \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + X_f(X_g(h)) - X_g(X_f(h)) \\ &\quad +X_g(X_h(f)) - X_h(X_g(f)) + X_h(X_f(g)) - X_f(X_h(g)) \\ &= 3(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}). \end{aligned}$$

□

Thus, any symplectic manifold is also a Poisson manifold, though the inverse is not true.

The classical ***Darboux theorem*** says that in the neighborhood of every point of (M, ω) there is a local system of coordinates $(p_1, q_1, \ldots, p_n, q_n)$, where $2n =$

$\dim M$, called ***Darboux coordinates*** or ***canonical coordinates***, such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \ . \tag{1.8}$$

A proof of Darboux theorem will be given in Section 1.4. In such a Darboux coordinate system one has the following expressions for the Poisson bracket and the Hamiltonian vector fields:

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}), \tag{1.9}$$

$$X_h = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^{n} \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{1.10}$$

The ***Hamiltonian equation*** of h (also called the ***Hamiltonian system*** of h), i.e. the ordinary differential equation for the integral curves of X_h, has the following form, which can be found in most textbooks on analytical mechanics:

$$\dot{q}_i = \frac{\partial h}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial h}{\partial q_i}. \tag{1.11}$$

In fact, to define the Hamiltonian vector field of a function, what one really needs is not a symplectic structure, but a Poisson structure: The Leibniz identity means that, for a given function f on a Poisson manifold M, the map $g \longmapsto \{f, g\}$ is a ***derivation***. Thus, there is a unique vector field X_f on M, called the ***Hamiltonian vector field*** of f, such that for any $g \in C^\infty(M)$ we have

$$X_f(g) = \{f, g\} \ . \tag{1.12}$$

EXERCISE 1.1.8. Show that, in the case of a symplectic manifold, Equation (1.5) and Equation (1.12) give the same vector field.

EXAMPLE 1.1.9. If N is a manifold, then its cotangent bundle T^*N has a unique natural symplectic structure, hence T^*N is a Poisson manifold with a natural Poisson bracket. The symplectic form on T^*N can be constructed as follows. Denote by $\pi : T^*N \to N$ the projection which assigns to each covector $p \in T^*_qN$ its base point q. Define the so-called ***Liouville 1-form*** θ on T^*N by

$$\langle \theta, X \rangle = \langle p, \pi_* X \rangle \ \ \forall \ X \in T_p(T^*N).$$

In other words, $\theta(p) = \pi^*(p)$, where on the left hand side p is considered as a point of T^*N and on the right hand side it is considered as a cotangent vector to N. Then $\omega = \mathrm{d}\theta$ is a symplectic form on N: ω is obviously closed; to see that it is nondegenerate take a local coordinate system $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on T^*N, where $(q_1, \ldots, q_n)$ is a local coordinate system on N and $(p_1, \ldots, p_n)$ are the coefficients of covectors $\sum p_i \mathrm{d}q_i(q)$ in this coordinate system. Then $\theta = \sum p_i \mathrm{d}q_i$ and $\omega = \mathrm{d}\theta = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$, i.e. $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is a Darboux coordinate system for ω. In classical mechanics, one often deals with Hamiltonian equations on a cotangent bundle T^*N equipped with the natural symplectic structure, where N is the ***configuration space***, i.e. the space of all possible configurations or positions; T^*N is called the ***phase space***.

A function g is called a ***first integral*** of a vector field X if g is constant with respect to X: $X(g) = 0$. Finding first integrals is an important step in the study of dynamical systems. Equation (1.12) means that a function g is a first integral of a Hamiltonian vector field X_f if and only if $\{f, g\} = 0$. In particular, every function h is a first integral of its own Hamiltonian vector field: $X_h(h) = \{h, h\} = 0$ due to the anti-symmetricity of the Poisson bracket. This fact is known in physics as the principle of ***conservation of energy*** (here h is the energy function).

The following classical theorem of Poisson [**171**] allows one sometimes to find new first integrals from old ones:

THEOREM 1.1.10 (Poisson). *If g and h are first integrals of a Hamiltonian vector field X_f on a Poisson manifold M then $\{g, h\}$ also is.*

Proof. Another way to formulate this theorem is

$$\left.\begin{array}{ll}\{g,f\} & = 0\\ \{h,f\} & = 0\end{array}\right\} \Rightarrow \{\{g,h\},f\} = 0 \tag{1.13}$$

But this is a corollary of the Jacobi identity. □

Another immediate consequence of the definition of Poisson brackets is the following lemma:

LEMMA 1.1.11. *Given a smooth Poisson manifold $(M, \{,\})$, the map $f \mapsto X_f$ is a homomorphism from the Lie algebra $\mathcal{C}^\infty(M)$ of smooth functions under the Poisson bracket to the Lie algebra of smooth vector fields under the usual Lie bracket. In other words, we have the following formula:*

$$[X_f, X_g] = X_{\{f,g\}} \tag{1.14}$$

Proof. For any $f, g, h \in \mathcal{C}^\infty(M)$ we have $[X_f, X_g]\, h = X_f\,(X_g h) - X_g\,(X_f h) = \{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\} = X_{\{f,g\}}h$. Since h is arbitrary, it means that $[X_f, X_g] = X_{\{f,g\}}$. □

1.2. Poisson tensors

In this section, we will express Poisson structures in terms of 2-vector fields which satisfy some special conditions.

Let M be a smooth manifold and q a positive integer. We denote by $\Lambda^q TM$ the space of tangent ***q-vectors*** of M: it is a vector bundle over M, whose fiber over each point $x \in M$ is the space $\Lambda^q T_x M = \Lambda^q(T_x M)$, which is the exterior (antisymmetric) product of q copies of the tangent space $T_x M$. In particular, $\Lambda^1 TM = TM$. If $(x_1, \dots, x_n)$ is a local system of coordinates at x, then $\Lambda^q T_x M$ admits a linear basis consisting of the elements $\dfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \dfrac{\partial}{\partial x_{i_q}}(x)$ with $i_1 < i_2 < \cdots < i_q$. A smooth ***q-vector field*** Π on M is, by definition, a smooth section of $\Lambda^q TV$, i.e. a map Π from V to $\Lambda^q TM$, which associates to each point x of M a q-vector $\Pi(x) \in \Lambda^q T_x M$, in a smooth way. In local coordinates, Π will have a local expression

$$\Pi(x) = \sum_{i_1 < \cdots < i_q} \Pi_{i_1 \dots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}} = \frac{1}{q!} \sum_{i_1 \dots i_q} \Pi_{i_1 \dots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}, \tag{1.15}$$

where the components $\Pi_{i_1 \dots i_q}$, called the ***coefficients*** of Π, are smooth functions. The coefficients $\Pi_{i_1 \dots i_q}$ are antisymmetric with respect to the indices, i.e. if we permute two indices then the coefficient is multiplied by -1. For example, $\Pi_{i_1 i_2 \dots} = -\Pi_{i_2 i_1 \dots}$. If $\Pi_{i_1 \dots i_q}$ are C^k-smooth, then we say that Π is C^k-smooth, and so on.

Smooth q-vector fields are dual objects to differential q-forms in a natural way. If Π is a q-vector field and α is a differential q-form, which in some local system of coordinates are written as $\Pi(x) = \sum_{i_1 < \dots < i_q} \Pi_{i_1 \dots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_q}}$ and $\alpha = \sum_{i_1 < \dots < i_q} a_{i_1 \dots i_q} \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_q}$, then their ***pairing*** $\langle \alpha, \Pi \rangle$ is a function defined by

$$\langle \alpha, \Pi \rangle = \sum_{i_1 < \dots < i_q} \Pi_{i_1 \dots i_q} a_{i_1 \dots i_q} \ . \tag{1.16}$$

EXERCISE 1.2.1. Show that the above definition of $\langle \alpha, \Pi \rangle$ does not depend on the choice of local coordinates.

In particular, smooth q-vector fields on a smooth manifold M can be considered as $\mathcal{C}^\infty(M)$-linear operators from the space of smooth differential q-forms on M to $\mathcal{C}^\infty(M)$, and vice versa.

A C^k-smooth q-vector field Π will define an $\mathbb{R}$-multilinear skewsymmetric map from $\mathcal{C}^\infty(M) \times \dots \times \mathcal{C}^\infty(M)$ (q times) to $\mathcal{C}^\infty(M)$ by the following formula:

$$\Pi(f_1, \dots, f_q) := \langle \Pi, \mathrm{d}f_1 \wedge \dots \wedge \mathrm{d}f_q \rangle \ . \tag{1.17}$$

Conversely, we have:

LEMMA 1.2.2. *A $\mathbb{R}$-multilinear map $\Pi : \mathcal{C}^\infty(M) \times \dots \times \mathcal{C}^\infty(M) \to \mathcal{C}^k(M)$ arises from a C^k-smooth q-vector field by Formula (1.17) if and only if Π is skewsymmetric and satisfies the Leibniz rule (or condition):*

$$\Pi(fg, f_2, \dots, f_q) = f\Pi(g, f_2, \dots, f_q) + g\Pi(f, f_2, \dots, f_q). \tag{1.18}$$

A map Π which satisfies the above conditions is called a ***multi-derivation***, and the above lemma says that multi-derivations can be identified with multi-vector fields.

Proof (sketch). The "only if" part is straightforward. For the "if" part, we have to check that the value of $\Pi(f_1, \dots, f_q)$ at a point x depends only on the value of $\mathrm{d}f_1, \dots, \mathrm{d}f_q$ at x. Equivalently, we have to check that if $\mathrm{d}f_1(x) = 0$ then $\Pi(f_1, \dots, f_q)(x) = 0$. If $\mathrm{d}f_1(x) = 0$ then we can write $f_1 = c + \sum_i x_i g_i$ where c is a constant and x_i and g_i are smooth functions which vanish at x. According to the Leibniz rule we have $\Pi(1 \times 1, f_2, \dots, f_q) = 1 \times \Pi(1, f_2, \dots, f_q) + 1 \times \Pi(1, f_2, \dots, f_q) = 2\Pi(1, f_2, \dots, f_q)$, hence $\Pi(1, f_2, \dots, f_q) = 0$. Now according to the linearity and the Leibniz rule we have $\Pi(f_1, \dots, f_q)(x) = c\Pi(1, f_2, \dots, f_q)(x) + \sum x_i(x)\Pi(g_i, f_2, \dots, f_q)(x) + \sum g_i(x)\Pi(x_i, f_2, \dots, f_q)(x) = 0$. $\square$

In particular, if Π is a Poisson structure, then it is skewsymmetric and satisfies the Leibniz condition, hence it arises from a 2-vector field, which we will also denote by Π:

$$\{f, g\} = \Pi(f, g) = \langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle \ . \tag{1.19}$$

A 2-vector field Π, such that the bracket $\{f, g\} := \langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle$ is a Poisson bracket (i.e. satisfies the Jacobi identity $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ for any smooth functions f, g, h), is called a ***Poisson tensor***, or also a ***Poisson structure***. The corresponding Poisson bracket is often denoted by $\{,\}_\Pi$. If the

Poisson tensor Π is a C^k-smooth 2-vector field, then we say that we have a C^k-smooth Poisson structure, and so on.

In a local system of coordinates $(x_1, \ldots, x_n)$ we have

$$\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ , \tag{1.20}$$

where $\Pi_{ij} = \langle \Pi, \mathrm{d}x_i \wedge \mathrm{d}x_j \rangle = \{x_i, x_j\}$, and

$$\{f, g\} = \langle \sum_{i<j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \rangle = \sum_{i,j} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \ . \tag{1.21}$$

EXAMPLE 1.2.3. The Poisson tensor corresponding to the standard symplectic structure $\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$ on $\mathbb{R}^{2n}$ is $\sum_{j=1}^n \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j}$.

NOTATION 1.2.4. : In this book, if functions $f_1, \ldots, f_p$ depend on variables $x_1, \ldots, x_p$, and maybe other variables, then we will denote by

$$\frac{\partial(f_1, \ldots, f_p)}{\partial(x_1, \ldots, x_p)} := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^p \tag{1.22}$$

the ***Jacobian determinant*** of $(f_1, \ldots, f_p)$ with respect to $(x_1, \ldots, x_p)$. For example,

$$\frac{\partial(f, g)}{\partial(x_i, x_j)} := \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \ . \tag{1.23}$$

With the above notation, we have the following local expression for Poisson brackets:

$$\{f, g\} = \sum_{i,j} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \sum_{i<j} \{x_i, x_j\} \frac{\partial(f, g)}{\partial(x_i, x_j)} \ . \tag{1.24}$$

Due to the Jacobi condition, not every 2-vector field will be a Poisson tensor.

EXERCISE 1.2.5. Show that the 2-vector field $\frac{\partial}{\partial x} \wedge (\frac{\partial}{\partial y} + x \frac{\partial}{\partial z})$ in $\mathbb{R}^3$ is *not* a Poisson tensor.

EXERCISE 1.2.6. Show that if $X_1, \ldots, X_m$ are pairwise commuting vector fields and a_{ij} are constants then $\sum_{ij} a_{ij} X_i \wedge X_j$ is a Poisson tensor.

To study the Jacobi identity, we will use the following lemma:

LEMMA 1.2.7. *For any C^1-smooth 2-vector field Π, one can associate to it a 3-vector field Λ defined by*

$$\Lambda(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} \tag{1.25}$$

where $\{k, l\}$ denotes $\langle \Pi, \mathrm{d}k \wedge \mathrm{d}l \rangle$ (i.e. the bracket of Π).

Proof. It is clear that the right-hand side of Formula (1.25) is $\mathbb{R}$-multilinear and antisymmetric. To show that it corresponds to a 3-vector field, one has to verify that it satisfies the Leibniz rule with respect to f, i.e.

$$\begin{aligned} &\{\{f_1 f_2, g\}, h\} + \{\{g, h\}, f_1 f_2\} + \{\{h, f_1 f_2\}, g\} = \\ &\quad = f_1(\{\{f_2, g\}, h\} + \{\{g, h\}, f_2\} + \{\{h, f_2\}, g\}) + \\ &\qquad + f_2(\{\{f_1, g\}, h\} + \{\{g, h\}, f_1\} + \{\{h, f_1\}, g\}). \end{aligned}$$

This is a simple direct verification, based on the Leibniz rule $\{ab, c\} = a\{b, c\} + b\{a, c\}$ for the bracket of the 2-vector field Π. It will be left to the reader as an exercise. $\square$

Direct calculations in local coordinates show that

$$\Lambda(f, g, h) = \sum_{ijk} \Big(\oint_{ijk} \sum_{s} \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} \Big) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k} \ , \tag{1.26}$$

where $\oint_{ijk} a_{ijk}$ means the cyclic sum $a_{ijk} + a_{jki} + a_{kij}$. In other words,

$$\Lambda = \sum_{i<j<k} \Big(\oint_{ijk} \sum_{s} \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} \Big) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} \ . \tag{1.27}$$

Clearly, the Jacobi identity for Π is equivalent to the condition that $\Lambda = 0$. Thus we have:

PROPOSITION 1.2.8. *A 2-vector field $\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ expressed in terms of a given system of coordinates $(x_1, \dots, x_n)$ is a Poisson tensor if and only if it satisfies the following system of equations:*

$$\oint_{ijk} \sum_{s} \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} = 0 \quad (\forall\ i, j, k) \ . \tag{1.28}$$

$\square$

An obvious consequence of the above proposition is that the condition for a 2-vector field to be a Poisson structure is a local condition. In particular, the restriction of a Poisson structure to an open subset of the manifold is again a Poisson structure.

EXAMPLE 1.2.9. Constant Poisson structures on $\mathbb{R}^n$: Choose arbitrary constants Π_{ij}. Then Equation (1.28) is obviously satisfied. The canonical Poisson structure on $\mathbb{R}^{2n}$, associated to the canonical symplectic form $\omega = \sum dq_i \wedge dp_i$, is of this type.

EXAMPLE 1.2.10. Any 2-vector field on a 2-dimensional manifold is a Poisson tensor. Indeed, the 3-vector field Λ in Lemma 1.2.7 is identically zero because there are no non-trivial 3-vectors on a 2-dimensional manifold. Thus the Jacobi identity is nontrivial only starting from dimension 3.

EXAMPLE 1.2.11. Let V be a finite-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$). A ***linear Poisson structure*** on V is a Poisson structure on V for which the Poisson bracket of two linear functions is again a linear function. Equivalently, in linear coordinates, the components of the corresponding Poisson tensor are linear functions. In this case, by restriction to linear functions, the operation $(f, g) \mapsto \{f, g\}$ gives rise to an operation $[\ ,\ ] : V^* \times V^* \longrightarrow V^*$, which is a Lie algebra structure on V^*, where V^* is the dual linear space of V.

Conversely, any Lie algebra structure on V^* determines a linear Poisson structure on V. Indeed, consider a finite-dimensional Lie algebra $(\mathfrak{g}, [\ ,\ ])$. For each linear function $f : \mathfrak{g}^* \longrightarrow \mathbb{R}$ we denote by $\tilde{f}$ the element of $\mathfrak{g}$ corresponding to it. If f and g are two linear functions on $\mathfrak{g}^*$ then we put $\{f, g\}(\alpha) = \langle \alpha, [\tilde{f}, \tilde{g}] \rangle$ for every α in $\mathfrak{g}^*$. If we choose a linear basis $e_1, \dots, e_n$ of $\mathfrak{g}$, with $[e_i, e_j] = \sum c_{ij}^k e_k$, then we have $\{x_i, x_j\} = \sum c_{ij}^k x_k$ where x_l is the function such that $\tilde{x}_l = e_l$. By taking $(x_1, \dots, x_n)$ as a linear system of coordinates on $\mathfrak{g}^*$, it follows from the Jacobi identity for $[\ ,\ ]$ that the functions $\Pi_{ij} = \{x_i, x_j\}$ verify Equation (1.28). Thus we get a Poisson structure on $\mathfrak{g}^*$. This Poisson structure can be defined intrinsically by the following formula:

$$\{f, g\}(\alpha) = \langle \alpha, [\mathrm{d}f(\alpha), \mathrm{d}g(\alpha)] \rangle \ , \tag{1.29}$$

where $\mathrm{d}f(\alpha)$ and $\mathrm{d}g(\alpha)$ are considered as elements of $\mathfrak{g}$ via the identification $(\mathfrak{g}^*)^* = \mathfrak{g}$. Thus, there is a natural bijection between finite-dimensional linear Poisson structures and finite-dimensional Lie algebras. One can even try to study Lie algebras by viewing them as linear Poisson structures (see, e.g., [**38**]).

REMARK 1.2.12. Multi-vector fields are also known as antisymmetric ***contravariant tensors***, because their coefficients change contravariantly under a change of local coordinates. In particular, the local expression of a Poisson bracket will change contravariantly under a change of local coordinates: Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two local coordinate systems on a same open subset of a Poisson manifold $(M, \{,\})$. Viewing y_i as functions of $(x_1, \dots, x_n)$, we have

$$\{y_i, y_j\} = \sum_{r<s} \frac{\partial(y_i, y_j)}{\partial(x_r, x_s)} \{x_r, x_s\} \ . \tag{1.30}$$

Denote $\Pi_{rs}(x) = \{x_r, x_s\}(x), \Pi'_{ij}(y) = \{y_i, y_j\}(y)$. Then the above equation can be rewritten as

$$\Pi'_{ij}(y(x)) = \sum_{r<s} \frac{\partial(y_i, y_j)}{\partial(x_r, x_s)}(x) \Pi_{rs}(x). \tag{1.31}$$

EXERCISE 1.2.13. Consider the Poisson structure on $\mathbb{R}^2$ defined by $\{x, y\} = e^x$. Show that in the new coordinates $(u, v) = (x, ye^{-x})$ the Poisson tensor will have the standard form $\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$.

EXERCISE 1.2.14. Let $\Pi = \sum \Pi_{ij} \partial/\partial x_i \wedge \partial/\partial x_j$ be a constant Poisson structure on $\mathbb{R}^n$, i.e. the coefficients Π_{ij} are constants. Show that there is a number $p \geq 0$ and a linear coordinate system $(y_1, \dots, y_n)$ in which the Poisson bracket has the form

$$\{f, g\} = \frac{\partial(f, g)}{\partial(y_1, y_2)} + \frac{\partial(f, g)}{\partial(y_3, y_4)} + \cdots + \frac{\partial(f, g)}{\partial(y_{2p-1}, y_{2p})} \ . \tag{1.32}$$

1.3. Poisson morphisms

DEFINITION 1.3.1. If $(M_1, \{,\}_1)$ and $(M_2, \{,\}_2)$ are two smooth Poisson manifolds, then a smooth map ϕ from M_1 to M_2 is called a smooth ***Poisson morphism*** or ***Poisson map*** if the associated pull-back map $\phi^* : \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$ is a Lie algebra homomorphism with respect to the corresponding Poisson brackets.

In other words, $\phi : (M_1, \{,\}_1) \to (M_2, \{,\}_2)$ is a Poisson morphism if

$$\{\phi^* f, \phi^* g\}_1 = \phi^* \{f, g\}_2 \ \ \forall\, f, g \in \mathcal{C}^\infty(M_2) \ . \tag{1.33}$$

Of course, Poisson manifolds together with Poisson morphisms form a category: the composition of two Poisson morphisms is again a Poisson morphism, and so on.

Notice that a Poisson morphism which is a diffeomorphism will automatically be a ***Poisson isomorphism***: the inverse map is also a Poisson map.

Similarly, a map $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is called a ***symplectic morphism*** if $\phi^*\omega_2 = \omega_1$. Clearly, a symplectic isomorphism is also a Poisson isomorphism. However, a symplectic morphism is *not* a Poisson morphism in general. For example, if M_1 is a point with a trivial symplectic form, and M_2 is a symplectic manifold of positive dimension, then any map $\phi : M_1 \to M_2$ is a symplectic morphism but not a Poisson morphism.

EXAMPLE 1.3.2. If $\phi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, then the linear dual map $\phi^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map, where $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are equipped with their respective linear Poisson structures. The proof of this fact will be left to the reader as an exercise. In particular, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the canonical projection $\mathfrak{g}^* \to \mathfrak{h}^*$ is Poisson.

EXAMPLE 1.3.3. If ϕ is a diffeomorphism of a manifold N, then it can be lifted naturally to a diffeomorphism $\phi_* : T^*N \to T^*N$ covering ϕ. By definition, ϕ_* preserves the Liouville 1-form θ (see Example 1.1.9), hence it preserves the symplectic form $\mathrm{d}\theta$. Thus, ϕ_* is a Poisson isomorphism.

EXAMPLE 1.3.4. Direct product of Poisson manifolds. Let $(M_1, \{,\}_1)$ and $(M_2, \{,\}_2)$ be two Poisson manifolds. Then their direct product $M_1 \times M_2$ can be equipped with the following natural bracket:

$$\{f(x_1,x_2), g(x_1,x_2)\} = \{f_{x_2}, g_{x_2}\}_1(x_1) + \{f_{x_1}, g_{x_1}\}_2(x_2) \tag{1.34}$$

where we use the notation $h_{x_1}(x_2) = h_{x_2}(x_1) = h(x_1, x_2)$ for any function h on $M_1 \times M_2$, $x_1 \in M_1$ and $x_2 \in M_2$. Using Equation (1.28), one can verify easily that this bracket is indeed a Poisson bracket on $M_1 \times M_2$. It is called the ***product Poisson structure***. With respect to this product Poisson structure, the projection maps $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

EXERCISE 1.3.5. Let $M_1 = M_2 = \mathbb{R}^n$ with trivial Poisson structure. Find a nontrivial Poisson structure on $M_1 \times M_2 = \mathbb{R}^{2n}$ for which the two projections $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

EXERCISE 1.3.6. Show that any Poisson map from a Poisson manifold to a symplectic manifold is a submersion.

A vector field X on a Poisson manifold (M, Π), is called a ***Poisson vector field*** if it is an ***infinitesimal automorphism*** of the Poisson structure, i.e. the Lie derivative of Π with respect to X vanishes:

$$\mathcal{L}_X \Pi = 0\ . \tag{1.35}$$

Equivalently, the local flow (φ_X^t) of X, i.e. the 1-dimensional pseudo-group of local diffeomorphisms of M generated by X, preserves the Poisson structure: $\forall t \in \mathbb{R}$, (φ_X^t) is a Poisson morphism wherever it is well-defined.

By the Leibniz rule we have $\mathcal{L}_X(\{f,g\}) = \mathcal{L}_X(\langle \Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle) = \langle \mathcal{L}_X\Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle + \langle \Pi, \mathrm{d}\mathcal{L}_X f \wedge \mathrm{d}g\rangle + \langle \Pi, \mathrm{d}f \wedge \mathrm{d}\mathcal{L}_X g\rangle = \langle \mathcal{L}_X\Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle + \{X(f), g\} + \{f, X(g)\}$. So another equivalent condition for X to be a Poisson vector field is the following:

$$\{Xf, g\} + \{f, Xg\} = X\{f,g\}\ . \tag{1.36}$$

When $X = X_h$ is a Hamiltonian vector field, then Equation (1.36) is nothing but the Jacobi identity. Thus any Hamiltonian vector field is a Poisson vector field. The inverse is not true in general, even locally. For example, if the Poisson structure is trivial, then any vector field is a Poisson vector field, while the only Hamiltonian vector field is the trivial one.

EXERCISE 1.3.7. Show that on $\mathbb{R}^{2n}$ with the standard Poisson structure $\sum \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$ any Poisson vector field is also Hamiltonian.

EXAMPLE 1.3.8. Infinitesimal version of Example 1.3.3. If X is a vector field on a manifold N, then X admits a unique natural lifting to a vector field $\hat{X}$ on T^*N which preserves the Liouville 1-form. In a local coordinate system $(p_1, \dots, p_n, q_1, \dots, q_n)$ on T^*N, where $(q_1, \dots, q_n)$ is a local coordinate system on N and the Liouville 1-form is $\theta = \sum_i p_i \mathrm{d}q_i$ (see Example 1.1.9), we have the following expression for $\hat{X}$:

$$\text{If } X = \sum_i \alpha_i(q) \frac{\partial}{\partial q_i} \text{ then } \hat{X} = \sum_i \alpha_i(q) \frac{\partial}{\partial q_i} - \sum_{i,j} \frac{\partial \alpha_i(q)}{\partial q_j} p_i \frac{\partial}{\partial p_j}.$$

The vector field $\hat{X}$ is in fact the Hamiltonian vector field of the function

$$\mathcal{X}(p_1, \dots, p_n, q_1, \dots, q_n) = \sum_i \alpha_i(q) p_i$$

on T^*N. This function $\mathcal{X}$ is nothing else than X itself, considered as a fiber-wise linear function on T^*N.

EXAMPLE 1.3.9. Let G be a connected Lie group, and denote by $\mathfrak{g}$ the Lie algebra of G. By definition, $\mathfrak{g}$ is isomorphic to the Lie algebra of left-invariant tangent vector fields of G (i.e. vector fields which are invariant under left translations $L_g : h \mapsto gh$ on G). Denote by e the neutral element of G. For each $X_e \in T_eG$, there is a unique left-invariant vector field X on G whose value at e is X_e (X obtained from X_e by left translations), so we may identify T_eG with $\mathfrak{g}$ via this association $X_e \mapsto X$. We will write $T_eG = \mathfrak{g}$, and $T_e^*G = \mathfrak{g}^*$ by duality. Consider the left translation map

$$L : T^*G \to \mathfrak{g}^* = T_e^*G, \quad L(p) = (L_g)^* p = L_{g^{-1}} p \ \ \forall\, p \in T_g^*G, \tag{1.37}$$

where $L_{g^{-1}}p$ means the push-forward $(L_{g^{-1}})_* p$ of p by $L_{g^{-1}}$ (we will often omit the subscript asterisk when writing push-forwards to simplify the notations).

THEOREM 1.3.10. *The above left translation map $L : T^*G \to \mathfrak{g}^*$ is a Poisson map, where T^*G is equipped with the standard symplectic structure, and $\mathfrak{g}^*$ is equipped with the standard linear Poisson structure (induced from the Lie algebra structure of $\mathfrak{g}$).*

Proof (sketch). It is enough to verify that, if x, y are two elements of $\mathfrak{g}$, considered as linear functions on $\mathfrak{g}^*$, then we have

$$\{L^*x, L^*y\} = L^*([x, y]).$$

Notice that L^*x is nothing else than x itself, considered as a left-invariant vector field on G and then as a left-invariant fiber-wise linear function on T^*G. By the formulas given in Example 1.3.8, the Hamiltonian vector field X_{L^*x} of L^*x is the natural lifting to T^*G of x, considered as a left-invariant vector field on G. Since the process of lifting of vector fields from N to T^*N preserves the Lie bracket for any manifold N, we have

$$[X_{L^*x}, X_{L^*y}] = X_{L^*[x,y]}.$$

It follows from the above equation and Lemma 1.1.11 that $\{L^*x, L^*y\}$ and $L^*([x, y])$ have the same Hamiltonian vector field on T^*G. Hence these two functions differ by a function which vanishes on the zero section of T^*G and whose Hamiltonian vector field is trivial on T^*G. The only such function is 0, so $\{L^*x, L^*y\} = L^*([x, y])$. $\square$

EXERCISE 1.3.11. Show that the right translation map $R : T^*G \to \mathfrak{g}^* = T^*_e G$, defined by $L(p) = (R_g)_* p \ \forall\ p \in T^*_g G$, is an anti-Poisson map. A map $\phi : (M, \Pi) \to (N, \Lambda)$ is called an ***anti-Poisson map*** if $\phi : (M, \Pi) \to (N, -\Lambda)$ is a Poisson map.

Given a subspace $V \in T_x M$ of a tangent space $T_x M$ of a symplectic manifold (M, ω), we will denote by $V^\perp$ the ***symplectic orthogonal*** to V: $V^\perp = \{X \in T_x M \mid \omega(X, Y) = 0 \ \forall\ Y \in V\}$. Clearly, $V = (V^\perp)^\perp$. V is called ***Lagrangian*** (resp. ***isotropic***, ***coisotropic***, ***symplectic***) if $V = V^\perp$ (resp. $V \subset V^\perp$, $V \supset V^\perp$, $V \cap V^\perp = 0$). A submanifold of a symplectic manifold is called ***Lagrangian*** (resp. ***isotropic***, ***coisotropic***, resp. ***symplectic***) if its tangent spaces are so. Lagrangian submanifolds play a central role in symplectic geometry, see, e.g., [**204, 142**]. In particular, we have the following characterization of symplectic isomorphisms in terms of Lagrangian submanifolds:

PROPOSITION 1.3.12. *A diffeomorphism $\phi : (M, \omega_1) \to (M_2, \omega_2)$ is a symplectic isomorphism if and only if its graph $\Delta = \{(x, \phi(x))\} \subset M_1 \times \overline{M_2}$ is a Lagrangian manifold of $M_1 \times \overline{M_2}$, where $\overline{M_2}$ means M_2 together with the opposite symplectic form $-\omega_2$.*

The proof is almost obvious and is left as an exercise. $\square$

A subspace $V \subset T_x M$ of a Poisson manifold (M, Π) is called ***coisotropic*** if for any $\alpha, \beta \in T^*_x M$ such that $\langle \alpha, X\rangle = \langle \beta, X\rangle = 0 \ \forall\ X \in V$ we have $\langle \Pi, \alpha \wedge \beta\rangle = 0$. In other words, $V^\circ \subset (V^\circ)^\perp$, where $V^\circ = \{\alpha \in T^*_x M \mid \langle \alpha, X\rangle = 0 \ \forall\ X \in V\}$ is the annulator of V and $(V^\circ)^\perp = \{\beta \in T^*_x M \mid \langle \Pi, \alpha \wedge \beta\rangle = 0 \ \forall\ \alpha \in V^\circ\}$ is the "Poisson orthogonal" of V°. A submanifold N of a Poisson manifold is called ***coisotropic*** if its tangent spaces are coisotropic.

PROPOSITION 1.3.13. *A map $\phi : (M_1, \Pi_1) \to (M_2, \Pi_2)$ between two Poisson manifolds is a Poisson map if and only if its graph $\Gamma(\phi) := \{(x, y) \in M_1 \times M_2;\ y = \phi(x)\}$ is a coisotropic submanifold of $(M_1, \Pi_1) \times (M_2, -\Pi_2)$.*

Again, the proof will be left as an exercise. $\square$

1.4. Local canonical coordinates

In this section, we will prove the ***splitting theorem*** of Alan Weinstein [**205**], which says that locally a Poisson manifold is a direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. This splitting theorem, together with Darboux theorem which will be proved at the same time, will give us local canonical coordinates for Poisson manifolds.

Given a Poisson structure Π (or more generally, an arbitrary 2-vector field) on a manifold M, we can associate to it a natural homomorphism

$$\sharp = \sharp_\Pi : T^*M \longrightarrow TM, \tag{1.38}$$

which maps each covector $\alpha \in T_x^*M$ over a point x to a unique vector $\sharp(\alpha) \in T_xM$ such that

$$\langle \alpha \wedge \beta, \Pi \rangle = \langle \beta, \sharp(\alpha) \rangle \tag{1.39}$$

for any covector $\beta \in T_x^*M$. We will call $\sharp = \sharp_\Pi$ the ***anchor map*** of Π.

The same notations $\sharp$ (or $\sharp_\Pi$) will be used to denote the operator which associates to each differential 1-form α the vector field $\sharp(\alpha)$ defined by $(\sharp(\alpha))(x) = \sharp(\alpha(x))$. For example, if f is a function, then $\sharp(\mathrm{d}f) = X_f$ is the Hamiltonian vector field of f.

The restriction of $\sharp_\Pi$ to a cotangent space T_x^*M will be denoted by $\sharp_x$ or $\sharp_{\Pi(x)}$. In a local system of coordinates $(x_1, \ldots, x_n)$ we have

$$\sharp\left(\sum_{i=1}^n a_i dx_i\right) = \sum_{ij} \{x_i, x_j\}\, a_i \frac{\partial}{\partial x_j} = \sum_{ij} \Pi_{ij} a_i \frac{\partial}{\partial x_j}.$$

Thus $\sharp_x$ is a linear operator, given by the matrix $[\Pi_{ij}(x)]$ in the linear bases $(dx_1, \ldots, dx_n)$ and $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$.

DEFINITION 1.4.1. Let (M, Π) be a Poisson manifold and x a point of M. Then the image $\mathcal{C}_x := \mathrm{Im}\sharp_x$ of $\sharp_x$ is called the ***characteristic space*** at x of the Poisson structure Π. The dimension $\dim \mathcal{C}_x$ of $\mathcal{C}_x$ is called the ***rank*** of Π at x, and $\max_{x \in M} \dim \mathcal{C}_x$ is called the ***rank*** of Π. When $\operatorname{rank} \Pi_x = \dim M$ we say that Π is ***nondegenerate*** at x. If $\operatorname{rank} \Pi_x$ is a constant on M, i.e. does not depend on x, then Π is called a ***regular Poisson structure***.

EXAMPLE 1.4.2. The constant Poisson structure $\sum_{i=1}^s \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+s}}$ on $\mathbb{R}^m$ ($m \geq 2s$) is a regular Poisson structure of rank $2s$.

EXERCISE 1.4.3. Show that $\operatorname{rank} \Pi_x$ is always an even number, and that Π is nondegenerate everywhere if and only if it is the associated Poisson structure of a symplectic structure.

The characteristic space $\mathcal{C}_x$ admits a unique natural antisymmetric nondegenerate bilinear scalar product, called the ***induced symplectic form***: if X and Y are two vectors of $\mathcal{C}_x$ then we put

$$(X, Y) := \langle \beta, X \rangle = \langle \Pi, \alpha \wedge \beta \rangle = -\langle \Pi, \beta \wedge \alpha \rangle = -\langle \alpha, Y \rangle = -(Y, X) \tag{1.40}$$

where $\alpha, \beta \in T_x^*M$ are two covectors such that $X = \sharp\alpha$ and $Y = \sharp\beta$.

EXERCISE 1.4.4. Verify that the above scalar product is anti-symmetric nondegenerate and is well-defined (i.e. does not depend on the choice of α and β). When Π is nondegenerate then the above formula defines the corresponding symplectic structure on M.

THEOREM 1.4.5 (Splitting theorem [**205**]). *Let x be a point of rank $2s$ of a Poisson m-dimensional manifold (M, Π): $\dim \mathcal{C}_x = 2s$ where $\mathcal{C}_x$ is the characteristic space at x. Let N be an arbitrary $(m-2s)$-dimensional submanifold of M which*

contains x and is transversal to $\mathcal{C}_x$ at x. Then there is a local system of coordinates $(p_1, \ldots, p_s, q_1, \ldots, q_s, z_1, \ldots, z_{m-2s})$ in a neighborhood of x, which satisfy the following conditions:
a) $p_i(N_x) = q_i(N_x) = 0$ *where* N_x *is a small neighborhood of* x *in* N.
b) $\{q_i, q_j\} = \{p_i, p_j\} = 0 \ \forall \ i, j$; $\{p_i, q_j\} = 0$ *if* $i \neq j$ *and* $\{p_i, q_i\} = 1 \ \forall \ i$.
c) $\{z_i, p_j\} = \{z_i, q_j\} = 0 \ \forall \ i, j$.
d) $\{z_i, z_j\}(x) = 0 \ \forall \ i, j$.

A local coordinate system which satisfies the conditions of the above theorem is called a system of local ***canonical cordinates***. In such canonical coordinates we have

$$\{f, g\} = \sum_{i,j} \{z_i, z_j\} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} + \sum_{i=1}^{s} \frac{\partial(f, g)}{\partial(p_i, q_i)} = \{f, g\}_N + \{f, g\}_S \ , \tag{1.41}$$

where

$$\{f, g\}_S = \sum_{i=1}^{s} \frac{\partial(f, g)}{\partial(p_i, q_i)} \tag{1.42}$$

defines the nondegenerate Poisson structure $\sum \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$ on the local submanifold $S = \{z_1 = \cdots = z_{m-2s} = 0\}$, and

$$\{f, g\}_N = \sum_{u,v} \{z_i, z_j\} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} \tag{1.43}$$

defines a Poisson structure on a neighborhood of x in N. Notice that, since $\{z_i, p_j\} = \{z_i, q_j\} = 0 \ \forall \ i, j$, the functions $\{z_i, z_j\}$ do not depend on the variables $(p_1, \ldots, p_s, q_1, \ldots, q_s)$. The equality $\{z_i, z_j\}(x) = 0 \ \forall \ i, j$ means that the Poisson tensor of $\{,\}_N$ vanishes at x.

Formula (1.41) means that the Poisson manifold (M, Π) is locally isomorphic (in a neighborhood of x) to the direct product of a symplectic manifold $(S, \sum_1^s \mathrm{d}p_i \wedge \mathrm{d}q_i)$ with a Poisson manifold $(N_x, \{,\}_N)$ whose Poisson tensor vanishes at x. That's why Theorem 1.4.5 is called the splitting theorem for Poisson manifolds: locally, we can split a Poisson structure in two parts – a regular part and a singular part which vanishes at a point.

Proof of Theorem 1.4.5. If $\Pi(x) = 0$ then $s = 0$ and there is nothing to prove. Suppose that $\Pi(x) \neq 0$. Let p_1 be a local function (defined in a small neighborhood of x in M) which vanishes on N and such that $\mathrm{d}p_1(x) \neq 0$. Since $\mathcal{C}_x$ is transversal to N, there is a vector $X_g(x) \in \mathcal{C}_x$ such that $\langle X_g(x), \mathrm{d}p_1(x) \rangle \neq 0$, or equivalently, $X_{p_1}(g)(x) \neq 0$, where X_{p_1} denotes the Hamiltonian vector field of p_1 as usual. Therefore $X_{p_1}(x) \neq 0$. Since $\mathcal{C}_x \ni \sharp(\mathrm{d}p_1)(x) = X_{p_1}(x) \neq 0$ and is not tangent to N, there is a local function q_1 such that $q_1(N) = 0$ and $X_{p_1}(q_1) = 1$ in a neighborhood of x, or

$$X_{p_1} q_1 = \{p_1, q_1\} = 1 \ . \tag{1.44}$$

Moreover, X_{p_1} and X_{q_1} are linearly independent ($X_{q_1} = \lambda X_{p_1}$ would imply that $\{p_1, q_1\} = -\lambda X_{p_1}(p_1) = 0$), and we have

$$[X_{p_1}, X_{q_1}] = X_{\{p_1, q_1\}} = 0 \ . \tag{1.45}$$

Thus X_{p_1} and X_{q_1} are two linearly independent vector fields which commute. Hence they generate a locally free infinitesimal $\mathbb{R}^2$-action in a neighborhood of x, which gives rise to a local regular 2-dimensional foliation. As a consequence, we can find a local system of coordinates $(y_1, \dots, y_m)$ such that

$$X_{q_1} = \frac{\partial}{\partial y_1}, \qquad X_{p_1} = \frac{\partial}{\partial y_2} \ . \tag{1.46}$$

With these coordinates we have $\{q_1, y_i\} = X_{q_1}(y_i) = 0$ and $\{p_1, y_i\} = X_{p_1}(y_i) = 0$, for $i = 3, \dots, m$. Poisson's Theorem 1.1.10 then implies that $\{p_1, \{y_i, y_j\}\} = \{q_1, \{y_i, y_j\}\} = 0$ for $i, j \geq 3$, whence

$$\begin{array}{l} \{y_i, y_j\} = \varphi_{ij}(y_3, \dots, y_m) \quad \forall\ i, j \geq 3 \ , \\ \{p_1, q_1\} = 1 \ , \\ \{p_1, y_j\} = \{q_1, y_j\} = 0 \quad \forall\ j \geq 3 \ . \end{array} \tag{1.47}$$

We can take $(p_1, q_1, y_3, \dots, y_m)$ as a new local system of coordinates. In fact, the Jacobian matrix of the map $\varphi : (y_1, y_2, y_3, \dots, y_m) \mapsto (p_1, q_1, y_3, \dots, y_m)$ is of the form

$$\begin{pmatrix} 0 & 1 & \\ -1 & 0 & * \\ & 0 & \mathrm{Id} \end{pmatrix} \tag{1.48}$$

(because $\frac{\partial q_1}{\partial y_1} = X_{q_1} q_1 = 0$, $\frac{\partial q_1}{\partial y_2} = X_{p_1} q_1 = \{q_1, p_1\} = 1, \dots$), which has a non-zero determinant (equal to 1). In the coordinates $(q_1, p_1, y_3, \dots, y_m)$, we have

$$\Pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{1}{2} \sum_{i,j \geq 3} \Pi'_{ij}(y_3, \dots, y_n) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \ . \tag{1.49}$$

The above formula implies that our Poisson structure is locally the product of a standard symplectic structure on a plane $\{(p_1, q_1)\}$ with a Poisson structure on a $(m-2)$-dimensional manifold $\{(y_3, \dots, y_m)\}$. In this product, N is also the direct product of a point (= the origin) of the plane $\{(p_1, q_1)\}$ with a local submanifold in the Poisson manifold $\{(y_3, \dots, y_m)\}$. The splitting theorem now follows by induction on the rank of Π at x. □

Remark 1.4.6. In the above theorem, when $m = 2s$, we recover Darboux theorem which gives local canonical coordinates for symplectic manifolds. If (M, Π) is a regular Poisson structure, then the Poisson structure of N_x in the above theorem must be trivial, and we get the following generalization of Darboux theorem: any regular Poisson structure is locally isomorphic to a standard constant Poisson structure.

Exercise 1.4.7. Prove the following generalization of Theorem 1.4.5. Let N be a submanifold of a Poisson manifold (M, Π), and x be a point of N such that $T_x N + \mathcal{C}_x = T_x M$ and $T_x N \cap \mathcal{C}_x$ is a symplectic subspace of $\mathcal{C}_x$, i.e. the restriction of the symplectic form on the characteristic space $\mathcal{C}_x$ to $T_z N \cap \mathcal{C}_x$ is nondegenerate. (Such a submanifold N is sometimes called *cosymplectic*). Then there is a coordinate system in a neighborhood of x which satisfy the conditions a), b), c) of Theorem 1.4.5, where $2s = \dim M - \dim N = \dim C_x - \dim(T_x N \cap \mathcal{C}_x)$.

1.5. Singular symplectic foliations

A smooth ***singular foliation*** in the sense of Stefan-Sussmann [**187, 192**] on a smooth manifold M is by definition a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of M into a disjoint union of smooth immersed connected submanifolds $\mathcal{F}_\alpha$, called ***leaves***, which satisfies the following ***local foliation property*** at each point $x \in M$: Denote the leaf that contains x by $\mathcal{F}_x$, the dimension of $\mathcal{F}_x$ by d and the dimension of M by m. Then there is a smooth local chart of M with coordinates $y_1, \ldots, y_m$ in a neighborhood U of x, $U = \{-\varepsilon < y_1 < \varepsilon, \ldots, -\varepsilon < y_m < \varepsilon\}$, such that the d-dimensional disk $\{y_{d+1} = \ldots = y_m = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}_x$ with U which contains x, and each d-dimensional disk $\{y_{d+1} = c_{d+1}, \ldots, y_m = c_m\}$, where $c_{d+1}, \ldots, c_m$ are constants, is wholly contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$. If all the leaves $\mathcal{F}_\alpha$ of a singular foliation $\mathcal{F}$ have the same dimension, then one says that $\mathcal{F}$ is a ***regular foliation***.

A ***singular distribution*** on a manifold M is the assignment to each point x of M a vector subspace D_x of the tangent space T_xM. The dimension of D_x may depend on x. For example, if $\mathcal{F}$ is a singular foliation, then it has a natural associated ***tangent distribution*** $D^{\mathcal{F}}$: at each point $x \in V$, $D_x^{\mathcal{F}}$ is the tangent space to the leaf of $\mathcal{F}$ which contains x.

A singular distribution D on a smooth manifold is called ***smooth*** if for any point x of M and any vector $X_0 \in D_x$, there is a smooth vector field X defined in a neighborhood U_x of x which is tangent to the distribution, i.e. $X(y) \in D_y \ \forall \, y \in U_x$, and such that $X(x) = X_0$. If, moreover, $\dim D_x$ does not depend on x, then we say that D is a smooth ***regular distribution***.

It follows directly from the local foliation property that the tangent distribution $D^{\mathcal{F}}$ of a smooth singular foliation is a smooth singular distribution.

An ***integral submanifold*** of smooth singular distribution D on a smooth manifold M is, by definition, a connected immersed submanifold W of M such that for every $y \in W$ the tangent space T_yW is a vector subspace of D_y. An integral submanifold W is called ***maximal*** if it is not contained in any other integral submanifold; it is said to be of ***maximum dimension*** if its tangent space at every point $y \in W$ is exactly D_y.

We say that a smooth singular distribution D on a smooth manifold M is an ***integrable distribution*** if every point of M is contained in a maximal integral manifold of maximum dimension of D.

Let C be a family of smooth vector fields on M. Then it gives rise to a smooth singular distribution D^C: for each point $x \in M$, D_x^C is the vector space spanned by the values at x of the vector fields of C. We say that D^C is generated by C.

A distribution D is called ***invariant*** with respect to a family of smooth vector fields C if it is invariant with respect to every element of C: if $X \in C$ and (φ_X^t) denotes the local flow of X, then we have $(\varphi_X^t)_* D_x = D_{\varphi_X^t(x)}$ wherever $\varphi_X^t(x)$ is well-defined.

The following result, due to Stefan [**187**] and Sussmann [**192**] (see also Dazord [**61**]), gives an answer to the following question: what are the conditions for a smooth singular distribution to be the tangent distribution of a singular foliation ?

