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Bihamiltonian structures of PDEs and Frobenius manifolds

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Abstract
These complementary lecture notes are based on the papers [50], [52], [53].
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1 Lecture 1

1.1 Brief summary of finite-dimensional Poisson geometry

1.1.1 Poisson brackets

Let $P$ be a $N$-dimensional smooth manifold. A Poisson bracket on $P$ is a structure of a Lie algebra on the ring of functions $\mathcal{F} := C^\infty(P)$

$$f, g \mapsto \{f, g\},$$

$$\{g, f\} = -\{f, g\}, \{af + bg, h\} = a\{f, h\} + b\{g, h\}, \ a, \ b \in \mathbb{R}, \ f, \ g, \ h \in \mathcal{F} \quad (1.1.1)$$

satisfying the Leibnitz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

for arbitrary three functions $f, g, h \in \mathcal{F}$. In a system of local coordinates $x^1, \ldots, x^N$ the Poisson bracket reads

$$\{f, g\} = \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad (1.1.3)$$

(summation over repeated indices will be assumed) where the bivector $\pi^{ij}(x) = -\pi^{ji}(x) = \{x^i, x^j\}$ satisfies the following system of equations equivalent to the Jacobi identity (1.1.2)

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} \equiv \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s} \pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s} \pi^{si} = 0 \quad (1.1.4)$$

for any $i, j, k$. Such a bivector satisfying (1.1.4) is called Poisson structure on $P$.

Clearly any bivector constant in some coordinate system is a Poisson structure. Vice versa, locally all solutions to (1.1.4) of the constant rank $2n = \text{rk}(\pi^{ij})$ can be reduced, by a change of coordinates, to the following normal form

$$\pi = \begin{pmatrix} \bar{\pi} & 0 \\ 0 & 0 \end{pmatrix} \quad (1.1.5)$$

with a constant nondegenerate antisymmetric $2n \times 2n$ matrix $\bar{\pi} = \bar{\pi}^{ab}$. That means that locally there exist coordinates $y^1, \ldots, y^{2n}, c^1, \ldots, c^k, 2n + k = N$, s.t.

$$\bar{\pi}^{ab} = \{y^a, y^b\}$$

and

$$\{f, c^j\} = 0, \ j = 1, \ldots, k \quad (1.1.6)$$

for an arbitrary function $f$. 

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For the case $2n = N$ the inverse matrix $(\pi_{ij}(x)) = (\pi^{ij}(x))^{-1}$ defines on $P$ a symplectic structure
\[
\Omega = \sum_{i<j} \pi_{ij}(x) dx^i \wedge dx^j, \quad \Omega^n \neq 0.
\]
For $2n < N$ one obtains on $P$ a structure of symplectic foliation $P = \cup_{c_0} P_{c_0}$, $c_0 = (c_0^1, \ldots, c_0^k)$, of the codimension $k = N - 2n$
\[
P_{c_0} := \{ x \mid c^1(x) = c_0^1, \ldots, c^k(x) = c_0^k \}.
\] (1.1.7)

The independent functions $c^1(x), \ldots, c^k(x)$ defined in (1.1.6) are called Casimir functions, or simply Casimirs of the Poisson structure. Every leaf $P_{c_0}$ is a symplectic manifold, and the restriction map $C^\infty(P) \to C^\infty(P_{c_0})$ is a homomorphism of Lie algebras.

**Example 1.1.1** Let $\mathfrak{g}$ be $n$-dimensional Lie algebra. The Lie - Poisson bracket on the dual space $P = \mathfrak{g}^*$ reads
\[
\{ x^i, x^j \} = c^i_j x^k.
\] (1.1.8)
Here $c^i_j$ are the structure constants of the Lie algebra. The Casimirs of this bracket are functions on $\mathfrak{g}^*$ invariant with respect to the co-adjoint action of the associated Lie group $G$. The symplectic leaves coincide with the orbits of the coadjoint action with the Berezin - Kirillov - Kostant symplectic structure on them.

1.1.2 Dirac bracket

An arbitrary foliation $P = \cup_{\phi_0} P_{\phi_0}$ of a codimension $m$ represented locally in the form
\[
P_{\phi_0} = \{ x \mid \phi^1(x) = \phi_0^1, \ldots, \phi^m(x) = \phi_0^m \}
\]
will be called cosymplectic if the $m \times m$ matrix $\{ \phi^a, \phi^b \}$ does not degenerate on the leaves. In this situation a new Poisson structure $\{ \cdot, \cdot \}_D$ can be defined on $P$ s.t. the functions $\phi^a(x)$ are Casimirs of $\{ \cdot, \cdot \}_D$. This is the Dirac bracket given explicitly by the formula
\[
\{ f, g \}_D = \{ f, g \} - \sum_{a,b} \{ f, \phi^a \} \{ \phi^a, \phi^b \}^{-1} \{ \phi^b, g \}.
\] (1.1.9)

It can be restricted in an obvious way to produce a Poisson structure on every leaf. The restriction map
\[
(C^\infty(P), \{ \cdot, \cdot \}) \to (C^\infty(P_{\phi_0}), \{ \cdot, \cdot \}_D)
\]
is a homomorphism of Lie algebras.
1.1.3 Hamiltonian vector fields, first integrals

A Poisson bracket defines an (anti)homomorphism
\[ \mathcal{F} \rightarrow Vect(P) \]
\[ H \mapsto X_H := \{ \cdot, H \}, \]
(1.1.10)
\[ [X_{H_1}, X_{H_2}] = -X_{\{H_1, H_2\}}. \]

\( X_H \) is called Hamiltonian vector field. The corresponding dynamical system
\[ \dot{x}^i = \pi_{ij}(x) \frac{\partial H}{\partial x^j} \]
(1.1.11)
is called Hamiltonian system with the Hamiltonian \( H(x) \). It is a symmetry of the Poisson bracket
\[ \text{Lie}_{X_H} \{ \cdot, \cdot \} = 0. \]
(1.1.12)

Any function \( F \) commuting with the Hamiltonian
\[ \{ F, H \} = 0 \]
is a first integral of the Hamiltonian system (1.1.11). The Hamiltonian vector fields \( X_H, X_F \) commute.

1.1.4 Poisson cohomology

The notion of Poisson cohomology of \((P, \pi)\) was introduced by Lichnerowicz [98]. We need to use the Schouten - Nijenhuis bracket. Denote

\[ \Lambda^k = H^0(P, \Lambda^k TP) \]

the space of multivectors on \( P \). The Schouten - Nijenhuis bracket is a bilinear pairing \( a, b \mapsto [a, b] \),

\[ \Lambda^k \times \Lambda^l \rightarrow \Lambda^{k+l-1} \]

uniquely determined by the properties of supersymmetry
\[ [b, a] = (-1)^{kl}[a, b], \quad a \in \Lambda^k, \quad b \in \Lambda^l \]
(1.1.13)

the graded Leibnitz rule
\[ [c, a \wedge b] = [c, a] \wedge b + (-1)^{lk+k}[c, b] \wedge [c, b], \quad a \in \Lambda^k, \quad c, b \in \Lambda^l \]
(1.1.14)

and the conditions \([f, g] = 0, f, g \in \Lambda^0 = \mathcal{F}, \]
\[ [v, f] = v^i \frac{\partial f}{\partial x^i}, \quad v \in \Lambda^1 = Vect(P), \quad f \in \Lambda^0 = \mathcal{F}, \]
\[ [v_1, v_2] = \text{commutator of vector fields for } v_1, v_2 \in \Lambda^1. \]
In particular for a vector field \( v \) and a multivector \( a \)
\[ [v, a] = \text{Lie}_v a. \]
Example 1.1.2 For two bivectors $\pi = (\pi^{ij})$ and $\rho = (\rho^{ij})$ their Schouten - Nijenhuis bracket is the following trivector

$$[\pi, \rho]_{ijk} = \frac{\partial \pi^{ij}}{\partial x^s} \rho^{sk} + \frac{\partial \rho^{ij}}{\partial x^s} \pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s} \rho^{si} + \frac{\partial \rho^{jk}}{\partial x^s} \pi^{si}. \quad (1.1.15)$$

Observe that the l.h.s. of the Jacobi identity (1.1.4) reads

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} = \frac{1}{2}[\pi, \pi]_{ijk}. \quad (1.1.16)$$

The Schouten - Nijenhuis bracket satisfies the graded Jacobi identity [126]

$$(−1)^{km}[\lbrack a, b\rbrack, c] + (−1)^{lm}[\lbrack c, a\rbrack, b] + (−1)^{kl}[\lbrack b, c\rbrack, a] = 0, \quad a ∈ \Lambda^k, b ∈ \Lambda^l, c ∈ \Lambda^m. \quad (1.1.17)$$

It follows that, for a Poisson bivector $\pi$ the map

$$\partial : \Lambda^k → \Lambda^{k+1}, \quad \partial a = [\pi, a]$$

is a differential, $\partial^2 = 0$. The cohomology of the complex $(\Lambda^∗, \partial)$ is called Poisson cohomology of $(P, \pi)$. We will denote it

$$H^k(P, \pi) = \oplus_{k≥0} H^k(P, \pi).$$

In particular, $H^0(P, \pi)$ coincides with the ring of Casimirs of the Poisson bracket, $H^1(P, \pi)$ is the quotient of the Lie algebra of infinitesimal symmetries

$$v ∈ Vect(P), \quad Lie_v \pi = 0$$

over the subalgebra of Hamiltonian vector fields, $H^2(P, \{pi\})$ is the quotient of the space of infinitesimal deformations of the Poisson bracket by those obtained by infinitesimal changes of coordinates (i.e., by those of the form $Lie_v \pi$ for a vector field $v$). (See details in the lecture notes of J.P.Dufour and N.T.Zung.)

On a symplectic manifold $(P, \pi)$ Poisson cohomology coincides with the de Rham one. The isomorphism is established by “lowering the indices”: for a cocycle $a = (a^{i_1...i_k}) ∈ \Lambda^k$ the $k$-form

$$\sum_{i_1<...<i_k} ω_{i_1...i_k} dx^{i_1} ∧ ... ∧ dx^{i_k}, \quad ω_{i_1...i_k} = \pi_{i_1j_1}...\pi_{i_kj_k} a^{j_1...j_k}$$

is closed. In particular, for $P = \text{ball}$ the Poisson cohomology is trivial. In the general case $\text{rk}(\pi^{ij}) < \text{dim } P$ the Poisson cohomology does not vanish even locally (see [98]). We will prove now a simple criterion of triviality of 1- and 2-cocycles.

Lemma 1.1.3 Let $\pi = (\pi^{ij}(x))$ be a Poisson structure of a constant rank $2n < N$ on a sufficiently small ball $U$. 1). A one-cocycle $v = (v^i(x)) ∈ H^1(U, \pi)$ is trivial iff the vector field $v$ is tangent to the leaves of the symplectic foliation (1.1.7). 2). A 2-cocycle $f = (f^{ij}(x)) ∈ H^2(U, \pi)$ is trivial iff

$$f(dc', dc'') = 0 \quad (1.1.18)$$

for arbitrary two Casimirs of $h$.
Of course, the statement of the lemma can be easily derived from the results of [98]. Nevertheless, we give a proof since we will use similar arguments also in the infinite dimensional situation.

Proof 1). For a coboundary \( v = \partial f \) and for any Casimir \( c \) of \( \pi \) we have
\[
\partial_v c = \{c, f\} = 0.
\]
This means that \( v \) is tangent to the symplectic leaves (1.1.7). To prove the converse statement let us choose the canonical coordinates \( x = (y^1, \ldots, y^{2n}, c^1, \ldots, c^k) \) reducing the bracket to the constant form (1.1.5). Here \( c^1, \ldots, c^k \) are independent Casimirs (1.1.6). In these coordinates \( v = (v^1, \ldots, v^{2n}, 0, \ldots, 0) \). The 1-form \( \omega = (\omega_1, \ldots, \omega_{2n}, 0, \ldots, 0) \) given by
\[
\omega_i = \sum_{j=1}^{2n} \pi_{ij} v^j
\]
has the property
\[
d\omega|_{P_{c_0}} = 0.
\]
Therefore a function \( g \) locally exists s.t.
\[
dg = \sum_{i=1}^{2n} \omega_i dy^i + \sum_{a=1}^k \phi_a dc^a
\]
for some functions \( \phi_1, \ldots, \phi_k \). This function is the Hamiltonian for the vector field \( v \).

2). We will again use the canonical coordinates for \( \pi \) as in the proof of the first part. For an exact 2-cocycle \( \rho = \partial v \) and arbitrary two functions \( c', c'' \)
\[
\rho(dc', dc'') = -\{c', v^i\} \partial_i c'' - \partial_i c' \{v^i, c''\}.
\]
This is equal to zero if \( c' \) and \( c'' \) are Casimirs of the bracket \( \pi \).

To prove the converse statement we first consider, for every \( a = 1, \ldots, k \), the vector field \( w \) (depending on \( a \))
\[
w^i = \rho^{ia}, \quad i = 1, \ldots, N.
\]
From (1.1.18) it follows that \( w \) is tangent to the symplectic leaves of \( \pi \). The cocycle condition
\[
0 = [\pi, \rho]^{aij} = \partial_k \rho^{ai} \pi^{kj} + \partial_k \rho^{ja} \pi^{ki} = (\partial w)^{ij}
\]
implies \( \partial w = 0 \). According to the first part of the lemma, there exists a function \( q^a(x) \) s.t. \( w = \partial q^a \):
\[
\rho^{ia} = \pi^{ik} \partial_k q^a, \quad a = 1, \ldots, k.
\]
Let us now change the cocycle by a coboundary
\[
\rho \mapsto \rho + \partial z
\]
where the vector field $z$ is given by

$$z = \sum_{a=1}^{k} q^a \frac{\partial}{\partial c^a}. \quad (1.1.22)$$

After such a change, due to (1.1.21), we obtain

$$\rho^{ia} = \rho^{ai} = 0, \quad i = 1, \ldots, N.$$  

The rest of the proof repeats the arguments of the first part. The 2-form

$$\omega_{ij} = \sum_{i,j=1}^{2n} \pi_{ik} \pi_{lj} \rho^{kl}$$

is closed along the symplectic leaves. Hence there exists a 1-form $\phi = (\phi_i)$ s.t.