Theorem 1.5.1 (Stefan-Sussmann). *Let D be a smooth singular distribution on a smooth manifold M. Then the following three conditions are equivalent:*
a) D is integrable.
b) D is generated by a family C of smooth vector fields, and is invariant with respect to C.
c) D is the tangent distribution $D^{\mathcal{F}}$ of a smooth singular foliation $\mathcal{F}$.

Proof (sketch). a) $\Rightarrow$ b). Suppose that D is integrable. Let C be the family of all smooth vector fields which are tangent to D. The smoothness condition of D implies that D is generated by C. It remains to show that if X is an arbitrary smooth vector field tangent to D, then D is invariant with respect to X. Let x be an arbitrary point in M, and denote by $\mathcal{F}(x)$ the maximal invariant submanifold of maximum dimension which contains x. Then by definition (the condition of maximum dimension), for every point $y \in \mathcal{F}(x)$ we have $T_y\mathcal{F}(x) = D_y$, which implies that the vector field X, when restricted to $\mathcal{F}(x)$, is tangent to $\mathcal{F}(x)$. In particular, the local flow φ_X^t can be restricted to $\mathcal{F}(x)$, i.e. $\mathcal{F}(x)$ is an invariant manifold for this local flow. Moreover, if $\varphi_X^\tau(x)$ is well-defined for some $\tau > 0$, then the point $\varphi_X^\tau(x)$ lies on $\mathcal{F}(x)$. This fact follows from the maximality condition on $\mathcal{F}(x)$. (Note that, the union of two invariant submanifolds of maximum dimension is again an invariant submanifold of maximum dimension if it is connected). Because X is tangent to $\mathcal{F}(x)$, we have $(\varphi_X^\tau)_*(T_x\mathcal{F}(x)) = T_{\varphi_X^\tau(x)}\mathcal{F}(x)$. But $T_x\mathcal{F}(x) = D_x$ and $T_{\varphi_X^\tau(x)}\mathcal{F}(x) = D_{\varphi_X^\tau(x)}$, hence $(\varphi_X^\tau)_* D_x = D_{\varphi_X^\tau(x)}$.

b) $\Rightarrow$ c). Suppose that D is generated by a family C of smooth vector fields, and is invariant with respect to C. Let x be an arbitrary point of M, denote by d the dimension of D_x, and choose d vector fields $X_1, \dots, X_d$ of C such that $X_1(x), \dots, X_d(x)$ span D_x. Denote by $\phi_1^t, \dots, \phi_d^t$ the local flow of $X_1, \dots, X_d$ respectively. The map

$$(s_1, \dots, s_d) \mapsto \phi_1^{s_1} \circ \dots \circ \phi_d^{s_d}(x) \tag{1.50}$$

is a local diffeomorphism from a d-dimensional disk to a d-dimensional submanifold containing x in M. The invariance of D with respect to C implies that this submanifold is an integral submanifold of maximum dimension. Gluing these local integral submanifolds together (wherever they intersect), we obtain a partition of M into a disjoint union of connected immersed integral submanifolds of maximum dimension, called leaves. To see that this partition satisfies the local foliation property of singular foliations, we can proceed by induction on the dimension of D_x: If $\dim D_x = 0$, then the local foliation property at x is empty. If $\dim D_x > 0$, then there is a vector field $X \in C$ such that $X(x) \neq 0$. Then the trajectories of X lie on the leaves, and we can take the quotient of a small neighborhood of x by the trajectories of X to reduce the dimension of M and of the leaves by 1. The invariance with respect to C and the local foliation property does not change under this reduction.

c) $\Rightarrow$ a): If $D = D^{\mathcal{F}}$ is the tangent distribution of a singular foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are maximal invariant submanifolds of maximum dimension for D. $\square$

Definition 1.5.2. An ***involutive distribution*** is a distribution D such that if X, Y are two arbitrary smooth vector fields which are tangent to D then their Lie bracket $[X, Y]$ is also tangent to D.

It is clear from Theorem 1.5.1 that if a singular distribution is integrable, then it is involutive. Conversely, for regular distributions we have:

Theorem 1.5.3 (Frobenius). *If a smooth regular distribution is involutive then it is integrable, i.e. it is the tangent distribution of a regular foliation.*

Proof (sketch). One can use Formula (1.50) to construct local invariant submanifolds of maximum dimension and then glue them together, just like in the proof of Theorem 1.5.1. One can also see Theorem 1.5.3 as a special case of Theorem 1.5.1, by first showing that a regular involutive distribution is invariant with respect to the family of all smooth vector fields which are tangent to it. □

Example 1.5.4. Consider the following singular foliation D on $\mathbb{R}^2$ with coordinates (x, y): $D_{(x,y)} = T_{(x,y)}\mathbb{R}^2$ if $x > 0$, and $D_{(x,y)}$ is spanned by $\frac{\partial}{\partial x}$ if $x \leq 0$. Then D is smooth involutive but not integrable.

The above example shows that, if in Frobenius theorem we omit the word *regular*, then it is false. The reason is that, though Formula (1.50) still provides us with local invariant submanifolds, they are not necessarily of maximum dimension. However, the situation in the finitely generated case is better. A smooth distribution D on a manifold M is called ***locally finitely generated*** if for any $x \in M$ there is a neighborhood U of x such that the $\mathcal{C}^\infty(U)$-module of smooth tangent vector fields to D in U is finitely generated: there is a finite number of smooth vector fields $X_1, \dots, X_n$ in U which are tangent to D, such that any smooth vector field Y in U which is tangent to D can be written as $Y = \sum_{i=1}^n f_i X_i$ with $f_i \in \mathcal{C}^\infty(U)$.

Theorem 1.5.5 (Hermann [**105**]). *Any locally finitely generated smooth involutive distribution on a smooth manifold is integrable.*

See [**215**] for a simple proof of Theorem 1.5.5. □

Consider now a smooth Poisson manifold (M, Π). Denote by $\mathcal{C}$ its characteristic distribution. Recall that

$$\mathcal{C}_x = \mathrm{Im}\sharp_x = \{X_f(x), f \in C^\infty(M)\} \;\; \forall\, x \in M\ . \tag{1.51}$$

Since the Hamiltonian vector fields preserve the Poisson structure, they also preserve the characteristic distribution. Thus, according to Stefan-Sussmann theorem, the characteristic foliation $\mathcal{C}$ is completely integrable and corresponds to a singular foliation, which we will denote by $\mathcal{F} = \mathcal{F}_\Pi$. For the reasons which will become clear below, this singular foliation is called the ***symplectic foliation*** of the Poisson manifold (M, Π).

For each point $x \in M$, denote by $\mathcal{F}(x)$ the leaf of $\mathcal{F}$ which contains x. Local charts of $\mathcal{F}(x)$ are readily provided by Theorem 1.4.5: If

$$(p_1, \dots, p_s, q_1, \dots, q_s, z_1, \dots, z_{m-2s})$$

is a local canonical system of coordinates at a point $x \in M$, then the submanifold $\{z_1 = \dots = z_{m-2s} = 0\}$ is a open subset of $\mathcal{F}(x)$, and it has a natural symplectic structure with Darboux coordinates (p_i, q_i). Notice that this symplectic structure does not depend on the choice of coordinates: at each point of $\{z_1 = \cdots = z_{m-2s} = 0\}$, it coincides with the symplectic form on the characteristic space. Thus, on each leaf $\mathcal{F}(x)$ we have a unique natural symplectic structure, which at each point coincides with the symplectic form on the corresponding characteristic space. It also follows from Assertions b), d) of Theorem 1.4.5 that the injection $i : \mathcal{F} \to M$ is a Poisson morphism: if f, g are two functions on M and $y \in \mathcal{F}(x)$, then

$$\{f, g\}(y) = \{f|_{\mathcal{F}(x)}, g|_{\mathcal{F}(x)}\}_x(y), \tag{1.52}$$

where $\{,\}_x$ is the Poisson bracket of the symplectic form on $\mathcal{F}(x)$. In other words, we have:

THEOREM 1.5.6 ([**205**]). *Every leaf $\mathcal{F}(x)$ of the symplectic foliation $\mathcal{F}_\Pi$ of a Poisson manifold (M, Π) is an immersed symplectic submanifold, the immersion being a Poisson morphism. The Poisson structure Π is completely determined by the symplectic structures on the leaves of $\mathcal{F}_\Pi$.*

EXAMPLE 1.5.7. Symplectic foliation of linear Poisson structures. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Recall that $\mathfrak{g}^*$ has a natural linear Poisson structure, also known as the ***Lie-Poisson structure***, defined by

$$\{f, h\}(\alpha) = \langle \alpha, [\mathrm{d}f(\alpha), \mathrm{d}h(\alpha)] \rangle. \tag{1.53}$$

Denote the neutral element of G by e, and identify $\mathfrak{g}$ with T_eG. G acts on $\mathfrak{g}$ by the adjoint action $\mathrm{Ad}_g(x) = (u \mapsto gug^{-1})_{*e}(x)$ and on $\mathfrak{g}^*$ by the coadjoint action (the induced dual action) $\mathrm{Ad}^*_g(\alpha)(x) = \alpha(\mathrm{Ad}_{g^{-1}}x)$, $\alpha \in \mathfrak{g}^*$, $x \in \mathfrak{g}$, $g \in G$. This action is generated infinitesimally by the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ defined by $ad^*_x(\alpha)(y) = \langle \alpha, [y, x] \rangle = -\langle \alpha, \mathrm{ad}_x y \rangle$.

THEOREM 1.5.8. *The symplectic leaves of the Lie-Poisson structure on the dual of an arbitrary finite-dimensional Lie algebra coincide with the orbits of the coadjoint representation on it.*

Proof. Due to the Leibniz rule, the tangent spaces to the symplectic leaves, i.e. the characteristic spaces, are generated by the Hamiltonian vector fields of linear functions. If f, h are two linear functions on $\mathfrak{g}^*$, also considered as two elements of $\mathfrak{g}$ by duality, and α is a point of $\mathfrak{g}^*$, then we have

$$X_f(h)(\alpha) = \langle \alpha, [f, h] \rangle = -\langle \mathrm{ad}^*_f(\alpha), h \rangle. \tag{1.54}$$

It implies that the tangent spaces of symplectic leaves are the same as the tangent spaces of coadjoint orbits. It follows that coadjoint orbits are open closed subsets of symplectic leaves, so they coincide with symplectic leaves because symplectic leaves are connected by definition. □

A corollary of the above theorem is that the orbits of the coadjoint representation of a finite-dimensional Lie algebra are of even dimension and equipped with a natural symplectic form. This symplectic form is also known as the ***Kirillov–Kostant–Souriau form***. Let us mention that coadjoint orbits play a very important role in the theory of unitary representations of Lie groups (the so-called *orbit method*), see, e.g., [**119**].

EXERCISE 1.5.9. Describe the symplectic leaves of $so^*(3)$ and $sl^*(2)$.

REMARK 1.5.10. A direct way to define the symplectic foliation of a Poisson manifold (M, Π) is as follows: two points x, y are said to belong to the same leaf if they can be connected by a piecewise-smooth curve consisting of integral curves of Hamiltonian vector fields. Then it is a direct consequence of the splitting theorem 1.4.5 that the corresponding partition of M into leaves satisfies the local foliation property. Thus, in fact, we can use the splitting theorem 1.4.5 instead of Stefan-Sussmann theorem 1.5.1 in order to show that on (M, Π) there is a natural associated foliation whose tangent distribution is the characteristic distribution.

1.6. Transverse Poisson structures

Let N be a smooth local (i.e. sufficiently small) disk of dimension $m - 2s$ of an m-dimensional Poisson manifold (M, Π), which intersects transversally a $2s$-dimensional leaf $\mathcal{F}(x)$ of the symplectic foliation $\mathcal{F}$ of (M, Π) at a point x. In other words, N contains x and is transversal to the characteristic space $\mathcal{C}_x$. Then according to the splitting theorem 1.4.5, there are local canonical coordinates in a neighborhood of x, which will define on N a Poisson structure. This Poisson structure on N is called the ***transverse Poisson structure*** at x of the Poisson manifold (M, Π).

To justify the above definition of transverse Poisson structures, we must show that the Poisson structure on N given by Theorem 1.4.5 does not depend on the choice of local canonical coordinates, nor on the choice of N itself, modulo local Poisson diffeomorphisms.

THEOREM 1.6.1. *With the above notations, we have:*
a) The local Poisson structure on N given by Theorem 1.4.5 does not depend on the choice of local canonical coordinates.
b) If x_0 and x_1 are two points on the symplectic leaf $\mathcal{F}(x)$, and N_0 and N_1 are two smooth local disks of dimension $m-2s$ which intersect $\mathcal{F}(x)$ transversally at x_0 and x_1 respectively, then there is a smooth local Poisson diffeomorphism from (N_0, x_0) to (N_1, x_1).

Proof. a) Theorem 1.4.5 implies that the local symplectic leaves near point x are direct products of the symplectic leaves of a neighborhood of x in N with the local symplectic manifold $\{(p_1, \ldots, p_s, q_1, \ldots, q_s)\}$. In particular, the symplectic leaves of N are connected components of intersections of the symplectic leaves of M with N, and the symplectic form on the symplectic leaves of N is the restriction of the symplectic form of the leaves of M to that intersections. This geometric characterization of the symplectic leaves of N and their corresponding symplectic forms shows that they do not depend on the choice of local canonical coordinates. Hence, according to Theorem 1.5.6, the Poisson structure of N does not depend on the choice of local canonical coordinates.

b) Let N_0 and N_1 be two local disks which intersect a symplectic leaf $\mathcal{F}(x)$ transversally at x_0 and x_1 respectively. Then there is a smooth 1-dimensional family of local submanifolds N_t $(0 \leq t \leq 1)$, connecting N_0 to N_1, such that N_t intersects $\mathcal{F}_\alpha$ transversely at a point x_t. Point x_t depends smoothly on t. According to (a parameterized version of) Theorem 1.4.5, there is a smooth family of local functions

$$(p_1^t, \ldots, p_s^t, q_1^t, \ldots, q_s^t, z_1^t, \ldots, z_{m-2s}^t),$$

such that for each $t \in [0,1]$, $(p_1^t, \ldots, p_s^t, q_1^t, \ldots, q_s^t, z_1^t, \ldots, z_{m-2s}^t)$ is a local canonical system of coordinates for (M, Π) in a neighborhood of x_t such that $p_i^t(N_t) = q_i^t(N_t) = 0$.

For each point $y \in N_t$ close enough to x_t, define a tangent vector $Y_t(y) \in T_y M$ as follows: For each τ near t, the disk N_τ intersects the local submanifold $\{z_1^t = z_1^t(y), \ldots, z_{m-2s}^t = z_{m-2s}^t(y)\}$ transversally at a unique point y_τ. The map $\tau \mapsto y_\tau$ is smooth. Vector $Y_t(y)$ is defined to be the derivation of this map at $\tau = t$. In particular, $Y_t(x_t)$ is the derivation of the map $\tau \mapsto x_\tau$ at $\tau = t$.

There is a unique cotangent vector $\beta_t(y) \in T_y^* M$ such that $\beta_t(y)$ annulates $T_y N_t$ (the tangent space of N_t at y), and $\sharp\beta_t(y) = Y_t(y)$. For each $t \in [0,1]$ we can choose a function f_t defined in a neighborhood of N_t, in such a way that f_t depends smoothly on t, and that $\mathrm{d}f_t(y) = \beta_t(y)\ \forall\ y \in N_t$. It implies that we have $X_t(y) = Y_t(y)\ \forall\ y \in N_t$, where $X_t = X_{f_t}$ denotes the Hamiltonian vector field of f_t.

Denote by φ_t the local flow of the time-dependent Hamiltonian vector field X_t (where t is considered as the time variable). Then of course φ_t preserves the Poisson structure of V (wherever φ_t is defined). From the construction of X_t, we also see that φ_t moves x_0 to x_t, and it moves a sufficiently small neighborhood of x_0 in N_0 into N_t. In particular, φ_1 defines a local diffeomorphism from N_0 to N_1. Since φ_1 preserves the Poisson structure of M, it also preserves the Poisson structure of N_0. (As explained in the first part of this theorem, the Poisson structure of N_0 depends only on Π and N_0, and does not depend on other things like local canonical coordinates). In other words, φ_1 defines a local Poisson diffeomorphism from (N_0, x_0) to (N_1, x_1). □

In practice, the transverse Poisson structure may be calculated by the following so-called Dirac's constrained bracket formula, or ***Dirac's formula***[1] for short.

Proposition 1.6.2 (Dirac's formula). *Let N be a local submanifold of a Poisson manifold (M, Π) which intersects a symplectic leaf transversely at a point z. Let $\psi_1, \ldots, \psi_{2s}$, where $2s = \text{rank}\ \Pi(z)$, be functions in a neighborhood U of z such that*

$$N = \{x \in U \mid \psi_i(x) = \text{constant}\}. \tag{1.55}$$

Denote by $P_{ij} = \{\psi_i, \psi_j\}$ and by (P^{ij}) the inverse matrix of $(P_{ij})_{i,j=1}^{2s}$. Then the bracket formula for the transverse Poisson structure on N is given as follows:

$$\{f, g\}_N(x) = \{\widetilde{f}, \widetilde{g}\}(x) - \sum_{i,j=1}^{2s} \{\widetilde{f}, \psi_i\}(x) P^{ij}(x) \{\psi_j, \widetilde{g}\}(x) \quad \forall\ x \in N, \tag{1.56}$$

where f, g are functions on N and $\widetilde{f}, \widetilde{g}$ are extensions of f and g to U. The above formula is independent of the choice of extensions $\widetilde{f}$ and $\widetilde{g}$.

Proof (sketch). If one replaces $\widetilde{f}$ by ψ_k ($\forall\ k = 1, \ldots, 2s$) in the above formula, then the right-hand side vanishes. If $\widetilde{f}$ and $\widehat{f}$ are two extensions of f, then we can write $\widehat{f} = \widetilde{f} + \sum_{i=1}^{2s} (\psi_i - \psi_i(z)) h_i$. Using the Leibniz rule, one verifies that

[1] According to Weinstein [**206**], Dirac's formula was actually found by T. Courant and R. Montgomery, who generalized a constraint procedure of Dirac.

the right hand side in Formula (1.56) does not depend on the choice of $\widetilde{f}$. By anti-symmetricity, the right hand side does not depend on the choice of $\widetilde{g}$ either. Finally, we can choose $\widetilde{f}$ and $\widetilde{g}$ to be independent of p_i, q_i in a canonical coordinate system $(p_1, \ldots, p_s, q_1, \ldots, q_s, z_1, \ldots, z_{m-2s})$ provided by the splitting theorem 1.4.5. For that particular choice we have $\{\widetilde{f}, \psi_i\}(x) = 0$ and $\{\widetilde{f}, \widetilde{g}\}(x) = \{f, g\}_N(x)$. $\square$

1.7. The Schouten bracket

1.7.1. Schouten bracket of multi-vector fields.

Recall that, if $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $B = \sum_i b_i \frac{\partial}{\partial x_i}$ are two vector fields written in a local system of coordinates $(x_1, \ldots, x_n)$, then the Lie bracket of A and B is

$$[A, B] = \sum_i a_i (\sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}) - \sum_i b_i (\sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j}) \ . \tag{1.57}$$

We will redenote $\frac{\partial}{\partial x_i}$ by ζ_i and consider them as *formal*, or *odd variables*[2] (formal in the sense that they don't take values in a field, but still form an algebra, and odd in the sense that $\zeta_i \zeta_j = -\zeta_j \zeta_i$, i.e. $\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}$). We can write $A = \sum_i a_i \zeta_i$ and $B = \sum_i b_i \zeta_i$ and consider them formally as functions of variables (x_i, ζ_i) which are linear in the odd variables (ζ_i). We can write $[A, B]$ formally as

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i} \ . \tag{1.58}$$

The above formula makes the Lie bracket of two vector fields look pretty much like the Poisson bracket of two functions in a Darboux coordinate system.

Now if $\Pi = \sum_{i_1 < \cdots < i_p} \Pi_{i_1 \ldots i_p} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}}$ is a p-vector field, then we will consider it as a homogeneous polynomial of degree p in the odd variables (ζ_i):

$$\Pi = \sum_{i_1 < \cdots < i_p} \Pi_{i_1 \ldots i_p} \zeta_{i_1} \ldots \zeta_{i_p} \ . \tag{1.59}$$

It is important to remember that the variables ζ_i do not commute. In fact, they anti-commute among themselves, and commute with the variables x_i:

$$\zeta_i \zeta_j = -\zeta_j \zeta_i; \ x_i \zeta_j = \zeta_j x_i; \ x_i x_j = x_j x_i. \tag{1.60}$$

Due to the anti-commutativity of (ζ_i), one must be careful about the signs when dealing with multiplications and differentiations involving these odd variables. The differentiation rule that we will adopt is as follows:

$$\frac{\partial(\zeta_{i_1} \ldots \zeta_{i_p})}{\partial \zeta_{i_p}} := \zeta_{i_1} \ldots \zeta_{i_{p-1}}. \tag{1.61}$$

[2]The name *odd variable* comes from the theory of supermanifolds, though it is not necessary to know what a supermanifold is in order to understand this section.

Equivalently, $\frac{\partial(\zeta_{i_1}\ldots\zeta_{i_p})}{\partial\zeta_{i_k}} = (-1)^{p-k}\zeta_{i_1}\ldots\widehat{\zeta_{i_k}}\ldots\zeta_{i_p}$, where the hat means that ζ_{i_k} is missing in the product ($1 \leq k \leq p$).

If

$$A = \sum_{i_1<\ldots<i_a} A_{i_1,\ldots,i_a}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_a}} = \sum_{i_1,\ldots,i_a} A_{i_1,\ldots,i_a}\zeta_{i_1}\ldots\zeta_{i_a} \tag{1.62}$$

is an a-vector field, and

$$B = \sum_{i_1<\ldots<i_b} B_{i_1,\ldots,i_b}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_b}} = \sum_{i_1,\ldots,i_a} B_{i_1,\ldots,i_a}\zeta_{i_1}\ldots\zeta_{i_b} \tag{1.63}$$

is a b-vector field, then generalizing Formula (1.58), we can define a bracket of A and B as follows:

$$[A,B] = \sum_i \frac{\partial A}{\partial\zeta_i}\frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)}\sum_i \frac{\partial B}{\partial\zeta_i}\frac{\partial A}{\partial x_i}\ . \tag{1.64}$$

Clearly, the bracket $[A,B]$ of A and B as defined above is a homogeneous polynomial of degree $a+b-1$ in the odd variables (ζ_i), so it is a $(a+b-1)$-vector field.

THEOREM 1.7.1 (Schouten–Nijenhuis). *The bracket defined by Formula (1.64) satisfies the following properties:*
a) Graded anti-commutativity: if A is an a-vector field and B is a b-vector field then

$$[A,B] = -(-1)^{(a-1)(b-1)}[B,A]\ . \tag{1.65}$$

b) Graded Leibniz rule: if A is an a-vector field, B is a b-vector field and C is a c-vector field then

$$[A,B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b}B\wedge[A,C]\ , \tag{1.66}$$

$$[A\wedge B,C] = A\wedge[B,C] + (-1)^{(c-1)b}[A,C]\wedge B\ . \tag{1.67}$$

c) Graded Jacobi identity:

$$\begin{aligned}(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(a-1)}&[B,[C,A]]+\\ &+(-1)^{(c-1)(b-1)}[C,[A,B]] = 0\ .\end{aligned} \tag{1.68}$$

d) If $A = X$ is a vector field then

$$[X,B] = \mathcal{L}_X B\ , \tag{1.69}$$

where $\mathcal{L}_X$ denotes the Lie derivative by X. In particular, if A and B are two vector fields then the Schouten bracket of A and B coincides with their Lie bracket. If $A = X$ is a vector field and $B = f$ is a function (i.e. a 0-vector field) then we have

$$[X,f] = X(f) = \langle \mathrm{d}f, X\rangle\ . \tag{1.70}$$

Proof. Assertion a) follows directly from the definition.

b) The differentiation rule (1.61) implies that

$$\frac{\partial(B\wedge C)}{\partial\zeta_i} = B\frac{\partial C}{\partial\zeta_i} + (-1)^c\frac{\partial B}{\partial\zeta_i}C.$$

Hence we have

$$
\begin{aligned}
[A, B \wedge C] &= \sum \frac{\partial A}{\partial \zeta_i} \frac{\partial (B \wedge C)}{\partial x_i} - (-1)^{(a-1)(b+c-1)} \sum \frac{\partial (B \wedge C)}{\partial \zeta_i} \frac{\partial A}{\partial x_i} \\
&= \sum \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} C + \sum \frac{\partial A}{\partial \zeta_i} B \frac{\partial C}{\partial x_i} - (-1)^{(a-1)(b+c-1)} \sum B \frac{\partial C}{\partial \zeta_i} \frac{\partial A}{\partial x_i} \\
&\quad -(-1)^{(a-1)(b+c-1)+c} \sum \frac{\partial B}{\partial \zeta_i} C \frac{\partial A}{\partial x_i} \\
&= \sum \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} C - (-1)^{(a-1)(b+c-1)+c+ac} \sum \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i} C \\
&\quad +(-1)^{(a-1)b} \left(-(-1)^{(a-1)(c-1)} \sum B \frac{\partial C}{\partial \zeta_i} \frac{\partial A}{\partial x_i} + \sum B \frac{\partial A}{\partial \zeta_i} \frac{\partial C}{\partial x_i} \right) \\
&= [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]
\end{aligned}
$$

The proof of Formula (1.67) is similar.

c) By direct calculations we have

$$(-1)^{(a-1)(c-1)}[A, [B, C]] = S_1 + S_2 + S_3 + S_4 \ ,$$

where

$$S_1 = (-1)^{(a-1)(c-1)} \sum_{i,j} \frac{\partial A}{\partial \zeta_j} \frac{\partial^2 B}{\partial x_j \partial \zeta_i} \frac{\partial C}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_{i,j} \frac{\partial B}{\partial \zeta_i} \frac{\partial^2 C}{\partial x_i \partial \zeta_j} \frac{\partial A}{\partial x_j} \ ,$$

$$S_2 = (-1)^{(a-1)(c-1)} \sum_{i,j} \frac{\partial A}{\partial \zeta_j} \frac{\partial B}{\partial \zeta_i} \frac{\partial^2 C}{\partial x_i \partial x_j} - (-1)^{(c-1)(b-1)} \sum_{i,j} \frac{\partial C}{\partial \zeta_i} \frac{\partial A}{\partial \zeta_j} \frac{\partial^2 B}{\partial x_i \partial x_j} \ ,$$

$$S_3 = (-1)^{(b-1)(a-1)} \sum_{i,j} \frac{\partial^2 B}{\partial \zeta_j \partial x_i} \frac{\partial C}{\partial \zeta_i} \frac{\partial A}{\partial x_j} - (-1)^{(c-1)(b-1)} \sum_{i,j} \frac{\partial^2 C}{\partial \zeta_i \partial x_j} \frac{\partial A}{\partial \zeta_j} \frac{\partial B}{\partial x_i} \ ,$$

$$
\begin{aligned}
S_4 = (-1)^{(b-1)(a+c)+b} &\sum_{i,j} \frac{\partial^2 C}{\partial \zeta_j \partial \zeta_i} \frac{\partial B}{\partial x_i} \frac{\partial A}{\partial x_j} - \\
&- (-1)^{(a-1)(b-1)+c} \sum_{i,j} \frac{\partial^2 B}{\partial \zeta_j \partial \zeta_i} \frac{\partial C}{\partial x_i} \frac{\partial A}{\partial x_j} = \\
= (-1)^{(b-1)(c-1)+a} \sum_{i,j} \frac{\partial^2 C}{\partial \zeta_j \partial \zeta_i} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} &- (-1)^{(a-1)(b-1)+c} \sum_{i,j} \frac{\partial^2 B}{\partial \zeta_j \partial \zeta_i} \frac{\partial C}{\partial x_i} \frac{\partial A}{\partial x_j} \ .
\end{aligned}
$$

(because $\frac{\partial^2 C}{\partial \zeta_i \partial \zeta_j} = -\frac{\partial^2 C}{\partial \zeta_j \partial \zeta_i}$). Each of the summands S_1, S_2, S_3, S_4 will become zero when adding similar terms from $(-1)^{(b-1)(a-1)}[B, [C, A]]$ and $(-1)^{(c-1)(b-1)}[C, [A, B]]$.

d) If f is a function and $X = \sum_i \xi_i \frac{\partial}{\partial x_i}$ is a vector field, then $\frac{\partial f}{\partial \zeta_i} = 0$, and $[X, f] = \sum \frac{\partial X}{\partial \zeta_i} \frac{\partial f}{\partial x_i} = \sum \xi_i \frac{\partial f}{\partial x_i} = X(f)$. When A and B are vector fields, Formula (1.64) clearly coincides with Formula 1.58. When B is a multi-vector field, Assertion d) can be proved by induction on the degree of B, using the Leibniz rules given by Assertion b). $\square$

A-priori, the bracket of an a-vector field A with a b-vector field B, as defined by Formula (1.64), may depend on the choice of local coordinates $(x_1, \ldots, x_n)$. However, the Leibniz rules (1.66) and (1.67) show that the computation of $[A, B]$

can be reduced to the computation of the Lie brackets of vector fields. Since the Lie bracket of vector fields does not depend on the choice of local coordinates, it follows that the bracket $[A, B]$ is in fact a well-defined $(a+b-1)$-vector field which does not depend on the choice of local coordinates.

DEFINITION 1.7.2. If A is an a-vector field and B is a b-vector field, then the uniquely defined $(a+b-1)$-vector field $[A, B]$, given by Formula (1.64) in each local system of coordinates, is called the ***Schouten bracket*** of A and B.

REMARK 1.7.3. Our sign convention in the definition of the Schouten bracket is the same as Koszul's [**123**], but different from Vaisman's [**195**] and some other authors.

The Schouten bracket was first discovered by Schouten [**182, 183**]. Theorem 1.7.1 is essentially due to Schouten [**182, 183**] and Nijenhuis [**162**]. The graded Jacobi identity (1.68) means that the Schouten bracket is a graded Lie bracket: the space $\mathcal{V}^\star(M) = \bigoplus_{p\geq 0} \mathcal{V}^p(M)$, where $\mathcal{V}^p(M)$ is the space of smooth p-vector fields on a manifold M, is a graded Lie algebra, also known as Lie super-algebra, under the Schouten bracket, if we define the grade of $\mathcal{V}^p(M)$ to be $p-1$. In other words, we have to shift the natural grading by -1 for $\mathcal{V}(M)$ together with the Schouten bracket to become a graded Lie algebra in the usual sense.

Another equivalent definition of the Schouten bracket, due to Lichnerowicz [**127**], is as follows. If A is an a-vector field, B is a b-vector field, and η is an $(a+b-1)$-form then

$$\langle \eta, [A, B]\rangle = (-1)^{(a-1)(b-1)}\langle \mathrm{d}(i_B\eta), A\rangle - \langle \mathrm{d}(i_A\eta), B\rangle + (-1)^a\langle \mathrm{d}\eta, A\wedge B\rangle. \tag{1.71}$$

In this formula, $i_A : \Omega^\star(M) \to \Omega^\star(M)$ denotes the ***inner product*** of differential forms with A, i.e. $\langle i_A\beta, C\rangle = \langle \beta, A\wedge C\rangle$ for any k-form β and $(k-a)$-vector field C. If $k < a$ then $i_A\beta = 0$.

More generally, we have the following useful formula, due to Koszul [**123**][3].

LEMMA 1.7.4. *For any $A \in \mathcal{V}^a(M), B \in \mathcal{V}^b(M)$ we have*

$$\begin{aligned} i_{[A,B]} = (-1)^{(a-1)(b-1)} i_A \circ \mathrm{d} \circ i_B - i_B \circ \mathrm{d} \circ i_A + \\ + (-1)^a i_{A\wedge B} \circ \mathrm{d} + (-1)^b \mathrm{d} \circ i_{A\wedge B}. \end{aligned} \tag{1.72}$$

Proof. By induction, using the Leibniz rule. □

Yet another equivalent definition of the Schouten bracket, via the so called curl operator, will be given in Section 2.5.

The Schouten bracket offers a very convenient way to characterize Poisson structures and Hamiltonian vector fields:

THEOREM 1.7.5. *A 2-vector field Π is a Poisson tensor if and only if the Schouten bracket of Π with itself vanishes:*

$$[\Pi, \Pi] = 0\ . \tag{1.73}$$

[3]Koszul [**123**] wrote (1.72) as $i_{[A,B]} = [[i_A, \mathrm{d}], i_B]$. But the brackets on the right-hand side must be understood as graded commutators of graded endomorphisms of $\Omega^\star(M)$, not usual commutators.

If Π *is a Poisson tensor and* f *is a function, then the corresponding Hamiltonian vector field* X_f *satisfies the equation*

$$X_f = -[\Pi, f]\ . \tag{1.74}$$

Proof. It follows directly from Formula (1.64) that Equation (1.73), when expressed in local coordinates, is the same as Equation (1.28). Thus the first part of the above theorem is a consequence of Proposition 1.2.8. The second part also follows directly from Formula (1.64) and the definition of X_f: $-[\Pi, f] = -[\sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, f] = -\sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} = \sum_{i,j} \frac{\partial f}{\partial x_i} \Pi_{ij} \frac{\partial}{\partial x_j} = X_f$. □

By abuse of language, we will call Equation (1.73) the ***Jacobi identity***, because it is equivalent to the usual Jacobi identity (1.2).

EXERCISE 1.7.6. Let Π be a smooth Poisson tensor on a manifold M. Using Theorem 1.7.5, show that the following two statements are equivalent: a) rank $\Pi \leq 2$; b) $f\Pi$ is a Poisson tensor for every smooth function f on M.

EXERCISE 1.7.7. Show that, if Λ is a p-vector field on a Poisson manifold (M, Π), then the Schouten bracket $[\Pi, \Lambda]$ can be given, in terms of multi-derivations, as follows:

$$\begin{aligned}[\Pi, \Lambda](f_1, \dots, f_{p+1}) = &\sum_{i=1}^{p+1} (-1)^{i+1} \{f_i, \Lambda(f_1, \dots \hat{f}_i \dots, f_{p+1})\} + \\ &+ \sum_{i<j} (-1)^{i+j} \Lambda(\{f_i, f_j\}, f_1, \dots \hat{f}_i \dots \hat{f}_j \dots, f_{p+1}),\end{aligned} \tag{1.75}$$

where the hat over f_i and f_j means that these terms are missing in the expression. (Hint: one can use Formula (1.71)).

1.7.2. Schouten bracket on Lie algebras.

The Schouten bracket on $\mathcal{V}^\star(M)$ extends the Lie bracket on $\mathcal{V}^1(M)$ by the graded Leibniz rule. Similarly, by the graded Leibniz rule (1.66,1.67), we can extend the Lie bracket on any Lie algebra $\mathfrak{g}$ to a natural graded Lie bracket on $\wedge^\star \mathfrak{g} = \bigoplus_{k=0}^\infty \wedge^k \mathfrak{g}$, where $\wedge^k \mathfrak{g}$ means $\mathfrak{g} \wedge \dots \wedge \mathfrak{g}$ (k times), which will also be called the ***Schouten bracket***. More precisely, we have:

LEMMA 1.7.8. *Given a Lie algebra* $\mathfrak{g}$ *over* $\mathbb{K}$*, there is a unique bracket on* $\wedge^\star \mathfrak{g} = \oplus_{k=0}^\infty \wedge^k \mathfrak{g}$ *which extends the Lie bracket on* $\mathfrak{g}$ *and which satisfies the following properties,* $\forall\ A \in \wedge^a \mathfrak{g}, B \in \wedge^b \mathfrak{g}, C \in \wedge^c \mathfrak{g}$*:*
a) Graded anti-commutativity:

$$[A, B] = -(-1)^{(a-1)(b-1)}[B, A] \tag{1.76}$$

b) Graded Leibniz rule:

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C] \tag{1.77}$$

$$[A \wedge B, C] = A \wedge [B, C] + (-1)^{(c-1)b}[A, C] \wedge B \tag{1.78}$$

c) Graded Jacobi identity:

$$\begin{aligned}(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + \\ +(-1)^{(c-1)(b-1)}[C, [A, B]] = 0\end{aligned} \tag{1.79}$$

d) The bracket of any element in $\wedge^\star \mathfrak{g}$ with an element in $\wedge^0 \mathfrak{g} = \mathbb{K}$ is zero.

Proof. The proof is straightforward and is left to the reader as an exercise. Remark that, another equivalent way to define the Schouten bracket on $\wedge^\star \mathfrak{g}$ is to identify $\wedge^\star \mathfrak{g}$ with the space of left-invariant multi-vector fields on G, where G is the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$, then restrict the Schouten bracket on $\mathcal{V}^\star(G)$ to these left-invariant multi-vector fields. □

If $\xi : \mathfrak{g} \to \mathcal{V}^1(M)$ is an action of a Lie algebra G on a manifold M, then it can be extended in a unique way by wedge product to a map

$$\wedge\xi : \wedge^\star \mathfrak{g} \to \mathcal{V}^\star(M).$$

For example, if $x, y \in \mathfrak{g}$ then $\wedge\xi(x \wedge y) = \xi(x) \wedge \xi(y)$.

Lemma 1.7.9. *If $\xi : \mathfrak{g} \to \mathcal{V}^1(M)$ is a Lie algebra homomorphism then its extension $\wedge\xi : \wedge^\star \mathfrak{g} \to \mathcal{V}^\star(M)$ preserves the Schouten bracket, i.e.*

$$\wedge\xi([\alpha, \beta]) = [\wedge\xi(\alpha), \wedge\xi(\beta)] \quad \forall\ \alpha, \beta \in \wedge\mathfrak{g}.$$

Proof. The proof is straightforward, by induction, based on the Leibniz rule. □

Notation 1.7.10. For an element $\alpha \in \wedge^\star \mathfrak{g}$, we will denote by α^+ the left-invariant multi-vector field on G whose value at the neutral element e of G is α, i.e. $\alpha^+(g) = L_g \alpha$, where L_g means the left translation by g. Similarly, α^- denotes the right-invariant multi-vector field $\alpha^-(g) = R_g \alpha$, where R_g means the right translation by g.

As a direct consequence of Lemma 1.7.9, we have:

Theorem 1.7.11. *For an element $r \in \mathfrak{g} \wedge \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, the following three conditions are equivalent:*
a) r satisfies the equation $[r, r] = 0$,
b) r^+ is a left-invariant Poisson structure on G,
c) r^- is a right-invariant Poisson structure on G.

Proof. Obvious. □

The equation $[r, r] = 0$ is called the ***classical Yang-Baxter equation***[4] [**86**], or ***CYBE*** for short. This equation will be discussed in more detail in Chapter **??**.

Example 1.7.12. If $x, y \in \mathfrak{g}$ such that $[x, y] = 0$ and $x \wedge y \neq 0$, then $r = x \wedge y$ satisfies the classical Yang-Baxter equation, and the corresponding left and right invariant Poisson structures on G have rank 2.

1.7.3. Compatible Poisson structures.

Definition 1.7.13. Two Poisson tensors Π_1 and Π_2 are called ***compatible*** if their Schouten bracket vanishes:

$$[\Pi_1, \Pi_2] = 0. \tag{1.80}$$

[4]The Yang-Baxter equation has its origins in integrable models in statistical mechanics, and is one of the main tools in the study of integrable systems (see, e.g., [**116**]).

Another equivalent definition is: two Poisson structures Π_1 and Π_2 are compatible if $\Pi_1+\Pi_2$ is also a Poisson structure. Indeed, we have $[\Pi_1+\Pi_2,\Pi_1+\Pi_2] = [\Pi_1,\Pi_1]+[\Pi_2,\Pi_2]+2[\Pi_1,\Pi_2] = 2[\Pi_1,\Pi_2]$, provided that $[\Pi_1,\Pi_1] = [\Pi_2,\Pi_2] = 0$. So Equation (1.80) is equivalent to $[\Pi_1+\Pi_2,\Pi_1+\Pi_2] = 0$.

If Π_1 and Π_2 are two compatible Poisson structures, then we have a whole 2-dimensional family of compatible Poisson structures (or projective 1-dimensional family): for any scalars c_1 and c_2, $c_1\Pi_1+c_2\Pi_2$ is a Poisson structure. Such a family of Poisson structures is often called a ***pencil of Poisson structures***.

EXAMPLE 1.7.14. The linear Poisson structure $x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}+x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}$ on $so^*(3)=\mathbb{R}^3$ can be decomposed into the sum of two compatible linear Poisson structures $(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1})\wedge\frac{\partial}{\partial x_3}$ and $x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}$.

EXAMPLE 1.7.15. If $r_1,r_2\in\mathfrak{g}\wedge\mathfrak{g}$ are solutions of the CYBE $[r,r]=0$, then r_1^+ and r_2^- form a pair of compatible Poisson structures on G, where G is Lie group whose Lie algebra is $\mathfrak{g}$.

EXAMPLE 1.7.16 ([**148**]). On the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, besides the standard Lie-Poisson structure $\{f,g\}_{LP}(x)=\langle[\mathrm{d}f(x),\mathrm{d}g(x)],x\rangle$, consider the following *constant* Poisson structure:

$$\{f,g\}_a(x)=\langle[\mathrm{d}f(x),\mathrm{d}g(x)],a\rangle, \tag{1.81}$$

where a is a fixed element of $\mathfrak{g}^*$. This constant Poisson structure $\{,\}_a$ and the Lie-Poisson structure $\{,\}_{LP}$ are compatible. In fact, their sum is the affine (i.e. nonhomogeneous linear) Poisson structure

$$\{f,g\}(x)=\langle[\mathrm{d}f(x),\mathrm{d}g(x)],x+a\rangle, \tag{1.82}$$

which can be obtained from the linear Poisson structure $\{,\}_{LP}$ by the pull-back of the translation map $x\mapsto x+a$ on $\mathfrak{g}^*$.

EXERCISE 1.7.17. Suppose that Π_1 is a nondegenerate Poisson structure, i.e. it corresponds to a symplectic form ω_1. For a Poisson structure Π_2, denote by ω_2 the differential 2-form defined as follows:

$$\omega_2(X,Y)=\langle\Pi_2,i_X\omega_1\wedge i_Y\omega_1\rangle \ \ \forall\ X,Y\in\mathcal{V}^1(M).$$

Show that $[\Pi_1,\Pi_2]=0$ if and only if $\mathrm{d}\omega_2=0$.

EXERCISE 1.7.18 ([**23**]). Consider a complex pencil of holomorphic Poisson structures $\lambda_1\Pi_1+\lambda_2\Pi_2$, $\gamma_1,\gamma_2\in\mathbb{C}$. Let S be the set of points $(\gamma_1,\gamma_2)\in\mathbb{C}^2$ such that the rank of $\gamma_1\Pi_1+\gamma_2\Pi_2$ is smaller than the rank of a generic Poisson structure in the pencil. Show that if $(\gamma_1,\gamma_2),(\delta_1,\delta_2)\in\mathbb{C}^2\setminus S$ are two arbitrary "regular" points of the pencil (which may coincide), f is a Casimir function for $\gamma_1\Pi_1+\gamma_2\Pi_2$ and g is a Casimir function for $\delta_1\Pi_1+\delta_2\Pi_2$, then $\{f,g\}_{\Pi_1}=\{f,g\}_{\Pi_2}=0$.

REMARK 1.7.19. A vector field X on a manifold is called a ***bi-Hamiltonian system*** if it is Hamiltonian with respect to two compatible Poisson structures: $X=X_{H_1}^{\Pi_1}=X_{H_2}^{\Pi_2}$. Bi-Hamiltonian systems often admit large sets of first integrals, which make them into integrable Hamiltonian systems. Conversely, a vast majority of known integrable systems turn out to be bi-Hamiltonian. The theory of biHamiltonian systems starts with Magri [**133**] and Mischenko–Fomenko [**148**], and there is now a very large amount of articles on the subject. See, e.g., [**2, 9, 10, 24, 66**] for an introduction to the theory of integrable Hamiltonian systems.

1.8. Symplectic realizations

We have seen in Section 1.5 that Poisson manifolds can be viewed as singular foliations by symplectic manifolds. In this section, we will discuss another way to look at Poisson manifolds, namely as quotients of symplectic manifolds.

DEFINITION 1.8.1. A ***symplectic realization*** of a Poisson manifold (P, Π) is a symplectic manifold (M, ω) together with a surjective Poisson submersion $\Phi : (M, \omega) \to (P, \Pi)$ (i.e. a submersion which is a Poisson map).

For example, Theorem 1.3.10 says that if G is a Lie group then T^*G together with the left translation map $L : T^*G \to \mathfrak{g}^*$ is a symplectic realization for $\mathfrak{g}^*$.

The existence of symplectic realizations for arbitrary Poisson manifolds is an important result due to Karasev [**118**] and Weinstein [**208**]:

THEOREM 1.8.2 (Karasev–Weinstein). *Any smooth Poisson manifold of dimension n admits a symplectic realization of dimension $2n$.*

In fact, the result of Karasev and Weinstein is stronger: any Poisson manifold can be realized by a local symplectic groupoid (see Section **??**). In this section, we will give a pedestrian proof of Theorem 1.8.2. First let us show a local version of it, which can be proved by an explicit formula. We will say that $\Phi : (M, \omega, L) \to (P, \Pi)$ is a ***marked symplectic realization*** of (P, Π), where L is a Lagrangian submanifold of M, if it is a symplectic realization such that $\Phi|_L : L \to P$ is a diffeomorphism. Note that in this case we automatically have $\dim M = 2 \dim L = 2 \dim P$.

THEOREM 1.8.3 ([**205**]). *Any point z of a smooth Poisson manifold (P, Π) has an open neighborhood U such that (U, Π) admits a marked symplectic realization.*

Proof. Denote by $(x_1, \ldots, x_n)$ a local system of coordinates at z. We will look for functions $w_i(x, y)$, $i = 1, \ldots, n$, $x = (x_1, \ldots, x_n)$, viewed as functions in a neighborhood of z which depend smoothly on n parameters $y = (y_1, \ldots, y_n)$, such that $w_i(x, 0) = x_i$, and if we denote by $x_i = x_i(w, y)$ $(w = (w_1, \ldots, w_n))$ the inverse functions, then the map $\Theta : (w, y) \mapsto x(w, y)$ is a Poisson submersion from a symplectic manifold M with coordinates (w, y) and standard symplectic structure $\omega = \sum_i \mathrm{d}w_i \wedge \mathrm{d}y_i$ to a neighborhood (U, Π) of z. We may also view (x, y) as a local coordinate system on M. The condition that Θ be a Poisson map can be written as:

$$\{x_i, x_j\}_\omega(x, y) = \{x_i, x_j\}_\Pi(x) \quad (\forall\ i, j = 1, \ldots, n), \tag{1.83}$$

or

$$\sum_{h=1}^{n} \left(\frac{\partial x_i}{\partial w_h}\frac{\partial x_j}{\partial y_h} - \frac{\partial x_i}{\partial y_h}\frac{\partial x_j}{\partial w_h}\right) = \{x_i, x_j\}_\Pi \quad (\forall\ i, j = 1, \ldots, n). \tag{1.84}$$

Viewing the above equation as a matrix equation, and multiplying it by $(\frac{\partial w_k}{\partial x_i})_{k,i}$ on the left and $(\frac{\partial w_l}{\partial x_j})_{j,l}$ on the right of each side, we get

$$\left(\frac{\partial w_k}{\partial x_i}\right)_{k,i}\left(\sum_{h=1}^{n}\left(\frac{\partial x_i}{\partial w_h}\frac{\partial x_j}{\partial y_h}-\frac{\partial x_i}{\partial y_h}\frac{\partial x_j}{\partial w_h}\right)\right)_{ij}\left(\frac{\partial w_l}{\partial x_j}\right)_{j,l}= \\ =\left(\frac{\partial w_k}{\partial x_i}\right)_{k,i}(\{x_i,x_j\}_\Pi)_{ij}\left(\frac{\partial w_l}{\partial x_j}\right)_{j,l}, \tag{1.85}$$

which means

$$\frac{\partial w_l}{\partial y_k}-\frac{\partial w_k}{\partial y_l}=\{w_k,w_l\}_\Pi \quad (\forall\ k,l=1,\ldots,n). \tag{1.86}$$

Equation (1.86) with the initial condition $w_i(x,0)=x_i$ has the following explicit local solution: denote by φ_y^t the local time-t flow of the local Hamiltonian vector field X_{f_y} of the local function $f_y=\sum_i y_i x_i$ on (P,Π). Then put (noting that φ_y^1 is well-defined in a neighborhood of z when y is small enough)

$$w_i(x,y)=\int_0^1 x_i\circ\varphi_y^t \mathrm{d}t\ . \tag{1.87}$$

A straightforward computation, which will be left as an exercise (see [**195, 205**]), shows that this is a solution of (1.86). The local Lagrangian submanifold in question can be given by $L=\{y=0\}$. □

PROPOSITION 1.8.4. *If $\Phi_1:(M_1,\omega_1,L_1)\to(P,\Pi)$ and $\Phi_2:(M_2,\omega_2,L_2)\to(P,\Pi)$ are two marked symplectic realizations of a Poisson manifold (P,Π), then there is a unique symplectomorphism $\Psi:U(L_1)\to U(L_2)$ from a neighborhood $U(L_1)$ of L_1 in (M_1,ω_1) to a neighborhood $U(L_2)$ of L_2 in (M_2,ω_2), which sends L_1 to L_2 and such that $\Phi_1|_{U(L_1)}=\Phi_2|_{U(L_2)}\circ\Psi$.*

Proof (sketch). Clearly, $\psi=(\Phi_2|_{L_2})^{-1}\circ\Phi_1|_{L_1}:L_1\to L_2$ is a diffeomorphism. We want to extend it to a symplectomorphism Ψ from a neighborhood of L_1 to a neighborhood of L_2 which satisfies the conditions of the theorem. Let f be a function on P. Then Ψ must send Φ_1^*f to Φ_2^*f, hence it sends the Hamiltonian vector field $X_{\Phi_1^*f}$ to $X_{\Phi_2^*f}$. If $x_1\in M_1$ is a point close enough to L_1, then there is a point $y_1\in L_1$ and a function f on P such that $x_1=\phi^1_{\Phi_1^*f}(y_1)$, where ϕ_g^t denotes the time-t flow of the Hamiltonian vector field X_g of the function g, and we must have

$$\Psi(x_1)=\phi^1_{\Phi_2^*f}(\psi(y_1)) \tag{1.88}$$

This formula shows the uniqueness of Ψ (if it can be defined) in a neighborhood of L_1. To show that this formula also defines Ψ unambiguously, we will find the graph of Ψ in $M_1\times M_2$. Consider the distribution D on $M_1\times M_2$, generated at each point $(x_1,x_2)\in M_1\times M_2$ by the tangent vectors of the type $(X_{\Phi_1^*f}(x_1),X_{\Phi_2^*f}(x_2))$. The fact that Φ_1 and Φ_2 are Poisson submersions imply that this distribution is regular involutive of dimension $n=\dim P$, so we have an n-dimensional foliation. The graph of Ψ is nothing but the union of the leaves which go through the n-dimensional submanifold $\{(y_1,\psi(y_1))\mid y_1\in L_1\}$.

It remains to show that Φ is symplectic, and sends Φ_1^*f to Φ_2^*f for any function f on P. To show that Φ is symplectic, it suffices to show that its graph in $M_1\times$

$\overline{M_2}$ is Lagrangian (see Proposition 1.3.12). Since the involutive distribution D is generated by Hamiltonian vector fields $(X_{\Phi_1^* f}, X_{\Phi_2^* f})$, and the property of being Lagrangian is invariant under Hamiltonian flow, it is enough to show that the tangent spaces to the graph of Ψ at points $(y_1, \psi(y_1))$, $y_1 \in L_1$, are Lagrangian. But this last fact can be verified immediately.

Since Ψ is symplectic and $\Psi_* X_{\Phi_1^* f} = X_{\Phi_2^* f}$ by construction, it means that $\Psi_*(\Phi_1^* f)$ is equal to $\Phi_2^* f$ up to a constant. But this constant is zero, because these two functions coincide on L_2 by the construction of Ψ. Thus $\Psi_*(\Phi_1^* f) = \Phi_2^* f$ for any function f on P, implying that $\Phi_1 = \Phi_2 \circ \Psi$. □

Theorem 1.8.2 is now a direct consequence of the local realization theorem 1.8.3 and the uniqueness proposition 1.8.4: there is a unique way to glue local marked symplectic realizations together, which glues the marked Lagrangian submanifolds together on their overlaps, to get a marked symplectic realization of a given Poisson manifold. □

Remark 1.8.5. Of course, (non-marked) symplectic realizations of a Poisson manifold (P, Π) of dimension n are not necessarily of dimension $2n$. For example, if (M, ω) is a symplectic realization of (P, Π) and (N, σ) is a symplectic manifold, then $M \times N$ is also a symplectic realization of P. And if (P, Π) is symplectic then it is a symplectic realization of itself. Proposition 1.8.4 can be generalized to an "essential uniqueness" result for non-marked local symplectic realizations (see [**205**]).

An important notion in symplectic geometry, directly related to symplectic realizations, is the following:

Definition 1.8.6 ([**126**]). A foliation $\mathcal{F}$ on a symplectic manifold (M, ω) is called a ***symplectically complete foliation*** if the symplectically orthogonal distribution $(T\mathcal{F})^\perp$ to $\mathcal{F}$ is integrable.

In other words, $\mathcal{F}$ is a symplectically complete foliation if there is another foliation $\mathcal{F}'$ such that $T_x\mathcal{F} = (T_x\mathcal{F}')^\perp \ \forall \ x \in M$. In this case, the pair $(\mathcal{F}, \mathcal{F}')$ is called a ***dual pair***. For example, any Lagrangian foliation is a symplectically complete foliation which is dual to itself.

Theorem 1.8.7 (Libermann [**126**]). *Let $\Phi : (M, \omega) \to P$ be a surjective submersion from a symplectic manifold (M, ω) to a manifold P, such that the level sets of Φ are connected. Denote by $\mathcal{F}$ the foliation whose leaves are level sets of Φ. Then there is a (unique) Poisson structure Π on P such that $\Phi : (M, \omega) \to (P, \Pi)$ is Poisson if and only if $\mathcal{F}$ is symplectically complete.*

Proof (sketch). The symplectically orthogonal distribution $(T\mathcal{F})^\perp$ to $\mathcal{F}$ is generated by Hamiltonian vector fields of the type $X_{\Phi^* f}$ where f is a function on P. The integrability of $(T\mathcal{F})^\perp$ is equivalent to the fact that $[X_{\Phi^* f}, X_{\Phi^* g}]$ is tangent to $(T\mathcal{F})^\perp$ for any functions f, g on P. In other words, $X_{\{\Phi^* f, \Phi^* g\}}$ is tangent to $(T\mathcal{F})^\perp$, i.e. $\{\Phi^* f, \Phi^* g\}$ is constant on the leaves of $\mathcal{F}$. Since the leaves of $\mathcal{F}$ are level sets of Φ, it means that there is a function h on P such that $\{\Phi^* f, \Phi^* g\} = \Phi^* h$. In other words, $\{f, g\} := h$ is a Poisson bracket on P such that Φ is Poisson. □

CHAPTER 2

Poisson cohomology

2.1. Poisson cohomology

2.1.1. Definition of Poisson cohomology.

Poisson cohomology was introduced by Lichnerowicz [**127**]. Its existence is based on the following simple lemma.

LEMMA 2.1.1. *If Π is a Poisson tensor, then for any multi-vector field A we have*

$$[\Pi, [\Pi, A]] = 0 \ . \tag{2.1}$$

Proof. By the graded Jacobi identity (1.68) for the Schouten bracket, if Π is a 2-vector field and A is an a-vector field then

$$(-1)^{a-1}[\Pi, [\Pi, A]] - [\Pi, [A, \Pi]] + (-1)^{a-1}[A, [\Pi, \Pi]] = 0 \ .$$

Moreover, $[A, \Pi] = -(-1)^{a-1}[\Pi, A]$ due to the graded anti-commutativity, hence $[\Pi, [\Pi, A]] = -\frac{1}{2}[A, [\Pi, \Pi]]$. Now if Π is a Poisson structure, then $[\Pi, \Pi] = 0$, and therefore $[\Pi, [\Pi, A]] = 0$. □

Let (M, Π) be a smooth Poisson manifold. Denote by $\delta = \delta_\Pi : \mathcal{V}^\star(M) \longrightarrow \mathcal{V}^\star(M)$ the $\mathbb{R}$-linear operator on the space of multi-vector fields on M, defined as follows:

$$\delta_\Pi(A) = [\Pi, A]. \tag{2.2}$$

Then Lemma 2.1.1 says that δ_Π is a ***differential operator*** in the sense that $\delta_\Pi \circ \delta_\Pi = 0$. The corresponding differential complex $(\mathcal{V}^\star(M), \delta)$, i.e.,

$$\ldots \longrightarrow \mathcal{V}^{p-1}(M) \xrightarrow{\delta} \mathcal{V}^p(M) \xrightarrow{\delta} \mathcal{V}^{p+1}(M) \longrightarrow \ldots \ , \tag{2.3}$$

will be called the ***Lichnerowicz complex***. The cohomology of this complex is called ***Poisson cohomology***.

By definition, Poisson cohomology groups of (M, Π), i.e. the cohomology groups of the Lichnerowicz complex (2.3), are the quotient groups

$$H^p_\Pi(M) = \frac{\ker(\delta : \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p+1}(M))}{\mathrm{Im}(\delta : \mathcal{V}^{p-1}(M) \longrightarrow \mathcal{V}^p(M))} \ . \tag{2.4}$$

The above Poisson cohomology groups are also denoted by $H^p(M, \Pi)$, or also $H^p_{LP}(M, \Pi)$, where LP stands for Lichnerowicz–Poisson.

REMARK 2.1.2. Poisson cohomology groups can be very big, infinite-dimensional. For example, when $\Pi = 0$ then $H^\star_\Pi(M) := \bigoplus_k H^k_\Pi(M) = \mathcal{V}^\star(M)$. Poisson cohomology groups of smooth Poisson manifolds have a natural induced topology from the Fréchet spaces of multi-vector fields, which make them into not-necessarily-separated locally convex topological vector spaces (see Ginzburg [**88, 89**]).

2.1.2. Interpretation of Poisson cohomology.

The zeroth Poisson cohomology group $H^0_\Pi(M)$ is the group of functions $f \in \mathcal{C}^\infty(M)$ such that $X_f = -[\Pi, f] = 0$. In other words, $H^0_\Pi(M)$ is the space of ***Casimir functions*** of Π, i.e. the space of first integrals of the associated symplectic foliation.

The first Poisson cohomology group $H^1_\Pi(M)$ is the quotient of the space of Poisson vector fields (i.e. vector fields X such that $[\Pi, X] = 0$) by the space of Hamiltonian vector fields (i.e. vector fields of the type $[\Pi, f] = X_{-f}$). Poisson vector fields are infinitesimal automorphisms of the Poisson structures, while Hamiltonian vector fields may be interpreted as *inner* infinitesimal automorphisms. Thus $H^1_\Pi(M)$ may be interpreted as the space of ***outer infinitesimal automorphisms*** of Π.

The second Poisson cohomology group $H^2_\Pi(M)$ is the quotient of the space of 2-vector fields Λ which satisfy the equation $[\Pi, \Lambda] = 0$ by the space of 2-vector fields of the type $\Lambda = [\Pi, Y]$. If $[\Pi, \Lambda] = 0$ and ε is a formal (infinitesimal) parameter, then $\Pi + \varepsilon\Lambda$ satisfies the Jacobi identity up to terms of order ε^2:

$$[\Pi + \varepsilon\Lambda, \Pi + \varepsilon\Lambda] = \varepsilon^2[\Lambda, \Lambda] = 0 \mod \varepsilon^2. \tag{2.5}$$

So one may view $\Pi + \varepsilon\Lambda$ as an infinitesimal deformation of Π in the space of Poisson tensors. On the other hand, up to terms of order ε^2, $\Pi + \varepsilon[\Pi, Y]$ is equal to $(\varphi^\varepsilon_Y)_*\Pi$, where φ^ε_Y denotes the time-ε flow of Y. Therefore $\Pi + \varepsilon[\Pi, Y]$ is a trivial infinitesimal deformation of Π up to a infinitesimal diffeomorphism. Thus, $H^2_\Pi(M)$ is the quotient of the space of all possible infinitesimal deformations of Π by the space of trivial deformations. In other words, $H^2_\Pi(M)$ may be interpreted as the moduli space of formal ***infinitesimal deformations*** of Π. For this reason, the second Poisson cohomology group plays a central role in the study of normal forms of Poisson structures.

The third Poisson cohomology group $H^3_\Pi(M)$ may be interpreted as the space of ***obstructions to formal deformation***. Suppose that we have an infinitesimal deformation $\Pi + \varepsilon\Lambda$, i.e. $[\Pi, \Lambda] = 0$. Then a-priori, $\Pi + \varepsilon\Lambda$ satisfies the Jacobi identity only modulo ε^2. To make it satisfy the Jacobi identity modulo ε^3, we have to add a term $\varepsilon^2\Lambda_2$ such that

$$[\Pi + \varepsilon\Lambda + \varepsilon^2\Lambda_2, \Pi + \varepsilon\Lambda + \varepsilon^2\Lambda_2] = 0 \mod \varepsilon^3. \tag{2.6}$$

The equation to solve is $2[\Pi, \Lambda_2] = -[\Lambda, \Lambda]$. This equation can be solved if and only if the cohomology class of $[\Lambda, \Lambda]$ in $H^3_\Pi(M)$ is trivial. Similarly, if (2.6) is already satisfied, to find a term $\varepsilon^3\Lambda_3$ such that

$$[\Pi + \varepsilon\Lambda + \varepsilon^2\Lambda_2 + \varepsilon^3\Lambda_3, \Pi + \varepsilon\Lambda + \varepsilon^2\Lambda_2 + \varepsilon^3\Lambda_3] = 0 \mod \varepsilon^4, \tag{2.7}$$

we have to make sure that the cohomology class of $[\Lambda, \Lambda_2]$ in $H^3_\Pi(M)$ vanishes, and so on.

The Poisson tensor Π is itself a cocycle in the Lichnerowicz complex. If the cohomology class of Π in $H^2_\Pi(M)$ vanishes, i.e. there is a vector field Y such that $\Pi = [\Pi, Y]$, then Π is called an ***exact Poisson structure***.

2.1.3. Poisson cohomology versus de Rham cohomology.

Recall that, the Poisson structure Π gives rise to a homomorphism

$$\sharp = \sharp_\Pi : T^*M \longrightarrow TM, \tag{2.8}$$

which associates to each covector α a unique vector $\sharp(\alpha)$ such that

$$\langle \alpha \wedge \beta, \Pi \rangle = \langle \beta, \sharp(\alpha) \rangle \tag{2.9}$$

for any covector β. This homomorphism is an isomorphism if and only if Π is nondegenerate, i.e., is a symplectic structure. By taking exterior powers of the above map, we can extend it to a homomorphism

$$\sharp : \Lambda^p T^*M \longrightarrow \Lambda^p TM, \tag{2.10}$$

and hence a $\mathcal{C}^\infty(M)$-linear homomorphism

$$\sharp : \Omega^p(M) \longrightarrow \mathcal{V}^p(M), \tag{2.11}$$

where $\Omega^p(M)$ denotes the space of smooth differential forms of degree p on M. Recall that $\sharp$ is called the ***anchor map*** of Π.

LEMMA 2.1.3. *For any smooth differential form η on a given smooth Poisson manifold (M, Π) we have*

$$\sharp(\mathrm{d}\eta) = -[\Pi, \sharp(\eta)] = -\delta_\Pi(\sharp(\eta)) \ . \tag{2.12}$$

Proof. By induction on the degree of η, using the Leibniz rule. If η is a function then $\sharp(\eta) = \eta$ and $\sharp(\mathrm{d}\eta) = -[\Pi, \eta] = X_\eta$, the Hamiltonian vector field of η. If $\eta = \mathrm{d}f$ is an exact 1-form then $\sharp(\mathrm{d}\eta) = 0$ and $[\Pi, \sharp(\eta)] = [\Pi, X_f] = 0$, hence Equation (2.12) is satisfied. If Equation (2.12) is satisfied for a differential p-form η and a differential q-form μ, then its also satisfied for their exterior product $\eta \wedge \mu$. Indeed, we have $\sharp(\mathrm{d}(\eta\wedge\mu)) = \sharp(\mathrm{d}\eta \wedge \mu + (-1)^p \eta \wedge \mathrm{d}\mu) = \sharp(\mathrm{d}\eta) \wedge \sharp(\mu) + (-1)^p \sharp(\eta) \wedge \sharp(\mathrm{d}\mu) = -[\Pi, \sharp(\eta)] \wedge \sharp(\mu) - (-1)^p \sharp(\eta) \wedge [\Pi, \sharp(\mu)] = -[\Pi, \sharp(\eta) \wedge \sharp(\mu)] = -[\Pi, \sharp(\eta \wedge \mu)]$. □

The above lemma means that, up to a sign, the operator $\sharp$ intertwines the usual differential operator d of the ***de Rham complex***

$$\ldots \longrightarrow \Omega^{p-1}(M) \xrightarrow{\mathrm{d}} \Omega^p(M) \xrightarrow{\mathrm{d}} \Omega^{p+1}(M) \longrightarrow \ldots \tag{2.13}$$

with the differential operator δ_Π of the Lichnerowicz complex. In particular, it induces a linear homomorphism of the corresponding cohomologies. In other words, we have:

THEOREM 2.1.4 ([**127**]). *For every smooth Poisson manifold (M, Π), there is a natural homomorphism*

$$\sharp^* : H^\star_{dR}(M) = \bigoplus_p H^p_{dR}(M) \longrightarrow H^\star_\Pi(M) = \bigoplus_p H^p_\Pi(M) \tag{2.14}$$

from its de Rham cohomology to its Poisson cohomology, induced by the map $\sharp = \sharp_\Pi$. If M is a symplectic manifold, then this homomorphism is an isomorphism.

When M is symplectic, $\sharp$ is an isomorphism, and that's why $\sharp^*$ is also an isomorphism. $\square$

REMARK 2.1.5. de Rham cohomology has a graded Lie algebra structure, given by the cap product (induced from the exterior product of differential forms). So does Poisson cohomology. The Lichnerowicz homomorphism $\sharp^* : H^\star_{dR}(M) \longrightarrow H^\star_\Pi(M)$ in the above theorem is not only a linear homomorphism, but also an algebra homomorphism.

REMARK 2.1.6. If (M, Π) is not symplectic then the map $\sharp^* : H^\star_{dR}(M) \to H^\star_\Pi(M)$ is not an isomorphism in general. In particular, while de Rham cohomology groups of manifolds of "finite type" (e.g. compact manifolds) are of finite dimensions, Poisson cohomology groups may have infinite dimension in general. An interesting and largely open question is: what are the conditions for the Lichnerowicz homomorphism to be injective or surjective ?

2.1.4. Other versions of Poisson cohomology.

If, in the Lichnerowicz complex, instead of *smooth* multi-vector fields, we consider other classes of multi-vector fields, then we arrive at other versions of Poisson cohomology. For example, if Π is an analytic Poisson structure, and one considers analytic multi-vector fields, then one gets *analytic Poisson cohomology*.

Recall that, a ***germ*** of an object (e.g., a function, a differential form, a Riemannian metric, etc.) at a point z is an object defined in a neighborhood of z. Two germs at z are considered to be the same if there is a neighborhood of z in which they coincide. When considering a germ of smooth (resp. analytic) Poisson structure Π at a point z, it is natural to talk about *germified Poisson cohomology*: the space $\mathcal{V}^\star(M)$ in the Lichnerowicz complex is replaced by the space of germs of smooth (resp. analytic) multi-vector fields. More generally, given any subset N of a Poisson manifold (M, Π), one can define germified Poisson cohomology at N. Similarly, one can talk about *formal Poisson cohomology*. By convention, the germ of a formal multi-vector field is itself. Viewed this way, formal Poisson cohomology is the formal version of germified Poisson cohomology.

If M is not compact, then one may be interested in *Poisson cohomology with compact support*, by restricting one's attention to multi-vector fields with compact support. Remark that Theorem 2.1.4 also holds in the case with compact support: if (M, Π) is a symplectic manifold then its de Rham cohomology with compact support is isomorphic to its Poisson cohomology with compact support.

If one considers only multi-vector fields which are tangent to the characteristic distribution, then one gets *tangential Poisson cohomology*. (A multi-vector field λ is said to be *tangent* to a distribution $\mathcal{D}$ on a manifold M if at each point $x \in M$ one can write $\Lambda(x) = \sum a_i v_{i1} \wedge \ldots \wedge v_{is}$ where v_{ij} are vectors lying in $\mathcal{D}$). It is easy to see that the homomorphism $\sharp^*$ in Theorem 2.1.4 also makes sense for tangential Poisson cohomology (and tangential de Rham cohomology).

The above versions of Poisson cohomology also have a natural interpretation, similar to the one given for smooth Poisson cohomology.

2.1.5. Computation of Poisson cohomology.

If a Poisson structure Π on a manifold M is nondegenerate (i.e. symplectic), then Poisson cohomology of Π is the same as de Rham cohomology of M. There are many tools for computing de Rham cohomology groups, and these groups have probably been computed for most "familiar" manifolds, see, e.g., [**27, 85**]. However, when Π is not symplectic, $H^\star_\Pi(M)$ is much more difficult to compute than $H^\star_{dR}(M)$ in general, and at the moment of writing of this book, there are few Poisson (non-symplectic) manifolds for which Poisson cohomology has been computed. For one thing, $H^\star_\Pi(M)$ can have infinite dimension even when M is compact, and the problem of determining whether $H^\star_\Pi(M)$ is finite dimensional or not is already a difficult open problem for most Poisson structures that we know of.

Nevertheless, various tools from algebraic topology and homological algebra can be adapted to the problem of computation of Poisson cohomology. One of them is the classical Mayer–Vietoris sequence (see, e.g., [**27**]). The following Poisson cohomology version of Mayer–Vietoris sequence is absolutely analogous to its de Rham cohomology version.

PROPOSITION 2.1.7 ([**195**]). *Let U and V be two open subsets of a smooth Poisson manifold (M, Π). Then*

$$0 \longrightarrow \mathcal{V}^\star(U \cup V) \xrightarrow{\alpha} \mathcal{V}^\star(U) \oplus \mathcal{V}^\star(V) \xrightarrow{\beta} \mathcal{V}^\star(U \cap V) \longrightarrow 0, \tag{2.15}$$

*where $\alpha(\Lambda) = (\Lambda|_U, \Lambda|_V)$ is the restriction map, and $\beta(\Lambda_1, \Lambda_2) = \Lambda_1|_{U\cap V} - \Lambda_2|_{U\cap V}$ is the difference map, is an exact sequence of smooth Lichnerowicz complexes, and the corresponding cohomological long exact sequence (called the **Mayer–Vietoris sequence**) has the form*

$$\begin{aligned} \ldots \longrightarrow H^k(U \cup V, \Pi) \xrightarrow{\alpha_*} H^k(U, \Pi) \oplus H^k(V, \Pi) \xrightarrow{\beta_*} \\ H^k(U \cap V, \Pi) \longrightarrow H^{k+1}(U \cup V, \Pi) \xrightarrow{\alpha_*} \ldots . \end{aligned} \tag{2.16}$$

The proof of Proposition 2.1.7 is also absolutely similar to the proof of its de Rham version. The above Mayer–Vietoris sequence reduces the computation of Poisson cohomology on a manifold to the computation of Poisson cohomology on small open sets (which contain singularities of the Poisson structure). To study (germified) Poisson cohomology of singularities of Poisson structures, one can try to use the tools from singularity theory. See, e.g., [**151**] for the case of dimension 2.

Another standard tool is the spectral sequence, which will be discussed in Section 2.4.

In the case of linear Poisson structures, Poisson cohomology is intimately related to Lie algebra cohomology, also known as Chevalley–Eilenberg cohomology, which will be discussed in Section 2.3.

Poisson cohomology can be viewed as a particular case of cohomology of Lie algebroids. This leads to a definition and study of Poisson cohomology from a purely algebraic point of view, as was done by Huebschmann [**111**].

Some other methods for computing and studying Poisson cohomology include: the use of *symplectic groupoids* to reduce the computation of Poison cohomology of certain Poisson manifolds to the computation of de Rham cohomology of other manifolds [**212**]; the *van Est map* which relates Lie algebroid cohomology with differentiable cohomology of Lie groupoids [**210, 56**]; comparison of Poisson cohomology of Poisson manifolds which are *Morita equivalent* [**92, 91, 90, 56**]; equivariant Poisson cohomology [**89**].

2.2. Normal forms of Poisson structures

Consider a Poisson structure Π on a manifold M. In a given system of coordinates $(x_1, \dots, x_m)$, Π has the expression

$$\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{2.17}$$

A priori, the coefficients Π_{ij} of Π may be very complicated, non-polynomial functions. The idea of normal forms is to simplify these coefficients in the expression of Π.

A (local) ***normal form*** of Π is a Poisson structure

$$\Pi' = \sum_{i<j} \Pi'_{ij} \frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j} = \frac{1}{2} \sum_{i,j} \Pi'_{ij} \frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j} \tag{2.18}$$

which is (locally) isomorphic to Π, i.e. there is a (local) diffeomorphism $\varphi : (x_i) \mapsto (x'_i)$ called a ***normalization*** such that $\varphi_* \Pi = \Pi'$, such that the functions Π'_{ij} are "simpler" than the functions Π_{ij}. The ideal would be that Π'_{ij} were constant functions. According to Remark 1.4.6, such a local normal form exists when Π is a (locally) regular Poisson structure.

Near a singular point of Π, we can use the splitting theorem 1.4.5 to write Π as the direct sum of a constant symplectic structure with a Poisson structure which vanishes at a point. The local normal form problem for Π is then reduced to the problem of local normal forms for a Poisson structure which vanishes at a point.

Having this in mind, we now assume that Π vanishes at the origin 0 of a given local coordinate system $(x_1, \dots, x_m)$. Denote by

$$\Pi = \Pi^{(k)} + \Pi^{(k+1)} + \ldots + \Pi^{(k+n)} + \ldots \quad (k \geq 1) \tag{2.19}$$

the Taylor expansion of Π in the coordinate system $(x_1, \dots, x_m)$, where for each $h \in \mathbb{N}$, $\Pi^{(h)}$ is a 2-vector field whose coefficients $\Pi^{(h)}_{ij}$ are homogeneous polynomial functions of degree h. $\Pi^{(k)}$, assumed to be non-trivial, is the term of lowest degree in Π, and is called the ***homogeneous part***, or ***principal part*** of Π. If $k = 1$ then $\Pi^{(1)}$ is called the ***linear part*** of Π, and so on. This homogeneous part can be defined intrinsically, i.e. it does not depend on the choice of local coordinates.

At the formal level, the Jacobi identity for Π can be written as

$$\begin{aligned} 0 &= [\Pi, \Pi] = [\Pi^{(k)} + \Pi^{(k+1)} + \ldots, \Pi^{(k)} + \Pi^{(k+1)} + \ldots] \\ &= [\Pi^{(k)}, \Pi^{(k)}] + 2[\Pi^{(k)}, \Pi^{(k+1)}] + 2[\Pi^{(k)}, \Pi^{(k+2)}] + [\Pi^{(k+1)}, \Pi^{(k+1)}] + \ldots, \end{aligned}$$

which leads to (by considering terms of the same degree):

$$\begin{aligned} &[\Pi^{(k)}, \Pi^{(k)}] = 0, \\ &2[\Pi^{(k)}, \Pi^{(k+1)}] = 0, \\ &2[\Pi^{(k)}, \Pi^{(k+2)}] + [\Pi^{(k+1)}, \Pi^{(k+1)}] = 0, \\ &\ldots \end{aligned} \tag{2.20}$$

In particular, the homogeneous part $\Pi^{(k)}$ of Π is a Poisson structure, and Π may be viewed as a deformation of $\Pi^{(k)}$. A natural homogenization question arises: is this deformation trivial ? In other words, is Π locally (or formally) isomorphic to its homogeneous part $\Pi^{(k)}$? That's where Poisson cohomology comes in, because, as explained in Subsection 2.1.2, Poisson cohomology governs (formal) deformations of Poisson structures.

When $k = 1$, one talks about the linearization problem, and when $k = 2$ one talks about the quadratization problem, and so on. These problems, for Poisson structures and related structures like Nambu structures, Lie algebroids and Lie groupoids, will be studied in detail in the subsequent chapters of this book. Here we will discuss, at the formal level, a more general problem of ***quasi-homogenization***.

Denote by

$$Z = \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}, \quad w_i \in \mathbb{N} \tag{2.21}$$

a given diagonal linear vector field with the following special property: its coefficients w_i are positive integers. Such a vector field is called a ***quasi-radial vector field***. (When $w_i = 1\ \forall i$, we get the usual ***radial vector field***).

A multi-vector field Λ is called ***quasi-homogeneous*** of degree d ($d \in \mathbb{Z}$) with respect to Z if

$$\mathcal{L}_Z \Lambda = d\Lambda. \tag{2.22}$$

For a function f, it means $Z(f) = df$. For example, a monomial k-vector field

$$(\prod_{i=1}^{n} x_i^{a_i}) \frac{\partial}{\partial x_{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{j_k}}, \quad a_i \in \mathbb{Z}_{\geq 0}, \tag{2.23}$$

is quasi-homogeneous of degree $\sum_{i=1}^{n} a_i w_i - \sum_{s=1}^{k} w_{j_s}$. In particular, monomial terms of high degree in the usual sense (i.e. with large $\sum a_i$) also have high quasi-homogeneous degree. As a consequence, quasi-homogeneous (smooth, formal or analytic) multi-vector fields are automatically polynomial in the usual sense. Note that the quasi-homogeneous degree of a monomial multi-vector field can be negative, though it is always greater or equal to $-\sum_{i=1}^{n} w_i$.

Given a Poisson structure Π with $\Pi(0) = 0$, by abuse of notation, we will now denote by

$$\Pi = \Pi^{(d_1)} + \Pi^{(d_2)} + \ldots, \quad d_1 < d_2 < \ldots \tag{2.24}$$

the quasi-homogeneous Taylor expansion of Π with respect to Z, where each term $\Pi^{(d_i)}$ is quasi-homogeneous of degree d_i. The term $\Pi^{(d_1)}$, assumed to be non-trivial, is called the ***quasi-homogeneous part*** of Π. Similarly to the case with usual homogeneous Taylor expansion, the Jacobi identity for Π implies the Jacobi identity for $\Pi^{(d_1)}$, which means that $\Pi^{(d_1)}$ is a quasi-homogeneous Poisson structure, and Π may be viewed as a deformation of $\Pi^{(d_1)}$. The quasi-homogenization problem is the following : is there a transformation of coordinates which sends Π to $\Pi^{(d_1)}$, i.e. which kills all the terms of quasi-homogeneous degree $> d_1$ in the expression of Π?

In order to treat this quasi-homogenization problem at the formal level, we will need the quasi-homogeneous graded version of Poisson cohomology.

Let $\Pi^{(d)}$ be a Poisson structure on an n-dimensional space $V = \mathbb{K}^n$, which is quasi-homogeneous of degree d with respect to a given quasi-radial vector field $Z = \sum_{i=1}^n w_i x_i \frac{\partial}{\partial x_i}$. For each $r \in \mathbb{Z}$, denote by $\mathcal{V}^k_{(r)} = \mathcal{V}^k_{(r)}(\mathbb{K}^n)$ the space of quasi-homogeneous polynomial k-vector fields on $\mathbb{K}^n$ of degree r with respect to Z. Of course, we have

$$\mathcal{V}^k = \oplus_r \mathcal{V}^k_{(r)}, \tag{2.25}$$

where $\mathcal{V}^k = \mathcal{V}^k(\mathbb{K}^n)$ is the space of all polynomial vector fields on $\mathbb{K}^n$. Note that, if $\Lambda \in \mathcal{V}^k_{(r)}$ then

$$\mathcal{L}_Z[\Pi^{(d)}, \Lambda] = [\mathcal{L}_Z \Pi^{(d)}, \Lambda] + [\Pi^{(d)}, \mathcal{L}_Z \Lambda] = (d+r)[\Pi^{(d)}, \Lambda],$$

i.e. $\delta_{\Pi^{(d)}} \Lambda = [\Pi^{(d)}, \Lambda] \in \mathcal{V}^{k+1}_{r+d}$. The group

$$H^k_{(r)}(\Pi^{(d)}) = \frac{\ker(\delta_{\Pi^{(d)}} : \mathcal{V}^k_{(r)} \longrightarrow \mathcal{V}^{k+1}_{(r+d)})}{\mathrm{Im}(\delta_{\Pi^{(d)}} : \mathcal{V}^{k-1}_{(r-d)} \longrightarrow \mathcal{V}^k_{(r)})} \tag{2.26}$$

is called the k-th quasi-homogeneous of degree r Poisson cohomology group of $\Pi^{(d)}$. Of course, there is a natural injection from $H^k_{(r)}(\Pi^{(d)})$ to the usual (formal, analytic or smooth) Poisson cohomology group $H^k(\Pi^{(d)})$ of $\Pi^{(d)}$ over $\mathbb{K}^n$. While $H^k(\Pi^{(d)})$ may be of infinite dimension, $H^k_{(r)}(\Pi^{(d)})$ is always of finite dimension (for each r).

Return now to the quasi-homogeneous Taylor series $\Pi = \Pi^{(d_1)} + \Pi^{(d_2)} + \ldots$. The Jacobi identity for Π implies that $[\Pi^{(d_1)}, \Pi^{(d_2)}] = 0$, i.e. $\Pi^{(d_2)}$ is a quasi-homogeneous cocycle in the Lichnerowicz complex of $\Pi^{(d_1)}$. If this term $\Pi^{(d_2)}$ is a coboundary, i.e. $\Pi^{(d_2)} = [\Pi^{(d_1)}, X^{(d_2-d_1)}]$ for some quasi-homogeneous vector field $X^{(d_2-d_1)} = X_i^{(d_2-d_1)} \partial/\partial x_i$, then the coordinate transformation $x'_i = x_i - X_i^{(d_2-d_1)}$ will kill the term $\Pi^{(d_2)}$ in the expression of Π. More generally, we have:

Proposition 2.2.1. *With the above notations, suppose that* $\Pi^{(d_k)} = [X, \Pi^{(d_1)}] + \Lambda^{(d_k)}$ *for some* $k > 1$*, where* $X = X_i \partial/\partial x_i$ *is a quasi-homogeneous vector field of degree* $d_k - d_1$*. Then the diffeomorphism (coordinate transformation)* $\phi : (x_i) \mapsto (x'_i) = (x_i - X_i)$ *transforms* Π *into*

$$\phi_* \Pi = \Pi^{(d_1)} + \ldots + \Pi^{(d_{k-1})} + \Lambda^{(d_k)} + \widetilde{\Pi}^{(d_{k+1})} \ldots . \tag{2.27}$$

In other words, this transformation suppresses the term $[X, \Pi^{(d_1)}]$ *without changing the terms of degree strictly smaller than* d_k*.*

Proof. Denote by $\Gamma = \phi_* \Pi$. For the Poisson structure Π we have

$$\{x'_i, x'_j\} = \sum_{uv} \frac{\partial x'_i}{\partial x_u} \frac{\partial x'_j}{\partial x_v} \{x_u, x_v\} = \sum_{uv} \frac{\partial x'_i}{\partial x_u} \frac{\partial x'_j}{\partial x_v} \Pi_{uv} =$$

$$= \sum_{uv} (\delta^u_i - \frac{\partial X_i}{\partial x_u})(\delta^v_j - \frac{\partial X_j}{\partial x_v})(\Pi^{(d_1)} + \Pi^{(d_2)} + \ldots)_{uv},$$

where δ^u_i is the Kronecker symbol, and the terms of degree smaller or equal to d_k in this expression give

$$(\Pi^{(d_1)} + \ldots + \Pi^{(d_k)})_{ij} - \sum_u \frac{\partial X_i}{\partial x_u} \Pi^{(d_1)}_{uj} - \sum_v \frac{\partial X_j}{\partial x_v} \Pi^{(d_1)}_{iv}.$$

On the other hand, by definition, $\{x'_i, x'_j\}$ is equal to $\Gamma_{ij} \circ \phi$. But the terms of degree smaller or equal to d_k in the expansion of $\Gamma_{ij} \circ \phi$ are

$$(\Gamma^{(d_1)} + \ldots + \Gamma^{(d_k)})_{ij} - \sum_s X_s \frac{\partial \Gamma^{(d_1)}_{ij}}{\partial x_s}.$$

Comparing the terms of degree $d_1, \ldots, d_{k-1}$, we get $\Gamma^{(d_1)}_{ij} = \Pi^{(d_1)}_{ij}, \ldots, \Gamma^{(d_{k-1})}_{ij} = \Pi^{(d_{k-1})}_{ij}$. As for the terms of degree d_k, they give

$$\Gamma^{(d_k)}_{ij} - \sum_s X_s \frac{\partial \Pi^{(d_1)}_{ij}}{\partial x_s} = \Pi^{(d_k)}_{ij} - \sum_u \frac{\partial X_i}{\partial x_u} \Pi^{(d_1)}_{uj} - \sum_v \frac{\partial X_j}{\partial x_v} \Pi^{(d_1)}_{iv}.$$

As we have

$$[X, \Pi^{(d_1)}]_{ij} = \sum_s X_s \frac{\partial \Pi^{(d_1)}_{ij}}{\partial x_s} - \sum_u \frac{\partial X_i}{\partial x_u} \Pi^{(d_1)}_{uj} - \sum_v \frac{\partial X_j}{\partial x_v} \Pi^{(d_1)}_{iv},$$

it follows that

$$\Gamma^{(d_k)}_{ij} = \Pi^{(d_k)}_{ij} + [X, \Pi^{(d_1)}]_{ij} = \Pi^{(d_k)}_{ij} - [\Pi^{(d_1)}, X]_{ij} = \Lambda^{(d_k)}_{ij}.$$

The proposition is proved. □

THEOREM 2.2.2. *If the quasi-homogeneous Poisson cohomology groups $H^2_{(r)}(\Pi^{(d)})$ of a quasi-homogeneous Poisson structure $\Pi^{(d)}$ of degree d are trivial for all $r > d$, then any Poisson structure admitting a formal quasi-homogeneous expansion $\Pi = \Pi^{(d)} + \Pi^{(d_2)} + \ldots$ is formally isomorphic to its quasi-homogeneous part $\Pi^{(d)}$.*

Proof. Use Proposition 2.2.1 to kill the terms of degree strictly greater than d in Π consecutively. □

EXAMPLE 2.2.3. One can use Theorem 2.2.2 to show that any Poisson structure on $\mathbb{K}^2$ of the form $\Pi = f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, where $f = x^2 + y^3+$ higher order terms, is formally isomorphic to $(x^2 + y^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. (This is a simple singularity studied by Arnold, see Theorem **??**). The quasi-radial vector field in this case is $Z = 3x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$.

2.3. Cohomology of Lie algebras

Let $\Pi^{(1)}$ be a linear Poisson structure on a vector space $\mathbb{K}^n$. Denote by $\mathfrak{g} = ((\mathbb{K}^n)^*, \{,\}_{\Pi^{(1)}})$ the Lie algebra corresponding to $\Pi^{(1)}$. We will see in this section that Poisson cohomology groups of $\Pi^{(1)}$ are special cases of Lie algebra cohomology of $\mathfrak{g}$.

2.3.1. Chevalley–Eilenberg complexes.

Let W be a $\mathfrak{g}$-module, i.e. a vector space together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to End(W)$ from $\mathfrak{g}$ to the Lie algebra of endomorphisms of W. In other words, ρ is a linear map such that $\rho([x, y]) = \rho(x).\rho(y) - \rho(y).\rho(x)\ \forall x, y \in \mathfrak{g}$. The action of an element $x \in \mathfrak{g}$ on a vector $v \in W$ is defined by

$$x.v = \rho(x)(v). \tag{2.28}$$

One associates to W the following complex, called ***Chevalley–Eilenberg complex*** of $\mathfrak{g}$ with coefficients in W [**51**]:

$$\ldots \xrightarrow{\delta} C^{k-1}(\mathfrak{g},\rho) \xrightarrow{\delta} C^{k}(\mathfrak{g},\rho) \xrightarrow{\delta} C^{k+1}(\mathfrak{g},\rho) \xrightarrow{\delta} \ldots, \tag{2.29}$$

where

$$C^k(\mathfrak{g},\rho) = (\wedge^k \mathfrak{g}^*) \otimes W \tag{2.30}$$

($k \geq 0$) is the space of k-multilinear antisymmetric maps from $\mathfrak{g}$ to W: an element $\theta \in C^k(\mathfrak{g},\rho)$ may be presented as a k-multilinear antisymmetric map from $\mathfrak{g}$ to W, or a linear map from $\wedge^k \mathfrak{g}$ to W:

$$\theta(x_1,\ldots,x_k) = \theta(x_1 \wedge \ldots \wedge x_k) \in W, \;\; x_i \in \mathfrak{g}. \tag{2.31}$$

The operator $\delta = \delta_{CE} : C^k(\mathfrak{g},\rho) \to C^{(k+1)}(\mathfrak{g},\rho)$ in the Chevalley–Eilenberg complex is defined as follows:

$$\begin{aligned}(\delta\theta)(x_1,\ldots,x_{k+1}) = &\sum_i (-1)^{i+1}\rho(x_i)(\theta(x_1,\ldots\widehat{x_i}\ldots,x_{k+1}))+ \\ &+\sum_{i<j}(-1)^{i+j}\theta([x_i,x_j],x_1,\ldots\widehat{x_i}\ldots\widehat{x_j}\ldots,x_{k+1}),\end{aligned} \tag{2.32}$$

the symbol $\widehat{}$ above a variable means that this variable is missing in the list.

It is a classical result [**51**], which follows directly from the Jacobi identity of $\mathfrak{g}$, that $\delta_{CE} \circ \delta_{CE} = 0$. It means that the Chevalley–Eilenberg complex is a differential complex with differential operator $\delta = \delta_{CE}$. Its cohomology groups

$$H^k(\mathfrak{g},\rho) = H^k(\mathfrak{g},W) = \frac{\ker(\delta : C^k(\mathfrak{g},\rho) \longrightarrow C^{k+1}(\mathfrak{g},\rho))}{\mathrm{Im}(\delta : C^{k-1}(\mathfrak{g},\rho) \longrightarrow C^{k}(\mathfrak{g},\rho))} \tag{2.33}$$

are called *cohomology groups of* $\mathfrak{g}$ with coefficients in W (or with respect to the representation ρ).

REMARK 2.3.1. Formula (2.32) is absolutely analogous to Cartan's formula (1.7). This construction of differential operators is sometimes referred to as ***Cartan–Chevalley–Eilenberg construction***. If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then the space $\Omega^\star_L(G)$ of left-invariant differential forms on G is a subcomplex of the de Rham complex of G which is naturally isomorphic to the Chevalley-Eilenberg $C^\star(G,\mathbb{R})$ for the trivial action of $\mathfrak{g}$ on $\mathbb{R}$, which implies that their cohomologies are also isomorphic:

$$H^\star_L(G) \cong H^\star(\mathfrak{g},\mathbb{R}). \tag{2.34}$$

(The isomorphism from $\Omega^\star_L(G)$ to $C^\star(G,\mathbb{R})$ associates to each left-invariant differential form on G its value at the neutral element e of G, after the identification of $\mathfrak{g}^*$ with T^*_eG). In particular, when G is compact, the averaging process $\alpha \mapsto \int_G L_g^*\alpha \mathrm{d}g$ (where α denotes a differential form on G, and L_g denotes the left translation by $g \in G$) induces an isomorphism from $H^\star_{dR}(G)$ to $H^\star_L(G)$, and we have $H^\star_{dR}(G) \cong H^\star(g,\mathbb{R})$.

EXERCISE 2.3.2. Show that, given a smooth Poisson manifold (M,Π), its Lichnerowicz complex can be identified with a subcomplex of the Chevalley-Eilenberg complex of the (infinite-dimensional) Lie algebra $C^\infty(M)$ with coefficients in $C^\infty(M)$

(with respect to the adjoint action given by the Poisson bracket), which consists of cochains which are multi-derivations. (Hint: use Formula (1.75)).

In general, the problem of computation of $H(\mathfrak{g}, W)$ for a finite-dimensional $\mathfrak{g}$-module W of a finite-dimensional Lie algebra $\mathfrak{g}$ is a problem of linear algebra: one simply has to deal with finite-dimensional systems of linear equations. However, even for low dimensional Lie algebras, these systems of linear equations often have high dimensions and require thousands or millions of computations, so it is not easy to do it by hand.

Fortunately, cohomology of semisimple Lie algebras is relatively simple, due in part to the following results, known as ***Whitehead's lemmas***.

THEOREM 2.3.3 (Whitehead). *If $\mathfrak{g}$ is semisimple, and W is a finite-dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, W) = 0$ and $H^2(\mathfrak{g}, W) = 0$.*

THEOREM 2.3.4 (Whitehead). *If $\mathfrak{g}$ is semisimple, and W is a finite-dimensional $\mathfrak{g}$-module such that $W^{\mathfrak{g}} = 0$, where $W^{\mathfrak{g}} = \{w \in W \mid x.w = 0 \ \forall\ x \in \mathfrak{g}\}$ denotes the set of elements in W which are invariant under the action of $\mathfrak{g}$, then $H^k(\mathfrak{g}, W) = 0 \ \forall\ k \geq 0$.*

See, e.g., [**115**] for the proof of Whitehead's lemmas. A refined (normed) version of Theorem 2.3.3 will be proved in Chapter 3. Let us also mention that if $\mathfrak{g}$ is simple then $\dim H^3(\mathfrak{g}, \mathbb{K}) = 1$. Combining the two Whitehead's lemmas with the fact that any finite-dimensional module W of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible, one gets the following formula:

$$H^\star(\mathfrak{g}, W) = H^\star(\mathfrak{g}, \mathbb{K}) \otimes W^{\mathfrak{g}} = \bigoplus_{k \neq 1,2} H^k(\mathfrak{g}, \mathbb{K}) \otimes W^{\mathfrak{g}}. \tag{2.35}$$

REMARK 2.3.5. If W is a smooth Fréchet module of a compact Lie group G and $\mathfrak{g}$ is the Lie algebra of G, then the formula $H^\star(\mathfrak{g}, W) = H^\star(\mathfrak{g}, \mathbb{R}) \otimes W^{\mathfrak{g}}$ is still valid, see Ginzburg [**89**]. In particular, if a compact Lie group G acts on a smooth manifold M, then $C^\infty(M)$ is a smooth Fréchet G-module, and we have

$$H^\star(\mathfrak{g}, C^\infty(M)) = H^\star(\mathfrak{g}, \mathbb{R}) \otimes (C^\infty(M))^{\mathfrak{g}}. \tag{2.36}$$

REMARK 2.3.6. Cohomology of Lie algebras is closely related to differentiable (or continuous) cohomology of Lie groups, via the so called van Est map and van Est spectral sequence. See, e.g., [**26, 97**].