$$\omega = d\phi + \tilde{\omega}$$

where every monomial in $\tilde{\omega}$ contains at least one $dc^a$ for some $a$. Therefore

$$\rho = \partial u$$

for the vector field

$$u^i = \sum_{k=1}^{2n} \pi^{ik} \phi_k, \quad i = 1, \ldots, 2n, \quad u^i = 0, \quad i > 2n.$$  

The lemma is proved. \hfill \Box

### 1.1.5 Formalism of supermanifolds

Consider the $(N|N)$ supermanifold $\mathbf{N} = \Pi T^* P$. Coordinates on $\mathbf{N}$:

$$x^1, \ldots, x^N, \theta_1, \ldots, \theta_N, \quad x^j x^i = x^i x^j, \quad \theta_j \theta_i = -\theta_i \theta_j$$

Bivector

$$\pi = \frac{1}{2} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \mapsto \frac{1}{2} \pi^{ij}(x) \theta_i \theta_j =: \hat{\pi}$$

(a superfunction on $\mathbf{N} = \Pi T^* P$). (Super)Poisson bracket on $\mathbf{N}$

$$\{P, Q\} = \frac{\partial P}{\partial \theta_i} \frac{\partial Q}{\partial x^i} + (-1)^{|P|} \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial \theta_i} \quad (1.1.23)$$

$|P|$ = parity of the superfunction $|P|$.

**Claim:** Jacobi identity for $\pi$ $\Leftrightarrow \{\hat{\pi}, \hat{\pi}\} = 0$
Proof

\[ \{ \hat{\pi}, \hat{\pi} \} = \pi^{sk} \theta_k \frac{\partial \pi^{ij}}{\partial x^s} \theta_i \theta_j \]

\[ = \frac{1}{3} \left( \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s} \pi^{sj} \right) \theta_i \theta_j \theta_k \]

More generally (see the lecture notes of P.Cartier), for multivectors \( a \in \Lambda^k \), \( b \in \Lambda^l \),

\[ \hat{a} = \frac{1}{k!} a^{i_1 \ldots i_k} \theta_{i_1} \ldots \theta_{i_k}, \quad \hat{b} = \frac{1}{l!} b^{j_1 \ldots j_l} \theta_{j_1} \ldots \theta_{j_l} \]

the super-Poisson bracket

\[ \{ \hat{a}, \hat{b} \} = [a, b] \quad (1.1.24) \]

where \([a, b] =\) Schouten - Nijenhuis bracket of multivectors \( a \) and \( b \).

1.2 Bihamiltonian structures

1.2.1 Magri chains

Definition. A bihamiltonian structure on the manifold \( P \) is a 2-dimensional linear subspace in the space of Poisson structures on \( P \).

Choosing two nonproportional Poisson structures \( \pi_1 \) and \( \pi_2 \) in the subspace we obtain that the linear combination

\[ a_1 \pi_1 + a_2 \pi_2 \]

with arbitrary constant coefficients \( a_1, a_2 \) is again a Poisson bracket. This reformulation is usually referred to as the compatibility condition of the two Poisson brackets. It is spelled out as vanishing of the Schouten - Nijenhuis bracket

\[ [\pi_1, \pi_2] = 0. \quad (1.2.26) \]

An importance of bihamiltonian structures for recursive constructions of integrable systems was discovered by F.Magri [107] in the analysis of the so-called Lenard scheme (see in [67]) of constructing the KdV integrals. The basic idea of these constructions is given by the following simple

Lemma 1.2.4 Let \( H_0, H_1, \ldots, \) be a sequence of functions on \( P \) satisfying the recursion relation

\[ \{ \ldots, H_{p+1} \} = \{ \ldots, H_p \}, \quad p = 0, 1, \ldots \quad (1.2.27) \]

Then

\[ \{ H_p, H_q \} = \{ H_p, H_q \} = 0, \quad p, q = 0, 1, \ldots \]
For convenience of the reader we reproduce the proof of the lemma. Let $p < q$ and $q - p = 2m$ for some $m > 0$. Using the recursion and antisymmetry of the brackets we obtain
\[
\{ H_p, H_q \}_1 = \{ H_p, H_{q-1} \}_2 = -\{ H_{q-1}, H_p \}_2 = -\{ H_{q-1}, H_{p+1} \}_1 = \{ H_{p+1}, H_{q-1} \}_1.
\]
Iterating we arrive at
\[
\{ H_p, H_q \}_1 = \ldots = \{ H_{p+m}, H_{q-m} \}_1 = 0
\]
since $p + m = q - m$. Doing similarly in the case $q - p = 2m + 1$ we obtain
\[
\{ H_p, H_q \}_1 = \ldots = \{ H_n, H_{n+1} \}_1 = \{ H_n, H_n \}_2 = 0
\]
where $n = p + m = q - m - 1$. The commutativity $\{ H_p, H_q \}_2 = 0$ easily follows from the recursion. The Lemma is proved.

### 1.2.2 Symplectic bihamiltonian structures. Recursion operator

We have not used yet the compatibility condition of the two brackets. It turns out to be crucial in constructing the Hamiltonians satisfying the recursion relation (1.2.27). There are two essentially different realizations of the recursive procedure.

The first one applies to the case when the bihamiltonian structure is *symplectic*, i.e. $N = 2n$ and the Poisson structures of the affine line (1.2.25) do not degenerate for generic $a_1, a_2$. Without loss of generality one may assume nondegeneracy of $\pi_1$. The recursion operator $R : TP \to TP$ is defined by
\[
R := \pi_2 \cdot \pi^{-1}_1.
\]  
(1.2.28)
Here we identify the tensor $\pi_2$ with the linear map
\[
\pi_2 : T^* P \to TP
\]
and $\pi^{-1}_1$ with
\[
\pi^{-1}_1 : TP \to T^* P.
\]
The main recursion relation (1.2.27) can be rewritten in the form
\[
dH_{p+1} = R^* dH_p, \quad p = 0, 1, \ldots
\]
(1.2.29)
where $R^* : T^* P \to T^* P$ is the adjoint operator.
Theorem 1.2.5 [107, 106] The Hamiltonians

\[ H_p := \frac{1}{p+1} \text{tr} R^{p+1}, \quad p \geq 0 \]

satisfy the recursion (1.2.29).

Clearly there are at most \( n \) independent of these commuting functions. We say that the bihamiltonian symplectic structure is \textit{generic} if exactly \( n \) of these functions are independent. Let us denote \( \lambda_i = \lambda_i(x) \) the eigenvalues of the recursion operator. Since the characteristic polynomial of \( R \) is a perfect square

\[ \det (R - \lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)^2. \]

only \( n \) of these eigenvalues can be distinct, say, \( \lambda_1 = \lambda_1(x), \ldots, \lambda_n = \lambda_n(x) \). For generic bihamiltonian symplectic structure these are independent functions on \( P \ni x \).