2.3.2. Cohomology of linear Poisson structures.

Consider now the case $W = \mathcal{S}^q\mathfrak{g}$, the q-symmetric power of $\mathfrak{g}$ together with the adjoint action of $\mathfrak{g}$:

$$\rho(x)(x_{i_1} \dots x_{i_q}) = \sum_{s=1}^{q} x_{i_1} \dots [x, x_{i_s}] \dots x_{i_q}. \tag{2.37}$$

Since $\mathfrak{g} = ((\mathbb{K}^n)^*, \{,\}_{\Pi^{(1)}})$, the space $W = \mathcal{S}^q\mathfrak{g}$ can be naturally identified with the space of homogeneous polynomials of degree q on $\mathbb{K}^n$, and we can write

$$\rho(x).f = \{x, f\}, \tag{2.38}$$

where $f \in \mathcal{S}^q\mathfrak{g}$, and $\{x, f\}$ denotes the Poisson bracket of x with f with respect to $\Pi^{(1)}$.

Denote by $\mathcal{V}^p_{(q)} = \mathcal{V}^p_{(q)}(\mathbb{K}^n)$ the space of homogeneous p-vector fields of degree q. (It is the same as the space of quasi-homogeneous p-vector fields of quasi-homogeneous degree $q-p$ with respect to the radial vector field $\sum x_i \partial/\partial x_i$). $\mathcal{V}^p_{(q)}$ can be identified with $C^p(\mathfrak{g}, \mathcal{S}^q\mathfrak{g})$ as follows: For

$$A = \sum_{i_1<\cdots<i_p} A_{i_1,\dots,i_p} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}} \in \mathcal{V}^p_{(q)} \tag{2.39}$$

define $\theta_A \in C^p(\mathfrak{g}, \mathcal{S}^q\mathfrak{g})$ by

$$\theta_A(x_{i_1}, \dots, x_{i_p}) = A_{i_1,\dots,i_p}. \tag{2.40}$$

Lemma 2.3.7. *With the above identification $A \leftrightarrow \theta_A$, the Lichnerowicz differential operator $\delta_{LP} = [\Pi^{(1)}, .] : \mathcal{V}^p_{(q)} \longrightarrow \mathcal{V}^{p+1}_{(q)}$ is identified with the Chevalley–Eilenberg differential operator $\delta_{CE} : C^p(\mathfrak{g}, \mathcal{S}^q\mathfrak{g}) \longrightarrow C^{p+1}(\mathfrak{g}, \mathcal{S}^q\mathfrak{g})$.*

Proof. We must show that $\theta_{[\Pi^{(1)},A]} = \delta_{CE}\theta_A$ for $A \in \mathcal{V}^p_{(q)}$. Write $\Pi^{(1)} = \frac{1}{2}\sum_{i,j,k} c^k_{ij} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, and $A = \frac{1}{p!}\sum_{i_1,\dots,i_p} A_{i_1,\dots,i_k} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}}$. Denote the Poisson bracket of $\Pi^{(1)}$ by $\{,\}$. By the Leibniz rule, we have

$$[\Pi^{(1)}, A] = E_1 + E_2$$

where

$$\begin{aligned} E_1 &= \frac{1}{p!}\sum [\Pi^{(1)}, A_{i_1,\dots,i_p}] \wedge \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}} \\ &= \frac{1}{p!}\sum_{i_1,\dots,i_p,i} \{x_i, A_{i_1,\dots,i_p}\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}} \end{aligned}$$

and

$$\begin{aligned} E_2 &= \frac{1}{p!}\sum A_{i_1,\dots,i_p} [\Pi^{(1)}, \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_p}}] \\ &= \frac{1}{2p!}\sum_{i_1,\dots,i_p,i,j,s} (-1)^s A_{i_1,\dots,i_p} c^{i_s}_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_{i_1}} \wedge \dots \widehat{\frac{\partial}{\partial x_{i_s}}} \dots \wedge \frac{\partial}{\partial x_{i_p}}. \end{aligned}$$

It means that

$$E_1(\mathrm{d}x_{i_1}, \dots, \mathrm{d}x_{i_{p+1}}) = \sum_s (-1)^{s+1} \{x_{i_s}, A_{i_1,\dots\widehat{i_s}\dots,i_{p+1}}\}$$

and

$$E_2(\mathrm{d}x_{i_1}, \dots, \mathrm{d}x_{i_{p+1}}) = \sum_{u<v;k} (-1)^{u+v} A_{k,i_1,\dots\widehat{i_u}\dots\widehat{i_v}\dots,i_{p+1}} c^k_{i_u i_v}.$$

On the other hand, we have

$$
\begin{aligned}
&\delta\theta_A(x_{i_1},\dots,x_{i_{p+1}}) = \\
&= \sum_u (-1)^{u+1}\rho(x_{i_u})A(x_{i_1},\dots\widehat{x_{i_u}}\dots,x_{i_{p+1}}) \\
&\quad + \sum_{u<v}(-1)^{u+v}A([x_{i_u},x_{i_v}],x_{i_1},\dots\widehat{x_{i_u}}\dots\widehat{x_{i_v}}\dots,x_{i_{p+1}}) \\
&= \sum_u (-1)^{u+1}\{x_{i_u},A_{i_1,\dots\widehat{i_u}\dots,i_{p+1}}\} \\
&\quad + \sum_{u<v;k}(-1)^{u+v}c^k_{i_u i_v}A(x_k,x_{i_1},\dots\widehat{x_{i_u}}\dots\widehat{x_{i_v}}\dots,x_{i_{p+1}}).
\end{aligned}
$$

It remains to compare the above formulas. □

An immediate consequence of Lemma 2.3.7 and Theorem 2.2.2 is the following:

THEOREM 2.3.8 ([**205**]). *If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $H^2(\mathfrak{g},\mathcal{S}^k\mathfrak{g}) = 0\ \forall\ k \geq 2$, then any formal Poisson structure Π which vanishes at a point and whose linear part $\Pi^{(1)}$ at that point corresponds to $\mathfrak{g}$ is formally linearizable. In particular, it is the case when $\mathfrak{g}$ is semisimple.*

REMARK 2.3.9. In Lemma 2.3.7, the fact that A is homogeneous is not so important. What is important is that the module W in question can be identified with a subspace of the space of functions on $\mathbb{K}^n$, where the action of $\mathfrak{g}$ is given by the Poisson bracket, i.e. by Formula (2.38). The following smooth (as compared to homogeneous) version of Lemma 2.3.7 is also true, with a similar proof (see, e.g., [**89, 93, 130, 131**]): if U is an Ad^*-invariant open subset of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of a connected Lie group G (or more generally, an open subset of a dual Poisson-Lie group G^* which is invariant under the dressing action of G – Poisson-Lie groups will be introduced later in the book), then

$$H^\star_\Pi(U) \cong H^\star(\mathfrak{g}, C^\infty(U)) \ , \tag{2.41}$$

where the action of $\mathfrak{g}$ on $C^\infty(U)$ is induced by the coadjoint (or dressing) action, and a natural isomorphism exists already at the level of cochain complexes. In particular, if G is compact semisimple, then this formula together with the Fréchet-module version of Whitehead's lemmas (Remark 2.3.5) leads to the following formula (see [**93**]):

$$H^\star_\Pi(U) = H^\star(\mathfrak{g}) \otimes (C^\infty(U))^G = \bigoplus_{k\neq 1,2} H^k(\mathfrak{g}) \otimes (C^\infty(U))^G. \tag{2.42}$$

2.3.3. Rigid Lie algebras.

Theorem 2.3.8 means that the second cohomology group

$$H^2(\mathfrak{g}, \mathcal{S}_{\geq 2}\mathfrak{g}) = \bigoplus_{k\geq 2} H^2(\mathfrak{g}, \mathcal{S}^k\mathfrak{g})$$

governs nonlinear deformations of the linear Poisson structure of $\mathfrak{g}^*$. Meanwhile, the group $H^2(\mathfrak{g},\mathfrak{g})$ governs deformations of $\mathfrak{g}$ itself (or equivalently, linear deformations of the Poisson structure on $\mathfrak{g}^*$ associated to $\mathfrak{g}$).

An n-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ can be determined by its structure constants c_{ij}^k with respect to a given basis (x_i): $[x_i, x_j]_{\mathfrak{g}} = \sum c_{ij}^k x_k$. The n^3-tuple of coefficients (c_{ij}^k) is called a Lie algebra structure of dimension n. The set $\mathcal{A}(n, \mathbb{K}) \subset \mathbb{K}^{n^3}$ of all Lie algebra structures of dimension n is an algebraic variety (the Jacobi identity and the anti-commutativity give the system of algebraic equations which determine this set). The full linear group $GL(n, \mathbb{K})$ acts naturally on $\mathcal{A}(n, \mathbb{K})$ by changes of basis, and two Lie algebra structures are isomorphic if and only if they lie on the same orbit of $GL(n, \mathbb{K})$. An n-dimensional Lie algebra $\mathfrak{g}$ is called ***rigid*** if the orbit of its structure is an open subset of $\mathcal{A}(n, \mathbb{K})$ with respect to the usual topology induced from the Euclidean topology of $\mathbb{K}^{n^3}$; equivalently, any Lie algebra $\mathfrak{g}'$ close enough to $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$.

THEOREM 2.3.10 (Nijenhuis–Richardson [**161**]). *If $\mathfrak{g}$ is a finite dimensional Lie algebra such that $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ then $\mathfrak{g}$ is rigid. In particular, semisimple Lie algebras are rigid.*

REMARK 2.3.11. The condition $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ is a sufficient but not a necessary condition for the rigidity of a Lie algebra. For example, Richardson [**173**] showed that, for any odd integer $n > 5$, the semi-direct product $\mathfrak{l}_n = sl(2, \mathbb{K}) \ltimes W^{2n+1}$, where W^{2n+1} is the $(2n+1)$-dimensional irreducible $sl(2, \mathbb{K})$-module, is rigid but has $H^2(\mathfrak{l}_n, \mathfrak{l}_n) \neq 0$. In fact, $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ means that there are non-trivial infinitesimal deformations, but not every infinitesimal deformation can be made into a true deformation. See, e.g., [**39, 40, 95**].

2.4. Spectral sequences

2.4.1. Spectral sequence of a filtered complex.

Spectral sequences are one of the main tools for computing cohomology groups. The general idea is as follows.

Let $(C = \oplus_{k \in \mathbb{Z}_+} C^k, \delta)$ be a differential complex. It means that C^k $(k \geq 0)$ are vector spaces (or more generally, Abelian groups), and $\delta : C^k \to C^{k+1}$ are linear operators such that $\delta \circ \delta = 0$.

Assume that (C, δ) admits a ***filtration*** $(C_h)_{h \in \mathbb{N}}$. It means that each C^k is filtered by subspaces

$$C^k = C_0^k \supset C_1^k \supset C_2^k \supset \dots, \tag{2.43}$$

such that $\delta C_h^k \subset C_h^{k+1}$ $\forall\, k, h$. In other words, $C_h = \bigoplus_k C_h^k$ is a differential subcomplex of C_{h-1} for $h \geq 1$, and $C_0 = C$. Put $C_h = C$ if $h < 0$ by convention. Using this filtration, one decomposes cohomology groups

$$H^k(C) = \frac{Z^k}{B^k} = \frac{\ker(\delta : C^k \longrightarrow C^{k+1})}{\mathrm{Im}(\delta : C^{k-1} \longrightarrow C^k)} \tag{2.44}$$

into smaller pieces $H_h^k(C)/H_{h+1}^k(C)$, where $H_h^k(C)$ consists of the elements of $H^k(C)$ which can be represented by cocycles lying in C_h^k. The group

$$\bigoplus_{h \geq 0} H_h^k(C)/H_{h+1}^k(C). \tag{2.45}$$

is called the graded version of $H^k(C)$; it is linearly isomorphic to $H^k(C)$ if, say, the filtration is finite, i.e. $C_n = 0$ for some $n \in \mathbb{N}$.

A way to compute $H_h^k(C)/H_{h+1}^k(C)$ and $H^k(C)$ is to use the ***spectral sequence*** $(E_r^{p,q})_{r\geq 0}$ of the above filtered complex. By definition,

$$E_r^{p,q} = \frac{Z_r^{p,q} + C_{p+1}^{p+q}}{B_r^{p,q} + C_{p+1}^{p+q}} \tag{2.46}$$

where

$$Z_r^{p,q} = \left\{ y \in C_p^{p+q} \mid \delta y \in C_{p+r}^{p+q+1} \right\} \tag{2.47}$$

and

$$B_r^{p,q} = \left\{ y \in C_p^{p+q} \mid y = \delta z, z \in C_{p-r+1}^{p+q-1} \right\}. \tag{2.48}$$

Clearly, $B_r^{p,q} \subset Z_r^{p,q}$, $B_r^{p,q} \subset B_{r+1}^{p,q}$ and $Z_r^{p,q} \supset Z_{r+1}^{p,q}$. Hence $E_r^{p,q}$ is well-defined, and is bigger than $E_{r+1}^{p,q}$. (There is a surjection from a subgroup of $E_r^{p,q}$ to $E_{r+1}^{p,q}$). As r tends to ∞, the group $E_r^{p,q}$ gets smaller and smaller, and it approximates better and better the group $H_p^{p+q}(C)/H_{p+1}^{p+q}(C)$. In fact, if the filtration is of finite length, i.e. $C_n = 0$ for some $n \in \mathbb{N}$, then

$$E_r^{p,q} = \frac{Z^{p+q} \cap C_p^{p+q} + C_{p+1}^{p+q}}{B^{p+q} \cap C_p^{p+q} + C_{p+1}^{p+q}} \cong H_p^{p+q}(C)/H_{p+1}^{p+q}(C) \quad \forall\ r \geq n, p. \tag{2.49}$$

In general, one says that $(E_r^{p,q})$ converges if its limit

$$E_\infty^{p,q} = \lim_{r\to\infty} E_r^{p,q} \tag{2.50}$$

is isomorphic to $H_p^{p+q}(C)/H_{p+1}^{p+q}(C)$.

The terms $E_r^{p,q}$ of the spectral sequence can be computed inductively on r (that's why they are useful for computing $H_p^{p+q}(C)/H_{p+1}^{p+q}(C)$). The zeroth term is:

$$E_0^{p,q} = C_p^{p+q}/C_{p+1}^{p+q}. \tag{2.51}$$

In other words, $E_0 = \oplus E_0^{p,q}$ is just the graded version of the complex C. For $r \geq 0$, the differential operator δ induces an operator on $(E_r^{p,q})$, denoted by δ_r:

$$\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1}. \tag{2.52}$$

(The image of $y \in Z_r^{p,q}$ mod $B_r^{p,q} + C_{p+1}^{p+q}$ under δ_r is $\delta y \in Z_r^{p+r,q-r+1}$ mod $B_r^{p+r,q-r+1} + C_{p+r+1}^{p+q}$. One verifies directly that δ_r is well-defined).

Since $\delta \circ \delta = 0$, we also have $\delta_r \circ \delta_r = 0$, i.e. δ_r is a differential operator. It turns out that $E_{r+1}^{p,q}$ is nothing but the cohomology of $E_r^{p,q}$ with respect to δ_r:

$$E_{r+1}^{p,q} = \frac{\ker(\delta_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})}{\operatorname{Im}(\delta_r : E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})}. \tag{2.53}$$

Exercise 2.4.1. Verify Formula (2.53), starting from (2.46), (2.47) and (2.48).

Remark 2.4.2. In the literature, Formula (2.53) is often used in the definition of spectral sequences, and Formulas (2.46), (2.47) and (2.48) only show up after or not at all.

2.4.2. Leray spectral sequence.

As a first example of spectral sequences, let us consider a locally trivial fibration $\pi : M \to N$ of a manifold M over a connected manifold N with fibers diffeomorphic to F. The de Rham complex $\Omega^\star(M)$ of differential forms on M admits a natural filtration with respect to this fibration: $\Omega^k_h(M)$ $(h \geq 0)$ is the subspace of $\Omega^k(M)$ consisting of k-forms ω which satisfy the following condition:

$$\omega_x(X_1, \ldots, X_k) = 0 \ \forall x \in M, X_1, \ldots X_k \in T_xM \ s.t. \ \pi_*X_1 = \ldots = \pi_*X_{k-h+1} = 0. \tag{2.54}$$

The associated spectral sequence of this filtration is known as the ***Leray spectral sequence***. Its zeroth term $E_0^{p,q}$ can be written as follows:

$$E_0^{p,q} \cong \Omega^p(N, \Omega^q(F)). \tag{2.55}$$

More precisely, $E_0^{p,q} = \Omega_p^{p+q}(M)/\Omega_{p+1}^{p+q}(M)$ is naturally isomorphic to the space of vector-valued p-forms on N with values in the vector bundle over N whose fiber over a point $y \in N$ is the space of q-forms on the fiber $F_y = \pi^{-1}(y)$ of the fibration of M over N. The first and second terms are

$$E_1^{p,q} = \Omega^p(N, H^q(F)) \tag{2.56}$$

and

$$E_2^{p,q} = H^p(N, H^q(F)). \tag{2.57}$$

In the above formulas, $H^q(F)$ must be understood as a vector bundle over N whose fiber over $y \in N$ is $H^q_{dR}(F_y)$, i.e., it is a local system of coefficients. If N is simply connected then this bundle is automatically trivial and we can write

$$E_2^{p,q} = H^p_{dR}(N) \otimes H^q_{dR}(F). \tag{2.58}$$

Example 2.4.3. The de Rham cohomology of the special unitary groups $SU(n)$ can be computed inductively on n with the help of the Leray spectral sequence associated to the natural fibration of $SU(n)$ over S^{2n-1} with fiber $SU(n-1)$ (this fibration is obtained via the natural action of $SU(n)$ on the unit sphere S^{2n-1} in $\mathbb{C}^n$). When $n = 2$, $SU(2)$ is diffeomorphic to the 3-dimensional sphere S^3, so we will simply write $H^\star_{dR}(SU(2)) = H^\star_{dR}(S^3)$. When $n = 3$, the second terms of the Leray spectral sequence of the fibration $SU(2) \to SU(3) \to S^5$ are as follows:

$$E_2^{0,3} = \mathbb{R}, \ E_2^{5,3} = \mathbb{R},$$
$$E_2^{0,0} = \mathbb{R}, \ E_2^{5,0} = \mathbb{R},$$

(the other second terms are zero). The differential δ_2 is automatically trivial, because, for example, it maps the nontrivial term $E_2^{0,3}$ to the trivial term $E_2^{2,2}$. Similarly, all the other differentials δ_r, $r \geq 2$, are trivial, because there are only 4 nontrivial cells $E^{0,3}, E^{5,3}, E^{0,0}, E^{5,0}$, and no differential δ_r connects two of these cells. It means that the Leray spectral sequence degenerates at E_2, implying that

$$H^\star_{dR}(SU(3)) \cong H^\star_{dR}(SU(2)) \otimes H^\star_{dR}(S^5) \cong H^\star_{dR}(S^3 \times S^5). \tag{2.59}$$

This isomorphism between $H^\star_{dR}(SU(3))$ and $H^\star_{dR}(S^3 \times S^5)$ is actually an algebra isomorphism, because the Leray spectral sequence is compatible with the product structure of de Rham cohomology in a natural sense. Using this compatibility, one can show inductively on n that the Leray spectral sequence for the fibration

$SU(n-1) \to SU(n) \to S^{n-1}$ degenerates at the second term E_2 for any $n \geq 3$, leading to the following algebra isomorphism:

$$H^{\star}_{dR}(SU(n)) \cong H^{\star}_{dR}(S^3 \times S^5 \times \ldots \times S^{2n-1}). \tag{2.60}$$

See, e.g., [**27, 85**] for details and other applications of Leray spectral sequences and other spectral sequences in topology.

2.4.3. Hochschild–Serre spectral sequence.

Given a Lie algebra $\mathfrak{l}$, a Lie subalgebra $\mathfrak{r} \subset \mathfrak{l}$, and an $\mathfrak{l}$-module W, the Chevalley–Eilenberg complex $C(\mathfrak{l}, W)$ has the following natural filtration with respect to $\mathfrak{r}$:

$$C^k_h(\mathfrak{l}, W) = \left\{ f \in C^k(\mathfrak{l}, W) \mid f(x_1, \ldots, x_k) = 0 \;\; if \;\; x_1, \ldots, x_{k-h+1} \in \mathfrak{r} \right\}. \tag{2.61}$$

By convention, $C^k_0(\mathfrak{l}, W) = C^k(\mathfrak{l}, W)$, and $C^k_h(\mathfrak{l}, W) = 0$ if $h > k$. One checks directly that $\delta C_h(\mathfrak{l}, W) \subset C_h(\mathfrak{l}, W)$, i.e. it is really a filtered complex. The corresponding spectral sequence is known as the ***Hochschild–Serre spectral sequence*** [**109**].

We will be mainly interested in a special case of this spectral sequence, when $\mathfrak{r}$ is an ideal of $\mathfrak{l}$, and the quotient Lie algebra $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is semisimple. (This is the case, for example, when $\mathfrak{r}$ is the ***radical***, i.e. the maximal solvable ideal of $\mathfrak{l}$). In this case, Hochschild–Serre spectral sequence leads to the following theorem.

Theorem 2.4.4 (Hochschild–Serre [**109**]). *Let $\mathfrak{l}$ be a finite dimensional Lie algebra over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\mathfrak{r}$ be an ideal of $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is semisimple, and W be a finite-dimensional $\mathfrak{l}$-module. Then*

$$H^k(\mathfrak{l}, W) \cong \bigoplus_{i+j=k} H^i(\mathfrak{g}, \mathbb{K}) \otimes H^j(\mathfrak{r}, W)^{\mathfrak{g}} \quad \forall\ k \geq 0. \tag{2.62}$$

In the above theorem, $H^j(\mathfrak{r}, W)$ has a natural structure of $\mathfrak{g}$-module which will be explained below, and $H^j(\mathfrak{r}, W)^{\mathfrak{g}}$ is the subspace of $H^j(\mathfrak{r}, W)$ consisting of elements which are invariant under the action of $\mathfrak{g}$.

Proof (sketch). Since $\mathfrak{g}$ is semisimple, by the classical Levi–Malcev theorem there is a Lie algebra injection $\mathfrak{g} \to \mathfrak{l}$ whose composition with the projection map $\mathfrak{l} \to \mathfrak{l}/\mathfrak{r} = \mathfrak{g}$ is identity. (See, e.g., [**28, 196**], and the beginning of Chapter 3). In other words, we may assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{l}$. As a vector space, $\mathfrak{l}$ is the direct sum of $\mathfrak{g}$ with $\mathfrak{r}$. As a Lie algebra, $\mathfrak{l}$ can be written a semi-direct product $\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}$.

By definition, the zeroth term $E^{p,q}_0$ of the spectral sequence is

$$E^{p,q}_0 = C^{p+q}_p(\mathfrak{l}, W)/C^{p+q}_{p+1}(\mathfrak{l}, W). \tag{2.63}$$

This space can be naturally identified with $C^p(\mathfrak{g}, C^q(\mathfrak{r}, W))$. Indeed, if we denote by $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ a basis of $\mathfrak{l}$ such that $(x_1, \ldots, x_m)$ span $\mathfrak{g}$ and $(y_1, \ldots, y_{n-m})$ span $\mathfrak{r}$, then an element $f \in C^{p+q}_p(\mathfrak{l}, W)$ mod $C^{p+q}_{p+1}(\mathfrak{l}, W)$ is completely determined by its value on elements of the type

$$x_{i_1} \wedge \ldots \wedge x_{i_p} \wedge y_{j_1} \wedge \ldots \wedge y_{j_q}.$$

The map

$$\theta_f : x_{i_1} \wedge \ldots \wedge x_{i_p} \mapsto \left(y_{j_1} \wedge \ldots \wedge y_{j_q} \mapsto f(x_{i_1} \wedge \ldots \wedge x_{i_p} \wedge y_{j_1} \wedge \ldots \wedge y_{j_q}) \right)$$

is a linear map from $\wedge^p \mathfrak{g}$ to $C^q(\mathfrak{r}, W)$, i.e. $\theta_f \in C^p(\mathfrak{g}, C^q(\mathfrak{r}, W))$. Note that $C^q(\mathfrak{r}, W) = \wedge^q \mathfrak{r} \otimes W$ is a $\mathfrak{g}$-module: $\mathfrak{g}$ acts on W by the restriction of the action of $\mathfrak{l}$; it acts on $\mathfrak{r}$ by the adjoint action of $\mathfrak{g}$ in $\mathfrak{l}$, and on $\mathfrak{r}^*$ by the dual action. It is clear that the correspondence $f \leftrightarrow \theta_f$ is one-to-one. Thus, we can write

$$E_0^{p,q} \cong C^p(\mathfrak{g}, C^q(\mathfrak{r}, W)). \tag{2.64}$$

The next step is to look at the first spectral term $E_1^{p,q}$, which is the cohomology of $E_0^{p,q}$ with respect to $\delta_0 : E_0^{p,q} \to E_0^{p,q+1}$. Using the identification $E_0^{p,q} \cong C^p(\mathfrak{g}, C^q(\mathfrak{r}, W))$, we can write δ_0 as

$$\delta_0 : C^p(\mathfrak{g}, C^q(\mathfrak{r}, W)) \to C^p(\mathfrak{g}, C^{q+1}(\mathfrak{r}, W)). \tag{2.65}$$

It follows that

$$E_1^{p,q} \cong C^p(\mathfrak{g}, H^q(\mathfrak{r}, W)). \tag{2.66}$$

Similarly, we have

$$E_2^{p,q} \cong H^p(\mathfrak{g}, H^q(\mathfrak{r}, W)). \tag{2.67}$$

Whitehead's lemma (see Formula (2.35)) implies that

$$E_2^{p,q} \cong H^p(\mathfrak{g}, H^q(\mathfrak{r}, W)^{\mathfrak{g}}) \cong H^p(\mathfrak{g}, \mathbb{K}) \otimes H^q(\mathfrak{r}, W)^{\mathfrak{g}}. \tag{2.68}$$

If $f \in \wedge^p \mathfrak{g}^*$ and $g \in \wedge^q \mathfrak{r}^* \otimes W$ are cocycles, and moreover g is invariant under the action of $\mathfrak{g}$ on $\wedge^q \mathfrak{r}^* \otimes W$, then their product

$$f \otimes g \in \wedge^p \mathfrak{g}^* \otimes \wedge^q \mathfrak{r}^* \otimes W \subset \wedge^{p+q} \mathfrak{l}^* \otimes W \tag{2.69}$$

is a cocycle. Equation (2.68) means that elements of $E_2^{p,q}$ can be written as linear combinations of such cocycles $f \otimes g$. In particular, any element of $E_2^{p,q}$ can be represented by a cocycle in $Z^{p+q}(\mathfrak{l}, W)$. It implies that $\delta_2 = \delta_3 = \ldots = 0$, and the Hochschild–Serre spectral sequence degenerates (stabilizes) at E_2, i.e.

$$E_2^{p,q} = E_3^{p,q} = \ldots = E_\infty^{p,q}.$$

Since the filtration is clearly of finite length, we have

$$H_p^{p+q}(\mathfrak{l}, W)/H_{p+1}^{p+q}(\mathfrak{l}, W) \cong E_\infty^{p,q} \cong H^p(\mathfrak{g}, \mathbb{K}) \otimes H^q(\mathfrak{r}, W)^{\mathfrak{g}}$$

and

$$H^k(\mathfrak{l}, W) \cong \bigoplus_p \frac{H_p^k(\mathfrak{l}, W)}{H_{p+1}^k(\mathfrak{l}, W)} \cong \bigoplus_{p+q=k} H^p(\mathfrak{g}, \mathbb{K}) \otimes H^q(\mathfrak{r}, W)^{\mathfrak{g}}.$$

□

2.4.4. Spectral sequence for Poisson cohomology.

Given a smooth Poisson manifold (M, Π), there is a natural filtration of the Lichnerowicz complex, induced by the characteristic distribution as follows. Denote by $\mathcal{V}_k^q(M, \Pi)$ the space of smooth q-vector fields Λ on M with the following property: $\Lambda(x) \in \wedge^k(\mathrm{Im}\sharp_x) \wedge \wedge^{q-k} T_x M \ \forall\, x \in M$. In other words, $\Lambda(x)$ is a linear combination of q-vectors of the type $Y_1 \wedge \ldots \wedge Y_q$ with $Y_1, \ldots, Y_k \in \mathrm{Im}\sharp_x$. It is clear that $\Pi \in \mathcal{V}_2^2(M, \Pi)$, $\mathcal{V}^\star(M) = \mathcal{V}_0^\star(M, \Pi) \supset \mathcal{V}_1^\star(M, \Pi) \supset \ldots \supset \mathcal{V}_{(\text{rank }\Pi)}^\star(M, \Pi) \supset \mathcal{V}_{(\text{rank }\Pi)+1}^\star(M, \Pi) = 0$, and $[\Pi, \Lambda] \in \mathcal{V}_k^{p+1}(M, \Pi)$ if $\Lambda \in \mathcal{V}_k^q(M, \Pi)$. So we have a filtration of finite length. The corresponding spectral sequence was first written down explicitly by Vaisman [**194, 195**]. It was also present implicitly in the work of Karasev and Vorobjev [**199**].

Let us mention that the zeroth column $E_0^{p,0}$ of the zeroth term of the above spectral sequence consists of multi-vector fields which are tangent to the characteristic distribution. Consequently, the zeroth column $E_1^{p,0}$ of the first term of the above spectral sequence consists of tangential Poisson cohomology groups, mentioned in Subsection 2.1.4.

The use of the above spectral sequence in the computation of Poisson cohomology has yielded only limited success so far, mainly in the case when the Poisson structure is regular and the characteristic symplectic foliation is a fibration [**199, 194**]. For this reason, we will not write down explicitly the above spectral sequence in the general case (the reader may try to do it as an exercise). Instead, we will give here a concrete simple example.

EXAMPLE 2.4.5. Let $M = P \times B^n$, where P is a closed manifold such that $H^1_{dR}(P) = 0$, and B^n is an open ball of dimension $n = \dim H^2_{dR}(P)$. Let Π be a regular Poisson structure on M, whose symplectic leaves are $P \times \{y\}, y \in B^n$, such that the map $B^n \to H^2_{dR}(P), y \mapsto [\omega_y]$, where ω_y is the symplectic form of the symplectic leaf $P \times \{y\}$ of Π, is a diffeomorphism from B to its image. Then the second Poisson cohomology of M vanishes: $H^2_\Pi(M) = 0$. To see this, decompose any 2-vector field Λ such that $[\Lambda, \Pi] = 0$ into the sum of three parts, $\Lambda = \Lambda_{xx} + \Lambda_{xy} + \Lambda_{yy}$, where Λ_{xx} is tangent to the symplectic leaves, $\Lambda_{xy} = \sum_{i=1}^n X_i \wedge \partial/\partial y_i$ with X_i being vector fields tangent to the symplectic leaves and (y_i) being a system of coordinates on B^n, and $\Lambda_{yy} = \sum f_{ij} \partial/\partial y_i \wedge \partial/\partial y_j$. The condition $[\Lambda, \Pi] = 0$ is equivalent to the following system of equations:

$$[\Lambda_{xx}, \Pi] = -\sum_i \alpha_i \wedge [\partial/\partial y_i, \Pi], \tag{2.70}$$

$$\sum_i [X_i, \Pi] \wedge \partial/\partial y_i = \sum_{i,j} f_{ij} [\partial/\partial y_j, \Pi] \wedge \partial/\partial y_i, \tag{2.71}$$

$$\sum_{i,j} [f_{ij}, \Pi] \wedge \partial/\partial y_i \wedge \partial/\partial y_j = 0. \tag{2.72}$$

The second equation means that $[X, \Pi] = \sum_j f_{ij} [\partial/\partial y_j, \Pi] \; \forall\, i$. If we fix a symplectic leaf $\{y = \mathit{constant}\}$, then $[X_i, \Pi]$ is exact on that leaf while $\sum_j f_{ij}[\partial/\partial y_j, \Pi]$ is not exact unless $f_{ij} = 0$ because of the hypothesis that $y \mapsto [\omega_y]$ is a diffeomorphism. Thus the equation $[\Lambda, \Pi] = 0$ implies that $f_{ij} = 0$, i.e. $\Lambda_{yy} = 0$, and $[X_i, \Pi] = 0 \; \forall\, i$. It follows from the hypothesis $H^1_{dR}(P) = 0$ that X_i is exact on each symplectic leaf, hence we can write $X_i = [g_i, \Pi]$. The 2-vector field $\Lambda' = \Lambda + \sum [g_i \partial/\partial y_i, \Pi]$ is tangent to the symplectic leaves (i.e. $\Lambda' = \Lambda'_{xx}$), and $[\Lambda', \Pi] = 0$. It follows again from the hypothesis that $y \mapsto [\omega_y]$ is a diffeomorphism that Λ' is exact, $\Lambda' = [Z, \Pi]$, and so is Λ. Thus, any two cocycle is a coboundary, and $H^2_\Pi(M) = 0$. In terms of spectral sequences, the decomposition $\Lambda = \Lambda_{xx} + \Lambda_{xy} + \Lambda_{yy}$ corresponds to the decomposition $H^2_\Pi(M) \cong E^{0,2}_\infty \oplus E^{1,1}_\infty \oplus E^{2,0}_\infty$, and we showed that each of the three summands in this cohomology decomposition is trivial.

EXERCISE 2.4.6. Write down more explicitly the spectral sequence for the Poisson cohomology of the above example.

REMARK 2.4.7. There are some other natural filtrations of the Lichnerowicz complex, e.g., the filtration associated to a momentum map, studied by Viktor Ginzburg [**88, 89**], and the filtration given by the powers of an ideal (usually the maximal ideal) of functions at a point where the Poisson structure vanishes). This last filtration is a general one, appearing in the study of local normal forms of many different objects – we already used it in Section 2.2, without even mentioning the spectral sequence.

2.5. The curl operator

2.5.1. Definition of the curl operator.

Recall that, if A is an a-vector field and ω is a differential p-form with $p \geq a$, then the ***inner product*** of ω by A is a unique $(p-a)$-form, denoted by $i_A\omega$ or $A \lrcorner \omega$, such that

$$\langle i_A\omega, B\rangle = \langle \omega, A \wedge B\rangle \tag{2.73}$$

for any $(p-a)$-vector field B. If $p < a$ then we put $i_A\omega = 0$ by convention.

For example, if X is a vector field then $i_X\omega(X_1, \dots, X_{p-1}) = \langle i_X\omega, X_1 \wedge \dots \wedge X_{p-1}\rangle = \langle \omega, X \wedge X_1 \wedge \dots \wedge X_{p-1}\rangle = \omega(X, X_1, \dots, X_{p-1})$.

Similarly, when $a \geq p$, then we can define the *inner product of an a-vector field A by a p-form η* to be a unique $(a-p)$-vector field, denoted by $i_\eta A$ or $\eta \lrcorner A$, such that

$$\langle \beta, i_\eta A\rangle = \langle \beta \wedge \eta, A\rangle \tag{2.74}$$

for any $(a-p)$-form β.

Warning: Due to the non-commutativity of the wedge product, one must be careful with the signs when dealing with inner products. Also, our sign convention may be different from some other authors.

EXERCISE 2.5.1. If f is a function and A a multi-vector field then

$$i_{\mathrm{d}f}A = [A, f]\ . \tag{2.75}$$

In particular, the Hamiltonian vector field of f with respect to a given Poisson structure Π is $X_f = \sharp_\Pi(\mathrm{d}f) = -i_{\mathrm{d}f}\Pi$.

Let Ω be a smooth ***volume form*** on a m-dimensional manifold M, i.e. a nowhere vanishing differential m-form. Then for every $p = 0, 1, \dots, m$, the map

$$\Omega^\flat : \mathcal{V}^p(M) \longrightarrow \Omega^{m-p}(M) \tag{2.76}$$

defined by $\Omega^\flat(A) = i_A\Omega$, is a $\mathcal{C}^\infty(M)$-linear isomorphism from the space $\mathcal{V}^p(M)$ of smooth p-vector fields to the space $\Omega^{m-p}(M)$ of smooth $(m-p)$-forms. The inverse map of $\Omega^\flat$ is denoted by $\Omega^\sharp : \Omega^{n-p}(M) \longrightarrow \mathcal{V}^p(M)$, which can be defined by $\Omega^\sharp(\eta) = i_\eta\widehat{\Omega}$, where $\widehat{\Omega}$ is the dual m-vector field of Ω, i.e. $\langle \Omega, \widehat{\Omega}\rangle = 1$.

EXERCISE 2.5.2. Prove the formula $(\eta \lrcorner \widehat{\Omega}) \lrcorner \Omega = \eta$.

Denote by $D_\Omega : \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p-1}(M)$ the linear operator defined by $D_\Omega = \Omega^\# \circ \mathrm{d} \circ \Omega^\flat$. Then we have the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{V}^p(M) \arrow[r, "\Omega^\flat"] \arrow[d, "D_\Omega"'] & \Omega^{m-p}(M) \arrow[d, "\mathrm{d}"] \\
\mathcal{V}^{p-1}(M) \arrow[r, "\Omega^\flat"] & \Omega^{m-p+1}(M)
\end{tikzcd}
\tag{2.77}
$$

Since $\mathrm{d} \circ \mathrm{d} = 0$, we also have $D_\Omega \circ D_\Omega = 0$.

DEFINITION 2.5.3. The above operator D_Ω is called the ***curl operator*** (with respect to the volume form Ω). If A is an a-vector field then $D_\Omega A$ is called the ***curl*** of A (with respect to Ω).

EXAMPLE 2.5.4. The curl $D_\Omega X$ of a vector field X is nothing but the divergence of X with respect to the volume form Ω: $(D_\Omega X)\Omega = \Omega^\flat(D_\Omega X) = \mathrm{d}i_X\Omega = \mathcal{L}_X\Omega = (\mathrm{Div}_\Omega X)\Omega$, which implies that $D_\Omega X = \mathrm{Div}_\Omega X$.

In a local system of coordinates $(x_1, \ldots, x_n)$ with $\Omega = dx_1 \wedge \ldots \wedge dx_n$, and denoting $\frac{\partial}{\partial x_i}$ by ζ_i as in Section 1.7, we have the following convenient formal formula for the curl operator:

$$
D_\Omega A = \sum_i \frac{\partial^2 A}{\partial x_i \partial \zeta_i}. \tag{2.78}
$$

The following proposition shows what happens to the curl when we change the volume form.

PROPOSITION 2.5.5. *If f is a non-vanishing function then we have*

$$
D_{f\Omega} A = D_\Omega A + [A, \ln|f|] \ . \tag{2.79}
$$

Proof. We have $D_{f\Omega}A - D_\Omega A = \Omega^\#\Omega^\flat(D_{f\Omega}A - D_\Omega A) = \Omega^\#(\frac{1}{f}\mathrm{d}i_A(f\Omega) - \mathrm{d}i_A\Omega)$ $= \Omega^\#(\mathrm{d}\ln|f| \wedge i_A\Omega) = i_{\mathrm{d}\ln|f|}\Omega^\#(i_A\Omega) = i_{\mathrm{d}\ln|f|}A = [A, \ln|f|]$. □

REMARK 2.5.6. It follows from the above proposition that, if we multiply the volume form by a non-zero constant, then the curl operator does not change. In particular, the curl operator D_Ω can be defined on non-orientable manifolds as well. Non-orientable manifolds don't admit global volume forms in the sense of non-vanishing differential forms of top degree, but they do admit measure-theoretic volume forms with smooth positive distribution. Such a measure-theoretic volume form is a non-oriented (or absolute) version of differential volume forms, and is also called a ***density***. Proposition 2.5.5 implies that one can replace a volume form by a density in the definition of the curl operator.

2.5.2. Schouten bracket via curl operator.

THEOREM 2.5.7 (Koszul [**123**]). *If A is an a-vector field, B is a b-vector field and Ω is a volume form then*

$$
[A, B] = (-1)^b D_\Omega(A \wedge B) - (D_\Omega A) \wedge B - (-1)^b A \wedge (D_\Omega B). \tag{2.80}
$$

Proof. By Formula (2.78) and Formula (1.64) we have:
$(-1)^b D_\Omega(A\wedge B) = (-1)^b \sum \frac{\partial^2(A\wedge B)}{\partial x_i \partial \zeta_i} = \sum \frac{\partial}{\partial x_i}\left(\frac{\partial A}{\partial \zeta_i} B + (-1)^b A \frac{\partial B}{\partial \zeta_i}\right)$
$= \sum \frac{\partial^2 A}{\partial x_i \partial \zeta_i} B + (-1)^b A \sum \frac{\partial^2 B}{\partial x_i \partial \zeta_i} + \frac{\partial A}{\partial \zeta_i}\frac{\partial B}{\partial x_i} + (-1)^b \sum \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial \zeta_i}$
$= (D_\Omega A)B + (-1)^b A (D_\Omega B) + \left(\frac{\partial A}{\partial \zeta_i}\frac{\partial B}{\partial x_i} - (-1)^{(b-1)(a-1)} \sum \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial \zeta_i}\right)$
$= (D_\Omega A)B + (-1)^b A(D_\Omega B) + [A,B].$ □

The curl operator is, up to a sign, a derivation of the Schouten bracket. More precisely, we have the following formula:

$$D_\Omega[A,B] = [A, D_\Omega B] + (-1)^{b-1}[D_\Omega A, B]. \tag{2.81}$$

EXERCISE 2.5.8. Prove the above formula, either by direct calculations, or by using Theorem 2.5.7 and the fact that $D_\Omega \circ D_\Omega = 0$.

2.5.3. The modular class.

A particular important application of the curl operator in Poisson geometry is the ***curl vector field*** $D_\Omega \Pi$, also called ***modular vector field***, of a Poisson structure Π with respect to a volume form Ω. This curl vector field is an infinitesimal automorphism of the Poisson structure, i.e. it is a Poisson vector field. Moreover, it also preserves the volume form:

LEMMA 2.5.9. *If* Π *is a Poisson tensor and* Ω *a volume form, then*

$$[D_\Omega \Pi, \Pi] = 0 \quad \text{and} \quad \mathcal{L}_{(D_\Omega \Pi)}\Omega = 0. \tag{2.82}$$

Proof. It follows from Formula (2.81) and the fact that $[\Pi,\Pi]=0$ that we have $0 = D_\Omega[\Pi,\Pi] = [\Pi, D_\Omega \Pi] - [D_\Omega \Pi, \Pi] = -2[D_\Omega \Pi, \Pi]$. Hence we have $[D_\Omega \Pi, \Pi] = 0$.

To prove the second equality, we don't even need the fact that Π is a Poisson structure. Indeed, we have $\mathcal{L}_{(D_\Omega \Pi)}\Omega = i_{(D_\Omega \Pi)} \mathrm{d}\Omega + \mathrm{d} i_{(D_\Omega \Pi)}\Omega = \mathrm{d}(\mathrm{d}(\Pi \lrcorner \Omega)) = 0$. □

EXERCISE 2.5.10. Show that the curl vector field of the linear Poisson structure $\Pi = y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ with respect to the volume forme $\mathrm{d}x \wedge \mathrm{d}y$ is $\frac{\partial}{\partial x}$, and it is *not* a Hamiltonian vector field.

Lemma 2.5.9 means that the curl vector field $D_\Omega \Pi$ is an 1-cocycle in the Lichnerowicz complex. Proposition 2.5.5 implies that if we change the volume form (or more precisely, the density, see Remark 2.5.6) then this cocycle changes by a coboundary. Thus the cohomology class of the curl vector field $D_\Omega \Pi$ in $H^1(M,\Pi)$ does not depend on the choice of the volume form Ω.

DEFINITION 2.5.11. If (M,Π) is a Poisson manifold and Ω a smooth density on M, then the cohomology class of the curl vector field $D_\Omega \Pi$ in $H^1(M,\Pi)$ is called the ***modular class*** of (M,Π). If this class is trivial, then (M,Π) is called a ***unimodular Poisson manifold***.

DEFINITION 2.5.12. A density Ω on a Poisson manifold (M,Π) is called an ***invariant density*** if it is preserved by all Hamiltonian vector fields on M: $\mathcal{L}_{X_f}\Omega = 0\ \forall\ f \in \mathcal{C}^\infty(M)$.

Lemma 2.5.13. *If Π is a Poisson structure and Ω is a smooth density, then $D_\Omega \Pi = 0$ if and only if Ω is an invariant density. In particular, a Poisson manifold is unimodular if and only if it admits a smooth invariant density.*

We will leave the proof of the above lemma as an exercise. □

Exercise 2.5.14. Show that, if (M^{2n}, ω) is a symplectic manifold of dimension $2n$, then it is unimodular as a Poisson manifold. Up to multiplication by a constant, the only invariant volume form on M is the so-called ***Liouville form***

$$\Omega = \frac{1}{n!} \wedge^n \omega. \tag{2.83}$$

(Don't confuse this Liouville volume form with the Liouville 1-form mentioned in Example 1.1.9).

Exercise 2.5.15. A ***unimodular Lie algebra*** is a Lie algebra $\mathfrak{g}$ such that for any $x \in \mathfrak{g}$, the linear operator $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is traceless. In other words, $\mathfrak{g}$ is called unimodular if its adjoint action preserves a standard volume form. Show that a linear Poisson structure is unimodular if and only if its corresponding Lie algebra is unimodular.

For more about the modular class, see, e.g., [**1, 77, 80, 91, 112**]. For the theory of (secondary) characteristic classes of Poisson manifolds (and Lie algebroids), of which the modular class is a particular case, see Fernandes [**80**] and Crainic [**56**].

2.5.4. The curl operator of an affine connection.

Recall that a ***linear connection*** on a vector bundle E over a manifold M is a $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{V}^1(M) \times \Gamma(E) \to \Gamma(E),\ (X, \xi) \mapsto \nabla_X \xi, \tag{2.84}$$

(where $\Gamma(E)$ denotes the space of sections of E), which is $C^\infty(M)$-linear with respect to X, i.e. $\nabla_{fX}\xi = f\nabla_X \xi\ \forall f \in C^\infty(M)$, and which satisfies the Leibniz rule with respect to ξ, i.e. $\nabla_X(f\xi) = f\nabla_X\xi + X(f)\xi$. A linear connection is also called a ***covariant derivation*** on E.

Let ∇ be an ***affine connection*** on a manifold M, i.e. a linear connection on the tangent bundle TM of M. By the Leibniz rule, one can extend ∇ to a map

$$\nabla : \mathcal{V}^1(M) \times \mathcal{V}^\star(M) \to \mathcal{V}^\star(M) \tag{2.85}$$

(and more generally, to a covariant derivation on all kinds of tensor fields on M). For example, $\nabla_X(Y \wedge Z) = (\nabla_X Y) \wedge Z + Y \wedge (\nabla_X Z)$. The operator

$$D_\nabla = \sum_k i_{\mathrm{d}x_k} \circ \nabla_{\partial/\partial x_k} : \mathcal{V}^\star(M) \to \mathcal{V}^\star(M), \tag{2.86}$$

where $(x_1, \ldots, x_m)$ denotes a system of coordinates on M, is called the ***curl operator*** of ∇.

Exercise 2.5.16. Show that the above definition of D_∇ does not depend on the choice of local coordinates.

Recall that, an affine connection ∇ on a manifold M is called ***torsionless*** if $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathcal{V}^1(M)$. We have the following statement, similar to Theorem 2.5.7:

THEOREM 2.5.17 (Koszul [**123**]). *If A is an a-vector field, B is a b-vector field and ∇ is a torsionless affine connection then*

$$[A, B] = (-1)^b D_\nabla(A \wedge B) - (D_\nabla A) \wedge B - (-1)^b A \wedge (D_\nabla B). \tag{2.87}$$

Proof. By induction, using the Leibniz identity. □

If ∇ is a flat torsionless connection with $(x_1, \dots, x_m)$ as a trivializing coordinate system, i.e. $\nabla_{\partial/\partial x_i} \partial/\partial x_j = 0 \ \forall i, j$, then Formula (2.86) coincides with Formula (2.78).

CHAPTER 3

Levi decomposition

In this chapter, we will discuss a type of local normal forms for Poisson structures which vanish at a point, called Levi normal forms, or Levi decompositions. A Levi normal form is a kind of partial linearization of a Poisson structure, and in "good" cases this leads to a true linearization. The name *Levi decomposition* comes from the analogy with the classical Levi decomposition for finite dimensional Lie algebras. Let us briefly recall here the classical theory (see, e.g., [**28, 196**]):

Let $\mathfrak{l}$ be a finite-dimensional Lie algebra. Denote by $\mathfrak{r}$ the radical of $\mathfrak{l}$, i.e. the maximal solvable ideal of $\mathfrak{l}$. Then the quotient Lie algebra $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is semi-simple, and we have the following exact sequence:

$$0 \to \mathfrak{r} \to \mathfrak{l} \to \mathfrak{g} \to 0. \tag{3.1}$$

The classical ***Levi-Malcev theorem*** says that the above sequence splits, i.e. there is an injective Lie algebra homomorphism $\imath : \mathfrak{g} \to \mathfrak{l}$ such that its composition with the projection map $\mathfrak{l} \to \mathfrak{g}$ is identity. The image $\imath(\mathfrak{g})$ of $\mathfrak{g}$ in $\mathfrak{l}$ is called a ***Levi factor*** of $\mathfrak{l}$. Up to conjugations in $\mathfrak{l}$, the Levi factor of $\mathfrak{l}$ is unique. We will identify $\mathfrak{g}$ with $\imath(\mathfrak{g})$. Then $\mathfrak{g}$ acts on $\mathfrak{r}$ by the adjoint action in $\mathfrak{l}$, and $\mathfrak{l}$ can be decomposed into a semi-direct product of $\mathfrak{g}$ with $\mathfrak{r}$:

$$\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}. \tag{3.2}$$

The above decomposition is called the ***Levi decomposition*** of $\mathfrak{l}$.

In the study of Poisson structures or other structures involving Lie brackets, we often have *infinite* dimensional Lie algebras. So the idea is to find analogs of the Levi-Malcev theorem which hold for these infinite dimensional Lie algebras. These analogs will give interesting information about Poisson structures.

In Section 3.1 we will give a formal infinite dimensional analog of the Levi-Malcev theorem, and illustrate its use in the example of singular foliations. Then in the rest of this chapter, we will discuss Levi decomposition for Poisson structures.

3.1. Formal Levi decomposition

Let $\mathcal{L}$ be a Lie algebra of infinite dimension. Suppose that $\mathcal{L}$ admits a filtration

$$\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \dots, \tag{3.3}$$

such that $\forall i, j \geq 0$, $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$ and $\dim(\mathcal{L}_i/\mathcal{L}_{i+1}) < \infty$. Then we say that $\mathcal{L}$ is a ***pro-finite Lie algebra***, and call the inverse limit

$$\widehat{\mathcal{L}} = \lim_{\infty \leftarrow i} \mathcal{L}/\mathcal{L}_i \tag{3.4}$$

the ***formal completion*** of $\mathcal{L}$ (with respect to a given pro-finite filtration).

EXAMPLE 3.1.1. Let $\mathcal{L}$ be the Lie algebra of smooth vector fields on $\mathbb{R}^n$ which vanish at the origin 0, and $\mathcal{L}_k$ be the ideal of $\mathcal{L}$ consisting of vector fields with zero k-jet at 0. Then $\mathcal{L}$ is pro-finite, and its formal completion is the algebra of formal vector fields at 0.

Given a pro-finite Lie algebra $\mathcal{L}$ as above, denote by $\mathfrak{r}$ the radical of $\mathfrak{l} = \mathcal{L}/\mathcal{L}_1$ and by $\mathfrak{g}$ the semisimple quotient $\mathfrak{l}/\mathfrak{r}$. Denote by $\mathcal{R}$ the preimage of $\mathfrak{r}$ under the projection $\mathcal{L} \to \mathfrak{l} = \mathcal{L}/\mathcal{L}_1$. Then $\mathcal{R}$ is an ideal of $\mathcal{L}$, called the ***pro-solvable radical***, and we have $\mathcal{L}/\mathcal{R} \cong \mathfrak{l}/\mathfrak{r} = \mathfrak{g}$. Denote by $\widehat{\mathcal{R}} = \lim_{\leftarrow} \mathcal{R}/\mathcal{L}_i$ the formal completion of $\mathcal{R}$. Then we have the following exact sequences:

$$0 \to \mathcal{R} \to \mathcal{L} \to \mathfrak{g} \to 0, \tag{3.5}$$

$$0 \to \widehat{\mathcal{R}} \to \widehat{\mathcal{L}} \to \mathfrak{g} \to 0. \tag{3.6}$$

The exact sequence (3.5) does not necessarily split, but its formal completion (3.6) always does:

THEOREM 3.1.2. *With the above notations, there is a Lie algebra injection $\imath : \mathfrak{g} \to \widehat{\mathcal{L}}$ whose composition with the projection map $\widehat{\mathcal{L}} \to \mathfrak{g}$ is the identity map. Up to conjugations in $\widehat{\mathcal{L}}$, such an injection is unique.*

Proof. By induction, for each $k \in \mathbb{N}$ we will construct an injection $\imath_k : \mathfrak{g} \to \mathcal{L}/\mathcal{L}_k$, whose composition with the projection map $\mathcal{L}/\mathcal{L}_k \to \mathfrak{g}$ is identity, and moreover the following compatibility condition is satisfied: the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\imath_{k+1}} & \mathcal{L}/\mathcal{L}_{k+1} \\ \mathrm{Id}\big\downarrow & & \big\downarrow \mathrm{proj.} \\ \mathfrak{g} & \xrightarrow{\imath_k} & \mathcal{L}/\mathcal{L}_k \end{array} \tag{3.7}$$

is commutative. Then $\imath = \lim_{\leftarrow} \imath_k$ will be the required injection. When $k = 1$, $\imath_1$ is given by the Levi-Malcev theorem. If we forget about the compatibility condition, then the other $\imath_k$, $k > 1$, can also be provided by the Levi-Malcev theorem. But to achieve the compatibility condition, we will construct $\imath_{k+1}$ directly from $\imath_k$.

Assume that $\imath_k$ has been constructed. Denote by $\rho : \mathfrak{g} \to \mathcal{L}/\mathcal{L}_{k+1}$ an arbitrary linear map which lifts the injective Lie algebra homomorphism $\imath_k : \mathfrak{g} \to \mathcal{L}/\mathcal{L}_k$. We will modify ρ into a Lie algebra injection.

Note that $\mathcal{L}_k/\mathcal{L}_{k+1}$ is a $\mathfrak{g}$-module. The action of $\mathfrak{g}$ on $\mathcal{L}_k/\mathcal{L}_{k+1}$ is defined as follows: for $x \in \mathfrak{g}, v \in \mathcal{L}_k/\mathcal{L}_{k+1}$, put $x.v = [\rho(x), v] \in \mathcal{L}_k/\mathcal{L}_{k+1}$. If $x, y \in \mathfrak{g}$ then $[\rho(x), \rho(y)] - \rho([x, y]) \in \mathcal{L}_k/\mathcal{L}_{k+1} \subset \mathcal{L}_1/\mathcal{L}_{k+1}$, and therefore $[[\rho(x), \rho(y)] - \rho([x, y]), v] = 0$ because $[\mathcal{L}_1/\mathcal{L}_{k+1}, \mathcal{L}_k/\mathcal{L}_{k+1}] = 0$. The Jacobi identity in $\mathcal{L}/\mathcal{L}_{k+1}$ then implies that $x.(y.v) - y.(x.v) = [x, y].v$, so $\mathcal{L}_k/\mathcal{L}_{k+1}$ is a $\mathfrak{g}$-module.

Define the following 2-cochain $f : \mathfrak{g} \wedge \mathfrak{g} \to \mathcal{L}_k/\mathcal{L}_{k+1}$:

$$x \wedge y \in \mathfrak{g} \wedge \mathfrak{g} \mapsto f(x, y) = [\rho(x), \rho(y)] - \rho([x, y]) \in \mathcal{L}_k/\mathcal{L}_{k+1}. \tag{3.8}$$

One verifies directly that f is a 2-cocycle of the corresponding Chevalley-Eilenberg complex: denoting by $\oint_{xyz}$ the cyclic sum in (x, y, z), we have

$$\delta f(x,y,z) = \oint_{xyz} \big(x.f(y,z) - f([y,z],x)\big) =$$
$$= \oint_{xyz} \Big(\big[\rho(x), [\rho(y), \rho(z)] - \rho([y,z])\big] - [\rho[y,z], \rho(x)] + \rho([[y,z],x])\Big)$$
$$= \oint_{xyz} \big[\rho(x), [\rho(y), \rho(z)]\big] + \oint_{xyz} \rho([[y,z],x]) = 0 + 0 = 0.$$

Since $\mathfrak{g}$ is semisimple, by Whitehead's lemma every 2-cocycle of $\mathfrak{g}$ is a 2-coboundary. In particular, there is an 1-cochain $\phi : \mathfrak{g} \to \mathcal{L}_k/\mathcal{L}_{k+1}$ such that $\delta\phi = f$, i.e.

$$[\rho(x), \phi(y)] - [\rho(y), \phi(x)] - [\phi(x), \phi(y)] = [\rho(x), \rho(y)] - \rho([x,y]). \tag{3.9}$$

It implies that the linear map $\imath_{k+1} = \rho - \phi$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathcal{L}/\mathcal{L}_{k+1}$. Since the image of ϕ lies in $\mathcal{L}_k/\mathcal{L}_{k+1}$, it is clear that $\imath_{k+1}$ is a lifting of $\imath_k$. Thus $\imath_{k+1}$ satisfies our requirements. By induction, the existence of $\imath$ is proved.

The uniqueness of $\imath$ up to conjugations in $\widehat{\mathcal{L}}$ is proved similarly. Suppose that $\imath_{k+1}, \imath'_{k+1} : \mathfrak{g} \to \mathcal{L}/\mathcal{L}_{k+1}$ are two different injections which lift $\imath_k$. Then $\imath'_{k+1} - \imath_{k+1}$ is an 1-cocycle, and therefore an 1-coboundary by Whitehead's lemma. Denote by α an element of $\mathcal{L}_k/\mathcal{L}_{k+1}$ such that $\delta\alpha$ is this 1-coboundary. Then the inner automorphism of $\mathcal{L}/\mathcal{L}_{k+1}$ given by

$$v \in \mathcal{L}/\mathcal{L}_{k+1} \mapsto \mathrm{Ad}_{\exp\alpha} v = v + [\alpha, v] \tag{3.10}$$

(because the other terms vanish) is a conjugation in $\mathcal{L}/\mathcal{L}_{k+1}$ which intertwines $\imath_{k+1}$ and $\imath'_{k+1}$, and which projects to the identity map on $\mathcal{L}/\mathcal{L}_k$. □

The image $\imath(\mathfrak{g})$ of $\mathfrak{g}$ in $\widehat{\mathcal{L}}$, where $\imath$ is given by Theorem 3.1.2, is called a ***formal Levi factor*** of $\mathcal{L}$.

REMARK 3.1.3. The above proof can be modified slightly to yield a proof of the classical Levi-Malcev theorem, pretty close to the one given in [**196**] (Put $\mathcal{L}_1 =$ the radical of $\mathcal{L}$ in the finite dimensional case).

REMARK 3.1.4. Every semisimple subalgebra of a finite dimensional Lie algebra is contained in a Levi factor. Similarly, each semisimple subalgebra of a pro-finite Lie algebra is formally contained in a formal Levi factor. These facts can also be proved by a slight modification of the uniqueness part of the proof of Theorem 3.1.2.

Relations between Levi decomposition and linearization problems were observed, for example, by Flato and Simon [**83**] in their work on linearization of field equations. Here we will show a simple example of such relations, involving singular foliations.

Let $\mathcal{F}$ be a singular holomorphic foliation in a neighborhood of 0 in $\mathbb{C}^n$. Holomorphic means that $\mathcal{F}$ is generated by holomorphic vector fields. We will assume that the rank of $\mathcal{F}$ at 0 is 0, i.e. $X(0) = 0$ for any tangent vector field X tangent to $\mathcal{F}$. Denote by $\mathcal{X}(\mathcal{F})$ the Lie algebra of germs at 0 of holomorphic vector fields tangent to $\mathcal{F}$. Denote by $\mathcal{X}^{(1)}(\mathcal{F})$ the Lie algebra consisting of linear parts of elements of $\mathcal{X}(\mathcal{F})$ at 0. Then $\mathcal{X}^{(1)}(\mathcal{F})$ is a Lie algebra of linear vector fields. Denote by $\mathcal{F}^{(1)}$ the singular foliation generated by $\mathcal{X}^{(1)}(\mathcal{F})$ and call it the linear part of $\mathcal{F}$.

THEOREM 3.1.5 (Cerveau [**46**]). *With the above notations, if $\mathcal{X}^{(1)}(\mathcal{F})$ is semisimple and $\dim \mathcal{F} = \dim \mathcal{F}^{(1)}$, then $\mathcal{F}$ is formally linearizable at 0, i.e. it is formally isomorphic to $\mathcal{F}^{(1)}$.*

Proof. $\mathcal{X}(\mathcal{F})$ is a pro-finite Lie algebra with the standard filtration given by the order of vanishing of vector fields at 0, hence it admits a formal Levi factor $\mathfrak{g}$. When $\mathcal{X}^{(1)}(\mathcal{F})$ is semisimple, then $\mathfrak{g}$ is isomorphic to $\mathcal{X}^{(1)}(\mathcal{F})$. Since $\mathfrak{g}$ is semisimple, its formal action on $\mathbb{C}^n$ is formally linearizable by a classical theorem of Hermann (Theorem 3.1.6). Suppose that the action of $\mathfrak{g}$ has been linearized. It means that $\mathfrak{g}$ consists of linear vector fields, hence it coincides with $\mathcal{X}^{(1)}(\mathcal{F})$. In other words, after the formal linearization, we have an inclusion $\mathcal{X}^{(1)}(\mathcal{F}) \subset \mathcal{X}(\mathcal{F})$, hence $\mathcal{F}^{(1)} \subset \mathcal{F}$. But $\mathcal{F}$ and $\mathcal{F}^{(1)}$ have the same dimension by assumptions, hence they must coincide. □

THEOREM 3.1.6 (Hermann [**106**]). *If $\mathfrak{g} \subset \mathcal{V}^1_{formal,0}(\mathbb{K}^n)$ is a finite dimensional semisimple subalgebra of the Lie algebra $\mathcal{V}^1_{formal,0}(\mathbb{K}^n)$ of formal vector fields on $\mathbb{K}^n$ which vanish at 0, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then there is a formal coordinate system $(z_1, \dots, z_n)$ of $\mathbb{K}^n$ at 0, with respect to which the elements of $\mathfrak{g}$ have linear coefficients.*

Proof (sketch). The proof follows the usual formal normalization procedure, and is based on Whitehead's lemma $H^1(\mathfrak{g}, W) = 0$. Let $X_1, \dots, X_d$ be a basis of $\mathfrak{g}$. Suppose that, in a coordinate system $(z_1, \dots, z_n)$, we have

$$X_i = X_i^{(1)} + X_i^{(s)} + X_i^{(s+1)} + \dots \tag{3.11}$$

with $s \geq 2$, where $X_i^{(s)}$ is a vector field whose coefficients are homogeneous of degree s, and so on. We want to kill the term $X_i^{(s)}$ in the expression of X_i by a coordinate transformation of the type $z'_i = z_i +$ terms of degree $\geq s$. Due to the Jacobi identity, the map $X_i \mapsto X_i^{(s)}$ is an 1-cocycle of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module of homogeneous vector fields of degree s. By Whitehead's lemma, this 1-cocycle is a coboundary, i.e. we can write

$$X_i^{(s)} = [X_i^{(1)}, Y], \tag{3.12}$$

where $Y = \sum_j f_j \partial/\partial z_j$ is homogeneous of degree s. Put $z'_i = z_i - f_i$. This coordinate transformation will kill the term of degree s in the Taylor expansion of X_i. □

3.2. Levi decomposition of Poisson structures

Let Π be a Poisson structure in a neighborhood of 0 in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which vanishes at 0: $\Pi(0) = 0$. Denote by $\Pi^{(1)}$ the linear part of Π at 0, and by $\mathfrak{l}$ the Lie algebra of linear functions on $\mathbb{K}^n$ under the linear Poisson bracket of $\Pi^{(1)}$. Let $\mathfrak{g} \subset \mathfrak{l}$ be a semisimple subalgebra of $\mathfrak{l}$. If Π is formal or analytic, we will assume that $\mathfrak{g}$ is a Levi factor of $\mathfrak{l}$. If Π is smooth (but not analytic), we will assume that $\mathfrak{g}$ is a maximal compact semisimple subalgebra of $\mathfrak{l}$, and we will call such a subalgebra a ***compact Levi factor***. Denote by $(x_1,\dots,x_m,y_1,\dots,y_{n-m})$ a linear basis of $\mathfrak{l}$, such that $x_1,\dots,x_m$ span $\mathfrak{g}$ ($\dim\mathfrak{g}=m$), and $y_1,\dots,y_{n-m}$ span a complement $\mathfrak{r}$ of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{l}$, i.e. $[\mathfrak{g},\mathfrak{r}]\subset\mathfrak{r}$. (In the formal and analytic cases, $\mathfrak{r}$ is the radical of $\mathfrak{l}$; in the smooth case it is not the radical in general). Denote by c_{ij}^k and a_{ij}^k the structural constants of $\mathfrak{g}$ and of the action of $\mathfrak{g}$ on $\mathfrak{r}$ respectively: $[x_i,x_j]=\sum_k c_{ij}^k x_k$ and $[x_i,y_j]=\sum_k a_{ij}^k y_k$.

DEFINITION 3.2.1. With the above notations, we will say that Π admits a formal (resp. analytic, resp. smooth) ***Levi decomposition*** or ***Levi normal form*** at 0, with respect to the (compact) Levi factor $\mathfrak{g}$, if there is a formal (resp. analytic, resp. smooth) coordinate system

$$(x_1^\infty,\dots,x_m^\infty,y_1^\infty,\dots,y_{n-m}^\infty),$$

with $x_i^\infty=x_i+$ higher order terms and $y_i^\infty=y_i+$ higher order terms, such that in this system of coordinates we have

$$\Pi=\sum_{i<j}c_{ij}^k x_k^\infty\frac{\partial}{\partial x_i^\infty}\wedge\frac{\partial}{\partial x_j^\infty}+\sum a_{ij}^k y_k^\infty\frac{\partial}{\partial x_i^\infty}\wedge\frac{\partial}{\partial y_j^\infty}+\sum_{i<j}P_{ij}\frac{\partial}{\partial y_i^\infty}\wedge\frac{\partial}{\partial y_j^\infty},\tag{3.13}$$

where P_{ij} are formal (resp. analytic, resp. smooth) functions.

REMARK 3.2.2. Another way to express Equation (3.13) is as follows:

$$\{x_i^\infty,x_j^\infty\}=\sum c_{ij}^k x_k^\infty \quad\text{and}\quad \{x_i^\infty,y_j^\infty\}=\sum a_{ij}^k y_k^\infty.\tag{3.14}$$

In other words, the Poisson brackets of x-coordinates with x-coordinates, and of x-coordinates with y-coordinates, are linear. Yet another way to say it is that the Hamiltonian vector fields of x_i^∞ are linear:

$$X_{x_i^\infty}=\sum c_{ij}^k x_k^\infty\frac{\partial}{\partial x_j^\infty}+\sum a_{ij}^k y_k^\infty\frac{\partial}{\partial y_j^\infty}.\tag{3.15}$$

In particular, the vector fields $X_{x_1^\infty},\dots,X_{x_m^\infty}$ form a Lie algebra isomorphic to $\mathfrak{g}$, and we have an infinitesimal linear Hamiltonian action of $\mathfrak{g}$ on $(\mathbb{K}^n,\Pi)$, whose momentum map $\mu:\mathbb{K}^n\to\mathfrak{g}^*$ is defined by $\langle\mu(z),x_i\rangle=x_i(z)$.

THEOREM 3.2.3 (Wade [**201**]). *Any formal Poisson structure* Π *in* $\mathbb{K}^n$ *(*$\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$*) which vanishes at* 0 *admits a formal Levi decomposition.*

Proof. Denote by $\mathcal{L}$ the algebra of formal functions in $\mathbb{K}^n$ which vanish at 0, under the Lie bracket of Π. Then it is a pro-finite Lie algebra, whose completion is itself. The Lie algebra $\mathcal{L}/\mathcal{L}_1$, where $\mathcal{L}_1$ is the ideal of $\mathcal{L}$ consisting of functions which vanish at 0 together with their first derivatives, is isomorphic to the Lie algebra $\mathfrak{l}$ of linear functions on $\mathbb{K}^n$ whose Lie bracket is given by the linear Poisson structure $\Pi^{(1)}$. By Theorem 3.1.2, $\mathcal{L}$ admits a Levi factor, which is isomorphic to the Levi factor $\mathfrak{g}$ of $\mathfrak{l}$. Denote by $x_1^\infty,\dots,x_m^\infty$ a linear basis of a Levi factor of $\mathcal{L}$, $\{x_i^\infty,x_j^\infty\}=\sum_k c_{ij}^k x_k^\infty$ where c_{ij}^k are structural constants of $\mathfrak{g}$. Then the Hamiltonian vector fields $X_{x_1^\infty},\dots,X_{x_m^\infty}$ gives a formal action of $\mathfrak{g}$ on $\mathbb{K}^n$. By Hermann's formal linearization theorem 3.1.6, this formal action can be linearized formally, i.e. there is a formal coordinate system $(x_1^0,\dots,y_{n-m}^0)$ in which we have

$$X_{x_i^\infty}=\sum c_{ij}^k x_k^0\frac{\partial}{\partial x_j^0}+\sum a_{ij}^k y_k^0\frac{\partial}{\partial y_j^0}.\tag{3.16}$$

A-priori, it may happen that $x_i^0 \neq x_i^\infty$, but in any case we have $x_i^0 = x_i^\infty +$ higher order terms, and $X_{x_i^\infty}(x_j^\infty) = \sum_k c_{ij}^k x_k^\infty, X_{x_i^\infty}(y_j^0) = \sum_k a_{ij}^k y_k^0$. Renaming y_i^0 by y_i^∞, we get a formal coordinate system $(x_1^\infty, \ldots, y_{n-m}^\infty)$ which puts Π in formal Levi normal form. □

REMARK 3.2.4. A particular case of Theorem 3.2.3 is the following formal linearization theorem of Weinstein [**205**] mentioned in Chapter 2: if the linear part of Π at 0 is semisimple (i.e. it corresponds to a semisimple Lie algebra $\mathfrak{l} = \mathfrak{g}$), then Π is formally linearizable at 0.

REMARK 3.2.5. As observed by Chloup [**52**], Theorem 3.2.3 may also be viewed as a consequence of Hochschild–Serre's Theorem 2.4.4. Indeed, according to Theorem 2.4.4 and Whitehead's lemma, we have

$$H^2(\mathfrak{l}, \mathcal{S}^p\mathfrak{l}) \cong H^0(\mathfrak{g}, \mathbb{K}) \otimes H^2(\mathfrak{r}, \mathcal{S}^p\mathfrak{l})^{\mathfrak{g}} \cong H^2(\mathfrak{r}, \mathcal{S}^p\mathfrak{l})^{\mathfrak{g}} \quad \forall\, p. \tag{3.17}$$

It means that any non-linear term in the Taylor expansion of Π, which is represented by a 2-cocycle of $\mathfrak{l}$ with values in $\mathcal{S}\mathfrak{l} = \oplus_p \mathcal{S}^p\mathfrak{l}$, can be "pushed to $\mathfrak{r}$", i.e. pushed to the "y-part" (consisting of terms $P_{ij}\partial/\partial y_i \wedge \partial/\partial y_j$) of Π.

In the analytic case, we have:

THEOREM 3.2.6 ([**218**]). *Any analytic Poisson structure* Π *in a neighborhood of* 0 *in* $\mathbb{K}^n$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, which vanishes at* 0*, admits an analytic Levi decomposition.*

REMARK 3.2.7. If in the above theorem, $\mathfrak{l} = ((\mathbb{K}^n)^*, \{.,.\}_{\Pi^{(1)}})$ is a semi-simple Lie algebra, i.e. $\mathfrak{g} = \mathfrak{l}$, then we recover the following analytic linearization theorem of Conn [**53**]: any analytic Poisson structure with a semi-simple linear part is locally analytically linearizable. When $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{K}$, then a Levi decomposition of Π is still automatically a linearization (because $\{y_1, y_1\} = 0$), and Theorem 3.2.6 implies the following result of Molinier [**149**] and Conn (unpublished): If the linear part of an analytic Poisson structure Π which vanishes at 0 corresponds to $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{K}$, where $\mathfrak{g}$ is semisimple, then Π is analytically linearizable in a neighborhood of 0.

REMARK 3.2.8. The existence of a local analytic Levi decomposition of Π is essentially equivalent to the existence of a Levi factor (and not just a formal Levi factor) for the Lie algebra $\mathcal{O}$ of germs at 0 of analytic functions under the Poisson bracket of Π. Indeed, if Π is in analytic Levi normal form with respect to a coordinate system $(x_1, \ldots, y_{n-m})$, then the functions $x_1, \ldots, x_m$ form a linear basis of a Levi factor of $\mathcal{O}$. Conversely, suppose that $\mathcal{O}$ admits a Levi factor with a linear basis $x_1, \ldots, x_m$. Then $X_{x_1}, \ldots, X_{x_m}$ generate a local analytic action of $\mathfrak{g}$ on $\mathbb{K}^n$. According to Kushnirenko-Guillemin–Sternberg analytic linearization theorem for analytic actions of semisimple Lie algebras [**99, 124**], we may assume that

$$X_{x_i} = \sum c_{ij}^k x_k^0 \frac{\partial}{\partial x_j^0} + \sum a_{ij}^k y_k^0 \frac{\partial}{\partial y_j^0} \tag{3.18}$$

in a local analytic system of coordinates $(x_1^0, \ldots, y_{n-m}^0)$, where $x_i^0 = x_i +$ higher order terms. Renaming y_i^0 by y_i, we get a local analytic system of coordinates $(x_1, \ldots, y_{n-m})$ which puts Π in Levi normal form.

In the smooth case, we have:

THEOREM 3.2.9 (Monnier–Zung [**152**]). *For any $n \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{\infty\}$ there is $p' \in \mathbb{N} \cup \{\infty\}$, $p' < \infty$ if $p < \infty$, such that the following statement holds: Let Π be a $C^{p'}$-smooth Poisson structure in a neighborhood of 0 in $\mathbb{R}^n$. Denote by $\mathfrak{l}$ the Lie algebra of linear functions in $\mathbb{R}^n$ under the Lie-Poisson bracket Π_1 which is the linear part of Π, and by $\mathfrak{g}$ a compact Levi factor of $\mathfrak{l}$. Then there exists a C^p-smooth Levi decomposition of Π with respect to $\mathfrak{g}$ in a neighborhood of 0 .*

REMARK 3.2.10. The condition that $\mathfrak{g}$ be compact in Theorem 3.2.9 is in a sense necessary, already in the case when $\mathfrak{l} = \mathfrak{g}$.

REMARK 3.2.11. Remark 3.2.7 and Remark 3.2.8 also apply to the smooth case (provided that $\mathfrak{g}$ is compact). In particular, when $\mathfrak{l} = \mathfrak{g}$, one recovers from Theorem 3.2.9 the following smooth linearization theorem of Conn [**54**]: any smooth Poisson structure whose linear part is compact semisimple is locally smoothly linearizable. When $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{R}$ with $\mathfrak{g}$ compact semisimple, Theorem 3.2.9 still gives a smooth linearization. And the existence of a local smooth Levi decomposition is equivalent to the existence of a compact Levi factor.

REMARK 3.2.12. There are analogs of the above Levi decomposition theorems for Lie algebroids.

In the rest of this chapter, we will give a full proof of Theorem 3.2.6, and then a sketch of the proof of Theorem 3.2.9, which is similar but more technical. These proofs of Theorem 3.2.6 and Theorem 3.2.9 are inspired by and based on Conn's work [**53, 54**], and use a normed version of Whitehead's lemma (on vanishing cohomology of semisimple Lie algebras) and the fast convergence method (of Kolmogorov in the analytic case and Nash–Moser in the smooth case) in order to show the convergence of a formal coordinate transformation putting the Poisson structure in Levi normal form.

3.3. Construction of Levi decomposition

In this section we will construct, by a recurrence process, a formal system of coordinates $(x_1^\infty, \ldots, x_m^\infty, y_1^\infty, \ldots, y_{n-m}^\infty)$ which satisfy Relations (3.14) for a given local analytic Poisson structure Π. We will later use analytic estimates to show that our construction actually yields a local analytic system of coordinates.

Each step in our recurrence process consists of 2 substeps: the first substep is to find an *almost Levi factor*. The second substep consists of *almost linearizing* this almost Levi factor.

We begin the first step with the original linear coordinate system

$$(x_1^0, \ldots, x_m^0, y_1^0, \ldots, y_{n-m}^0) = (x_1, \ldots, x_m, y_1, \ldots, y_{n-m}) \ . \tag{3.19}$$

For each positive integer l, after Step l we will find a local coordinate system $(x_1^l, \ldots, x_m^l, y_1^l, \ldots, y_{n-m}^l)$ with the following properties (3.20), (3.21), (3.24):

$$(x_1^l, \ldots, x_m^l, y_1^l, \ldots, y_{n-m}^l) = (x_1^{l-1}, \ldots, x_m^{l-1}, y_1^{l-1}, \ldots, y_{n-m}^{l-1}) \circ \phi_l \ , \tag{3.20}$$

where ϕ_l is a local analytic diffeomorphism of $(\mathbb{K}^n, 0)$ of the type

$$\phi_l(z) = z + \text{terms of order} \geq 2^{l-1} + 1 \ . \tag{3.21}$$

The space $(\mathbb{K}^n, 0)$ above is fixed (our local Poisson manifold). The functions x_1^{l-1}, x_1^l, etc. are local functions on that fixed space.

Denote by

$$X_i^l = X_{x_i^l} \ (i = 1, \dots, m) \tag{3.22}$$

the Hamiltonian vector field of x_i^l with respect to our Poisson structure Π. Then we have

$$X_i^l = \hat{X}_i^l + Y_i^l \ , \tag{3.23}$$

where

$$\hat{X}_i^l = \sum_{jk} c_{ij}^k x_k^l \frac{\partial}{\partial x_j^l} + \sum_{jk} a_{ij}^k y_k^l \frac{\partial}{\partial y_j^l} \ , \ Y_i^l \in O(|z|^{2^l+1}) \ , \tag{3.24}$$

i.e., $\hat{X}_i^l$ is the linear part of $X_i^l = X_{x_i^l}$ in the coordinate system $(x_1^l, \dots, y_{n-m}^l)$, c_{ij}^k and a_{ij}^k are structural constants as appeared in Theorem 3.2.6, and $Y_i^l = X_i^l - \hat{X}_i^l$ does not contain terms of order $\leq 2^l$.

Condition (3.24) may be rewritten as

$$\{x_i^l, x_j^l\} = \sum_k c_{ij}^k x_k^l \text{ modulo terms of order } \geq 2^l + 1 \ , \tag{3.25}$$

$$\{x_i^l, y_j^l\} = \sum_k a_{ij}^k y_k^l \text{ modulo terms of order } \geq 2^l + 1 \ . \tag{3.26}$$

So we may say that the functions $(x_1^l, \dots, x_m^l)$ form an ***almost Levi factor***, and their corresponding Hamiltonian vector fields are ***almost linearized***, up to terms of order $2^l + 1$.

Of course, when $l = 0$, then Relation (3.24) is satisfied by the assumptions of Theorem 3.2.6. Let us show how to construct the coordinate system $(x_1^{l+1}, \dots, y_{n-m}^{l+1})$ from the coordinate system $(x_1^l, \dots, y_{n-m}^l)$. Denote

$$\mathcal{O}_l = \{\text{local analytic functions in } (\mathbb{K}^{\mathrm{n}}, 0) \text{ without terms of order} \leq 2^l\} \ . \tag{3.27}$$

Due to Relations (3.20) and (3.21), it doesn't matter if we use the original coordinate system $(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ or the new one $(x_1^l, \dots, x_m^l, y_1^l, \dots, y_{n-m}^l)$ in the above definition of $\mathcal{O}_l$. It follows from Relation (3.24) that

$$f_{ij}^l := \{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l = Y_i^l(x_j^l) \in \mathcal{O}_l \ . \tag{3.28}$$

Denote by $(\xi_1, \dots, \xi_m)$ a fixed basis of the semi-simple algebra $\mathfrak{g}$, with

$$[\xi_i, \xi_j] = \sum_k c_{ij}^k \xi_k \ . \tag{3.29}$$

Then $\mathfrak{g}$ acts on $\mathcal{O}$ via vector fields $\hat{X}_1^l, \dots, \hat{X}_m^l$, and this action induces the following linear action of $\mathfrak{g}$ on the finite-dimensional vector space $\mathcal{O}_l/\mathcal{O}_{l+1}$: if $g \in \mathcal{O}_l$, considered modulo $\mathcal{O}_{l+1}$, then we put

$$\xi_i \cdot g := \hat{X}_i^l(g) = \sum_{jk} c_{ij}^k x_k^l \frac{\partial g}{\partial x_j^l} + \sum_{jk} a_{ij}^k y_k^l \frac{\partial g}{\partial y_j^l} \;\; mod \; \mathcal{O}_{l+1} \; . \tag{3.30}$$

Notice that if $g \in \mathcal{O}_l$ then $Y_i^l(g) \in \mathcal{O}_{l+1}$, and hence we have

$$\xi_i \cdot g = X^l(g) \; mod \; \mathcal{O}_{l+1} \;\; = \{x_i^l, g\} \; mod \; \mathcal{O}_{l+1} \; . \tag{3.31}$$

The functions f_{ij}^l in (3.28) form a 2-cochain f^l of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathcal{O}_l/\mathcal{O}_{l+1}$:

$$\begin{array}{c} f^l : \mathfrak{g} \wedge \mathfrak{g} \to \mathcal{O}_l/\mathcal{O}_{l+1} \\ f^l(\xi_i \wedge \xi_j) := f_{ij}^l \; mod \; \mathcal{O}_{l+1} \;\; = \{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l \; mod \; \mathcal{O}_{l+1} \; . \end{array} \tag{3.32}$$

In other words, if we denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$, and by $(\xi_1^*, \ldots, \xi_m^*)$ the basis of $\mathfrak{g}^*$ dual to $(\xi_1, \ldots, \xi_m)$, then we have

$$f^l = \sum_{i<j} \xi_i^* \wedge \xi_j^* \otimes (f_{ij}^l \; mod \; \mathcal{O}_{l+1}) \in \wedge^2 \mathfrak{g}^* \otimes \mathcal{O}_l/\mathcal{O}_{l+1} \; . \tag{3.33}$$

It follows from (3.28), and the Jacobi identity for the Poisson bracket of Π and the algebra $\mathfrak{g}$, that the above 2-cochain is a 2-cocycle. Because $\mathfrak{g}$ is semi-simple, we have $H^2(\mathfrak{g}, \mathcal{O}_l/\mathcal{O}_{l+1}) = 0$, i.e. the second cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$-module $\mathcal{O}_l/\mathcal{O}_{l+1}$ vanishes, and therefore the above 2-cocycle is a coboundary. In other words, there is an 1-cochain

$$w^l \in \mathfrak{g}^* \otimes \mathcal{O}_l/\mathcal{O}_{l+1} \tag{3.34}$$

such that

$$f^l(\xi_i \wedge \xi_j) = \xi_i \cdot w^l(\xi_j) - \xi_j \cdot w^l(\xi_i) - w^l(\sum_k c_{ij}^k \xi_k) \; . \tag{3.35}$$

Denote by w_i^l the element of $\mathcal{O}_l$ which is a polynomial of order $\leq 2^{l+1}$ in variables $(x_1^l, \ldots, x_m^l, y_1^l, \ldots, y_{n-m}^l)$ such that the projection of w_i^l in $\mathcal{O}_l/\mathcal{O}_{l+1}$ is $w^l(\xi_i)$.

Remark 3.3.1. Remember that w_i^l are local functions on our fixed space $(\mathbb{K}^n, 0)$. They are not functions of variables $(x_1^l, \ldots, x_m^l, y_1^l, \ldots, y_{n-m}^l)$ per se, but when expressed in terms of these variables they become polynomial functions.

Define x_i^{l+1} as follows:

$$x_i^{l+1} = x_i^l - w_i^l \;\; \forall \; i = 1, \ldots, m \; . \tag{3.36}$$

Then it follows from (3.28) and (3.35) that we have

$$\{x_i^{l+1}, x_j^{l+1}\} - \sum_k c_{ij}^k x_k^{l+1} \in \mathcal{O}_{l+1} \; for \; i,j \leq m \; . \tag{3.37}$$

This concludes our first substep (the (x_i^{l+1}) form a better "almost Levi factor" than (x_i^l)). Let us now proceed to the second substep.

Denote by $\mathcal{Y}^l$ the space of local analytic vector fields of the type $u = \sum_{i=1}^{n-m} u_i \partial/\partial y_i^l$ (with respect to the coordinate system $(x_1^l, \ldots, y_{n-m}^l)$), with u_i being local analytic functions. For each natural number k, denote by $\mathcal{Y}_k^l$ the following subspace of $\mathcal{Y}^l$:

$$\mathcal{Y}^l_k = \left\{ u = \sum_{i=1}^{n-m} u_i \partial/\partial y^l_i \;\middle|\; u_i \in \mathcal{O}_k \right\} \ . \tag{3.38}$$

Then $\mathcal{Y}^l$, as well as $\mathcal{Y}^l_l/\mathcal{Y}^l_{l+1}$, are $\mathfrak{g}$-modules under the following action :

$$\xi_i \cdot \sum_j u_j \partial/\partial y^l_j := [\hat{X}^l_i, u] = \Big[\sum_{jk} c^k_{ij} x^l_k \frac{\partial}{\partial x^l_j} + \sum_{jk} a^k_{ij} y^l_k \frac{\partial}{\partial y^l_j} \ , \ \sum_j u_j \partial/\partial y^l_j \Big] \ . \tag{3.39}$$

The above linear action of $\mathfrak{g}$ on $\mathcal{Y}_l/\mathcal{Y}_{l+1}$ can also be written as follows :

$$\xi_i \cdot \sum_j u_j \partial/\partial y^l_j = \sum_j (\{x^l_i, u_j\} - \sum_k a^k_{ij} u_k) \partial/\partial y^l_j \ \ mod \ \ \mathcal{Y}^l_{l+1} \ . \tag{3.40}$$

Define the following 1-cochain of $\mathfrak{g}$ with values in $\mathcal{Y}^l_l/\mathcal{Y}^l_{l+1}$:

$$\sum_{i=1}^{m} \big(\xi^*_i \otimes \big(\sum_{j=1}^{n-m} (\{x^{l+1}_i, y^l_j\} - \sum_k a^k_{ij} y^l_k) \partial/\partial y^l_j \ mod \ \mathcal{Y}^l_{l+1} \big) \big) \ \in \ \mathfrak{g}^* \otimes \mathcal{Y}^l_l/\mathcal{Y}^l_{l+1} \ . \tag{3.41}$$

Due to Relation (3.37), the above 1-cochain is an 1-cocycle. Since $\mathfrak{g}$ is semisimple, we have $H^1(\mathfrak{g}, \mathcal{Y}^l_l/\mathcal{Y}^l_{l+1}) = 0$, and the above 1-cocycle is an 1-coboundary. In other words, there exists a vector field

$$\sum_{j=1}^{n-m} v^l_j \partial/\partial y^l_j \in \mathcal{Y}^l_l \ , \tag{3.42}$$

with v^l_j being polynomial functions of degree $\leq 2^{l+1}$ in variables $(x^l_1, \dots, y^l_{n-m})$, such that for every $i = 1, \dots, m$ we have

$$\sum_j (\{x^{l+1}_i, y^l_j\} - \sum a^k_{ij} y^l_k) \partial/\partial y^l_j = \sum_j (\{x^l_i, v^l_j\} - \sum a^k_{ij} v^l_k) \partial/\partial y^l_j \ \ mod \ \ \mathcal{Y}^l_{l+1} \ . \tag{3.43}$$

We now define the new system of coordinates as follows :

$$\begin{array}{l} x^{l+1}_i = x^l_i - w^l_i \ (i = 1, \dots, m), \\ y^{l+1}_i = y^l_i - v^l_i \ (i = 1, \dots, n-m), \end{array} \tag{3.44}$$

where functions $w^l_i, v^l_i \in \mathcal{O}_l$ are chosen as above. In particular, Relations (3.37) and (3.43) are satisfied, which means that

$$\begin{array}{l} \{x^{l+1}_i, x^{l+1}_j\} - \sum c^k_{ij} x^{l+1}_k \in \mathcal{O}_{l+1} \ , \\ \{x^{l+1}_i, y^{l+1}_j\} - \sum a^k_{ij} y^{l+1}_k \in \mathcal{O}_{l+1} \ , \end{array} \tag{3.45}$$

i.e. Relation (3.24) is satisfied with l replaced by $l+1$. Of course, Relations (3.20) and (3.21) are also satisfied with l replaced by $l+1$, and with ϕ_{l+1} being the map which when written in the variables $(x^l_1, \dots, y^l_{n-m})$ has the following form:

$$\phi_{l+1} = \mathrm{Id} + \psi_{l+1} \ , \tag{3.46}$$

where

$$\psi_{l+1} = -(w_1^l, \dots, w_m^l, v_1^l, \dots, v_{n-m}^l) \in (O_l)^n \ . \tag{3.47}$$

REMARK 3.3.2. We stress here the fact that Formula (3.46) is valid with respect to the coordinate system $(x_1^l, \dots, y_{n-m}^l)$ only. In particular, the sum there is taken with respect to the local linear structure given by the coordinate system $(x_1^l, \dots, y_{n-m}^l)$ and not by the original coordinate system $(x_1, \dots, y_{n-m})$. If we want to express ϕ_{l+1} in terms of the original coordinate system then it will be much more complicated.

Recall that, by the above construction, $w_1^l, \dots, w_m^l, v_1^l, \dots, v_{n-m}^l$ are polynomial functions of degree $\leq 2^{l+1}$ in variables $(x_1^l, \dots, y_{n-m}^l)$, which do not contain terms of degree $\leq 2^l$.

Define the following limits

$$\begin{array}{l} (x_1^\infty, \dots, y_{n-m}^\infty) = \lim_{l\to\infty}(x_1^l, \dots, y_{n-m}^l) \ , \\ \Phi_\infty = \lim_{l\to\infty} \Phi_l \ \text{ where } \Phi_l = \phi_1 \circ \ldots \circ \phi_l \ . \end{array} \tag{3.48}$$

It is clear that the above limits exist in the formal category,

$$(x_1^\infty, \dots, y_{n-m}^\infty) = (x_1^0, \dots, y_{n-m}^0) \circ \Phi_\infty, \tag{3.49}$$

and the formal coordinate system $(x_1^\infty, \dots, y_{n-m}^\infty)$ satisfies Relation (3.14).

The above construction works not only for local analytic Poisson structures, but also for formal Poisson structures, so it gives us another proof of Theorem 3.2.3. To prove Theorem 3.2.6, it remains to show that, when Π is analytic, we can choose functions w_i^l, v_i^l in such a way that $(x_1^\infty, \dots, y_{n-m}^\infty)$ is in fact a local analytic system of coordinates.

REMARK 3.3.3. The above construction of formal Levi decomposition differs from the construction of Wade [**201**] and Weinstein [**209**]. Their construction is simpler (they don't almost linearize the almost Levi factor at each step, and they kill only one term at each step), and is good enough to show the existence of a formal Levi decomposition. However, in order to prove the existence of an *analytic* Levi decomposition, using Kolmogorov's fast convergence method, one needs to kill a bunch of terms at each step, and that's why the second substep (almost linearizing an almost Levi factor) is important.

3.4. Normed vanishing of cohomology

In this section, using *normed vanishing* of first and second cohomology groups of $\mathfrak{g}$, we will obtain some estimates on $w_i^l = x_i^l - x_i^{l+1}$ and $v_i^l = y_i^l - y_i^{l+1}$. For some basic results on semi-simple Lie algebras and their representations which will be used below, one may consult a book on Lie algebras, e.g., [**115, 196**].

We will denote by $\mathfrak{g}_\mathbb{C}$ the algebra $\mathfrak{g}$ if $\mathbb{K} = \mathbb{C}$, and the complexification of $\mathfrak{g}$ if $\mathbb{K} = \mathbb{R}$. So $\mathfrak{g}_\mathbb{C}$ is a complex semi-simple Lie algebra of dimension m. Denote by $\mathfrak{g}_0$ the compact real form of $\mathfrak{g}_\mathbb{C}$, and identify $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{C}$. Fix an orthonormal basis $(e_1, \dots, e_m)$ of $\mathfrak{g}_\mathbb{C}$ with respect to the Killing form: $\langle e_i, e_j \rangle = \delta_{ij}$. We may assume that $e_1, \dots, e_m \in \sqrt{-1}\mathfrak{g}_0$. Denote by $\Gamma = \sum_i e_i^2$ the ***Casimir element*** of $\mathfrak{g}_\mathbb{C}$: Γ lies in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ and does not depend on the choice of the basis (e_i). When $\mathbb{K} = \mathbb{R}$ then Γ is real, i.e., $\Gamma \in \mathcal{U}(\mathfrak{g})$.

Let W be a finite dimensional complex linear space endowed with a Hermitian metric denoted by $\langle , \rangle$. If $v \in W$ then its norm is denoted by $\|v\| = \sqrt{\langle v, v\rangle}$. Assume that W is a Hermitian $\mathfrak{g}_0$-module. In other words, the linear action of $\mathfrak{g}_0$ on W is via infinitesimal unitary (i.e. skew-adjoint) operators. W is a $\mathfrak{g}_{\mathbb{C}}$-module via the identification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. We have the decomposition $W = W_0 + W_1$, where $W_1 = \mathfrak{g}_{\mathbb{C}} \cdot W$ (the image of the representation), and $\mathfrak{g}_{\mathbb{C}}$ acts trivially on W_0. Since W_1 is a $\mathfrak{g}_{\mathbb{C}}$-module, it is also a $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$-module. The action of Γ on W_1 is invertible: $\Gamma \cdot W_1 = W_1$, and we will denote by Γ^{-1} the inverse mapping.