Theorem 1.2.6 [107, 108, 106] Let \( \pi_{1,2} \) be a generic symplectic bihamiltonian structure. Then

1) All the commuting Hamiltonians

\[ H_p = \frac{1}{p+1} \text{tr} R^{p+1} = \frac{1}{p+1} \sum_{i=1}^{n} \lambda_i^{p+1}(x), \quad p = 0, 1, \ldots, n - 1 \]

generate completely integrable systems on \( P \).

2) The eigenvalues \( \lambda_i(x) \) can be included in a coordinate system \( \lambda_1, \mu_1, \ldots, \lambda_n, \mu_n \) in order to reduce the two Poisson structures to a block diagonal form where the \( i \)-th block in \( \pi_1 \) and in \( \pi_2 \) reads, respectively

\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \quad i = 1, \ldots, n. \]

The last formula gives the normal form of a generic symplectic bihamiltonian structure. Therefore all such structures are equivalent w.r.t. the group of local diffeomorphisms.

1.2.3 Poisson pencils of constant rank. Construction of hierarchies

Let us now consider the degenerate situation. We assume that the Poisson structure (1.2.25) has constant rank for generic \( a_1 \) and \( a_2 \). Without loss of generality we may assume that

\[ k = \text{corank} \pi_1 \equiv \text{corank}(\pi_1 + \epsilon \pi_2) \quad (1.2.30) \]

for an arbitrary sufficiently small \( \epsilon \).

Let us first prove the following useful property of bihamiltonian structures of the constant rank.

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Lemma 1.2.7 Let the bihamiltonian structure satisfy (1.2.30). Then the Casimirs of \( \pi_1 \) commute w.r.t. \( \pi_2 \).

Proof Let \( 2m \) be the rank of \( \pi_1 \). We first reduce the matrix of this bracket to the canonical constant block diagonal form. Denote \((\pi_{ab})\) the matrix of the second Poisson bracket in these coordinates. Let us now choose two integers \( i, j \) such that \( 2m < i < j \leq N = 2m + k \) and form a \((2m + 1) \times (2m + 1)\) minor of the matrix \( \pi_1 + \epsilon \pi_2 \) by adding \( i \)-th column and \( j \)-th row to the principal \( 2m \times 2m \) minor standing in the first \( 2m \) columns and first \( 2m \) rows. The condition (1.2.30) is equivalent to vanishing of the determinants of all these minors. It is easy to see that the determinant in question is equal to \(- \epsilon \pi_{ij} + O(\epsilon^2)\). Therefore \( \pi_{ij} = 0 \) for all pairs \((i,j)\) greater than \(2m\). The lemma is proved.

Corollary 1.2.8 For a compatible pair of Poisson brackets of the constant rank \((\pi_2 - \lambda \pi_1) = \text{rank} \pi_1, \lambda \to \infty\)

\[ \pi_2 \in H^2(P, \pi_1) \]

is a trivial cocycle.

Proof What \( \pi_2 \) is a cocycle w.r.t. the Poisson cohomology of \((P, \pi_1)\) follows from (1.2.26). To prove triviality use commutativity of the Casimirs of the first Poisson bracket and also Lemma 1.1.3.

A bihamiltonian structure with a marked line \( \lambda \pi_1 \) is called Poisson pencil. Choosing another Poisson bracket \( \pi_2 \) of the pencil one can represent the brackets of the bihamiltonian structure in the form

\[ \pi_\lambda := \pi_2 - \lambda \pi_1. \quad (1.2.31) \]

The representation (1.2.31) is well-defined up to an affine change of the parameter \( \lambda \)

\[ \lambda \mapsto a \lambda + b. \]

In the case of Poisson pencils of constant rank the corank of \( \pi_\lambda \) equals \( k \) for \( \lambda \to \infty \). The recursive construction of the commuting flows in this case is given by the following simple statement (cf. [107, 70, 132]).

Theorem 1.2.9 Under the assumption (1.2.30) the coefficients of the Taylor expansion

\[ c^\alpha(x, \lambda) = c^\alpha_0(x) + \frac{c^\alpha_0(x)}{\lambda} + \frac{c^\alpha_0(x)}{\lambda^2} + \ldots, \quad \lambda \to \infty \quad (1.2.32) \]

of the Casimirs \( c^\alpha(x, \lambda), \alpha = 1, \ldots, k \) of the Poisson bracket \( \pi_\lambda \) commute with respect to both the Poisson brackets

\[ \{c^\alpha_p, c^\beta_q\}_{1, 2} = 0, \quad \alpha, \beta = 1, \ldots, k, \quad p, q \geq -1. \]
Proof Spelling out the definition of the Casimirs
\[
\{ , \, c^\alpha \}_\lambda = 0
\]
for the coefficients of the expansion (1.2.32) we must have first that
\[
\{ , \, c^\alpha_{-1} \}_1 = 0. \quad (1.2.33)
\]
That is, the leading coefficients of the Taylor expansions are Casimirs of \{ , \}_1. For the subsequent coefficients we get the recursive relations
\[
\{ , \, c^\alpha_{p+1} \}_1 = \{ , \, c^\alpha_p \}_2, \quad p = -1, 0, 1, \ldots \quad (1.2.34)
\]
From (1.2.34) and Theorem 1 it follows that
\[
\{ c^\alpha_p, c^\alpha_q \}_{1,2} = 0, \quad p, q \geq -1.
\]
The commutativity \{ c^\alpha_p, c^\beta_q \}_{1,2} = 0 for \( \alpha \neq \beta \) easily follows from the same recursion trick and from commutativity of the Casimirs
\[
\{ c^\alpha_{-1}, c^\beta_{-1} \}_2 = 0 \quad (1.2.35)
\]
proved in Lemma 1.2.7. The theorem is proved. \( \square \)

1.2.4 Remark about the “method of argument translation”

Example 1.2.10 According to Corollary 1.2.8 there exists a vector field \( Z \) such that
\[
\text{Lie}_Z \pi_1 = \pi_2.
\]
We say, following [11] that the bihamiltonian structure is exact if the vector field \( Z \) can be chosen in such a way that
\[
(\text{Lie}_Z)^2 \pi_1 = 0. \quad (1.2.36)
\]
For an exact bihamiltonian structure the generating functions (1.2.32) of the commuting Hamiltonians \( c^\alpha_p(x) \) have the form
\[
c^\alpha(x; \lambda) = \exp (-Z/\lambda) c^\alpha_{-1}(x) = c^\alpha_{-1}(x) - \frac{1}{\lambda} \partial_Z c^\alpha_{-1}(x) + \frac{1}{\lambda^2} \partial^2_Z c^\alpha_{-1}(x) \ldots \quad (1.2.37)
\]
for every \( \alpha = 1, \ldots, k \).

This formula can be easily proved by choosing a system of local coordinates \( x^1, \ldots, x^N \) on the phase space \( P \) such that the vector field \( Z \) corresponds to the shift along \( x^1 \). In these coordinates the tensor of the first Poisson bracket depends linearly on \( x^1 \) and the second Poisson bracket is \( x^1 \)-independent. The Poisson pencil \( \pi_\lambda \) is obtained from \( \pi_1 \) by the shift \( x^1 \mapsto x^1 - 1/\lambda \) and by multiplication by \( -\lambda \).
Conversely, if, in a given coordinate system, \( \pi_1 \) depends linearly on one of the coordinates and \( \pi_2 \) does not depend on this coordinate then the bihamiltonian structure is exact. In particular this trick can be applied to the standard linear Lie - Poisson structures on the dual spaces to Lie algebras (cf \[?\]). In this case it was called in \[118\] the method of argument translation.

All our bihamiltonian structures on the loop spaces to be studied below will be exact. However, at the moment we do not see their Lie algebraic origin.

1.2.5 From Poisson pencils of constant rank to commuting hierarchies

The construction of Theorem 1.2.9 for a bihamiltonian structure of the constant corank \( k \) produces \( k \) chains of pairwise commuting bihamiltonian flows

\[
\frac{dx}{dt^{\alpha,p}} = \{x, c_p^\alpha\}_1 = \{x, c_p^{\alpha-1}\}_2, \quad \alpha = 1, \ldots, k, \quad p = 0, 1, 2, \ldots
\] (1.2.38)

The chains are labeled by the Casimirs \( c_{p-1}^\alpha \) of the first Poisson bracket. The level \( p \) in each chain corresponds to the number of iterations of the recursive procedure (we will keep using this expression although the recursion operator is not defined in the degenerate case). All the family of commuting flows organized by the above recursion procedure is called the hierarchy determined by the bihamiltonian structure.

The hierarchy structure of the constructed family of commuting flows depends non-trivially on the choice of \( \{ , \}_1 \) in the Poisson pencil (1.2.25). On the contrary, a different choice of the second Poisson bracket in the pencil produces a triangular linear transformation of the commuting Hamiltonians, i.e., to the Hamiltonians of the level \( p \) it will be added a linear combination of the Hamiltonians of the lower levels.

In the finite dimensional case we are discussing now all the chains of the hierarchy will be finite. In other words, the generating functions (1.2.32) of the commuting flows will become polynomials after multiplication by a suitable power of \( \lambda \) (the degrees of an appropriate system of these polynomials correspond to the type of the bihamiltonian structure [132]). A simple necessary and sufficient condition of complete integrability of the flows of the hierarchy was found by A.Brailov and A.Bolsinov (see in [7]). The problem of normal forms of degenerate bihamiltonian structures has been studied by I.M.Gelfand and I.Zakharevich [70], [71] for the case of the corank 1 and by I.Zakharevich [159] and A.Panasyuk [132] for higher coranks.

2 Lecture 2.

2.1 Formal loop spaces

Let \( M \) be a \( n \)-dimensional smooth manifold. Our aim is to describe an appropriate class of Poisson brackets on the loop space

\[
\mathcal{L}(M) = \{S^1 \to M\}.
\]
In our definitions we will treat $\mathcal{L}(M)$ formally in the spirit of formal variational calculus of [20, 17]. We define the formal loop space $\mathcal{L}(M)$ in terms of ring of functions on it. We also describe calculus of differential forms and vector fields on the formal loop space. In the next section we will also deal with multivectors on the formal loop space.

Let $U \subset M$ be a chart on $M$ with the coordinates $u^1, \ldots, u^n$. Denote $\mathcal{A} = \mathcal{A}(U)$ the space of polynomials in the independent variables $u^i, s = 1, 2, \ldots$

$$f(x; u; u_x, u_{xx}, \ldots) := \sum_{m \geq 0} f_{i_1 s_1; \ldots; i_m s_m}(x; u) u^{i_1 s_1} \ldots u^{i_m s_m} \quad (2.1.1)$$

with the coefficients $f_{i_1 s_1; \ldots; i_m s_m}(x; u)$ being smooth functions on $S^1 \times M$. Such an expression will be called differential polynomial. We will often use an alternative notation for the independent variables

$$u_x^i = u^{i_1}, \quad u_{xx}^i = u^{i_2}, \ldots$$

Observe that polynomiality w.r.t. $u = (u^1, \ldots, u^n)$ is not assumed.

The operator $\partial_x$ is defined as follows

$$\partial_x f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u^i} u_x^i + \ldots + \frac{\partial f}{\partial u_i^s} u_{xx}^{i,s} + \ldots \quad (2.1.2)$$

We will often use the notation

$$f^{(k)} := \partial_x^k f. \quad (2.1.3)$$

The following identities will be useful

$$\frac{\partial}{\partial u^i} \partial_x = \partial_x \frac{\partial}{\partial u^i} \quad (2.1.4)$$

$$\frac{\partial}{\partial u_i^s} \partial_x = \partial_x \frac{\partial}{\partial u_i^{s-1}} \quad + \frac{\partial}{\partial u_i^{s-2}} \quad (2.1.5)$$

(here $\frac{\partial}{\partial u^0} := \frac{\partial}{\partial u^0}$).

We define the space

$$\mathcal{A}_{0,0} = \mathcal{A}/\mathbb{R}, \quad \mathcal{A}_{0,1} = \mathcal{A}_{0,0} \, dx,$$

the operator

$$d : \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}, \quad df := \partial_x f \, dx \quad (2.1.6)$$

and the quotient

$$\Lambda_0 = \mathcal{A}_{0,1}/d\mathcal{A}_{0,0}. \quad (2.1.7)$$

The elements of the space $\Lambda_0$ will be written as integrals over the circle $S^1$

$$\bar{f} := \int f(x; u; u_x, u_{xx}, \ldots) \, dx \in \Lambda_0 \quad (2.1.8)$$

We will use below the following simple statement.
Lemma 2.1.1 If \( \int fg \, dx = 0 \) for an arbitrary \( g \in \mathcal{A} \) then \( f \in \mathcal{A} \) is equal to zero.

The expressions (2.1.8) are also called local functionals with the density \( f \). The space of local functionals is the main building block of the “space of functions” on the formal loop space. The full ring \( \mathcal{F} = \mathcal{F}(\mathcal{L}(U)) \) of functions on the formal loop space by definition coincides with the suitably completed symmetric tensor algebra of \( \Lambda_0 \)

\[
\mathcal{F} = \mathbb{R} \oplus \Lambda_0 \oplus \hat{S}^2\Lambda_0 \oplus \hat{S}^3\Lambda_0 \oplus \ldots
\]  

(2.1.9)

Elements of the (completed) \( k \)-th symmetric power \( \hat{S}^k\Lambda_0 \) will be written as multiple integrals of differential polynomials of \( k \) copies of the variables that we denote \( u^i(x_1), \ldots, u^i(x_k) \) etc.

\[
\int f(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); u_z(x_1), \ldots, u_z(x_k), \ldots) \, dx_1 \ldots dx_k \in \hat{S}^k\Lambda_0
\]  

(2.1.10)

(they are also called \( k \)-local functionals, cf. [153]). The short exact sequence

\[
0 \to \mathcal{A}_{0,0} \overset{d}{\to} \mathcal{A}_{0,1} \overset{\pi}{\to} \Lambda_0 \to 0
\]