Denote by $\mathfrak{g}^*_{\mathbb{C}}$ the dual of $\mathfrak{g}_{\mathbb{C}}$, and by $(e_1^*, \ldots, e_m^*)$ the basis of $\mathfrak{g}^*_{\mathbb{C}}$ dual to $(e_1, \ldots, e_m)$. If $w \in \mathfrak{g}^*_{\mathbb{C}} \otimes W$ is an 1-cochain and $f : \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W$ is a 2-cochain with values in W, then we will define the norm of f and w as follows :

$$\|w\| = \max_i \|w(e_i)\| \ , \ \|f\| = \max_{i,j} \|f(e_i \wedge e_j)\| \ . \tag{3.50}$$

Since $H^2(\mathfrak{g}, \mathbb{K}) = 0$, there is a (unique) linear map $h_0 : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ such that if $u \in \wedge^2 \mathfrak{g}^*$ is a 2-cocycle for the trivial representation of $\mathfrak{g}$ in $\mathbb{K}$ (i.e. $u([x, y], z) + u([y, z], x) + u([z, x], y) = 0$ for any $x, y, z \in \mathfrak{g}$), then $u = \delta h_0(u)$, i.e. $u(x, y) = h_0(u)([x, y])$. By complexifying h_0 if $\mathbb{K} = \mathbb{R}$, and taking its tensor product with the projection map $P_0 : W \to W_0$, we get a map

$$h_0 \otimes P_0 : \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W \to \mathfrak{g}^*_{\mathbb{C}} \otimes W_0 \ . \tag{3.51}$$

Define another map

$$h_1 : \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W \to \mathfrak{g}^*_{\mathbb{C}} \otimes W_1 \tag{3.52}$$

as follows : if $f \in \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W$ then we put

$$h_1(f) = \sum_i e_i^* \otimes (\Gamma^{-1} \cdot \sum_j (e_j \cdot f(e_i \wedge e_j))) \ . \tag{3.53}$$

Then the map

$$h = h_0 \otimes P_0 + h_1 : \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W \to \mathfrak{g}^*_{\mathbb{C}} \otimes W \tag{3.54}$$

is an explicit ***homotopy operator***, in the sense that if $f \in \wedge^2 \mathfrak{g}^*_{\mathbb{C}} \otimes W$ is a 2-cocycle (i.e. $\delta f = 0$ where δ denotes the differential of the Chevalley–Eilenberg complex $\ldots \to \wedge^k \mathfrak{g}^*_{\mathbb{C}} \otimes W \to \wedge^{k+1} \mathfrak{g}^*_{\mathbb{C}} \otimes W \to \ldots$), then $f = \delta(h(f))$.

Similarly, the map $h : \mathfrak{g}^*_{\mathbb{C}} \otimes W \to W$ defined by

$$h(w) = \Gamma^{-1} \cdot (\sum_i e_i \cdot w(e_i)) \tag{3.55}$$

is also a homotopy operator, in the sense that if $w \in \mathfrak{g}^*_{\mathbb{C}} \otimes W$ is an 1-cocycle then $w = \delta(h(w))$.

When $\mathbb{K} = \mathbb{R}$, i.e. when $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, then the above homotopy operators h are real, i.e. they map real cocycles into real cochains.

The above formulas make it possible to control the norm of a primitive of a 1-cocycle w or a 2-cocycle f in terms of the norm of w or f. More precisely, we have the following lemma about ***normed vanishing of cohomology***, which is a normed version of Whitehead's lemma which says that $H^1(\mathfrak{g}, W) = H^2(\mathfrak{g}, W) = 0$.

LEMMA 3.4.1 (Conn). *There is a positive constant D (which depends on $\mathfrak{g}$ but does not depend on W) such that with the above notations we have*

$$\|h(f)\| \leq D\|f\| \text{ and } \|h(w)\| \leq D\|w\| \tag{3.56}$$

for any 1-cocycle w and any 2-cocycle f of $\mathfrak{g}_{\mathbb{C}}$ with values in W.

REMARK 3.4.2. The above lemma is essentially due to Conn (see Proposition 2.1 of [**53**]). Conn stated the result only for some particular modules that he needed, but his proof, which we give below, works without any change for other Hermitian modules.

Proof (sketch). We can decompose W into an orthogonal sum (with respect to the Hermitian metric of W) of irreducible modules of $\mathfrak{g}_0$. The above homotopy operators decompose correspondingly, so it is enough to prove the above lemma for the case when W is a non-trivial irreducible module, which we will now suppose. Let $\lambda \neq 0$ denote the highest weight of the irreducible $\mathfrak{g}_0$-module W, and by δ one-half the sum of positive roots of $\mathfrak{g}_0$ (with respect to a fixed Cartan subalgebra and Weyl chamber). Then Γ acts on W by multiplication by the scalar $\langle \lambda, \lambda + 2\delta \rangle$, which is greater or equal to $\|\lambda\|^2$. Denote by $\mathcal{J}$ the weight lattice of $\mathfrak{g}_0$, and $D = m(\min_{\gamma \in \mathcal{J}} \|\gamma\|)^{-1}$. Then $D < \infty$ does not depend on W, and $\|\lambda\|^2 > \frac{m\|\lambda\|}{D}$, which implies that the norm of the inverse of the action of Γ on W is smaller or equal to $\frac{D}{m\|\lambda\|}$. On the other hand, the norm of the action of e_i on W is smaller or equal to $\|\lambda\|$ for each $i = 1, \ldots, m$ (recall that $\sqrt{-1}e_i \in \mathfrak{g}_0$ and $\langle e_i, e_i \rangle = 1$), hence the norm of the operator $\sum_{i=1}^m e_i \cdot \Gamma^{-1} : W \to W$ is smaller or equal to D. Now apply Formulas (3.53) and (3.55). The lemma is proved. □

Let us now apply the above lemma to $\mathfrak{g}$-modules $\mathcal{O}_l/\mathcal{O}_{l+1}$ and $\mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l$ introduced in the previous section. Recall that $\mathfrak{g}$ is a Levi factor of $\mathfrak{l}$, the space of linear functions in $\mathbb{K}^n$, which is a Lie algebra under the linear Poisson bracket $\Pi^{(1)}$. $\mathfrak{g}$ acts on $\mathfrak{l}$ by the (restriction of the) adjoint action, and on $\mathbb{K}^n$ by the coadjoint action. By complexifying these actions if necessary, we get a natural action of $\mathfrak{g}_{\mathbb{C}}$ on $(\mathbb{C}^n)^*$ (the dual space of $\mathbb{C}^n$) and on $\mathbb{C}^n$. The elements $x_1, \ldots, x_m, y_1, \ldots, y_{n-m}$ of the original linear coordinate system in $\mathbb{K}^n$ may be view as a basis of $(\mathbb{C}^n)^*$. Notice that the action of $\mathfrak{g}_{\mathbb{C}}$ on $(\mathbb{C}^n)^*$ preserves the subspace spanned by $(x_1, \ldots, x_m)$ and the subspace spanned by $(y_1, \ldots, y_{n-m})$. Fix a basis $(z_1, \ldots, z_n)$ of $(\mathbb{C}^n)^*$, such that the Hermitian metric of $(\mathbb{C}^n)^*$ for which this basis is orthonormal is preserved by the action of $\mathfrak{g}_0$, and such that

$$z_i = \sum_{j \leq m} A_{ij} x_j + \sum_{j \leq n-m} A_{i,j+m} y_j \ , \tag{3.57}$$

with the constant transformation matrix (A_{ij}) satisfying the following condition :

$$A_{ij} = 0 \text{ if } (i \leq m < j \text{ or } j \leq m < i) \ . \tag{3.58}$$

Such a basis $(z_1, \ldots, z_n)$ always exists, and we may view $(z_1, \ldots, z_n)$ as a linear coordinate system on $\mathbb{C}^n$. We will also define local complex analytic coordinate systems $(z_1^l, \ldots, z_n^l)$ as follows :

$$z_i^l = \sum_{j \leq m} A_{ij} x_j^l + \sum_{j \leq n-m} A_{i,j+m} y_j^l \ . \tag{3.59}$$

Let l be a natural number, ρ a positive number, and f a local complex analytic function of n variables. Define the following ball $B_{l,\rho}$ and L^2-norm $\|f\|_{l,\rho}$, whenever it makes sense :

$$B_{l,\rho} = \left\{ x \in \mathbb{C}^n \mid \sqrt{\sum |z_i^l(x)|^2} \leq \rho \right\} , \tag{3.60}$$

$$\|f\|_{l,\rho} = \sqrt{\frac{1}{V_\rho} \int_{S_{l,\rho}} |f(x)|^2 \mathrm{d}\mu_l} , \tag{3.61}$$

where $\mathrm{d}\mu_l$ is the standard volume form on the boundary $S_{l,\rho} = \partial B_{l,\rho}$ of the complex ball $B_{l,\rho}$ with respect to the coordinate system $(z_1^l, \ldots, z_n^l)$, and V_ρ is the volume of $S_{l,\rho}$, i.e. of a $(2n-1)$-dimensional sphere of radius ρ.

We will say that the ball $B_{l,\rho}$ is well-defined if it is analytically diffeomorphic to the standard ball of radius ρ via the coordinate system $(z_1^l, \ldots, z_n^l)$, and will use $\|f\|_{l,\rho}$ only when $B_{l,\rho}$ is well-defined. When $B_{l,\rho}$ is not well-defined we simply put $\|f\|_{l,\rho} = \infty$. We will write B_ρ and $\|f\|_\rho$ for $B_{0,\rho}$ and $\|f\|_{0,\rho}$ respectively. If f is a real analytic function (the case when $\mathbb{K} = \mathbb{R}$), we will complexify it before taking the norms.

It is well-known (see., e.g., Chapter 1 of [**178**]) that the L^2-norm $\|f\|_\rho$ is given by a Hermitian metric, in which the monomial functions form an orthogonal basis : if $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \prod_i z_i^{\alpha_i}$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha \prod_i z_i^{\alpha_i}$ then the scalar product $\langle f, g \rangle_\rho$ is given by

$$\langle f, g \rangle_\rho = \sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!(n-1)!}{(|\alpha| + n - 1)!} \rho^{2|\alpha|} a_\alpha \bar{b}_\alpha , \tag{3.62}$$

(where $\alpha! = \prod_i \alpha_i!, |a| = \sum \alpha_i$, and $\bar{b}$ is the complex conjugate of b), and the norm $\|f\|_\rho$ is given by

$$\|f\|_\rho = \left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!(n-1)!}{(|\alpha| + n - 1)!} |a_\alpha|^2 \rho^{2|\alpha|} \right)^{1/2} . \tag{3.63}$$

The above scalar product turns $\mathcal{O}_l / \mathcal{O}_{l+1}$ into a Hermitian space, if we consider elements of $\mathcal{O}_l / \mathcal{O}_{l+1}$ as polynomial functions of degree less or equal to 2^{l+1} and which do not contain terms of order $\leq 2^l$. Of course, when $\mathbb{K} = \mathbb{R}$ we will have to complexify $\mathcal{O}_l / \mathcal{O}_{l+1}$, but will redenote $(\mathcal{O}_l / \mathcal{O}_{l+1})_\mathbb{C}$ by $\mathcal{O}_l / \mathcal{O}_{l+1}$, for simplicity.

For the space $\mathcal{Y}^l$ of local vector fields of the type $u = \sum_{i=1}^{n-m} u_i \partial / \partial z_{i+m}^l$ (due to (3.58) and (3.59), this is the same as the space of vector fields of the type $\sum_{i=1}^{n-m} u_i' \partial / \partial y_i^l$ defined in the previous section, up to a complexification if $\mathbb{K} = \mathbb{R}$), we define the L^2-norms in a similar way:

$$\|u\|_{l,\rho} = \sqrt{\frac{1}{V_\rho} \int_{S_{l,\rho}} \sum_{i=1}^{n-m} |u_i(x)|^2 \mathrm{d}\mu_l} . \tag{3.64}$$

These L^2-norms are given by Hermitian metrics similar to (3.62), which make $\mathcal{Y}_l^l / \mathcal{Y}_{l+1}^l$ into Hermitian spaces.

Remark that if $u = (u_1, \dots, u_{n-m})$ then

$$\sum_i \|u_i\|_{l,\rho} \geq \|u\|_{l,\rho} \geq \max_i \|u_i\|_{l,\rho} \ . \tag{3.65}$$

It is an important observation that, since the action of $\mathfrak{g}_0$ on $\mathbb{C}^n$ preserves the Hermitian metric of $\mathbb{C}^n$, its actions on $\mathcal{O}_l/\mathcal{O}_{l+1}$ and $\mathcal{Y}^l_l/\mathcal{Y}^l_{l+1}$, as given in the previous section, also preserve the Hermitian metrics corresponding to the norms $\|f\|_{l,\rho}$ and $\|u\|_{l,\rho}$ (with the same l). Thus, applying Lemma 3.4.1 to these $\mathfrak{g}_{\mathbb{C}}$-modules, we get:

Lemma 3.4.3. *There is a positive constant D_1 such that for any $l \in \mathbb{N}$ and any positive number ρ there exist local analytic functions $w^l_1, \dots, w^l_m, v^l_1, \dots, v^l_{n-m}$, which satisfy the relations of the previous section (in particular Relation (3.35) and Relation (3.43)), and which have the following additional property whenever $B_{l,\rho}$ is well-defined :*

$$\max_i \|w^l_i\|_{l,\rho} \leq D_1 . \max_{i,j} \left\| \{x^l_i, x^l_j\} - \sum_k c^k_{ij} x^l_k \right\|_{l,\rho} \tag{3.66}$$

and

$$\max_i \|v^l_i\|_{l,\rho} \leq D_1 . \max_{i,j} \left\| \{x^l_i - w^l_i, y^l_j\} - \sum_k a^k_{ij} y^l_k \right\|_{l,\rho} \ . \tag{3.67}$$

□

3.5. Proof of analytic Levi decomposition theorem

Besides the L^2-norms defined in the previous section, we will need the following L^∞-norms : If f is a local function then put

$$|f|_{l,\rho} = \sup_{x \in B_{l,\rho}} |f(x)| \ , \tag{3.68}$$

where the complex ball $B_{l,\rho}$ is defined by (3.60). Similarly, if $g = (g_1, \dots, g_N)$ is a vector-valued local map then put $|g|_{l,\rho} = \sup_{x \in B_{l,\rho}} \sqrt{\sum_i |g_i(x)|^2}$. For simplicity, we will write $|f|_\rho$ for $|f|_{0,\rho}$.

For the Poisson structure Π, we will use the following norms :

$$|\Pi|_{l,\rho} := \max_{i,j=1,\dots,n} \left| \{z^l_i, z^l_j\} \right|_{l,\rho} \ . \tag{3.69}$$

Due to the following lemma, we will be able to use the norms $|f|_\rho$ and $\|f\|_\rho$ interchangeably for our purposes, and control the norms of the derivatives:

Lemma 3.5.1. *For any $\varepsilon > 0$ there is a finite number $K < \infty$ depending on ε such that for any integer $l > K$, positive number ρ, and local analytic function $f \in \mathcal{O}_l$ we have*

$$|f|_{(1+\varepsilon/l^2)\rho} \geq \exp(2^{l/2}) |f|_{(1+\varepsilon/2l^2)\rho} \geq \rho |\mathrm{d}f|_\rho \ , \tag{3.70}$$

and

$$|f|_{(1-\varepsilon/l^2)\rho} \leq \|f\|_\rho \leq |f|_\rho \ . \tag{3.71}$$

We will postpone the proof of Lemma 3.5.1 a little bit. Now we want to show a key proposition which, together with a simple lemma, will imply Theorem 3.2.6.

Proposition 3.5.2. *Under the assumptions of Theorem 3.2.6, there exists a constant C, such that for any positive number $\varepsilon < 1/4$, there is a natural number $K = K(\varepsilon)$ and a positive number $\rho = \rho(\varepsilon)$, such that for any $l \geq K$ we can construct a local analytic coordinate system $(x_1^l, \dots, y_{n-m}^l)$ as in the previous sections, with the following additional properties (using the previous notations) :*

$(i)_l$ *(Chains of balls) The ball $B_{l,\exp(1/l)\rho}$ is well-defined, and if $l > K$ we have*

$$B_{l-1,\exp(\frac{1}{l}-\frac{2\varepsilon}{l^2})\rho} \subset B_{l,\exp(1/l)\rho} \subset B_{l-1,\exp(\frac{1}{l}+\frac{2\varepsilon}{l^2})\rho} \ . \tag{3.72}$$

$(ii)_l$ *(Norms of changes) If $l > K$ then we have*

$$|\psi_l|_{l-1,\exp(\frac{1}{l-1}-\frac{\varepsilon}{(l-1)^2})\rho} < \rho \ . \tag{3.73}$$

$(iii)_l$ *(Norms of the Poisson structure) :*

$$|\Pi|_{l,\exp(1/l)\rho} \leq C.\exp(-1/\sqrt{l})\rho \ . \tag{3.74}$$

Theorem 3.2.6 follows immediately from the first part of Proposition 3.5.2 and the following lemma:

Lemma 3.5.3. *If there is a finite number K such that Condition $(i)_l$ of Proposition 3.5.2 is satisfied for all $l \geq K$, then the formal coordinate system*

$$(x_1^\infty, \dots, x_m^\infty, y_1^\infty, \dots, y_{n-m}^\infty)$$

is convergent (i.e. locally analytic).

The main idea behind Lemma 3.5.3 is that, if Condition $(i)_l$ is true for any $l \geq K$, then the infinite intersection $\bigcap_{l=K}^\infty B_{l,\exp(1/l)\rho}$ contains an open neighborhood of 0, implying a positive radius of convergence.

The second and third parts of Proposition 3.5.2 are needed for the proof of the first part. Proposition 3.5.2 will be proved by recurrence : By taking ρ small enough, we can obviously achieve Conditions $(iii)_K$ and $(i)_K$ (Condition $(ii)_K$ is void). Then provided that K is large enough, when $l \geq K$ we have that Condition $(ii)_l$ implies Conditions $(i)_l$ and $(iii)_l$, and Condition $(iii)_l$ in turn implies Condition $(ii)_{l+1}$. In other words, Proposition 3.5.2 is a direct consequence of the following three technical lemmas:

Lemma 3.5.4. *There exists a finite number K (depending on ε) such that if Condition $(ii)_{l+1}$ is satisfied and $l \geq K$ then Condition $(i)_{l+1}$ is also satisfied.*

Lemma 3.5.5. *There exists a finite number K (depending on ε) such that if Condition $(iii)_l$ (of Proposition 3.5.2) is satisfied and $l \geq K$ then Condition $(ii)_{l+1}$ is also satisfied.*

Lemma 3.5.6. *There exists a finite number K (depending on ε) such that if Conditions $(ii)_{l+1}$ and $(iii)_l$ are satisfied and $l \geq K$ then Condition $(iii)_{l+1}$ is also satisfied.*

The lemmas of this section will be proved now, one by one. But first let us mention here the main ingredients behind the last three ones: The proof of Lemma 3.5.4 and Lemma 3.5.6 is straightforward and uses only the first part of Lemma 3.5.1. Lemma 3.5.5 (the most technical one) follows from the estimates on the primitives of cocycles as provided by Lemma 3.4.3.

Proof of Lemma 3.5.1. Let f be a local analytic function in $(\mathbb{C}^n, 0)$. To make an estimate on df, we use the Cauchy integral formula. For $z \in B_\rho$, denote by γ_i the following circle : $\gamma_i = \{v \in \mathbb{C}^n \mid v_j = z_j \text{ if } j \neq i \ , \ |v_i - z_i| = \varepsilon\rho/2l^2\}$. Then $\gamma_i \subset B_{(1+\varepsilon/l^2)\rho}$, and we have

$$\left|\frac{\partial f}{\partial z_i}(z)\right| = \frac{1}{2\pi}\left|\oint_{\gamma_i} \frac{f(v)dv}{(v-z)^2}\right| \leq \frac{2l^2}{\varepsilon\rho}|f|_{(1+\varepsilon/2l^2)\rho} \ ,$$

which implies that $\exp(2^{l/2})|f|_{(1+\varepsilon/2l^2)\rho} \geq \rho|df|$ when l is large enough.

Now let $f \in \mathcal{O}_l$ such that $|f|_{(1+\varepsilon/l^2)\rho} < \infty$. We want to show that if $x \in B_{(1+\varepsilon/2l^2)\rho}$ then $|f(x)| \leq \exp(2^{l/2})|f|_{(1+\varepsilon/l^2)\rho}$ (provided that l is large enough compared to $1/\varepsilon$). Fix a point $x \in B_{(1+\varepsilon/2l^2)\rho}$ and consider the following holomorphic function of one variable : $g(z) = f(\frac{x}{|x|}z)$. This function is holomorphic in the complex 1-dimensional disk $B^1_{(1+\varepsilon/l^2)\rho}$ of radius $(1+\varepsilon/l^2)\rho$, and is bounded by $|f|_{(1+\varepsilon/l^2)\rho}$ in this disk. Because $f \in \mathcal{O}_l$, we have that $g(z)$ is divisible by z^{2^l}, that is $g(z)/z^{2^l}$ is holomorphic in $B^1_{(1+\varepsilon/l^2)\rho}$. By the maximum principle we have

$$\frac{|f(x)|}{|x|^{2^l}} = \left|\frac{g(|x|)}{|x|^{2^l}}\right| \leq \max_{|z|=(1+\varepsilon/l^2)\rho}\left|\frac{g(z)}{z^{2^l}}\right| \leq \frac{|f|_{(1+\varepsilon/l^2)\rho}}{((1+\varepsilon/l^2)\rho)^{2^l}} \ ,$$

which implies that

$$|f(x)| \leq \left(\frac{1+\varepsilon/2l^2}{1+\varepsilon/l^2}\right)^{2^l}|f|_{(1+\varepsilon/l^2)\rho} \approx \exp(-\frac{2^l}{2\varepsilon l^2})|f|_{(1+\varepsilon/l^2)\rho} \leq$$
$$\leq \exp(-2^{l/2})|f|_{(1+\varepsilon/l^2)\rho}$$

(when l is large enough). Thus we have proved that there is a finite number K depending on ε such that

$$|f|_{(1+\varepsilon/l^2)\rho} \geq \exp(2^{l/2})|f|_{(1+\varepsilon/2l^2)\rho}$$

for any $l > K$ and any $f \in \mathcal{O}_l$.

To compare the norms of f, we use Cauchy-Schwartz inequality: for $f = \sum_{\alpha\in\mathbb{N}^k} c_\alpha \prod_i z_i^{\alpha_i}$ and $|z| = (1-\varepsilon/2l^2)\rho$ we have

$$|f(z)| \leq \sum_{\alpha\in\mathbb{N}^k}|c_\alpha|\prod_i |z_i|^{\alpha_i} \leq$$
$$\leq \left(\sum_\alpha |c_\alpha|^2\frac{\alpha!(n-1)!}{(|\alpha|+n-1)!}\rho^{2|\alpha|}\right)^{1/2}\left(\sum_\alpha \frac{(|\alpha|+n-1)!}{\alpha!(n-1)!}\rho^{-2|\alpha|}\prod_i |z_i|^{2\alpha}\right)^{1/2} =$$
$$= \|f\|_\rho\left(1-\sum_i\frac{|z_i|^2}{\rho^2}\right)^{-n/2} = \|f\|_\rho\big(1-(1-\varepsilon/2l^2)^2\big)^{-n/2} \leq \frac{(2l)^n}{\varepsilon^{n/2}}\|f\|_\rho \ .$$

It means that for any local analytic function f we have

$$|f|_{(1-\varepsilon/2l^2)\rho} \leq \frac{(2l)^n}{\varepsilon^{n/2}}\|f\|_\rho \ .$$

Now if $f \in \mathcal{O}_l$, we can apply Inequality (3.70) to get

$$|f|_{(1-\varepsilon/l^2)\rho} \leq \exp(-2^{l/2})|f|_{(1-\varepsilon/2l^2)\rho} \leq \frac{(2l)^n}{\varepsilon^{n/2}}\exp(-2^{l/2})\|f\|_\rho \leq \|f\|_\rho \ ,$$

provided that l is large enough compared to $1/\varepsilon$. Lemma 3.5.1 is proved. $\square$

Proof of Lemma 3.5.3. The main point is to show that the limit $\bigcap_{l=K}^{\infty} B_{l,\rho}$ contains a ball B_r of positive radius centered at 0. Then for $x \in B_r$, we have $x \in B_{l,\rho}$, implying $\|(z_1^l(x), \dots, z_n^l(x))\| < \rho$ is uniformly bounded, which in turn implies that the formal functions $z_i^\infty = \lim_{l\to\infty} z_i^l$ are analytic functions inside B_r (recall that $(z_1^l, \dots, z_n^l)$ is obtained from $(x_1^l, \dots, y_{n-m}^l)$ by a constant linear transformation (A_{ij}) which does not depend on l).

Recall the following fact of complex analysis, which is a consequence of the maximum principle : if g is a complex analytic map from a complex ball of radius ρ to some linear Hermitian space such that $g(0) = 0$ and $|g(x)| \leq C$ for all $|x| < \rho$ and some constant C, then we have $|g(x)| \leq C|x|/\rho$ for all x such that $|x| < \rho$. If $l_1, l_2 \in \mathbb{N}$ and $r_1, r_2 > 0, s > 1$, then applying this fact we get:

$$\text{If } B_{l_1,r_1} \subset B_{l_2,r_2} \text{ then } B_{l_1,r_1/s} \subset B_{l_2,r_2/s}\ . \tag{3.75}$$

(Here r_1 plays the role of ρ, r_2 plays the role of C, and the coordinate transformation from $(z_1^{l_1}, \dots, z_n^{l_1})$ to $(z_1^{l_2}, \dots, z_n^{l_2})$ plays the role of g in the previous statement).

Using Formula (3.75) and Condition $(i)_l$ recursively, we get

$$\begin{aligned} B_{l,\rho} \supset B_{l-1,\exp(-1/l^2)\rho} \supset B_{l-2,\exp(-1/l^2-1/(l-1)^2)\rho} \supset \\ \supset \dots \supset B_{K,\exp(-\sum_{k=K}^{l} 1/k^2)\rho}\ . \end{aligned}$$

Since $c = \exp(-\sum_{k=K}^{\infty} 1/k^2)$ is a positive number, we have $\bigcap_{l=K}^{\infty} B_{l,\rho} \supset B_{K,c\rho}$, which clearly contains an open neighborhood of 0. Lemma 3.5.3 is proved. $\square$

Proof of Lemma 3.5.4. Suppose that Condition $(ii)_{l+1}$ is satisfied. For simplicity of exposition, we will assume that the coordinate system $(z_1^l, \dots, z_n^l)$ coincides with the coordinate system $(x_1^l, \dots, y_{n-m}^l)$ (The more general case, when $(z_1^l, \dots, z_n^l)$ is obtained from $(x_1^l, \dots, y_{n-m}^l)$ by a constant linear transformation, is essentially the same). Suppose that we have

$$|\psi_{l+1}|_{l,\exp(1/l-\varepsilon/l^2)\rho} < \rho\ .$$

Then it follows from Lemma 3.5.1 that, provided that l is large enough:

$$|d\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} < \frac{1}{2n}\ .$$

(In order to define $|d\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho}$, consider $d\psi_{l+1}$ as an n^2-vector valued function in variables $(z_1^l, \dots, z_n^l)$). Hence the map $\phi_{l+1} = \mathrm{Id} + \psi_{l+1}$ is injective in $B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$: if $x, y \in B_{l,\rho_l}, x \neq y$, then $\|\phi_{l+1}(x) - \phi_{l+1}(y)\| \geq \|x-y\| - \|\psi_{l+1}(x) - \psi_{l+1}(y)\| \geq \|x-y\| - n|d\psi_{l+1}|_{\exp(1/l-2\varepsilon/l^2)\rho}\|x-y\| \geq (1-1/2)\|x-y\| > 0$. (Here $(x-y)$ means the vector $(z_1^l(x) - z_1^l(y), \dots, z_n^l(x) - z_n^l(y))$, i.e. their difference is taken with respect to the coordinate system $(z_1^l, \dots, z_n^l)$).

It follows from Lemma 3.5.1 that

$$\begin{aligned} &|\phi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} = |\mathrm{Id} + \psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \leq \\ &\leq |\mathrm{Id}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} + |\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} < \\ &< \exp(1/l - 2\varepsilon/l^2)\rho + \frac{\varepsilon}{4l^2}\exp(1/l - 2\varepsilon/l^2)\rho < \exp(1/l - \varepsilon/l^2)\rho\ . \end{aligned}$$

In other words, we have

$$\phi_{l+1}(B_{l,\exp(1/l-2\varepsilon/l^2)\rho}) \subset B_{l,\exp(1/l-\varepsilon/l^2)\rho} \ . \tag{3.76}$$

Applying Formula (3.75) to the above relation, noticing that $1/l - 2\varepsilon/l^2 > 1/(l+1)$, and simplifying the obtained formula a little bit, we get

$$\phi_{l+1}(B_{l,\exp(1/(l+1)-2\varepsilon/(l+1)^2)\rho}) \subset B_{l,\exp(1/(l+1))\rho} \ . \tag{3.77}$$

We will show that ϕ_{l+1}^{-1} is well-defined in $B_{l,\exp(1/(l+1))\rho}$, and

$$\phi_{l+1}^{-1}(B_{l,\exp(1/(l+1))\rho}) = B_{l+1,\exp(1/(l+1))\rho} \subset B_{l,\exp(1/(l+1)+2\varepsilon/(l+1)^2)\rho} \ . \tag{3.78}$$

Indeed, denote by $S_{l,\exp(1/l-2\varepsilon/l^2)\rho}$ the boundary of $B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$. Then

$$\phi_{l+1}(S_{l,\exp(1/l-2\varepsilon/l^2)\rho}) \subset B_{l,\exp(1/l-\varepsilon/l^2)\rho}$$

and is homotopic to $S_{l,\exp(1/l-2\varepsilon/l^2)\rho}$ via a homotopy which does not intersect $B_{l,\exp(1/(l+1))\rho}$. It implies, via the classical Brower's fixed point theorem, that $\phi_{l+1}(B_{l,\exp(1/l-2\varepsilon/l^2)\rho})$ must contain $B_{l,\exp(1/(l+1))\rho}$. Because ϕ_{l+1} is injective in $(B_{l,\exp(1/l-2\varepsilon/l^2)\rho})$, it means that the inverse map is well-defined in $B_{l,\exp(1/(l+1))\rho}$, with

$$\phi_{l+1}^{-1}(B_{l,\exp(1/(l+1))\rho}) \subset B_{l,\exp(1/l-2\varepsilon/l^2)\rho} \ .$$

In particular, $B_{l+1,\exp(1/(l+1))\rho} = \phi_{l+1}^{-1}(B_{l,\exp(1/(l+1))\rho})$ is well-defined. Lemma 3.5.4 then follows from (3.77) and (3.78). $\square$

Proof of Lemma 3.5.5. Suppose that Condition $(iii)_l$ is satisfied. Then according to (3.28) we have :

$$\|f_{ij}^l\|_{l,\exp(1/l)\rho} \leq |f_{ij}^l|_{l,\exp(1/l)\rho} = \Big|\{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l\Big|_{l,\exp(1/l)\rho} \leq \tag{3.79}$$

$$\leq C_1|\Pi|_{l,\exp(1/l)\rho} + \sum_k |c_{ij}^k| |x_k^l|_{l,\rho} \leq C_1.C.\rho + C_2.\exp(1/l)\rho \sum_k |c_{ij}^k| < C_3\rho \ ,$$

where C_3 is some positive constant (which does not depend on l).

We can apply the above inequality $\|f_{ij}^l\|_{l,\exp(1/l)\rho} < C_3\rho$ and Lemma 3.4.3 to find a positive constant C_4 (which does not depend on l) and a solution w_i^l of (3.37), such that

$$\left\|w_i^l\right\|_{l,\exp(1/l)\rho} < C_4\rho \ . \tag{3.80}$$

Together with Lemma 3.5.1, the above inequality yields

$$\left|\mathrm{d}w_i^l\right|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_4, \tag{3.81}$$

provided that l is large enough. Applying Lemma 3.5.1 and the assumption that $|\Pi|_{l,\exp(1/l)\rho} < C\rho$ to the above inequality, we get

$$\left|\{w_i^l, y_j^l\}\right|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_5\rho$$

for some constant C_5 (which does not depend on l). Using this inequality, and inequalities similar to (3.79), we get that the norm $\|.\|_{l,\exp(1/l-\varepsilon/2l^2)\rho}$ of the 1-cocycle given in Formula (3.41) is bounded from above by $C_6\rho$, where C_6 is some constant which does not depend on L. Using Lemma 3.4.3, we find a solution v_i^L to Equation 3.43 such that

$$\left\|v_i^l\right\|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_6\rho \ , \tag{3.82}$$

where C_6 is some constant which does not depend on l. Lemma 3.5.5 (fr l large enough compared to C_6) now follows directly from Inequalities (3.80), (3.82) and Lemma 3.5.1. □

Proof of Lemma 3.5.6. Suppose that Condition $(ii)_{l+1}$ is satisfied. By Lemma 3.5.4, Condition $(i)_{l+1}$ is also satisfied. In particular,

$$B_{l+1,\exp(1/(l+1))\rho} \subset B_{l,\exp(1/(l+1)+2\varepsilon/(l+1)^2)\rho} \subset B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$$

(for $\varepsilon < 1/4$ and l large enough). Thus we have

$$|\{z_i^{l+1}, z_j^{l+1}\}|_{l+1,\exp(1/(l+1))\rho} \leq |\{z_i^{l+1}, z_j^{l+1}\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \leq T^1 + T^2 + T^3 + T^4$$

where

$$\begin{aligned}
T^1 &= |\{z_i^l, z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \ , \\
T^2 &= |\{z_i^{l+1} - z_i^l, z_j^{l+1}\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \ , \\
T^3 &= |\{z_i^{l+1}, z_j^{l+1} - z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \ , \\
T^4 &= |\{z_i^{l+1} - z_i^l, z_j^{l+1} - z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \ .
\end{aligned}$$

For the first term, we have

$$T^1 \leq |\{z_i^l, z_j^l\}|_{l,\exp(1/l)\rho} \leq |\Pi|_{l,\exp(1/l)\rho} \leq C.\exp(-1/\sqrt{l})\rho \ .$$

Notice that $C\exp(-1/\sqrt{l+1})\rho - C\exp(-1/\sqrt{l})\rho > \frac{C}{l^2}\rho$ (for l large enough). So to verify Condition $(iii)_{l+1}$, it suffices to show that $T^2 + T^3 + T^4 < \frac{C}{l^2}\rho$. But this last inequality can be achieved easily (provided that l is large enough) by Conditions $(ii)_{l+1}$, $(iii)_l$ and Lemma 3.5.1. Lemma 3.5.6 is proved. □

3.6. The smooth case

In this section we will give a sketch of the proof of Theorem 3.2.9, referring the reader to [**152**] for the details, which are quite long. Or rather, we will show what modifications to be made to the proof of the analytic Levi decomposition theorem 3.2.6 in order to obtain the proof of Theorem 3.2.9.

In this section, $\mathfrak{g}$ will be a *compact* semisimple Lie algebra. Denote by $(\xi_1, \dots, \xi_m)$ a fixed basis of $\mathfrak{g}$, which is orthonormal with respect to a fixed positive definite invariant metric on $\mathfrak{g}$. Denote by c_{ij}^k the structural constants of $\mathfrak{g}$ with respect to this basis:

$$[\xi_i, \xi_j] = \sum_k c_{ij}^k \xi_k \, . \tag{3.83}$$

Since $\mathfrak{g}$ is compact, we may extend $(\xi_1, \dots, \xi_m)$ to a basis

$$(\xi_1, \dots, \xi_m, y_1, \dots, y_{n-m})$$

of $\mathfrak{l}$ such that the corresponding Euclidean metric is preserved by the adjoint action of $\mathfrak{g}$. The algebra $\mathfrak{g}$ acts on $\mathfrak{l}^* = \mathbb{R}^n$ via the coadjoint action of $\mathfrak{l}$ $\zeta(z) := \mathrm{ad}_\zeta^*(z)$ for

$\zeta \in \mathfrak{g} \subset \mathfrak{l}, z \in \mathbb{R}^n = \mathfrak{l}^*$. The basis $(\xi_1, \dots, \xi_m, y_1, \dots, y_{n-m})$ of $\mathfrak{l}$ may be viewed as a coordinate system $(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ on $\mathbb{R}^n$ (with $x_i = \xi_i$).

Denote by G the compact simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. Then the above action of $\mathfrak{g}$ on $\mathbb{R}^n$ integrates into an action of G on $\mathbb{R}^n$ (the coadjoint action). The action of G on $\mathbb{R}^n$ preserves the Euclidean metric of $\mathbb{R}^n$ given by $\|z\|^2 = \sum |x_i(z)|^2 + \sum |y_j(z)|^2$.

For each positive number $r > 0$, denote by B_r the closed ball of radius r in $\mathbb{R}^n$ centered at 0. The group G (and hence the algebra $\mathfrak{g}$) acts linearly on the space of functions on B_r via its action on B_r: for each function F and element $g \in G$ we put

$$g(F)(z) := F(g^{-1}(z)) = F(\mathrm{Ad}_{(g^{-1})} z). \tag{3.84}$$

In the smooth case, we will use C^k-norms and Sobolev norms. For each non-negative integer $k \geq 0$ and each pair of real-valued functions F_1, F_2 on B_r, we will define the Sobolev inner product of F_1 with F_2 with respect to the Sobolev H_k-norm as follows:

$$\langle F_1, F_2 \rangle^H_{k,r} := \sum_{|\alpha| \leq k} \int_{B_r} \left(\frac{|\alpha|!}{\alpha!} \right) \left(\frac{\partial^{|\alpha|} F_1}{\partial z^\alpha}(z) \right) \left(\frac{\partial^{|\alpha|} F_2}{\partial z^\alpha}(z) \right) \mathrm{d}\mu(z). \tag{3.85}$$

The Sobolev H_k-norm of a function F on B_r is

$$\|F\|^H_{k,r} := \sqrt{\langle F, F \rangle^H_{k,r}} \ . \tag{3.86}$$

We will denote by $\mathcal{C}_r$ the subspace of the space $\mathcal{C}^\infty(B_r)$ of smooth real-valued functions on B_r, which consists of functions vanishing at 0 whose first derivatives also vanish at 0. Then the action of G on $\mathcal{C}_r$ defined by (3.84) preserves the Sobolev inner products (3.85).

Denote by $\mathcal{Y}_r$ the space of smooth vector fields on B_r of the type

$$u = \sum_{i=1}^{n-m} u_i \partial / \partial y_i \ , \tag{3.87}$$

such that u_i vanish at 0 and their first derivatives also vanish at 0. Then $\mathcal{Y}_r$ is a $\mathfrak{g}$-module under the following action :

$$\xi_i \cdot \sum_j u_j \partial/\partial y_j := \Big[\sum_{jk} c^k_{ij} x_k \frac{\partial}{\partial x_j} + \sum_{jk} a^k_{ij} y_k \frac{\partial}{\partial y_j} \ , \ \sum_j u_j \partial/\partial y_j \Big] \ , \tag{3.88}$$

where $X_i = \sum_{jk} c^k_{ij} x_k \partial/\partial x_j + \sum_{jk} a^k_{ij} y_k \partial/\partial y_j$ are the linear vector fields which generate the linear orthogonal coadjoint action of $\mathfrak{g}$ on $\mathbb{R}^n$.

Equip $\mathcal{Y}_r$ with Sobolev inner products:

$$\langle u, v \rangle^H_{k,r} := \sum_{i=1}^{n-m} \langle u_i, v_i \rangle_{k,r} \ , \tag{3.89}$$

and denote by $\mathcal{Y}^H_{k,r}$ the completion of $\mathcal{Y}_r$ with respect to the corresponding $H_{k,r}$-norm. Then $\mathcal{Y}^H_{k,r}$ is a separable real Hilbert space on which $\mathfrak{g}$ and G act orthogonally.

The C^k-norms can be defined as follows:

$$\|F\|_{k,r} := \sup_{|\alpha|\leq k} \sup_{z\in B_r} |D^\alpha F(z)| \tag{3.90}$$

for $F \in \mathcal{C}_r$, where the sup runs over all partial derivatives of degree $|\alpha|$ at most k. Similarly, for $u = \sum_{i=1}^{n-m} u_i \partial/\partial y_i \in \mathcal{Y}_r$ we put

$$\|u\|_{k,r} := \sup_i \sup_{|\alpha|\leq k} \sup_{z\in B_r} |D^\alpha u_i(z)| \tag{3.91}$$

The C^k norms $\|.\|_{k,r}$ are related to the Sobolev norms $\|.\|^H_{k,r}$ as follows:

$$\|F\|_{k,r} \leq C\|F\|^H_{k+s,r} \text{ and } \|F\|^H_{k,r} \leq C(n+1)^k \|F\|_{k,r} \tag{3.92}$$

for any F in $\mathcal{C}_r$ or $\mathcal{Y}_r$ and any $k \geq 0$, where $s = [\frac{n}{2}]+1$ and C is a positive constant which does not depend on k. In other words, C^k norms and Sobolev norms are "tamely equivalent". A priori, the constant C depends on r, but later on we will always assume that $1 \leq r \leq 2$, and so may assume C to be independent of r. The above inequality is a version of the classical Sobolev's lemma for Sobolev spaces.

Similarly to the analytic case, we will need the following normed version of Whitehead's lemma (cf. Proposition 2.1 of [**54**]):

Lemma 3.6.1 (Conn). *For any given positive number r, and $W = \mathcal{C}_r$ or $\mathcal{Y}_r$ with the above action of $\mathfrak{g}$, consider the (truncated) Chevalley-Eilenberg complex*

$$W \xrightarrow{\delta_0} W \otimes \wedge^1\mathfrak{g}^* \xrightarrow{\delta_1} W \otimes \wedge^2\mathfrak{g}^* \xrightarrow{\delta_2} W \otimes \wedge^3\mathfrak{g}^*$$

Then there is a chain of operators

$$W \xleftarrow{h_0} W \otimes \wedge^1\mathfrak{g}^* \xleftarrow{h_1} W \otimes \wedge^2\mathfrak{g}^* \xleftarrow{h_2} W \otimes \wedge^3\mathfrak{g}^*$$

such that

$$\begin{aligned} \delta_0 \circ h_0 + h_1 \circ \delta_1 &= \mathrm{Id}_{W\otimes\wedge^1\mathfrak{g}^*} \ , \\ \delta_1 \circ h_1 + h_2 \circ \delta_2 &= \mathrm{Id}_{W\otimes\wedge^2\mathfrak{g}^*} \ . \end{aligned} \tag{3.93}$$

Moreover, there exist a constant $C > 0$, which is independent of the radius r of B_r, such that

$$\|h_j(u)\|^H_{k,r} \leq C\|u\|^H_{k,r} \quad j = 0, 1, 2 \tag{3.94}$$

for all $k \geq 0$ and $u \in W \otimes \wedge^{j+1}\mathfrak{g}^*$. If u vanishes to an order $l \geq 0$ at the origin, then so does $h_j(u)$.

Strictly speaking, Conn [**54**] proved the above lemma only in the case when $\mathfrak{g} = \mathfrak{l}$ and for the module $\mathcal{C}_r$, but his proof is quite general and works perfectly in our situation without any modification. In fact, in order to prove Lemma 3.6.1, it is sufficient to show that W is a infinite direct sum of finite-dimensional orthogonal modules, and then repeat the proof of Lemma 3.4.1. □

For simplicity, in the sequel we will denote the homotopy operators h_j in the above lemma simply by h. Homotopy relation (3.93) will be rewritten simply as follows:

$$\mathrm{Id} - \delta \circ h = h \circ \delta \ . \tag{3.95}$$

The meaning of the last equality is as follows: if u is an 1-cocycle or 2-cocycle, then it is also a coboundary, and $h(u)$ is an explicit primitive of u: $\delta(h(u)) = u$. If u is a "near cocycle" then $h(u)$ is also a "near primitive" for u.

Combining Inequality (3.94) with Sobolev inequalities, we get the following estimate for the homotopy operators h with respect to C^k norms:

$$\|h(u)\|_{k,r} \leq C(n+1)^{k+s}\|u\|_{k+s,r} \quad \forall\ j = 0,1,2 \tag{3.96}$$

for all $k \geq 0$ and $u \in W \otimes \wedge^{j+1}\mathfrak{g}^*$ where $W = \mathcal{C}_r$ or $\mathcal{Y}_r$. Here $s = [\frac{n}{2}] + 1$, C is a positive constant which does not depend on k (and r provided that $1 \leq r \leq 2$).

It is well-known that the space $C^\infty(B_r)$ with C^k norms (3.90) is a ***tame Fréchet space*** (see, e.g., [**103**] for the theory of tame Fréchet spaces). Since $\mathcal{C}_r$ is a tame direct summand of $C^\infty(B_r)$, it is also a tame Fréchet space. Similarly, $\mathcal{Y}_r$ with norms (3.91) is a tame Frechet space as well. What we will use here is the fact that tame Fréchet spaces admit smoothing operators and interpolation inequalities:

For each $t > 1$ there is a linear operator $S(t) = S_r(t)$ from $\mathcal{C}_r$ to itself, called a ***smoothing operator***, with the following properties:

$$\|S(t)F\|_{p,r} \leq C_{p,q}t^{(p-q)}\|F\|_{q,r} \tag{3.97}$$

and

$$\|(\mathrm{Id} - S(t))F\|_{q,r} \leq C_{p,q}t^{(q-p)}\|F\|_{p,r} \tag{3.98}$$

for any $F \in \mathcal{C}_r$, where p, q are any nonnegative integers such that $p \geq q$, Id denotes the identity map, and $C_{p,q}$ denotes a constant which depends on p and q.

The second inequality means that $S(t)$ is close to identity and tends to identity when $t \to \infty$. The first inequality means that F becomes "smoother" when we apply $S(t)$ to it. That's why $S(t)$ is called a smoothing operator. A priori, the constants $C_{p,q}$ also depend on the radius r. But later on, we will always have $1 \leq r \leq 2$, and so we may choose $C_{p,q}$ to be independent of r.

There is a similar smoothing operator from $\mathcal{Y}_r$ to itself, which by abuse of language we will also denote by $S(t)$ or $S_r(t)$. We will assume that inequalities (3.97) and (3.98) are still satisfied when F is replaced by an element of $\mathcal{Y}_r$.

For any F in $\mathcal{C}_r$ or $\mathcal{Y}_r$, and nonnegative integers $p_1 \geq p_2 \geq p_3$, we have the following ***interpolation inequality***:

$$(\|F\|_{p_2,r})^{p_3-p_1} \leq C_{p_1,p_2,p_3}(\|F\|_{p_1,r})^{p_3-p_2}(\|F\|_{p_3,r})^{p_2-p_1}, \tag{3.99}$$

where C_{p_1,p_2,p_3} is a positive constant which may depend on p_1, p_2, p_3.

Similarly to the analytic case, in order to prove Theorem 3.2.9, we will construct by recurrence a sequence of local smooth coordinate systems $(x^d, y^d) := (x_1^d, \ldots, x_m^d, y_1^d, \ldots, y_{n-m}^d)$, which converges to a local coordinate system $(x^\infty, y^\infty) = (x_1^\infty, \ldots, x_m^\infty, y_1^\infty, \ldots, y_{n-m}^\infty)$, in which the Poisson structure Π has the desired form. Here $(x^0, y^0) = (x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ is the original linear coordinate system.

For simplicity of exposition, we will assume that Π is C^∞-smooth. However, in every step of the proof of Theorem 3.2.9, we will only use differentiability of Π up to some finite order, and that's why our proof will also work for finitely (sufficiently highly) differentiable Poisson structures.

We will denote by Θ_d the local diffeomorphisms of $(\mathbb{R}^n, 0)$ such that

$$(x^d, y^d)(z) = (x^0, y^0) \circ \Theta_d(z)\,, \tag{3.100}$$

where z denotes a point of $(\mathbb{R}^n, 0)$.