(\( \pi \) is the projection) is included in the variational bicomplex

\[
\begin{array}{cccccc}
0 & \to & \mathcal{A}_{2,0} & \overset{d}{\to} & \mathcal{A}_{2,1} & \overset{\pi}{\to} \Lambda_2 & \to 0 \\
\uparrow{\delta} & & \uparrow{\delta} & & \uparrow{\delta} & & \\
0 & \to & \mathcal{A}_{1,0} & \overset{d}{\to} & \mathcal{A}_{1,1} & \overset{\pi}{\to} \Lambda_1 & \to 0 \\
\uparrow{\delta} & & \uparrow{\delta} & & \uparrow{\delta} & & \\
0 & \to & \mathcal{A}_{0,0} & \overset{d}{\to} & \mathcal{A}_{0,1} & \overset{\pi}{\to} \Lambda_0 & \to 0 \\
\uparrow{\delta} & & \uparrow{\delta} & & \uparrow{\delta} & & \\
0 & & 0 & & 0 & & 0
\end{array}
\]

Here \( \mathcal{A}_{k,l} \) are elements of the total degree \( k + l \) in the Grassman algebra with the generators \( \delta u^{i,s}, i = 1, \ldots, n, s = 0, 1, 2, \ldots \) (observe the difference in the range of the second index of \( u^{i,s} \) and \( \delta u^{i,s} \)) and \( dx \) with the coefficients in \( \mathcal{A} \) having the degree \( l \) in \( dx \). We will often identify

\[
\delta u^{i,0} = \delta u^i.
\]

For example, every \( k \)-form \( \omega \in \mathcal{A}_{k,0} \) is a finite sum

\[
\omega = \frac{1}{k!} \omega_{i_1 s_1; \ldots; i_k s_k} \delta u^{i_1 s_1} \wedge \ldots \wedge \delta u^{i_k s_k}
\]  

(2.1.11)

where the coefficients \( \omega_{i_1 s_1; \ldots; i_k s_k} \in \mathcal{A} \) are assumed to be antisymmetric w.r.t. permutations of pairs \( i_p, s_p \leftrightarrow i_q, s_q \).
The exterior differential in the Grassman algebra is decomposed into a sum \( d + \delta \). The horizontal differential
\[
d : A_{k,0} \rightarrow A_{k,1}
\]
is defined by
\[
d \omega = dx \wedge \partial_x \omega \tag{2.1.12}
\]
where the derivation \( \partial_x \)
\[
\partial_x(\omega_1 \wedge \omega_2) = \partial_x \omega_1 \wedge \omega_2 + \omega_1 \wedge \partial_x \omega_2
\]
is given by (2.1.2) on the coefficients of the differential form and by
\[
\partial_x \delta u^{i,s} = \delta u^{i,s+1}.
\]

The elements of the quotient
\[
\Lambda_k = A_{k,1}/dA_{k,0}
\]
will be called (local) \( k \)-forms on the loop space. \( k \)-forms will also be written by integrals
\[
\int dx \wedge \omega \in \Lambda_k, \ \omega \in A_{k,0}.
\]

**Example 2.1.2** Any one-form has a unique representative
\[
\phi = \int dx \wedge \phi_i \delta u^i \tag{2.1.13}
\]
(use integration by parts).

More generally, for every \( k \)-form \( \omega \) written as in (2.1.11)
\[
dx \wedge \omega \sim dx \wedge \tilde{\omega} \pmod{d(A_{k,0})}
\]
where
\[
\tilde{\omega} = \frac{1}{(k-1)!} \tilde{\omega}_{i_1;i_2:s_2;\ldots;i_k:s_k} \delta u^{i_1} \wedge \delta u^{i_2:s_2} \wedge \ldots \wedge \delta u^{i_k:s_k} \tag{2.1.14}
\]
\[
\tilde{\omega}_{i_1;i_2:s_2;\ldots;i_k:s_k} = \frac{1}{k} \sum_{0 \leq r_1 \leq s_l} \sum_{s \geq r_2 + \ldots + r_k} (-1)^s \binom{s}{r_2 \ldots r_k} \omega_{i_1;i_2:s_2;\ldots;i_k:s_k}(s-r_2-\ldots-r_k)
\]
here
\[
\binom{s}{r_2 \ldots r_k} = \frac{s!}{r_2! \ldots r_k!(s-r_2-\ldots-r_k)!} \tag{2.1.15}
\]
stands for the multinomial coefficients. The coefficients \( \tilde{\omega}_{i_1, s_1} \ldots i_k, s_k \) will be called \textit{reduced components} of \( \omega \). They are antisymmetric w.r.t. pairs \( i_2, s_2, \ldots, i_k, s_k \) but with the permutation of \( i_1 \) and \( i_2 \) they behave as

\[
\tilde{\omega}_{i_2, i_1, s_2} \ldots i_k, s_k = \sum_{0 \leq t_1 \leq s_1} \sum_{l \leq t_2} (-1)^{t_1+1} \begin{pmatrix} t \\ s_2 \\ t_3 \\ \ldots \\ t_k \end{pmatrix} \omega_{i_1; i_2; s_2; i_3; s_3; \ldots; i_k, s_k - t_k}
\]

(2.1.16)

We now define vertical arrows of the bicomplex. For a monomial

\[
\omega = f \delta u^{i_1, s_1} \wedge \ldots \wedge \delta u^{i_k, s_k}
\]

put

\[
\delta \omega = \sum_{t \geq 0} \frac{\partial f}{\partial u^{j,t}} \delta u^{j,t} \wedge \delta u^{i_1, s_1} \wedge \ldots \wedge \delta u^{i_k, s_k},
\]

(2.1.17)

where we denote

\[
\frac{\partial}{\partial u^{j,t}} := \frac{\partial}{\partial u^j}.
\]

This gives vertical differential \( \delta : A_{k,0} \to A_{k+1,0} \),

\[\delta^2 = 0.\]

The map \( \delta \) on \( A_{k,1} \) is defined by essentially same formula, \( \delta dx = 0 \). Anticommutativity

\[\delta d = -d\delta\]

justifies action of \( \delta \) on the quotient \( \Lambda_k \).

\textbf{Example 2.1.3} On \( \Lambda_0 \) the differential \( \delta \) acts as follows

\[
\delta \int f \, dx = \int dx \wedge \left( \sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}} \right) \delta u^i
\]

(\textit{the Euler - Lagrange differential}). We will use the notation

\[
\frac{\delta \bar{f}}{\delta u^i(x)} := \sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}}
\]

(2.1.19)

for the components of the 1-form, \( \bar{f} = \int f \, dx \).

\textbf{Theorem 2.1.4} \cite{17} For \( M = \text{ball} \) both arrows and columns of the variational bicomplex are exact.
Example 2.1.5 A necessary and sufficient condition for

\[
\frac{\delta \bar{f}}{\delta u^i(x)} = 0, \quad i = 1, \ldots, n.
\]

is the existence of a differential polynomial \( g = g(x; u; u_x; \ldots) \) such that \( f = \partial_x g \).

Let us now consider the space \( \Lambda^1 \) of vector fields on the formal loop space. These will be formal infinite sums

\[
\xi = \xi^0 \frac{\partial}{\partial x} + \sum_{k \geq 0} \xi^{i,k} \frac{\partial}{\partial u^{i,k}}, \quad \xi^{i,k} \in A
\]  

(2.1.20)

where we denote

\[
\frac{\partial}{\partial u^{i,0}} := \frac{\partial}{\partial u^i}.
\]

The derivative of a functional \( \bar{f} = \int f(x; u; u_x; \ldots) dx \in \Lambda_0 \) along \( \xi \) reads

\[
\xi \bar{f} := \int \left( \xi^0 \frac{\partial f}{\partial x} + \sum \xi^{i,k} \frac{\partial f}{\partial u^{i,k}} \right) dx.
\]

The Lie bracket of two vector fields is defined by

\[
[\xi, \eta] = (\xi^0 \eta^0_x - \eta^0 \xi^0_x + \xi^{j,t} \frac{\partial \eta^0}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^0}{\partial u^{j,t}}) \frac{\partial}{\partial x} + \sum_{s \geq 0} \left( \xi^0 \frac{\partial \eta^{i,s}}{\partial x} - \eta^0 \frac{\partial \xi^{i,s}}{\partial x} + \xi^{j,t} \frac{\partial \eta^{i,s}}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^{i,s}}{\partial u^{j,t}} \right) \frac{\partial}{\partial u^{i,s}}
\]  

(2.1.21)

Evolutionary vector fields \( a \) are defined by the conditions of vanishing of the \( \partial/\partial x \)-component and the commutativity

\[
[\partial_x, a] = 0
\]

They are parameterized by \( n \)-tuples \( a^1, \ldots, a^n \) of elements of \( A \) as follows

\[
a = \sum_{s \geq 0} \partial_s a^i \frac{\partial}{\partial u^{i,s}}.
\]  

(2.1.22)

The corresponding system of evolutionary PDEs reads

\[
u_x^i = a^i(x; u; u_x, u_{xx}, \ldots), \quad i = 1, \ldots, n.
\]  

(2.1.23)

In particular, an evolutionary vector field \( a \) is called translation invariant if the coefficients \( a^i \) do not depend explicitly on \( x \),

\[
\frac{\partial a^i}{\partial x} = 0, \quad i = 1, \ldots, n.
\]
The contraction $i_{\xi}\omega$ of a $k$-form $\omega \in \mathcal{A}_{k,0}$ given by (2.11) and a vector field $\xi$ is a $(k-1)$-form defined by
\begin{equation}
   i_{\xi}\omega = \frac{1}{(k-1)!} \xi^{j_1}\omega_{j_1i_1s_1: \ldots :i_{k-1}s_{k-1}} \delta u^{i_1s_1} \wedge \ldots \wedge \delta u^{i_{k-1}s_{k-1}}.
\end{equation}
(2.1.24)

As usual
\begin{equation*}
   i_{\xi}i_{\eta} = -i_{\eta}i_{\xi}
\end{equation*}
for two vector fields $\xi, \eta$. For a form $\omega \in \mathcal{A}_{k,1}$ the contraction $i_{\xi}\omega \in \mathcal{A}_{k-1,1}$ is defined by essentially same formula provided the vector field $\xi$ contains no $\partial/\partial x$-term. It is an easy exercise to check, using (2.1.5), that for an evolutionary vector field $a$
\begin{equation}
   da + ia = 0.
\end{equation}
(2.1.25)

It readily follows that contraction with evolutionary vector fields is well-defined on the quotient $i_{\xi} : \Lambda_k \to \Lambda_{k-1}$. A more strong statement holds true

**Lemma 2.1.6** Let $\omega \in \mathcal{A}_{k,1}$. It belongs to $d(\mathcal{A}_{k,0})$ iff $i_{a}\omega \in d(\mathcal{A}_{k-1,0})$ for an arbitrary evolutionary vector field $a$.

**Proof** We use induction in $k$. For $k = 1$ we can choose a representative of the class of $\omega \wedge dx \in \Lambda_1$ with the 1-form $\omega$ given by (2.13). The contraction reads
\begin{equation*}
   i_{a}(dx \wedge \omega) = -\int dx a^i \omega_i.
\end{equation*}

Using Lemma 2.1.1 we obtain $\omega_i = 0$ for all $i$.

Let us assume validity of the lemma for any $(k-1)$-form. We will prove that the condition $i_{a}\omega \wedge dx = 0 \in \Lambda_{k-1}$ implies vanishing of all the reduced components (2.14). By induction the above condition is equivalent to
\begin{equation*}
   i_{b_1} \ldots i_{b_k} i_{a}\omega \wedge dx \in d(\mathcal{A}_{0,0})
\end{equation*}
for arbitrary evolutionary vector fields $b_1, \ldots, b_k$. Integrating by parts we rewrite the last line in the form
\begin{equation*}
   \int a^i \phi_i dx = 0
\end{equation*}
(2.1.26)
where
\begin{equation*}
   \phi_i = k \omega_{i_1 \ldots i_k s_1 : \ldots : i_{k-1}s_{k-1}} \partial_{x}^{s_1}b_1^{i_1} \ldots \partial_{x}^{s_k}b_k^{i_k}.
\end{equation*}
From (2.1.26) it follows that $\phi_i = 0$ for all $i$. Since $b_1^i, \ldots, b_k^i$ are arbitrary differential polynomials, this implies $\omega_{i_1 \ldots i_k s_1 : \ldots : i_{k-1}s_{k-1}} = 0$. That means that the form $\omega \wedge dx$ is equivalent to zero, modulo $d(\mathcal{A}_{k,0})$. The lemma is proved.

**Corollary 2.1.7** A form $\omega \in \mathcal{A}_{k,1}$ belongs to $d(\mathcal{A}_{k,0})$ iff
\begin{equation*}
   i_{a_1} \ldots i_{a_k}\omega \in d(\mathcal{A})
\end{equation*}
for arbitrary evolutionary vector fields $a_1, \ldots, a_k$. 

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Example 2.1.8 For the one-form $\omega = \delta \int f \, dx$ the contraction $i_a \omega$ reads

$$i_a \omega = \int a^i \frac{\delta \tilde{f}}{\delta u^i(x)} \, dx \in \Lambda_0.$$  
This is the time derivative of the functional $\tilde{f} = \int f \, dx$ w.r.t. the evolutionary system (2.1.23).

Example 2.1.9 For a one-form $\omega = \omega^i \delta u^i \wedge dx \in \Lambda_1$ the condition of closedness

$$\delta \omega = 0 \in \Lambda_2$$
reads

$$\frac{\partial \omega_i}{\partial u^j} = \sum_{t \geq s} (-1)^t \binom{t}{s} \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{s+1}}$$

(2.1.27)
for any $i, j = 1, \ldots, n$, $s = 0, 1, \ldots$. This is the classical Volterra’s criterion [153] for the system of ODEs

$$\omega_1(x; u; u_x, u_{xx}, \ldots) = 0, \ldots, \omega_n(x; u; u_x, u_{xx}, \ldots) = 0$$
to be locally representable in the Euler-Lagrange form

$$\omega_i = \frac{\delta \tilde{f}}{\delta u^i(x)}, \quad i = 1, \ldots, n$$
(use exactness of the variational bicomplex).