Denote by Π^d the Poisson structure obtained from Π by the action of Θ_d:

$$\Pi^d = (\Theta_d)_* \Pi\,. \tag{3.101}$$

Of course, $\Pi^0 = \Pi$. Denote by $\{.,.\}_d$ the Poisson bracket with respect to the Poisson structure Π^d. Then we have

$$\{F_1, F_2\}_d(z) = \{F_1 \circ \Theta_d, F_2 \circ \Theta_d\}({\Theta_d}^{-1}(z))\,. \tag{3.102}$$

Assume that we have constructed $(x^d, y^d) = (x, y) \circ \Theta_d$. Let us now construct $(x^{d+1}, y^{d+1}) = (x, y) \circ \Theta_{d+1}$. Similarly to the analytic case, this construction consists of two steps : 1) find an "almost Levi factor", i.e. coordinates x_i^{d+1} such that the error terms $\{x_i^{d+1}, x_j^{d+1}\} - \sum_k c_{ij}^k x_k^{d+1}$ are small, and 2) "almost linearize" it, i.e. find the remaining coordinates y^{d+1} such that in the coordinate system (x^{d+1}, y^{d+1}) the Hamiltonian vector fields of the functions x_i^{d+1} are very close to linear ones. In fact, we will define a local diffeomorphism θ_{d+1} of $(\mathbb{R}^n, 0)$ and then put $\Theta_{d+1} = \theta_{d+1} \circ \Theta_d$. In particular, we will have $\Pi^{d+1} = (\theta_{d+1})_* \Pi^d$ and $(x^{d+1}, y^{d+1}) = (x^d, y^d) \circ (\Theta_d)^{-1} \circ \theta_{d+1} \circ \Theta_d$.

Similarly to the analytic case, consider the 2-cochain

$$f^d = \sum_{ij} f_{ij}^d \otimes \xi_i^* \wedge \xi_j^* \tag{3.103}$$

of the Chevalley-Eilenberg complex associated to the $\mathfrak{g}$-module $\mathcal{C}_r$, where now

$$f_{ij}^d(x, y) = \{x_i, x_j\}_d - \sum_{k=1}^m c_{ij}^k x_k, \tag{3.104}$$

and $r = r_d$ depends on d and can be chosen as follows:

$$r_d = 1 + \frac{1}{d+1}. \tag{3.105}$$

In particular, $r_0 = 2$, $r_d/r_{d+1} \sim 1 + \frac{1}{d^2}$, and $\lim_{d\to\infty} r_d = 1$ is positive. This choice of radii r_d means in particular that we will be able to arrange so that the Poisson structure $\Pi^d = (\Theta_d)_* \Pi$ is defined in the closed ball of radius r_d. (For this to hold, we will have to assume that Π is defined in the closed ball of radius 2, and show by recurrence that $B_{r_d} \subset \theta_d(B_{r_{d-1}})$ for all $d \in \mathbb{N}$).

Put

$$\varphi^{d+1} := \sum_i \varphi_i^{d+1} \otimes \xi_i^* = S(t_d)\big(h(f^d)\big)\,, \tag{3.106}$$

where h is the homotopy operator as given in Lemma 3.6.1, S is the smoothing operator and the parameter t_d is chosen as follows: take a real constant $t_0 > 1$

(which later on will be assumed to be large enough) and define the sequence $(t_d)_{d\geq 0}$ by $t_{d+1} = t_d^{3/2}$. In other words, we have

$$t_d = \exp\left(\left(\frac{3}{2}\right)^d \ln t_0\right) \ , \ \ln t_0 > 0 \, . \tag{3.107}$$

The above choice of smoothing parameter t_d is a standard one in problems involving the Nash-Moser method, see, e.g., [**102, 103**]. The number $\frac{3}{2}$ in the above formula is just a convenient choice. The main point is that this number is greater than 1 (so we have a very fast increasing sequence) and smaller than 2 (where 2 corresponds to the fact that we have a fast convergence algorithm which "quadratizes" the error term at each step, i.e. go from an "ε-small" error term to an "ε^2-small" error term).

According to Inequality (3.96), in order to control the C^k-norm of $h(f^d)$ we need to control the C^{k+s}-norm of f^d, i.e. we face a "loss of differentiability". That's why in the above definition of φ^{d+1} we have to use the smoothing operator S, which will allow us to compensate for this loss of differentiability. This is a standard trick in the Nash-Moser method.

Next, consider the 1-cochain

$$\hat{g}^d = \sum_i \left(\sum_\alpha \{x_i - h(f^d)_i, y_\alpha\}_d - \sum_{\beta=1}^{n-m} a_{i\alpha}^\beta y_\beta \frac{\partial}{\partial y_\alpha} \right) \otimes \xi_i^* \tag{3.108}$$

of the Chevalley-Eilenberg complex associated to the $\mathfrak{g}$-module $\mathcal{Y}_r$, where $r = r_d = 1 + \frac{1}{d+1}$, and put

$$\psi^{d+1} := \sum_\alpha \psi_\alpha^{d+1} \frac{\partial}{\partial y_\alpha} = S(t_d)\big(h(\hat{g}^d)\big) \, , \tag{3.109}$$

where h is the homotopy operator as given in Lemma 3.6.1, and $S(t_d)$ is the smoothing operator (with the same t_d as in the definition of φ^{d+1}).

Now define θ_{d+1} to be a local diffeomorphism of $\mathbb{R}^n$ given by

$$\theta_{d+1} := Id - (\varphi_1^{d+1}, \ldots, \varphi_m^{d+1}, \psi_1^{d+1}, \ldots, \psi_{n-m}^{d+1}) \, . \tag{3.110}$$

This finishes our construction of $\Theta_{d+1} = \theta_{d+1} \circ \Theta_d$ and $(x^{d+1}, y^{d+1}) = (x, y) \circ \Theta_{d+1}$. This construction is very similar to the analytic case, except mainly for the use of the smoothing operator. Another difference is that, for technical reasons, in the smooth case we use the original coordinate system and the transformed Poisson structures Π^d for determining the error terms, while in the analytic case the original Poisson structure and the transformed coordinate systems are used. In particular, the closed balls used here are always balls with respect to the original coordinate system – this allows us to easily compare the Sobolev norms of functions on them, i.e. bigger balls correspond to bigger norms.

The technical part of the proof (see [**152**]) consists of a series of lemmas which show that the above construction actually yields a smooth Levi normalization in the limit, provided that Π is defined on the closed ball of radius 2 and is sufficiently close to its linear part there. If Π does not satisfy these conditions, then we may use the following homothety trick to make it satisfy: replace Π by $\Pi^t = \frac{1}{t}G(t)_*\Pi$ where $G(t) : z \mapsto tz$ is a homothety, $t > 0$. The limit $\lim_{t\to\infty} \Pi^t$ is equal to the linear part of Π. So by choosing t high enough, we may assume that Π^t is defined on the closed ball of radius 2 and is sufficiently close to its linear part there. If Θ is a local smooth Levi normalization for Π^t, then $G(1/t) \circ \Theta \circ G(t)$ will be a local smooth Levi normalization for Π.

REMARK 3.6.2. In [**152**] there is also an abstract *Nash–Moser normal form theorem*, which can be applied to the problem of smooth Levi decomposition of Poisson structures, and hopefully to other smooth normal form problems as well.

CHAPTER 4

Linearization of Poisson structures

4.1. Nondegenerate Lie algebras

Let Π be a Poisson structure which vanishes at a point z: $\Pi(z) = 0$. Denote by

$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \dots, \tag{4.1}$$

the Taylor expansion of Π in a local coordinate system centered at z, where $\Pi^{(k)}$ is a homogeneous 2-vector field of degree k. Recall that, the terms of degree k of the equation $[\Pi, \Pi] = 0$ give

$$\sum_{i=1}^{k} \left[\Pi^{(i)}, \Pi^{(k+1-i)}\right] = 0. \tag{4.2}$$

In particular, $[\Pi^{(1)}, \Pi^{(1)}] = 0$, i.e. the linear part $\Pi^{(1)}$ of Π is a linear Poisson structure. One says that Π is locally smoothly (resp. analytically, resp. formally) ***linearizable*** if there is a local smooth (resp. analytic, resp. formal) diffeomorphism ϕ (a coordinate transformation) such that $\phi_* \Pi = \Pi^{(1)}$.

Definition 4.1.1 ([**205**]). A finite-dimensional Lie algebra $\mathfrak{g}$ is called formally (resp. analytically, resp. smoothly) ***nondegenerate*** if any formal (resp. analytic, resp. smooth) Poisson structure Π which vanishes at a point and whose linear part at that point corresponds to $\mathfrak{g}$ is formally (resp. analytically, resp. smoothly) linearizable.

In other words, a Lie algebra is nondegenerate if any Poisson structure, whose linear part corresponds to this algebra, is completely determined by its linear part up to local isomorphisms.

The above definition begs the question: which Lie algebras are nondegenerate and which are degenerate? This question is the main topic of this chapter. One of the main tools for studying it is the Levi decomposition, treated in the previous chapter. The question is still largely open, though we now know of several series of nondegenerate Lie algebras, and many degenerate ones (it is much easier in general to find degenerate Lie algebras than to find nondegenerate ones).

As explained in Chapter 2, formal deformations of Poisson structures are governed by Poisson cohomology, and for linear Poisson structures Poisson cohomology is a special case of Chevalley-Eilenberg cohomology of the corresponding Lie algebras. In particular, if $\mathfrak{l}$ is a Lie algebra such that $H^2(\mathfrak{l}, \mathcal{S}^k \mathfrak{l}) = 0 \ \forall\ k \geq 2$, then it is formally nondegenerate (Theorem 2.3.8). Let us recall here a special case of this result:

Theorem 4.1.2 ([**205**]). *Any semisimple Lie algebra is formally nondegenerate.*

In general, it is much more difficult to study smooth or analytic nondegeneracy of Lie algebras than to study their formal nondegeneracy, because the former problem involves not only algebra (cohomology of Lie algebras) but also geometry and analysis (to show the analyticity or smoothness of coordinate transformations).

The first significant results about analytic and smooth nondegenerate Lie algebras are due to Conn [**53, 54**], and are already mentioned in Chapter 3 as special cases of Levi decomposition theorems. Let us recall here Conn's results.

THEOREM 4.1.3 ([**53**]). *Any semisimple Lie algebra is analytically nondegenerate.*

THEOREM 4.1.4 ([**54**]). *Any compact semisimple Lie algebra is smoothly nondegenerate.*

On the other hand, most non-compact real semisimple Lie algebras are smoothly degenerate (see Section [**207**]).

Related to the notion of (formal) nondegeneracy is the notion of ***rigidity*** of Lie algebras, mentioned in Subsection 2.3.3, and also the notion of ***strong rigidity*** [**25**]. Recall that $H^2(\mathfrak{g},\mathfrak{g})$ is the cohomology group which governs infinitesimal deformations of a Lie algebra $\mathfrak{g}$. This group is somehow related to the group $\oplus_{k\geq 2}H^2(\mathfrak{g},S^k\mathfrak{g})$, but they are not the same. Not surprisingly, there are Lie algebras which are rigid but degenerate (e.g. $\mathfrak{saff}(2)$, see Example 4.1.5), Lie algebras which are non-rigid but nondegenerate (e.g. a 3-dimensional solvable Lie algebra $\mathbb{K}\ltimes_A\mathbb{K}^2$, where A is a nonresonant 2×2 matrix, see Theorem 4.2.2), Lie algebras which are both rigid and nondegenerate (e.g. semisimple Lie algebras), and Lie algebras which are both non-rigid and degenerate (e.g. Abelian Lie algebras).

EXAMPLE 4.1.5. Denote by $\mathfrak{saff}(2,\mathbb{K}) = sl(2,\mathbb{K})\ltimes\mathbb{K}^2$ the Lie algebra of infinitesimal area-preserving affine transformations on $\mathbb{K}^2$. Then $\mathfrak{saff}(2,\mathbb{K})$ is rigid but degenerate. The linear Poisson structure corresponding to $\mathfrak{saff}(2)$ has the form $\Pi^{(1)} = 2e\partial h\wedge\partial e - 2f\partial h\wedge\partial f + h\partial e\wedge\partial f + y_1\partial h\wedge\partial y_1 - y_2\partial h\wedge\partial y_2 + y_1\partial e\wedge\partial y_2 + y_2\partial f\wedge\partial y_1$ in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \tilde{\Pi}$ with $\tilde{\Pi} = (h^2+4ef)\partial y_1\wedge\partial y_2$. Then Π is a Poisson structure, vanishing at the origin, with a linear part corresponding to $\mathfrak{saff}(2)$. For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a codimension 2 linear subspace (given by the equations $y_1 = 0$ and $y_2 = 0$). For Π the set where the rank is less or equal to 2 is a 2-dimensional cone (the cone given by the equations $y_1 = 0$, $y_2 = 0$ and $h^2+4ef = 0$). So these two Poisson structures are not isomorphic, even formally. The rigidity of $\mathfrak{saff}(2)$ will be left to the reader as an exercise (see [**173, 40**]).

EXAMPLE 4.1.6. The Lie algebra $\mathfrak{e}(3) = so(3)\ltimes\mathbb{R}^3$ of rigid motions of the Euclidean space $\mathbb{R}^3$ is degenerate and non-rigid. The linear Poisson structure corresponding to $\mathfrak{e}(3)$ has the form $\Pi^{(1)} = x_1\partial x_2\wedge\partial x_3 + x_2\partial x_3\wedge\partial x_1 + x_3\partial x_1\wedge\partial x_2 + y_1\partial x_2\wedge\partial y_3 + y_2\partial x_3\wedge\partial y_1 + y_3\partial x_1\wedge\partial y_2$ in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \tilde{\Pi}$ with $\tilde{\Pi} = (x_1^2+x_2^2+x_3^2)(x_1\partial x_2\wedge\partial x_3 + x_2\partial x_3\wedge\partial x_1 + x_3\partial x_1\wedge\partial x_2)$. For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a dimension 3 subspace (given by the equation $y_1 = y_2 = y_3 = 0$), while for Π the set where the rank is less or equal to 2 is the origin. A Lie algebra not isomorphic to $\mathfrak{e}(3)$ but adjacent to $\mathfrak{e}(3)$ is $so(3,1)$, the Lie algebra of infinitesimal linear automorphisms of the Minkowski space. Here the adjective *adjacent* means that, in the variety of all Lie algebra structures of dimension 6 (see Subsection 2.3.3), the $GL(6)$-orbit which corresponds to $\mathfrak{e}(3)$ lies in the closure (with respect to the Euclidean topology) of the $GL(6)$-orbit which corresponds to $so(3,1)$. One also says that $e(3)$ is a contraction of $so(3,1)$. If a Lie algebra is a contraction of another Lie algebra, then it is not rigid.

A ***strongly rigid Lie algebra*** is a Lie algebra $\mathfrak{g}$ whose universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is rigid as an associative algebra [**25**]. It is easy to see that if $\mathfrak{g}$ is strongly rigid then it is rigid. A sufficient condition for $\mathfrak{g}$ to be strongly rigid is $H^2(\mathfrak{g}, S^k\mathfrak{g}) = 0 \ \forall\, k \geq 0$, and if this condition is satisfied then $\mathfrak{g}$ is called ***infinitesimally strongly rigid*** [**25**]. Obviously, if $\mathfrak{g}$ is infinitesimally strongly rigid, then it is formally nondegenerate. In fact, we have the following result, due to Bordemann, Makhlouf and Petit:

THEOREM 4.1.7 ([**25**]). *If $\mathfrak{g}$ is a strongly rigid Lie algebra then it is formally nondegenerate.*

We refer to [**25**] for the proof of the above theorem, which is based on Kontsevich's theorem [**121**] on the existence of deformation quantization of Poisson structures.

4.2. Linearization of low-dimensional Poisson structures

4.2.1. Two-dimensional case.

Up to isomorphisms, there are only two Lie algebras of dimension 2: the trivial, i.e. Abelian one, and the solvable Lie algebra $\mathbb{K} \ltimes \mathbb{K}$, which has a basis (e_1, e_2) with $[e_1, e_2] = e_1$. This Lie algebra is isomorphic to the Lie algebra of infinitesimal affine transformations of the line, so we will denote it by $\mathfrak{aff}(1)$.

The Abelian Lie algebra of dimension 2 is of course degenerate. For example, the quadratic Poisson structure $(x_1^2 + x_2^2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ is non-trivial and is not locally isomorphic to its linear part, which is trivial.

On the other hand, we have:

THEOREM 4.2.1 ([**5**]). *The Lie algebra $\mathfrak{aff}(1)$ is formally, analytically and smoothly nondegenerate.*

Proof. We begin with $\{x, y\} = x + \ldots$. Putting $x' = \{x, y\}, y' = y$, we obtain $\{x', y'\} = \frac{\partial x'}{\partial x}\{x, y\} = x'a(x', y')$, where a is a function such that $a(0) = 1$. We finish with a second change of coordinates $x'' = x', y'' = f(x', y')$, where f is a function such that $\frac{\partial f}{\partial y'} = 1/a$. $\square$

4.2.2. Three dimensional case.

Every Lie algebra of dimension 3 over $\mathbb{R}$ or $\mathbb{C}$ is of one of the following three types, where (e_1, e_2, e_3) denote a basis:

- $so(3)$ with brackets $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$.
- $sl(2)$ with brackets $[e_1, e_2] = e_3, [e_1, e_3] = e_1, [e_2, e_3] = -e_2$. (Recall that $so(3, \mathbb{R}) \not\cong sl(2, \mathbb{R})$, $so(3, \mathbb{C}) \cong sl(2, \mathbb{C})$).

- semi-direct products $\mathbb{K} \ltimes_A \mathbb{K}^2$ where $\mathbb{K}$ acts linearly on $\mathbb{K}^2$ by a matrix A. In other words, we have brackets $[e_2, e_3] = 0, [e_1, e_2] = ae_2 + be_3, [e_1, e_3] = ce_2 + de_3$, and A is the 2×2-matrix with coefficients a, b, c and d. (Different matrices A may correspond to isomorphic Lie algebras).

The Lie algebras $sl(2)$ and $so(3)$ are simple, so they are formally and analytically nondegenerate, according to Weinstein's and Conn's theorems.

The fact that the compact simple Lie algebra $so(3,\mathbb{R})$ is smoothly nondegenerate (it is a special case of Conn's Theorem 4.1.4) is due to Dazord [**62**]. In Chapter **??**, we will extend this result of Dazord to the case of elliptic singularities of Nambu structures, using arguments similar to his.

On the other hand, $sl(2,\mathbb{R})$ is known to be smoothly degenerate (see [**205**]). A simple construction of a smooth non-linearizable Poisson structure whose linear part corresponds to $sl(2,\mathbb{R})$ is as follows: In a linear coordinate system (y_1, y_2, y_3), write

$$\Pi^{(1)} = y_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_3} - y_1 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} = X \wedge Y,$$

where $X = y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2}$, and $Y = \frac{\partial}{\partial y_1} + \frac{y_1}{y_2^2+y_3^2}\left(y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3}\right)$. This linear Poisson structure corresponds to $sl(2,\mathbb{R})$ and has $C = y_2^2 + y_3^2 - y_1^2$ as a Casimir function. Denote by Z a vector field on $\mathbb{R}^3$ such that $Z = 0$ when $y_2^2 + y_3^2 - y_1^2 \geq 0$, and $Z = \frac{\sqrt{G(C)}}{\sqrt{y_2^2+y_3^2}}\left(y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3}\right)$ when $y_2^2 + y_3^2 - y_1^2 > 0$, where G is a flat function such that $G(0) = 0$ and $G(C) > 0$ when $C > 0$. Then Z is a flat vector field such that $[Z, X] = [Z, Y] = 0$. Hence $\Pi = X \wedge (Y - Z)$ is a Poisson structure whose linear part is $\Pi^{(1)} = X \wedge Y$. While Y is a periodic vector field, the integral curves of $Y - Z$ in the region $\{y_2^2 + y_3^2 - y_1^2 > 0\}$ are spiraling towards the cone $\{y_2^2 + y_3^2 - y_1^2 = 0\}$. Thus, while almost all the symplectic leaves of $\Pi^{(1)}$ are closed, the symplectic leaves of Π in the region $\{y_2^2 + y_3^2 - y_1^2 > 0\}$ contain the cone $\{y_2^2 + y_3^2 - y_1^2 = 0\}$ in their closure (also locally in a neighborhood of 0). This implies that the symplectic foliation of Π is not locally homeomorphic to the symplectic foliation of $\Pi^{(1)}$. Hence Π can't be locally smoothly equivalent to $\Pi^{(1)}$.

For solvable Lie algebras $\mathbb{K} \ltimes_A \mathbb{K}^2$, we have the following result:

THEOREM 4.2.2 ([**69**]). *The Lie algebra $\mathbb{R}^2 \times_A \mathbb{R}$ is smoothly (or formally) nondegenerate if and only if A is nonresonant in the sense that there are no relations of the type*

$$\lambda_i = n_1\lambda_1 + n_2\lambda_2 \quad (i = 1 \text{ or } 2), \tag{4.3}$$

where λ_1 and λ_2 are the eigenvalues of A, n_1 and n_2 are two nonnegative integers with $n_1 + n_2 > 1$.

Proof. Let Π be a Poisson structure on a 3-dimensional manifold which vanishes at a point with a linear part corresponding to $\mathbb{R}^2 \times_A \mathbb{R}$ with a nonresonant A. In a system of local coordinates (x, y, z), centered at the considered point, we have

$$\{z, x\} = ax + by + O(2), \quad \{z, y\} = cx + dy + O(2), \quad \{x, y\} = O(2), \tag{4.4}$$

where a, b, c, d are the coefficients of A, and $O(2)$ means terms of degree at least 2. It follows that the curl vector field $D_\Omega \Pi$ (see Section 2.5), with respect to any volume form Ω, has the form $(a + c)\partial/\partial z + Y$, where Y is a vector field vanishing at the origin. But the non-resonance hypothesis imposes that the trace of A is not zero; so $D_\Omega\Pi$ doesn't vanish in a neighborhood of the origin. We can straighten it and suppose that the coordinates (x, y, z) are chosen such that $D_\Omega\Pi = \partial z$.

Now the 3-dimensional hypothesis implies $\Pi \wedge \Pi = 0$ and, using formula

$$[\Pi, \Pi] = D_\Omega(\Pi \wedge \Pi) \pm D_\Omega(\Pi) \wedge \Pi$$

(see Section 2.5), we obtain

$$D_\Omega(\Pi) \wedge \Pi = 0.$$

In the above coordinates this gives $\partial/\partial z \wedge \Pi = 0$ and, so,

$$\Pi = \partial/\partial z \wedge X,$$

where X is a vector field. Now we recall the basic formula

$$[D_\Omega(\Pi), \Pi] = 0$$

which have the consequence that we can suppose that X depends only on the coordinates x and y.

Because of the form of the linear part of Π, X is a vector field which vanishes at the origin but with a nonresonant linear part. Hence, up to a smooth (or formal) change of coordinates x and y, we can linearize X (see Appendix **??**). This gives the smooth (or formal) nondegeneracy of $\mathbb{R}^2 \times_A \mathbb{R}$.

To prove the "only if" part, we start with a linear Poisson structure

$$\Pi^{(1)} = \partial/\partial z \wedge X^{(1)},$$

where $X^{(1)}$ is a linear resonant vector field. Every resonance relation permits the construction of a polynomial perturbation X of $X^{(1)}$ which is not smoothly isomorphic to $X^{(1)}$, even up to a product with a function (see Appendix **??**). Then it is not difficult to prove that $\partial/\partial z \wedge X$ is a polynomial perturbation of $\Pi^{(1)}$ which is not equivalent to it. □

Remark 4.2.3. The same proof shows that algebras of type $\mathbb{K}^2 \times_A \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are analytic nondegenerate if we add to the non-resonance condition a Diophantine condition on the eigenvalues of A (see Appendix **??**).

4.2.3. Four-dimensional case.

The results on (non)degeneracy of 4-dimensional Lie algebras presented in this subsection are taken from Molinier's thesis [**149**].

According to [**170**], every 4-dimensional Lie algebra over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, belongs to (at least) one of the following four types:

Type 1: direct products $\mathbb{K} \times L_3$ where L_3 is a 3-dimensional algebra (see the preceding paragraph for a classification)

Type 2: semi-direct products $\mathbb{K} \ltimes_A \mathbb{K}^3$, where $\mathbb{K}^3$ is the commutative Lie algebra of dimension 3, and $\mathbb{K}$ acts on $\mathbb{K}^3$ by a matrix A.

Type 3: semi-direct products $\mathbb{K} \ltimes_A H_3$, where H_3 is the 3-dimensional Heisenberg Lie algebra: H_3 has a basis (x, y, z) such that $[x, y] = z, [x, z] = [y, z] = 0$.

Type 4: semi-direct products $\mathbb{K}^2 \ltimes \mathbb{K}^2$.

In the first type we have $\mathbb{K} \times sl(2)$ and $\mathbb{K} \times so(3)$, which are the same when $\mathbb{K} = \mathbb{C}$, and the cases where L_3 is a semi-direct product $\mathbb{K} \ltimes \mathbb{K}^2$. The Levi decomposition theorems from Chapter 3 imply that $\mathbb{K} \times sl(2)$ and $\mathbb{K} \times so(3)$ are formally and analytically nondegenerate, and that $\mathbb{R} \times so(3, \mathbb{R})$ is smoothly nondegenerate. However, $\mathbb{R} \times sl(2, \mathbb{R})$ is smoothly degenerate: just repeat the proof, given in the previous subsection, of the fact that $sl(2)$ is smoothly degenerate. The case where L_3 is a semi-direct product is degenerate (formally, analytically and smoothly): if we choose coordinates (u, v, x, y) such that the corresponding linear Poisson tensor has the form

$$(ax + by)\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial v} \wedge \frac{\partial}{\partial y}, \tag{4.5}$$

then we can add a quadratic term $v^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$ to get a non-linearizable Poisson tensor.

Every algebra of the second type is degenerate. To prove this we choose coordinates (u, x_1, x_2, x_3) such that in these coordinates the corresponding linear Poisson structure has brackets $\{u, x_i\} = \sum_j a_i^j x_j$, (the others brackets are trivial), where a_i^j are the coefficients of the matrix A. Up to isomorphisms, we can suppose that A is in Jordan form and we can also replace A by λA where λ is any non vanishing constant. So we have the following list of cases:

$$\{u, x_1\} = 0, \quad \{u, x_2\} = x_1, \quad \{u, x_3\} = x_2; \tag{4.6}$$

$$\{u, x_1\} = ax_1, \quad \{u, x_2\} = x_2, \quad \{u, x_3\} = x_2 + x_3; \tag{4.7}$$

$$\{u, x_1\} = x_1, \quad \{u, x_2\} = 0, \quad \{u, x_3\} = x_2; \tag{4.8}$$

$$\{u, x_1\} = x_1, \quad \{u, x_2\} = x_1 + x_2, \quad \{u, x_3\} = x_2 + x_3; \tag{4.9}$$

$$\{u, x_1\} = x_1, \quad \{u, x_2\} = ax_2, \quad \{u, x_3\} = bx_3; \tag{4.10}$$

$$\{u, x_1\} = ax_1, \quad \{u, x_2\} = bx_2 - x_3, \quad \{u, x_3\} = x_2 + bx_3. \tag{4.11}$$

Each of the above cases can be perturbed to a non-linearizable Poisson structure by adding a quadratic term: $\partial x_3 \wedge (x_1^2 \partial x_1 + x_1 x_2 \partial x_2)$ for (4.6); $x_2^2 \partial x_3 \wedge \partial x_2$ for (4.7) and (4.8); $x_1^2 \partial x_3 \wedge \partial x_2$ for (4.9); $x_2 x_3 \partial x_3 \wedge \partial x_2$ for (4.10); and $(x_2^2 + x_3^2)\partial x_3 \wedge \partial x_2$ for (4.11).

Similarly, every algebra of the third type is also degenerate. To prove this we choose coordinates (u, x_1, x_2, x_3) such that the corresponding linear Poisson structure has brackets $\{x_3, x_2\} = x_1$, $\{u, x_i\} = \sum_j a_i^j x_j$, (the other brackets are zero), where a_i^j are the coefficients of the matrix A. Up to isomorphisms, we have the following cases:

$$\{u, x_1\} = 2x_1, \quad \{u, x_2\} = x_2, \quad \{u, x_3\} = x_2 + x_3; \tag{4.12}$$

$$\{u, x_1\} = (1 + b)x_1, \quad \{u, x_2\} = x_2, \quad \{u, x_3\} = bx_3; \tag{4.13}$$

$$\{u, x_1\} = 2ax_1, \quad \{u, x_2\} = ax_2 - x_3, \quad \{u, x_3\} = x_2 + ax_3. \tag{4.14}$$

Each case can be perturbed to a non-linearizable Poisson structure by adding a quadratic term: $x_2^2 \partial x_3 \wedge \partial x_2$ for (4.12); $x_2 x_3 \partial x_3 \wedge \partial x_2$ for (4.13); and $(x_2^2 + x_3^2)\partial x_3 \wedge \partial x_2$ for (4.14).

Finally, if $\mathfrak{g}$ is a Lie algebra of the last type, which does not belong to the previous three types, then it admits a basis (u, v, x, y), with either the brackets

$$[u, x] = x, \quad [v, y] = y; \tag{4.15}$$

or the brackets

$$[u,x]=x, \quad [u,y]=y \quad ,[v,x]=-y, \quad [v,y]=x. \tag{4.16}$$

When $\mathbb{K}=\mathbb{C}$ then the above two cases are the same and are isomorphic to $\mathfrak{aff}(1,\mathbb{C})\times\mathfrak{aff}(1,\mathbb{C})$. When $\mathbb{K}=\mathbb{R}$, these are the two real versions of $\mathfrak{aff}(1,\mathbb{C})\times\mathfrak{aff}(1,\mathbb{C})$. We will see in Section 4.3 that (4.15) is formally, analytically and smoothly nondegenerate, and therefore (4.16) is also formally and analytically nondegenerate. We don't know whether (4.16) is smoothly nondegenerate.

4.3. Nondegeneracy of $\mathfrak{aff}(n)$

As an application of the Levi decomposition, we will prove the following:

THEOREM 4.3.1 ([**75**]). *For any natural number n, the Lie algebra $\mathfrak{aff}(n,\mathbb{K})=\mathfrak{gl}(n,\mathbb{K})\ltimes\mathbb{K}^n$ of affine transformations of $\mathbb{K}^n$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, is formally and analytically nondegenerate.*

Proof. We will prove the above theorem in the analytic case. The formal case is absolutely similar, if not simpler. Denote by $\mathfrak{l}=\mathfrak{g}\ltimes\mathfrak{r}$ a Levi decomposition for a (real or complex) Lie algebra $\mathfrak{l}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{r}$ is the solvable radical of $\mathfrak{l}$. Let Π be an analytic Poisson structure vanishing at a point 0 in a manifold whose linear part at 0 corresponds to $\mathfrak{l}$. According to the analytic Levi decomposition theorem 3.2.6, there exists a local analytic system of coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_d)$ in a neighborhood of 0, where $m=\dim\mathfrak{g}$ and $d=\dim\mathfrak{r}$, such that in these coordinates we have

$$\{x_i,x_j\}=\sum c_{ij}^k x_k \ , \quad \{x_i,y_r\}=\sum a_{ir}^s y^s \ , \tag{4.17}$$

where c_{ij}^k are structural constants of $\mathfrak{s}$ and a_{ir}^s are constants. This gives what we call a semi-linearization for Π. Note that the remaining Poisson brackets $\{y_r,y_s\}$ are nonlinear in general.

We now restrict our attention to the case where $\mathfrak{l}=\mathfrak{aff}(n)$, $m=n^2-1$, $d=n+1$, $\mathfrak{g}=\mathfrak{sl}(n),\mathfrak{r}=\mathbb{R}(\mathrm{Id})\ltimes\mathbb{K}^n$ where Id acts on $\mathbb{K}^n$ by the identity map. The following lemma says that we may have a semi-linearization associated to the decomposition $\mathfrak{aff}(n)=\mathfrak{gl}(n)\ltimes\mathbb{K}^n$ (which is slightly better than the Levi decomposition).

LEMMA 4.3.2. *There is a local analytic coordinate system*

$$(x_1,\ldots,x_{n^2-1},y_0,y_1,\ldots,y_n)$$

which satisfy Relations (4.17), with the following extra properties: $\{y_0,y_r\}=y_r$ for $r=1,\ldots,n$; $\{x_i,y_0\}=0\ \forall i$.

Proof. We can assume that the coordinates y_r are chosen so that Relations (4.17) are already satisfied, and y_0 corresponds to Id in $\mathbb{R}(\mathrm{Id})\ltimes\mathbb{K}^n$. Then the Hamiltonian vector fields X_{x_i} are linear and form a linear action of $\mathfrak{sl}(n)$. Because of (4.17), we have that $\{x_i,y_0\}=0$, which implies that $[X_{x_i},X_{y_0}]=0$, i.e. X_{y_0} is invariant under the $\mathfrak{sl}(n)$ action. Moreover we have $X_{y_0}(x_i)=0$ (i.e. X_{y_0} does not contain components $\partial/\partial x_i$), and $X_{y_0}=\sum_1^n y_i\partial/\partial y_i+$ nonlinear terms. Hence we can use (the parametrized equivariant version of) Poincaré linearization theorem to linearize X_{y_0} in a $\mathfrak{sl}(n)$-invariant way. After this linearization, we have that $X_{y_0} = \sum_1^n y_i \partial/\partial y_i$. In other words, Relations (4.17) are still satisfied, and moreover we have $\{y_0, y_i\} = X_{y_0}(y_i) = y_i$. □

REMARK 4.3.3. Lemma 4.3.2 still holds if we replace $\mathfrak{aff}(n)$ by any Lie algebra of the type $(\mathfrak{g} \oplus \mathbb{K}e_0) \ltimes \mathfrak{n}$ where $\mathfrak{g}$ is semisimple and e_0 acts on $\mathfrak{n}$ by the identity map (or any matrix whose corresponding linear vector field is nonresonant and satisfies a Diophantine condition).

We will redenote y_0 in Lemma 4.3.2 by x_{n^2}. Then Relations (4.17) are still satisfied. We will work in a coordinate system $(x_1, \ldots, x_{n^2}, y_1, \ldots, y_n)$ provided by this lemma . We will fix the variables $x_1, \ldots, x_{n^2}$, and consider them as linear functions on $\mathfrak{gl}^*(n)$ (they give a Poisson projection from our (n^2+n)-dimensional space to $\mathfrak{gl}^*(n)$). Denote by $F_1, \ldots, F_n$ the n basic Casimir functions for $\mathfrak{gl}^*(n)$. (If we identify $\mathfrak{gl}(n)$ with its dual via the Killing form, then $F_1, \ldots, F_n$ are basic symmetric functions of the eigenvalues of $n \times n$ matrices). We will consider $F_1(x), \ldots, F_n(x)$ as functions in our (n^2+n)-dimensional space, which do not depend on variables y_i. Denote by $X_1, \ldots, X_n$ the Hamiltonian vector fields of $F_1, \ldots, F_n$.

LEMMA 4.3.4. *The vector fields $X_1, \ldots, X_n$ do not contain components $\partial/\partial x_i$. They form a system of n linear commuting vector fields on $\mathbb{K}^n$ (the space of $y = (y_1, \ldots, y_n)$) with coefficients which are polynomial in $x = (x_1, \ldots, x_{n^2})$. The set of x such that they are linearly dependent everywhere in $\mathbb{K}^n$ is an analytic space of complex codimension strictly greater than 1 (when $\mathbb{K} = \mathbb{C}$).*

Proof. The fact that the X_i are y-linear with x-polynomial coefficients follows directly from Relations (4.17). Since F_i are Casimir functions for $\mathfrak{gl}(n)$, we have $X_i(x_k) = \{F_i, x_k\} = 0$, and $[X_i, X_j] = X_{\{F_i,F_j\}} = 0$.

One checks that, for a given x, $X_1 \wedge \cdots \wedge X_n = 0$ identically on $\mathbb{K}^n$ if and only if x is a singular point for the map $(F_1, \ldots, F_n)$ from $\mathfrak{gl}^*(n)$ to $\mathbb{K}^n$. The set of singular points of the map $(F_1, \ldots, F_n)$ in the complex case is of codimension greater than 1 (in fact, it is of codimension 3). □

LEMMA 4.3.5. *Write the Poisson structure Π in the form $\Pi = \Pi^{(1)} + \tilde{\Pi}$, where $\Pi^{(1)}$ is the linear part and $\tilde{\Pi}$ denote the higher order terms. Then $\tilde{\Pi}$ is a Poisson structure which can be written in the form*

$$\tilde{\Pi} = \sum_{i<j} f_{ij} X_i \wedge X_j \ , \tag{4.18}$$

where the functions f_{ij} are analytic functions which depend only on the variables x, and they are Casimir functions for $\mathfrak{gl}^*(n)$ (if we consider the variables x as linear functions on $\mathfrak{gl}^*(n)$).

Proof. We work first locally near a point (x, y) where the vector fields X_k are linearly independent point-wise. As $\tilde{\Pi}$ is a 2-vector field in $\mathbb{K}^n = \{y\}$ (with coefficient depending on x) we have a local formula $\tilde{\Pi} = \sum_{i<j} f_{ij} X_i \wedge X_j$ where f_{ij} are analytic functions in variables (x, y). Since X_k are Hamiltonian vector fields for Π and also for $\Pi^{(1)}$, we have $[X_k, \tilde{\Pi}] = [X_k, \Pi] - [X_k, \Pi^{(1)}] = 0$ for $k = 1, \ldots, n$. This leads to $X_k(f_{ij}) = 0 \, \forall \, k, i, j$. Hence, because the X_k generate $\mathbb{K}^n$, the functions f_{ij} are locally independent of y. Using analytic extension, Hartog's theorem and

the fact that the set of x such that $X_1, \dots, X_n$ are linearly dependent point-wise everywhere in $\mathbb{K}^n$ is of complex codimension greater than 1, we obtain that f_{ij} are local analytic functions in a neighborhood of 0 which depend only on the variables x. The fact that $\tilde{\Pi}$ is a Poisson structure, i.e. $[\tilde{\Pi}, \tilde{\Pi}] = 0$, is now evident, because $X_k(f_{ij}) = 0$ and $[X_i, X_j] = 0$.

Relations $[X_{x_k}, \tilde{\Pi}] = [X_{x_i}, \Pi] - [X_{x_i}, \Pi^{(1)}] = 0$ imply that $X_{x_k}(f_{ij}) = 0$, which means that f_{ij} are Casimir functions for $\mathfrak{gl}^*(n)$. □

REMARK 4.3.6. Lemma 4.3.5 is still valid in the formal case. In fact, every homogeneous component of $\tilde{\Pi}$ satisfies a relation of type (4.18).

LEMMA 4.3.7. *There exists a vector field Y of the form $Y = \sum_{i=1}^n \alpha_i X_i$, where the analytic functions α_i depend only on the variables x and are Casimir functions for $\mathfrak{gl}^*(n)$, such that*

$$[Y, \Pi^{(1)}] = -\tilde{\Pi}\ , \quad [Y, \tilde{\Pi}] = 0. \tag{4.19}$$

Proof. Since the functions f_{ij} of Lemma 4.3.5 are analytic Casimir functions for $\mathfrak{gl}(n)$, we have $f_{ij} = \phi_{ij}(F_1, \dots, F_n)$ where $\phi_{ij}(z_1, \dots, z_n)$ are analytic functions of n variables. On the other hand, since $\Pi^{(1)}, \tilde{\Pi}$ and $\Pi = \Pi^{(1)} + \tilde{\Pi}$ are Poisson structures, they are compatible, i.e. we have $[\Pi^{(1)}, \tilde{\Pi}] = 0$. Decomposing this relation, we get $\frac{\partial \phi_{ij}}{\partial z_k} + \frac{\partial \phi_{jk}}{\partial z_i} + \frac{\partial \phi_{ki}}{\partial z_j} = 0\ \forall\ i, j, k$. This is equivalent to the fact that the 2-form $\phi := \sum_{ij} \phi_{ij} dz_i \wedge dz_j$ is closed. By Poincaré's lemma we get $\phi = d\alpha$ with an 1-form $\alpha = \sum_i \alpha_i dz_i$. Then we put $Y := \sum_i \alpha_i(F_1, \dots, F_n) X_i$. An elementary calculation proves that Y is the desired vector field. □

Return now to the proof of Theorem 4.3.1. Consider a path of Poisson structures given by $\Pi_t := \Pi^{(1)} + t\tilde{\Pi}$. As we have $[Y, \Pi_t] = \tilde{\Pi} = \frac{d}{dt}\Pi_t$, the time-1 map of the vector field Y moves $\Pi^{(1)} = \Pi_0$ into $\Pi = \Pi_1$. This shows that Π is locally analytically linearizable, thus proving our theorem. □

REMARK 4.3.8. For any $n \in \mathbb{N}$, the algebra $\mathfrak{aff}(n)$ is a ***Frobenius Lie algebra***, in the sense that its coadjoint representation has an open orbit. In other words, its corresponding linear Poisson structure has rank equal to the dimension of the algebra almost everywhere. One may think that there must be some links between the nondegeneracy and the property of being a Frobenius Lie algebra. Unfortunately, the search for new nondegenerate Lie algebras among Frobenius Lie algebras, carried out by Wade and Zung [**202**], didn't bring up any new nondegenerate example so far, though some of the degenerate Frobenius Lie algebras turn out to be finitely determined (see Subsection **??**).

In the above proof of Theorem 4.3.1, we implicitly showed that

$$H^2_{CE}(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0\ \ \forall\ k \geq 2, \tag{4.20}$$

where $\mathcal{S}^k$ denotes the symmetric product of order k and H_{CE} denotes the Chevalley–Eilenberg cohomology (it is hidden in the last two lemmas). A purely algebraic proof of this fact was obtained independently by Bordemann, Makhlouf and Petit in [**25**], who showed that $\mathfrak{aff}(n)$ is infinitesimally strongly rigid, i.e.

$$H^2_{CE}(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0\ \ \forall\ k \geq 0. \tag{4.21}$$

They also verified that

$$H^1_{CE}(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0 \ \forall\ k \geq 1 \tag{4.22}$$

and

$$H^1_{CE}(\mathfrak{aff}(n), \mathbb{K}) = \mathbb{K}. \tag{4.23}$$

Since $\mathfrak{aff}(n)$ is a Frobenius Lie algebra, we also have

$$H^0_{CE}(\mathfrak{aff}(n), \mathcal{S}^k(\mathfrak{aff}(n))) = 0 \ \forall\ k \geq 1 \tag{4.24}$$

(geometrically, it means that $\mathfrak{aff}^*(n)$ can't admit a homogeneous Casimir function of degree $k \geq 1$, because it has an open symplectic leaf), and

$$H^0_{CE}(\mathfrak{aff}(n), \mathbb{K}) = \mathbb{K}. \tag{4.25}$$

THEOREM 4.3.9. *Any finite direct sum $\mathfrak{l} = \bigoplus \mathfrak{l}_i$, where each $\mathfrak{l}_i$ is either simple or isomorphic to $\mathfrak{aff}(n_i)$ for some $n_i \in \mathbb{N}$, is formally nondegenerate.*

Proof (sketch). Using the above formulas, Whitehead's lemmas, and Hochschild–Serre spectral sequence, one can show that $H^2_{CE}(\mathfrak{l}, S^k\mathfrak{l}) = 0$ for any $k \geq 1$. □

REMARK 4.3.10. The Lie algebra $\mathfrak{aff}(n) \oplus \mathfrak{g}$ ($n \in \mathbb{N}$, $\mathfrak{g}$ semisimple) is infinitesimally strongly rigid, but the Lie algebra $\mathfrak{aff}(n_1) \oplus \mathfrak{aff}(n_2)$ ($n_1, n_2 \in \mathbb{N}$) is not infinitesimally strongly rigid, because $H^2_{CE}(\mathfrak{aff}(n_1) \oplus \mathfrak{aff}(n_2), \mathbb{K}) = \mathbb{K}$ (see [**25**]).

CONJECTURE 4.3.11. Any finite direct sum $\mathfrak{l} = \bigoplus \mathfrak{l}_i$, where each $\mathfrak{l}_i$ is either simple or isomorphic to $\mathfrak{aff}(n_i)$ for some $n_i \in \mathbb{N}$, is analytically nondegenerate.

Theorem 4.3.1 shows that the above conjecture is true when $\mathfrak{l} = \mathfrak{aff}(n)$. It is also true when $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{aff}(n)$, $\mathfrak{g}$ being semisimple, with the same proof. Another case where we know that the conjecture is true is the following:

THEOREM 4.3.12 (Dufour–Molinier [**71**]). *The direct product*

$$\mathfrak{aff}(1, \mathbb{K}) \times \ldots \times \mathfrak{aff}(1, \mathbb{K})$$

of n copies of $\mathfrak{aff}(1, \mathbb{K})$ is formally and analytically nondegenerate for any natural number n.

Proof (sketch). Denote $\mathfrak{l} = \mathfrak{aff}(1) \times \ldots \times \mathfrak{aff}(1)$ (n times). As discussed about, simple direct computations show that $H^2_{CE}(\mathfrak{l}, \mathcal{S}^k\mathfrak{l}) = 0 \ \forall\ k \geq 2$, so $\mathfrak{l}$ is formally nondegenerate.

Consider now an analytic Poisson structure Π whose linear part $\Pi^{(1)}$ corresponds to $\mathfrak{l}$. Consider the set

$$\Sigma = \{x \in (\mathbb{K}^{2n}, 0) \mid \operatorname{rank} \Pi(x) < 2n\}$$

of singular points of Π. This set is given by the analytic equation

$$\det(\Pi_{ij}(x))_{i,j=1}^{2n} = 0\ ,$$

where Π_{ij} are the coefficients of Π in a coordinate system.

When Π is linear, Σ is just a union of n hyperplanes in $\mathbb{K}^{2n}$ in generic position. Since Π is formally linearizable, there are n formal hyperplanes in generic position which are formal solutions of $\det(\Pi_{ij}(x))_{i,j=1}^{2n} = 0$. Applying Artin's theorem [**8**] about approximation of formal solutions of analytic equations by analytic solutions, we obtain that the equation $\det(\Pi_{ij}(x))_{i,j=1}^{2n} = 0$ admits n local hypersurfaces in generic position near 0 as its solutions. Thus, locally, Σ is a union of n analytic hypersurfaces. So there is a local analytic coordinate system $(x_1, y_1, \dots, x_n, y_n)$ such that

$$\Sigma = \bigcup_{i=1}^{n} \{x_i = 0\} .$$

It is easy to see that in such a coordinate system we have

$$\{x_i, x_j\} = x_i x_j \sigma_{ij}(x, y), \{x_i, y_j\} = x_i \beta_{ij}(x, y) .$$

A first change of coordinates of the type $x'_i = x_i \nu_i(x, y)$, $y'_j = y_j$ leads to $\{x_i, x_j\} = 0$. Then another change of coordinates of the type $x'_i = x_i$, $y'_j = y_j + b_j(x, y)$ gives $\{x_i, y_j\} = \delta_{ij} x_i$. Finally, we obtain $\{y_i, y_j\} = 0$ after a change of coordinates of the type $x'_i = x_i$, $y'_j = y_j + c_j(x)$. $\square$

REMARK 4.3.13. It is also shown in [**71**] that $\mathfrak{aff}(1, \mathbb{R}) \times \mathfrak{aff}(1, \mathbb{R})$ is C^∞-nondegenerate. We don't know if other Lie algebras of the type $\bigoplus_i \mathfrak{aff}(n_i)$ are smoothly nondegenerate or not.

APPENDIX A

A.1. Moser's path method

A smooth time dependent vector field $X = (X_t)_{t\in]a,b[}$ on a manifold M is a smooth path of smooth vector fields X_t on M, parametrized by a parameter t (the *time parameter*) taken in some interval $]a,b[$. Any smooth path $(\phi_t)_{t\in]a,b[}$ of diffeomorphisms determines a time dependent vector field X by the formula

$$X_t(\phi_t(x)) = \frac{\partial \phi_t}{\partial t}(x). \tag{A.1}$$

Conversely, the classical theory of ordinary differential equations says that, given a smooth time dependent vector field $X = (X_t)_{t\in]a,b[}$, in a neighborhood of any (x_0, t_0) in $M\times]a,b[$, we can define a unique smooth map $(x,t) \mapsto \phi_t(x)$, called the ***flow*** of X starting at time t_0, with $\phi_{t_0}(x) \equiv x$ and which satisfies Equation (A.1). Because ϕ_{t_0} is locally the identity, ϕ_t are local diffeomorphisms for t sufficiently near t_0. In some circumstances, for example when X has compact support, these local diffeomorphisms extend to global diffeomorphisms. Taking then $t_0 = 0$, we can get by this procedure a path of (local) diffeomorphisms $(\phi_t)_t$ with $\phi_0 = \mathrm{Id}$.

Suppose that we want to prove that two tensors Λ and Λ', on a manifold M, are isomorphic, i.e. we want show the existence of a diffeomorphism ϕ of the ambient manifold such that

$$\phi^*(\Lambda') = \Lambda. \tag{A.2}$$

Sometimes, this isomorphism problem can be solved with the help of the ***Moser's path method***, which consists of the following:

- First, construct an adapted smooth path $(\Lambda_t)_{t\in[0,1]}$ of such tensors, with $\Lambda_0 = \Lambda$ and $\Lambda_1 = \Lambda'$.
- Second, try to construct a smooth path $(\phi_t)_{t\in[0,1]}$ of diffeomorphisms of M with $\phi_0 =$Id and

$$\phi_t^*(\Lambda_t) = \Lambda_0 \quad \forall\ t \in [0,1], \tag{A.3}$$

or equivalently,

$$\partial(\phi_t^*(\Lambda_t))/\partial t = 0 \quad \forall\ t \in [0,1]. \tag{A.4}$$

One tries to construct a time-dependent vector field $X = (X_t)$ whose flow (starting at time 0) gives ϕ_t. Equation (A.4) is then translated to the following equation on (X_t):

$$\mathcal{L}_{X_t}(\Lambda_t) = -\frac{\partial \Lambda_t}{\partial t}. \tag{A.5}$$

If this last equation can be solved, then one can define ϕ_t by integrating X, and $\phi = \phi_1$ will solve Equation (A.2).

The Moser's path method works best for local problems because time-dependent vector fields can always be integrated locally. In global problems, one usually needs an additional compactness condition to assure that $\phi = \phi_1$ is globally defined.

The method is named after Jurgen Moser, who first used it to prove the following result:

THEOREM A.1.1 ([**153**]). *Let ω and ω' be two volume forms on a manifold M which coincide everywhere except on a compact subset K. If we have $\omega - \omega' = \mathrm{d}\alpha$ where the form α has its support in K, then there is a diffeomorphism ϕ of M, which is identity on $M \setminus K$, such that $\phi^*(\omega') = \omega$.*

Proof. We have $\omega' = f\omega$ where f is a strictly positive function on M ($f = 1$ on $M \setminus K$). Then $\omega_t = f_t\omega$, with $f_t = tf + 1 - t$, $t \in [0, 1]$, is a path of volume forms on M. Now Equation (A.5) reduces to

$$\mathrm{d}i_{X_t}\omega_t = \omega - \omega'. \tag{A.6}$$

The hypothesis of the theorem allows us to replace Equation (A.6) by

$$i_{X_t}\omega_t = \alpha. \tag{A.7}$$

But this last equation has a unique solution X_t. Since the support of X_t lies in K, we can integrate this time-dependent vector field to a path of diffeomorphisms ϕ_t on M which is identity on $M \setminus K$, and such that $\phi_1^*(\omega') = \omega$. □

In particular, when $M = K$ we get the following corollary:

COROLLARY A.1.2 ([**153**]). *Two volume forms on a compact manifold are isomorphic if and only if they have the same total volume.*

It was pointed out by Weinstein [**203, 204**] that the path method works very well in the local study of symplectic manifolds. A basic result in that direction is the following.

THEOREM A.1.3. *Let $(\omega_t)_{t\in[0,1]}$ be a smooth path of symplectic forms on a manifold M. If we have $\partial\omega_t/\partial t = \mathrm{d}\gamma_t$ for a smooth path γ_t of 1-forms with compact support, then there is a diffeomorphism ϕ of M with $\phi^*\omega_1 = \omega_0$.*

Proof. Equation (A.5) follows from the equation $i_{X_t}\omega_t = -\gamma_t$, which has a solution $(X_t)_t$ because ω_t is nondegenerate, so the path method works in this case. □

THEOREM A.1.4 (see [**142**]). *Let K be a compact submanifold of a manifold M. Suppose that ω_0 and ω_1 are two symplectic forms on M which coincide at each point of K. Then there exist neighborhoods N_0 and N_1 of K and a diffeomorphism $\phi : N_0 \longrightarrow N_1$, which fixes K, such that $\phi^*\omega_1 = \omega_0$.*

Proof. Consider the path $\omega_t = (1 - t)\omega_0 + t\omega_1$ of symplectic forms in a neighborhood of K. Similarly to the proof of the previous theorem, it is sufficient to find an 1-form γ such that $\gamma(x) = 0$ for any $x \in K$ and

$$\mathrm{d}\gamma = \partial\omega_t/\partial t = \omega_1 - \omega_0. \tag{A.8}$$

The existence of such a γ is a generalization of Poincaré's lemma which says that a closed form is exact on any contractible open subset of a manifold. It can be proved by the following method, inspired by the Moser's path method. Choose a sufficiently small tubular neighborhood T of K in M and denote by ψ_t the mapping from T into T which is the linear contraction $v \mapsto tv$ along the fibers of T. With $\theta = \omega_1 - \omega_0$, we have

$$\theta = \psi_1^*(\theta) - \psi_0^*(\theta) = \int_0^1 \frac{\partial \psi_t^* \theta}{\partial t} \mathrm{d}t, \tag{A.9}$$

because ψ_1 is the identity and ψ_0 has its range in K. Now, if we denote by Y_t the time dependent vector field associated by Formula (A.1) to the path $(\psi_t)_{t\in]0,1]}$, we get

$$\frac{\partial \psi_t^* \theta}{\partial t} = \psi_t^* \mathcal{L}_{Y_t} \theta = \mathrm{d}\gamma_t, \tag{A.10}$$

with $\gamma_t = \psi_t^* \iota_{Y_t} \theta$. The path γ_t is, a priori, defined only for $t > 0$, but it extends clearly to $t = 0$; also it vanishes on K. So Equation (A.9) gives $\theta = \mathrm{d}\gamma$ with $\gamma = \int_0^1 \gamma_t \mathrm{d}t$, and leads to the conclusion. □

A direct corollary of Theorem A.1.4 is the following result of Weinstein:

THEOREM A.1.5 ([**203**]). *Let L be a compact Lagrangian submanifold of the symplectic manifold (M, ω). There is a neighborhood N_1 of L in M, a neighborhood N_0 of L (identified with the zero section) in T^*L and a diffeomorphism $\phi : N_0 \longrightarrow N_1$, which fixes L, such that $\phi^*\omega$ is the canonical symplectic form on T^*L.*

Proof (sketch). Choose a Lagrangian complement E_x to each T_xL in T_xM in order to get a fiber bundle E over L complement to TL in $TM|_L$. We construct a fiber bundle isomorphism $f : E \longrightarrow T^*L$ by $f(v)(w) = \omega(v, w)$. As E realizes a normal bundle to L, we can consider that f gives a diffeomorphism from a tubular neighborhood of L in M to a neighborhood of L in T^*L which sends $\omega|_L$ to the canonical symplectic form of T^*L (restricted to L). So we can suppose that ω is defined on a neighborhood of L in T^*L and is equal to the canonical symplectic form at every point on L. Then we achieve our goal using Theorem A.1.4. □

When the compact submanifold K is just one point, we recover from Theorem A.1.4 the classical Darboux's theorem:

THEOREM A.1.6 (Darboux). *Every point of a symplectic manifold admits a neighborhood with a local system of coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ (called **Darboux coordinates** or **canonical coordinates**) in which the symplectic form has the standard form $\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$.*

Proof. We can suppose that we work with a symplectic form ω near the origin 0 in $\mathbb{R}^{2n}$. Moreover we can suppose, up to a linear change, that a first system of coordinates is chosen such that $\omega(0) = \omega_0(0)$ where $\omega_0 = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$. Then we apply Theorem A.1.4 to the case $K = \{0\}$. □

In fact, Moser's path method gives a simple proof of the following *equivariant* version of Darboux's theorem [**204**], which would be very hard (if not impossible) to prove by the classical method of coordinate-by-coordinate construction.

THEOREM A.1.7 (Equivariant Darboux theorem). *Let G be a compact Lie group which acts symplectically on a symplectic manifold (M, ω) and which fixes a point $z \in M$. Then there is a local canonical system of coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ in a neighborhood of z in M, with respect to which the action of G is linear.*

Proof. One first linearize the action of G near z using Bochner's Theorem **??**. Then, after a linear change, one arrive at a system of coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ in which the action of G is linear, and such that $\omega(0) = \omega_0(0)$ where $\omega_0 = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$. One then uses the path method to move ω to ω_0 in a G-equivariant way. To do this, one must find an 1-form γ such that $\omega - \omega_0 = \mathrm{d}\gamma$ as in the proof of Theorem A.1.3, and moreover γ must be G-invariant in order to assure that the resulting flow ϕ_t is G-invariant (i.e. it preserves the action of G). Note that both ω and ω_0 are G-invariant. In order to find such a G-invariant 1-form γ, one starts with an arbitrary 1-form $\hat{\gamma}$ such that $\omega - \omega_0 = \mathrm{d}\hat{\gamma}$, and then average it by the action of G:

$$\gamma = \int_G \rho(g)^* \hat{\gamma} \mathrm{d}\mu, \tag{A.11}$$

where ρ denotes the action of G, and $\mathrm{d}\mu$ denotes the Haar measure on G. □

The Moser's path method has also become an essential tool in the study of singularities of smooth maps. In that domain we often have to construct equivalences between two maps f_0 and f_1, e.g. relations $g \circ \phi = f$ (right equivalence), or $g \circ \phi = \psi \circ f$ (right-left equivalence), or more general equivalences (contact equivalence, etc.), which are given by some diffeomorphisms (ϕ, ψ, etc.). To use the path method we first construct an appropriate path (f_t) which connects f_0 to f_1. Then we try to find a path of diffeomorphisms which gives the corresponding equivalence between f_t and f_0. Differentiating the equation with respect to t, we get a version of Equation (A.5). For example, in the case of right-left equivalence we fall on the following equation:

$$\mathrm{d}f_t(X_t(x)) + Y_t(f_t(x)) = -\frac{\partial f_t(x)}{\partial t}, \tag{A.12}$$

where the unknown are time-dependent vector fields X_t and Y_t on the source and the target spaces respectively. The singularists have developed various methods to solve these equations, such as the celebrated preparation theorem (see, e.g., [**94, 7**]). The Tougeron's theorem that we present below is typical of the use of the path method in this domain.

Let $\mathcal{C}_0^\infty(\mathbb{R}^n)$ be the algebra of germs at the origin of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. We denote by the same letter such a function and its germ at the origin. Let $\Delta(f) \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ be the ideal generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ of f. The ***codimension*** of $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is, by definition, the dimension of the real vector space $\mathcal{C}_0^\infty(\mathbb{R}^n)/\Delta(f)$.

We say that $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is ***k-determinant*** if every $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, such that f and g have the same Taylor expansion at the origin up to order k, is right equivalent to f.

THEOREM A.1.8 (Tougeron [**193**]). *If $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ has finite codimension k then it is $(k+1)$-determinant.*

Proof. Denote by $\mathfrak{M} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ the ideal of germs of functions vanishing at the origin. Consider the following sequence of inequalities:

$$\begin{aligned}\dim \mathcal{C}_0^\infty(\mathbb{R}^n)/(\mathfrak{M} + \Delta(f)) &\leq \dim \mathcal{C}_0^\infty(\mathbb{R}^n)/(\mathfrak{M}^2 + \Delta(f)) \leq \cdots \\ \leq \dim \mathcal{C}_0^\infty(\mathbb{R}^n)/(\mathfrak{M}^m + \Delta(f)) &\leq \dim \mathcal{C}_0^\infty(\mathbb{R}^n)/(\mathfrak{M}^{m+1} + \Delta(f)) \leq \cdots \leq k.\end{aligned} \tag{A.13}$$

It follows that there is a number $q \in \mathbb{Z}, 0 \leq q \leq k$, such that $\mathfrak{M}^q + \Delta(f) = \mathfrak{M}^{q+1} + \Delta(f)$, and hence $\mathfrak{M}^q \subset \mathfrak{M}^{q+1} + \Delta(f)$. Applying Nakayama's lemma (Lemma A.1.9) to this relation, we get $\mathfrak{M}^q \subset \Delta(f)$, which implies that

$$\mathfrak{M}^{k+2} \subset \mathfrak{M}^2 \Delta(f). \tag{A.14}$$

Let $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with the same $(k+1)$-Taylor expansion as f. We consider the path $f_t = (1-t)f + tg$ and, following the path method, try to construct a path of local diffeomorphisms $(\phi_t)_t$ which fixes the origin and such that $f_t \circ \phi_t = f$. Here the equation to solve (Equation (A.5)) becomes

$$\sum_{i=1}^n \frac{\partial f_t}{\partial x_i}(x) X_i(t,x) = (f-g)(x), \tag{A.15}$$

where the unknown functions $X_i(t,x)$ must be such that $X_i(t,0) = 0$. In fact, we will try to find X_i such that $X_i(t,.) \in \mathfrak{M}^2$ for any t, so the differential of the resulting diffeomorphisms ϕ_t at 0 will be equal to identity.

By compactness of the interval $[0,1]$, we need only be able to construct $(\phi_t)_t$ for t near any fixed $t_0 \in [0,1]$,, i.e. we need to solve Equation (A.15) only for t near t_0 (and x near 0). We denote by A the ring of germs at $(t_0, 0)$ of smooth functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$; $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is considered naturally as a subring of A. Then Equation (A.15) (with $X_i \in \mathfrak{M}^2$, where $\mathfrak{M}$ is now the ideal of A generated by germs of functions f such that $f(t,0) = 0$ for any t near t_0) can be replaced by

$$\mathfrak{M}^{k+2} \subset \mathfrak{M}^2 \Delta(f_t), \tag{A.16}$$

where $\Delta(f_t)$ is now the ideal of A generated by the functions $h(t,x) = \frac{\partial f_t(x)}{\partial x_i}$ (recall that $f - g$ belongs to $\mathfrak{M}^{k+2}$).

Now, because $\frac{\partial f_t}{\partial x_i} = \frac{\partial f}{\partial x_i}$ modulo $\mathfrak{M}^{k+1}A$, we have $\Delta(f) \subset \Delta(f_t) + \mathfrak{M}^{k+1}A$, and Relation (A.14) leads to

$$\mathfrak{M}^{k+2}A \subset \mathfrak{M}^2 \Delta(f_t) + \mathfrak{M}^{k+3}A, \tag{A.17}$$

or

$$\mathfrak{M}^{k+2}A \subset \mathfrak{M}^2 \Delta(f_t) + I \cdot \mathfrak{M}^{k+2}A, \tag{A.18}$$

where I is the ideal of A consisting of germs of functions vanishing at $(t_0, 0)$. Applying again Nakayama's lemma, we get $\mathfrak{M}^{k+2}A \subset \mathfrak{M}^2 \Delta(f_t)$, i.e. Relation (A.16), which leads to the result. □

LEMMA A.1.9 (Nakayama's lemma). *Let R be a commutative ring with unit and I an ideal of R such that, for any x in I, $1 + x$ is invertible. If M and N are two R-modules, M being finitely generated, then the relation $M \subset N + IM$ implies that $M \subset N$.*

Proof. Choose a system of generators $m_1, \dots, m_q$ for M. Then $M \subset N + IM$ gives a linear system $\sum_{j=1\dots q} a_i^j m_j = n_i$ where n_i are elements of N, $i = 1, \dots, q$ and a_i^j are elements of R of the form $a_i^j = \delta_i^j + \nu_i^j$, where δ_i^j is the Kronecker symbol and ν_i^j are in I. As the matrix (a_i^j) is invertible, we can write the m_i as linear combinations of the n_j and obtain $M \subset N$. □

A special case of the preceding Tougeron's theorem with $k = 1$ is the Morse lemma which says that, near any singular point with invertible Hessian matrix, a smooth function is right equivalent to $(x_1, \dots, x_n) \mapsto c + \sum_{i=1\dots n} \pm x_i^2$.

The Moser's path method is also useful in the study of contravariant tensors, e.g. vector fields and Poisson structures. In the case of vector fields, we fall on the equation

$$[X_t, Z_t] = -\frac{\partial Z_t}{\partial t}, \tag{A.19}$$

where $(Z_t)_t$ is a given path of vector fields and $(X_t)_t$ is the unknown (see, e.g., [**176**]).

For Poisson structures, we get a similar equation:

$$[X_t, \Pi_t] = -\frac{\partial \Pi_t}{\partial t}, \tag{A.20}$$

where $(\Pi_t)_t$ is a path of Poisson structures. However, the use of Moser's path method in the study of Poisson structures is rather tricky, because it is not easy to find a path of Poisson structures which connects two given Poisson structures, even locally, due to the Jacobi condition. One needs to make some preparatory work first, for example to make the two original Poisson structures have the same characteristic foliation (then it will become easier to find a path connecting the two structures). The *equivariant splitting theorem* for Poisson structures can be proved using this approach, see [**146**].

A.2. The neighborhood of a symplectic leaf

In this section, following Vorobjev [**200**], we will give a description of a Poisson structure in the neighborhood of a symplectic leaf in terms of *geometric data*, and then use these geometric data to study the problem of linearization of Poisson structures along a symplectic leaf.

A.2.1. Geometric data and coupling tensors.

First let us recall the notion of an Ehresmann (nonlinear) connection. Let $p : E \longrightarrow S$ be a submersion over a manifold S. Denote by $T_V E$ the vertical subbundle of the tangent bundle TE of E, and by $\mathcal{V}_V^1(E)$ the space of vertical tangent vector fields (i.e. vector fields tangent to the fibers of the fibration) of E. An ***Ehresmann connection*** on E is a splitting of TE into the direct sum of $T_V E$ and another tangent subbundle $T_H E$, called the ***horizontal subbundle*** of E. It can be defined by a $\mathcal{V}_V^1(E)$-valued 1-form $\Gamma \in \Omega^1(E) \otimes \mathcal{V}_V^1(E)$ on E such that $\Gamma(Z) = Z$ for every $Z \in T_V E$. Then the horizontal subbundle is the kernel of Γ: $T_H E := \{X \in TE, \Gamma(X) = 0\}$. For every vector field $u \in \mathcal{V}^1(S)$ on S, there is a unique lifting of u to a horizontal vector field $Hor(u) \in \mathcal{V}_H^1(E)$ on E

(whose projection to S is u). The ***curvature*** of an Ehresmann connection is a $\mathcal{V}^1_V(E)$-valued 2-form on S, $Curv_\Gamma \in \Omega^2(S) \otimes \mathcal{V}^1_V(E)$, defined by

$$Curv_\Gamma(u,v) := [Hor(u), Hor(v)] - Hor([u,v]),\ u,v \in \mathcal{V}^1(S), \tag{A.21}$$

and the associated covariant derivative $\partial_\Gamma : \Omega^i(S) \otimes \mathcal{C}^\infty(E) \longrightarrow \Omega^{i+1}(S) \otimes \mathcal{C}^\infty(E)$ is defined by an analog of Cartan's formula:

$$\begin{aligned} \partial_\Gamma K(u_1,\dots,u_{k+1}) = &\sum_i (-1)^{i+1} \mathcal{L}_{Hor(u_i)}(K(u_1,\dots,\widehat{u_i},\dots,u_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} K([u_i,u_j],u_1,\dots,\widehat{u_i},\dots,\widehat{u_j},\dots,u_{k+1}). \end{aligned} \tag{A.22}$$

Remark that $\partial_\Gamma \circ \partial_\Gamma = 0$ if and only if Γ is a flat connection, i.e. $Curv_\Gamma = 0$.

Suppose now that S is a symplectic leaf in a Poisson manifold (M,Π), and E is a small tubular neighborhood of S with a projection $p : E \longrightarrow S$. Then there is a natural Ehresmann connection $\Gamma \in \Omega^1(E) \otimes \mathcal{V}^1_V(E)$ on E, whose horizontal subbundle is spanned by the Hamiltonian vector fields $X_{f\circ p}$, $f \in \mathcal{C}^\infty(S)$. The Poisson structure Π splits into the sum of its horizontal part and its vertical part,

$$\Pi = \mathcal{V} + \mathcal{H}, \tag{A.23}$$

where $\mathcal{V} = \Pi_V \in \mathcal{V}^2_V(E)$ and $\mathcal{H} = \Pi_H \in \mathcal{V}^2_H(E)$ (there is no mixed part). The horizontal 2-vector field $\mathcal{H}$ is nondegenerate on $T_H E$. Denote by $\mathbb{F}$ its dual 2-form; it is a section of $\wedge^2 T^*_H E$ which can be defined by the following formula:

$$\mathbb{F}(X_{f\circ p}, X_{g\circ p}) = \langle \Pi, p^*\mathrm{d}f \wedge p^*\mathrm{d}g\rangle, \quad f,g \in \mathcal{C}^\infty(S), \tag{A.24}$$

(recall that $X_{f\circ p}, X_{g\circ p} \in \mathcal{V}^1_H(E)$). Via the horizontal lifting of vector fields, $\mathbb{F}$ may be viewed as a nondegenerate $C^\infty(E)$-valued 2-form on S, $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$.

The above triple $(\mathcal{V},\Gamma,\mathbb{F})$ is called a set of ***geometric data*** for (M,Π) in a neighborhood of S.

Conversely, given a set of geometric data $(\mathcal{V},\Gamma,\mathbb{F})$, one can define a 2-vector field Π on E by the formula $\Pi = \mathcal{V} + \mathcal{H}$, where $\mathcal{H}$ is the horizontal 2-vector field dual to $\mathbb{F}$. A natural question arises: how to express the condition $[\Pi,\Pi] = 0$, i.e. Π is a Poisson structure, in terms of geometric data $(\mathcal{V},\Gamma,\mathbb{F})$? The answer to this question is given by the following theorem:

THEOREM A.2.1 (Vorobjev [**200**]). *A triple of geometric data $(\mathcal{V},\Gamma,\mathbb{F})$ on a fibration $p : E \longrightarrow S$, where Γ is an Ehresmann connection on E, $\mathcal{V} \in \mathcal{V}^2_V(E)$ is a vertical 2-vector field, and $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$ is a nondegenerate $\mathcal{C}^\infty(E)$-valued 2-form on S, determines a Poisson structure on E (by the above formulas) if and only if it satisfies the following four compatibility conditions:*

$$[\mathcal{V},\mathcal{V}] = 0, \tag{A.25}$$

$$\mathcal{L}_{Hor(u)}\mathcal{V} = 0 \ \ \forall\, u \in \mathcal{V}^1(S), \tag{A.26}$$

$$\partial_\Gamma \mathbb{F} = 0, \tag{A.27}$$

$$Curv_\Gamma(u,v) = \mathcal{V}^\sharp(\mathrm{d}(\mathbb{F}(u,v))) \ \ \forall\, u,v \in \mathcal{V}^1(S), \tag{A.28}$$

where $\mathcal{V}^\sharp$ means the map from T^*E to TE defined by $\langle \mathcal{V}^\sharp(\alpha),\beta\rangle = \langle \mathcal{V}, \alpha\wedge\beta\rangle$.

REMARK A.2.2. Equations (A.25) and (A.26) mean that the vertical part $\mathcal{V}$ of Π is a Poisson structure (on each fiber of E) which is preserved under parallel transport. This gives another proof of Theorem 1.6.1 which says that the transverse Poisson structure to a symplectic leaf is unique up to local isomorphisms.

REMARK A.2.3. In the above theorem, E is not necessarily a tubular neighborhood of S. The symplectic case (E is a symplectic manifold) of the above theorem was obtained by Guillemin, Lerman and Sternberg in [**98**]. In fact, the proof of the symplectic case can be easily adapted to the Poisson case because a Poisson manifold is just a singular foliation by symplectic manifolds. The Poisson structure Π is called the ***coupling tensor*** of $(\mathcal{V}, \Gamma, \mathbb{F})$ (it couples a horizontal tensor with a vertical tensor via a connection).

Proof. Consider a local system of coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ on E, where $y_1, \ldots, y_{n-m}$ are local functions on a fiber and $x_1, \ldots, x_m$ are local functions on S ($m = \dim S$ is even). Denote the horizontal lifting of the vector field $\partial x_i := \partial/\partial x_i$ from S to E by $\overline{\partial x_i}$. Then we can write $\Pi = \mathcal{V} + \mathcal{H}$, where

$$\mathcal{V} = \frac{1}{2}\sum_{ij} a_{ij}\partial y_i \wedge \partial y_j \quad (a_{ij} = -a_{ji}), \tag{A.29}$$

and

$$\mathcal{H} = \frac{1}{2}\sum_{ij} b_{ij}\overline{\partial x_i} \wedge \overline{\partial x_j} \quad (b_{ij} = -b_{ji}) \tag{A.30}$$

is the dual horizontal 2-vector field of $\mathbb{F}$.

The condition $[\Pi, \Pi] = 0$ is equivalent to

$$0 = [\mathcal{V}, \mathcal{V}] + 2[\mathcal{V}, \mathcal{H}] + [\mathcal{H}, \mathcal{H}] = A + B + C + D, \tag{A.31}$$

where

$$A = [\mathcal{V}, \mathcal{V}], \tag{A.32}$$

$$B = 2\sum_i [\mathcal{V}, \overline{\partial x_i}] \wedge X_i, \text{ where } X_i = \sum_j b_{ij}\overline{\partial x_j}, \tag{A.33}$$

$$C = \sum_{ij} [\mathcal{V}, b_{ij}] \wedge \overline{\partial x_i} \wedge \overline{\partial x_j} + \sum_{ij} \overline{\partial x_i} \wedge \overline{\partial x_j} \wedge (\sum_{kl} b_{ik} b_{jl} [\overline{\partial x_k}, \overline{\partial x_l}]), \tag{A.34}$$

$$D = \sum_{ijkl} b_{ij}\overline{\partial x_j}(b_{kl})\ \overline{\partial x_i} \wedge \overline{\partial x_k} \wedge \overline{\partial x_l}. \tag{A.35}$$

Notice that A, B, C, D belong to complementary subspaces of $\mathcal{V}^3(E)$, so the condition $A + B + C + D = 0$ means that $A = B = C = D = 0$.

The equation $A = 0$ is nothing but Condition (A.25): $[\mathcal{V}, \mathcal{V}] = 0$.

The equation $B = 0$ means that $[\mathcal{V}, \overline{\partial x_i}] = 0\ \forall\ i$, i.e. $\mathcal{L}_{\overline{\partial x_i}}\mathcal{V} = 0\ \forall\ i$, which is equivalent to Condition (A.26).

The equation $D = 0$ means that $\oint_{ikl} \sum_j b_{ij}\overline{\partial x_j}(b_{kl}) = 0 \quad \forall\ i, k, l$, where $\oint_{ikl}$ denotes the cyclic sum. Let us show that this condition is equivalent to Condition (A.27). Notice that $\mathbb{F}(\partial x_i, \partial x_j) = c_{ij}$, where (c_{ij}) is the inverse matrix of (b_{ij}), and

$\partial_\Gamma \mathbb{F}(\partial x_i, \partial x_j, \partial x_k) = \oint_{ijk} \overline{\partial x_i}(c_{jk})$. By direct computations, we have

$$\partial_\Gamma \mathbb{F}(\sum_\alpha b_{i\alpha}\partial x_\alpha, \sum_\beta b_{j\beta}\partial x_\beta, \sum_\gamma b_{k\gamma}\partial x_\gamma) = 2\oint_{ijk} \sum_l b_{il}\overline{\partial x_l}(b_{jk}). \tag{A.36}$$

Thus the condition $D = 0$ is equivalent to the condition

$$\partial_\Gamma \mathbb{F}(\sum_\alpha b_{i\alpha}\partial x_\alpha, \sum_\beta b_{j\beta}\partial x_\beta, \sum_\gamma b_{k\gamma}\partial x_\gamma) = 0 \quad \forall\ i, j, k. \tag{A.37}$$

Since the matrix (b_{ij}) is invertible, the last condition is equivalent to $\partial_\Gamma \mathbb{F} = 0$.

Similarly, by direct computations, one can show that the condition $C = 0$ is equivalent to Condition (A.28). □

THEOREM A.2.4 (Vorobjev [**200**]). *Let E be a sufficiently small neighborhood E of a symplectic leaf S of a Poisson manifold (M, Π), together with a given projection $p : E \longrightarrow S$. Denote by $(\mathcal{V}, \Gamma, \mathbb{F})$ the associated geometric data in E. Consider an arbitrary tensor field $\phi \in \Omega^1(S) \otimes \mathcal{C}^\infty(E)$ whose restriction to $\Omega^1(S) = \Omega^1(S) \otimes \mathcal{C}^\infty(S)$ via the inclusion $S \hookrightarrow E$ is trivial, and the following new set of geometric data:*

$$\mathcal{V}' = \mathcal{V}, \tag{A.38}$$

$$\Gamma' = \Gamma - \mathcal{V}^\sharp(\mathrm{d}p^*\phi), \tag{A.39}$$

$$\mathbb{F}' = \mathbb{F} - \partial_\Gamma \phi - \{\phi, \phi\}_{\mathcal{V}}. \tag{A.40}$$

Then the coupling tensor Π' of $(\mathcal{V}', \Gamma', \mathbb{F}')$ is also a Poisson tensor, and there is a diffeomorphism f between neighborhoods of S, which fixes every point of S and such that $f_*\Pi = \Pi'$.

In the above theorem, $\mathcal{V}^\sharp(\mathrm{d}p^*\phi)$ means an element of $\Omega^1(E) \otimes \mathcal{V}^1_V(E)$ defined by the formula $\mathcal{V}^\sharp(\mathrm{d}p^*\phi)(w) = \mathcal{V}^\sharp(\mathrm{d}(\phi(p_*w)), w \in TE$, where p_*w is the projection of w to S, $\phi(p_*w)$ is viewed as a function on the fiber T_xE over the origin x of p_*w, and $\mathcal{V}^\sharp(\mathrm{d}(\phi(p_*w))$ is the Hamiltonian vector field with respect to $\mathcal{V}$ on T_xE of the function $\phi(p_*w)$. Similarly, $\{\phi, \phi\}_{\mathcal{V}}$ means an element of $\Omega^2(S) \otimes C^\infty(E)$ defined by $\{\phi, \phi\}_{\mathcal{V}}(u, v) = \{\phi(u), \phi(v)\}_{\mathcal{V}}$, where $u, v \in \mathcal{V}^1(S)$, and the bracket is taken with respect to $\mathcal{V}$.

Proof (sketch). We will use Moser's path method. Consider the following family of geometric data,

$$\begin{aligned}
\mathcal{V}_t &= \mathcal{V}, \\
\Gamma_t &= \Gamma - t\mathcal{V}^\sharp(\mathrm{d}p^*\phi), \\
\mathbb{F}_t &= \mathbb{F} - t\partial_\Gamma\phi - t^2\{\phi, \phi\}_{\mathcal{V}},
\end{aligned}$$

and the corresponding family of coupling tensors Π_t, $t \in [0, 1]$. Define a time-dependent vector field $X = (X_t)_{t\in[0,1]}$ as follows: X_t is the unique horizontal vector field with respect to Γ_t which satisfies the equation

$$X_t \lrcorner \mathbb{F}_t = -\phi$$

(where ϕ and $\mathbb{F}_t$ are considered as differential forms on E by lifting). One verifies directly that we have

$$[X_t, \Pi_t] = -\frac{\partial \Pi_t}{\partial t}. \tag{A.41}$$

It implies that the time-1 flow φ_X^1 of $X = (X_t)$ moves $\Pi = \Pi_0$ to $\Pi' = \Pi_1$. As a consequence, Π' is automatically a Poisson tensor. Note that X vanishes on S, so φ_X^1 fixes every point of S. $\square$

Remark A.2.5. The flow φ_X^t in the above proof preserves the symplectic leaves of Π (so Π and Π' have the same foliation though not the same symplectic forms on the leaves). What the flow does is to change the projection map p. It also allows us to compare different geometric data of the same Poisson structure but with respect to different projection maps.

A.2.2. Linear models.

Consider the geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ of a Poisson structure in a neighborhood E of a symplectic leaf S with respect to a projection $p : E \to S$. We will embed E in the normal bundle NS of S by a fiber-wise embedding which maps S to the zero section in NS and which projects to the identity map on S. Then we can view $(\mathcal{V}, \Gamma, \mathbb{F})$ as geometric data in a neighborhood of S (identified with the zero section) in NS.

Denote by $\mathcal{V}^{(1)}$ the fiber-wise linear part of $\mathcal{V}$, $\Gamma^{(1)}$ the fiber-wise linear part of Γ, and $\mathbb{F}^{(1)}$ the fiber-wise affine part of $\mathbb{F}$ in NS. For example, if $X, Y \in T_xS$, then $\mathbb{F}(X, Y)$ is a function on a neighborhood of zero in N_xS, and $\mathbb{F}^{(1)}(X, Y)$ is the sum of the constant part and the linear part of $\mathbb{F}(X, Y)$ on N_x. By looking at the fiber-wise linear terms of the equations in Theorem A.2.1, we obtain immediately that $(\mathcal{V}^{(1)}, \Gamma^{(1)}, \mathbb{F}^{(1)})$ also satisfy these equations, which implies that the coupling tensor $\Pi^{(1)}$ of $(\mathcal{V}^{(1)}, \Gamma^{(1)}, \mathbb{F}^{(1)})$ is also a Poisson structure, defined in a neighborhood of S in NS. We will call $\Pi^{(1)}$ the ***Vorobjev linear model*** of Π along the symplectic leaf S.

Theorem A.2.6. *Up to isomorphisms, the Vorobjev linear model of a Poisson structure* Π *along a symplectic leaf* S *is uniquely determined by* Π *and* S *(and does not depend on the choice of the projection).*

Proof. We will fix a projection $p : E \to S$, and linearize the fibers of E by an embedding from E to NS which is compatible with p. This way we may consider the linear model $\Pi^{(1)}$ of Π with respect to p as living in E. Consider now another arbitrary projection p_1. We can find a smooth path of projections p_t with $p_0 = p$ and $p_1 = p_1$. There is a unique time-dependent vector field $Y = (Y_t)$ in a neighborhood of S which satisfies the following properties: Y_t is tangent to the symplectic leaves of Π, is symplectically orthogonal to the intersections of the fibers of p_t with the symplectic leaves, vanishes on S, and the flow φ_Y^t of Y moves the fibers of p_0 to the fibers of p_t: $p_t \circ \varphi_Y^t = p_0$. Denote $\Pi_t = (\varphi_Y^t)^{-1}\Pi$. Denote by $\Pi_t^{(1)}$ the Vorobjev linear model of Π_t with respect to the projection p ($\Pi_t^{(1)}$ also lives in E via the fixed linearization of E). To prove the theorem, it is sufficient to show that $\Pi_1^{(1)}$ is isomorphic to $\Pi^{(1)}$ by a diffeomorphism in a neighborhood of S.

Denote by $(\mathcal{V}_t, \Gamma_t, \mathbb{F}_t)$ the geometric data of Π_t with respect to p. Note that $\frac{\partial \Pi_t}{\partial t} = -[Y_t, \Pi_t]$ by construction. Similarly to the proof of Theorem A.2.4, we have

$$\mathcal{V}_t = \mathcal{V},\ \frac{\partial \Gamma_t}{\partial t} = -\mathcal{V}^\sharp(\mathrm{d}p^*\phi_t),\ \frac{\partial \mathbb{F}_t}{\partial t} = -\partial_{\Gamma_t}\phi_t,$$

where ϕ_t a family of elements of $\Omega^1(S) \otimes \mathcal{C}^\infty(E)$ defined by $\phi_t = -Y_t \lrcorner \mathbb{F}_t$. Looking only at the fiber-wise linear terms of the above equations, we get

$$\mathcal{V}_t^{(1)} = \mathcal{V}^{(1)}, \ \frac{\partial \Gamma_t^{(1)}}{\partial t} = -(\mathcal{V}^{(1)})^\sharp(\mathrm{d}p^* \phi_t^{(1)}), \ \frac{\partial \mathbb{F}_t^{(1)}}{\partial t} = -\partial_{\Gamma_t^{(1)}} \phi_t^{(1)},$$

which implies that

$$\frac{\partial \Pi_t^{(1)}}{\partial t} = -[Z_t, \Pi_t^{(1)}],$$

where $Z = (Z_t)$ is the time-dependent vector field defined by the formula $\phi_t^{(1)} = -Z_t \lrcorner \mathbb{F}_t^{(1)}$. As a consequence, the time-1 flow of (Z_t) moves $\Pi^{(1)}$ to $\Pi_1^{(1)}$. □

Remark A.2.7. The linear model of a Poisson structure along a symplectic leaf can also be constructed from the transitive Lie algebroid which is the restriction of the cotangent algebroid to the symplectic leaf, see [**200**]. We will leave it as an exercise for the reader to show that Vorobjev's original construction via transitive Lie algebroids is equivalent to the above construction.

The following simple example shows that, in general, one can't hope to find a local isomorphism between a Poisson structure and its Vorobjev linear model along a symplectic leaf, even if the leaf is simply-connected, the normal bundle is trivial and the transverse Poisson structure is linearizable.

Example A.2.8. Put $M = S^2 \times \mathbb{R}^3$ with Poisson structure $\Pi = f\Pi_1 + \Pi_2$, where Π_1 is a nondegenerate Poisson structure on S^2, $\Pi_2 = x\partial y \wedge \partial z + y\partial z \wedge \partial x + y\partial x \wedge \partial y$ is the Lie-Poisson structure on $\mathbb{R}^3$ corresponding to $so(3)$, and $f = f(x^2 + y^2 + z^2)$ is a Casimir function on $(\mathbb{R}^3, \Pi_2)$. Since the linear part of f on $\mathbb{R}^3$ is trivial, the linear model of Π is $f(0)\Pi_1 + \Pi_2$. If f is not a constant then Π can't be isomorphic to $\Pi^{(1)}$ near S for homological reasons: the regular symplectic leaves are $S^2 \times S^2$, the integral of the symplectic form over the first component S^2 is a constant (does not depend on the leaf) in the linear model $\Pi^{(1)}$, but is not a constant when the symplectic form comes from Π.

If one wants to linearize only $\mathcal{V}$ and Γ but not $\mathbb{F}$, then the situation becomes more reasonable. See [**30**] for some results in that direction.

A.3. Dirac structures

An ***almost Dirac structure*** on a manifold M is a subbundle L of the bundle $TM \oplus T^*M$, which is isotropic with respect to the natural indefinite symmetric scalar product on $TM \oplus T^*M$,

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle := \frac{1}{2}(\langle \alpha_1, X_2 \rangle + \langle \alpha_2, X_1 \rangle) \tag{A.42}$$

for $(X_1, \alpha_1), (X_2, \alpha_2) \in \Gamma(TM \oplus T^*M)$, and such that the rank of L is maximal possible, i.e. equal to the dimension of M.

For example, if ω is an arbitrary differential 2-form on M, then its graph $L_\omega = \{(X, i_X\omega) \mid X \in TM\}$ is an almost Dirac structure. Furthermore, an almost Dirac structure L is the graph of a 2-form if and only if $L_x \cap (\{0\} \oplus T_x^*M) = \{0\}$ for any $x \in M$. Similarly, if Λ is an arbitrary 2-vector field on M, then the set $L_\Lambda = \{(i_\alpha\Lambda, \alpha) \mid \alpha \in T^*M\}$ is also an almost Dirac structure.

A Dirac structure is an almost Dirac structure plus an integrability condition. To formulate this condition, consider the following bracket on $\Gamma(TM \oplus T^*M)$, called the ***Courant bracket*** [**55**]:

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_C = ([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - i_{X_2}\mathrm{d}\alpha_1). \tag{A.43}$$

An almost Dirac structure L is called a ***Dirac structure*** if it is close under the Courant bracket, i.e. $[(X_1, \alpha_1), (X_2, \alpha_2)]_C \in \Gamma(L)$ for any $(X_1, \alpha_1), (X_2, \alpha_2) \in \Gamma(L)$. In this case, the pair (M, L) is called a ***Dirac manifold***.

Example A.3.1. If ω is a 2-form on M then the almost Dirac structure $L_\omega = \{(X, X \lrcorner \omega) \mid X \in TM\}$ is a Dirac structure if and only if ω is closed. Similarly, if Λ is a 2-vector field on M then the almost Dirac structure $L_\Lambda = \{(\alpha \lrcorner \Lambda, \alpha) \mid \alpha \in T^*M\}$ is a Dirac structure if and only if Λ is a Poisson structure. In other words, Dirac structures generalize both presymplectic structures and Poisson structures.

Example A.3.2. If L is a Dirac structure on M such that its canonical projection $pr_1 : L \to M$ to M vanishes at a point $x_0 \in M$, then for x near x_0 we have $L_x \cap (T_xM \oplus \{0\}) = \{0\}$, which implies that locally $L = L_\Lambda$ is the graph of a 2-vector field Λ, and the integrability of L means that Λ satisfies the Jacobi identity. Thus locally a Dirac structure whose projection to TM vanishes at a point is the same as a Poisson structure which vanishes at that point.

The Courant bracket (A.43) is not anti-symmetric nor does it satisfy the Jacobi identity on $\Gamma(TM \oplus T^*M)$. But if L is a Dirac structure, then one can verify easily that the restriction of the Courant bracket to $\Gamma(L)$ is anti-symmetric and satisfies the Jacobi identity, and it turns L into a Lie algebroid over M whose anchor map is the canonical projection $pr_1 : L \to TM$ from L to TM, see [**55**]. For example, when $L = L_\Lambda$ comes from a Poisson structure, then this Lie algebroid is naturally isomorphic to the cotangent algebroid associated to Λ.

In particular, if L is a Dirac structure, then its associated distribution $\mathcal{D}_L$ on M, $(\mathcal{D}_L)_x = pr_1(L_x)$, is integrable and gives rise to the associated foliation $\mathcal{F}_L$ on M. Furthermore, there is a 2-form ω_L defined on each leaf of this foliation by the formula

$$\Omega_L(X, Y) = \langle \alpha, Y \rangle \quad \forall\ (X, \alpha), (Y, \beta) \in L_x. \tag{A.44}$$

The fact that L is isotropic assures that Ω_L is well-defined and skew-symmetric. Moreover, we have:

Theorem A.3.3 ([**55**]). *If L is a Dirac structure on M then $\mathrm{d}\Omega_L = 0$ on any leaf of the associated singular foliation $\mathcal{F}_L$ of L.*

The meaning of the above proposition is that, roughly speaking, a Dirac structure is a singular foliation by *presymplectic leaves*. Its proof is a straightforward verification similar to the Poisson case. Note that L is completely determined by $\mathcal{D}_L$ and Ω_L.

A submanifold of a Poisson manifold is not a Poisson manifold in general, but is a Dirac manifold under some mild assumptions. More generally, we have:

Proposition A.3.4 ([**55**]). *Let Q be a submanifold of a Dirac manifold (M, L). If $L_q \cap (T_qQ \oplus T_q^*M)$ has constant dimension (i.e. its dimension does not depend* *on $q \in Q$), then there is a natural induced Dirac structure L_Q on Q defined by the formula*

$$(L_Q)_q = \frac{L_q \cap (T_q Q \oplus T_q^* M)}{L_q \cap (\{0\} \oplus (T_Q)^0)}. \tag{A.45}$$

A special case of the above proposition is when $Q = N$ is a slice, i.e. a local transversal to a presymplectic leaf $\mathcal{O}$ at a point x_0. Then the condition of the theorem is satisfied, so N admits an induced Dirac structure, whose projection to TN vanishes at x_0, thus in fact N admits a Poisson structure which vanishes at x_0, and one can talk about the transverse Poisson structure to a presymplectic leaf in a Dirac manifold – provided that it is unique up to local isomorphisms.

Vorobjev's (semi)local description of Poisson structures via coupling tensors (see Subsection A.2.1) can be naturally extended to the case of Dirac structures. More precisely, given a triple of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ on a manifold E with a submersion $p : E \to S$, where Γ is an Ehresmann connection, $\mathcal{V}$ is a vertical 2-vector field, and $\mathbb{F}$ is a (maybe degenerate) $C^\infty(E)$-valued 2-form on S, denote by $L = L(\mathcal{V}, \Gamma, \mathbb{F})$ the associated subbundle of $TE \oplus T^*E$, which is generated by sections of the types $(\alpha, i_\alpha \mathcal{V})$ and $(X, i_X \mathbb{F})$, where $X \in \mathcal{V}^1_H E$ is a horizontal vector field and α is a vertical 1-form, i.e. $\alpha|_{T_H E} = 0$. Here $i_X \mathbb{F}$ means the contraction of $\mathbb{F}$, considered as a horizontal 2-form on E, with X. Then L is an almost Dirac structure on E.

Theorem A.3.5 ([**74**]). *Given a set of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ for a submersion $p : E \to S$ such as above, the corresponding almost Dirac structure $L(\mathcal{V}, \Gamma, \mathbb{F})$ is a Dirac structures if and only if the following four conditions (the same as in Theorem A.2.1) are satisfied:*

$$[\mathcal{V}, \mathcal{V}] = 0, \tag{A.46}$$

$$\mathcal{L}_{Hor(u)} \mathcal{V} = 0 \quad \forall\ u \in \mathcal{V}^1(S), \tag{A.47}$$

$$\partial_\Gamma \mathbb{F} = 0, \tag{A.48}$$

$$Curv_\Gamma(u, v) = \mathcal{V}^\sharp(\mathrm{d}(\mathbb{F}(u, v))) \quad \forall\ u, v \in \mathcal{V}^1(S). \tag{A.49}$$

Conversely, if E is a sufficiently small tubular neighborhood of a presymplectic leaf S with a projection map $p : E \to S$ in a Dirac manifold (M, L), then there is a unique triple of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ on (E, p) such that $L = L(\mathcal{V}, \Gamma, \mathbb{F})$ on E. Moreover, the vertical Poisson structure $\mathcal{V}$ vanishes on S, and the restriction of $\mathbb{F}$ to S is the presymplectic form of S induced from L.

The proof of Theorem A.3.5 is absolutely similar to the Poisson case. (The only difference between the Dirac case and the Poisson case is that the horizontal 2-form $\mathbb{F}$ may be degenerate in the Dirac case). A direct consequence of Theorem A.3.5 is that, similarly to the Poisson case, the transverse Poisson structure to a presymplectic leaf in a Dirac manifold is well-defined, i.e. up to local isomorphisms it does not depend on the choice of the slice. Another simple consequence is that the dimensions of the presymplectic leaves of a Dirac structure have the same parity.

Dirac and almost Dirac structures provide a convenient setting in which to study dynamical systems with constraints (holonomic and non-holonomic) and control theory, and there is a theory of symmetry and reduction of (almost) Dirac structures, which generalizes the theory for symplectic and Poisson structures. See, e.g., [**55, 68, 59, 19, 20, 21**] and references therein.

For a generalization of the notion of Dirac structures to Lie algebroids, see [**129**]. In a different development, the complex version of Dirac structures (L is a complex subbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$ which satisfies the same conditions as in the real case) leads to *generalized complex structures*, see, e.g., [**107, 96**].

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