Example 2.1.10 For a 2-form

$$\omega = \frac{1}{2} \omega_{is,jr} \delta u^i \wedge \delta u^j,$$
the contraction with two evolutionary vector fields $a$ and $b$ can be represented in the form

$$i_a i_b \omega = -2 \int a^i \tilde{\omega}_{is} b^j \partial_x^{s+1} \, dx$$

(2.1.28)
where

$$\tilde{\omega}_{is} = \frac{1}{2} \sum_{r=0}^{s} \sum_{t \geq s-r} (-1)^t \binom{t}{s-r} \partial_x^{t-s+r} \omega_{is;jr}$$
are the reduced components (2.1.14). The tilde will be omitted in the subsequent formulae. According to this we will often represent 2-forms in the reduced form

$$dx \wedge \omega = \omega_{is} dx \wedge \delta u^i \wedge \delta u^is.$$  
(2.1.29)
The reduced coefficients must satisfy the antisymmetry conditions (2.1.16). They are spelled out as follows

$$\omega_{is} = \sum_{t \geq s} (-1)^{t+1} \binom{t}{s} \partial_x^{t-s} \omega_{is;u}$$

(2.1.30)
(integrate by parts in (2.1.28) and use arbitrariness of $a$ and $b$).
Example 2.1.11 A 2-form $dx \wedge \omega = \delta dx \wedge \phi$ for

$$\phi = \phi_i \delta u^i$$

has the reduced representative (2.1.29) with

$$\omega_{ijs} = \frac{1}{2} \left( \frac{\partial \phi_i}{\partial u^j} + \sum_{t \geq s} (-1)^{t+1} \left( \begin{array}{c} t \\ s \end{array} \right) \frac{\partial \phi_j}{\partial u^t} \frac{\partial \phi_l}{\partial u^s} \right). \tag{2.1.31}$$

Example 2.1.12 A 2-form (2.1.29) is closed, $\delta \omega = 0$, iff

$$\left( \sum \sum_{m-s} + \sum \sum_{m \geq l} \right) (-1)^m \left( \begin{array}{c} m \\ r s \end{array} \right) \partial_{x^{m-r-s}} \omega_{jkl} - \frac{\partial \omega_{jkl}}{\partial u^m} - \frac{\partial \omega_{jkl}}{\partial u^r} = 0 \tag{2.1.32}$$

for any $i, j, k = 1, \ldots, n$, $s = 0, 1, 2, \ldots$.

Proof By definition

$$\delta(\omega) = \sum \frac{\partial \omega_{ijs}}{\partial u^k} \delta u^i \wedge \delta u^j \wedge \delta u^k \wedge dx.$$ So $\delta \omega = 0$ means that for any three evolutionary vector fields

$$a = \sum (a^i)^{(s)} \frac{\partial}{\partial u^i}, \quad b = \sum (b^i)^{(s)} \frac{\partial}{\partial u^i}, \quad c = \sum (c^i)^{(s)} \frac{\partial}{\partial u^i}$$

the contraction $i_a i_b i_c \delta(dx \wedge \omega) \in d(A_0, \omega)$, i.e.,

$$\int \frac{\partial \omega_{ijs}}{\partial u^k} \left[ a^i (b^k)(l) (c^j)^{(s)} - a^i (b^j)^{(s)} (c^k)(l) - (a^k)(l) b^j (c^j)^{(s)} - (a^k)(l) (b^k)(s) c^j \right] dx = 0 \tag{2.1.33}$$

Using integration by parts we get

$$\int \frac{\partial \omega_{ijs}}{\partial u^k} \left[ a^i (b^k)(l) (c^j)^{(s)} - a^i (b^j)^{(s)} (c^k)(l) \right]$$

$$+ \sum (-1)^{m+1} \left( \begin{array}{c} m \\ l r \end{array} \right) \partial_{x^{m-l-r}} \omega_{ijs} \frac{\partial}{\partial u^k} \left( a^k (b^j)^{(l)} (c^j)^{(s+r)} - a^k (b^j)^{(s+l)} (c^j)^{(r)} \right)$$

$$+ \sum (-1)^m \left( \begin{array}{c} m \\ s r \end{array} \right) \partial_{x^{m-s-r}} \omega_{jkl} \frac{\partial}{\partial u^k} \left( a^j (b^i)^{(s)} (c^k)^{(l+r)} - a^j (b^k)^{(l+s)} (c^i)^{(r)} \right) dx = 0$$

The above identity is equivalent to

$$\frac{\partial \omega_{ijs}}{\partial u^k} = \frac{\partial \omega_{ijs}}{\partial u^r}$$
components of a covector are described by their reduced components \( V \) such objects comprise the space \( \Lambda^0 \) the formal loop space taking the tensor algebra of \( \Lambda^0 \). Remark 2.1.13 The equations (2.1.32) were derived by Dorfman in the theory of the so-called local symplectic structures [28], see also the book [29].

Corollary 2.1.14 Any solution to (2.1.32) satisfying (2.1.30) can be locally represented in the form (2.1.31).

This follows from exactness of the variational bicomplex.

We will briefly outline necessary points of the global picture of functionals, differential forms and vector fields on the formal loop space \( \mathcal{L}(M) \) for a general smooth manifold \( M \) (i.e., not only for a ball). The corresponding objects must be defined for any chart \( U \subset M \) as it was explained above. On the intersections \( U \cap V \) they must satisfy certain consistency conditions. For example, functionals in the charts \( U, V \) with the coordinates \( u^1, \ldots, u^n \) and \( v^1, \ldots, v^n \) are defined by densities \( f_U(x; u; u_x, \ldots) \) and \( f_V(x; v; v_x, \ldots) \) s.t.

\[
f_V(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots) dx = f_U(x; u; u_x, \ldots) dx \pmod{\text{Im} d}.
\]

Such objects comprise the space \( \Lambda_0(M) \). As above, we obtain the ring of functions on the formal loop space taking the tensor algebra of \( \Lambda_0(M) \). One-forms in the charts \( U, V \) are described by their reduced components \( \omega^U_i \) and \( \omega^V_i \) s.t., on \( U \cap V \) transform as components of a covector

\[
\omega^V_a(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots) = \omega^U_i(x; u; u_x, \ldots) \frac{\partial u^i}{\partial u^a}
\]

etc. The components of evolutionary vector fields transform like vectors

\[
a^V_a(x; v(u); \frac{\partial v}{\partial u} u_x, \ldots) = \frac{\partial v^k}{\partial u^a} a^U_i(x; u; u_x, \ldots).
\]

Now by using the antisymmetry condition (2.1.30) we see that the third term in the above sum is equal to the fourth term, and by using the identity (2.1.5) and the antisymmetry condition (2.1.30) again we see that the last two terms equal to the second and first term respectively. Thus we arrive at the proof of (2.1.32). \( \square \)
The contraction \( i_a dx \wedge \omega \) of a 1-form with an evolutionary vector field is well-defined as an element of \( \Lambda_0(M) \).

The global theory of the variational bicomplex was developed in [146], [152].

## 2.2 Local multivectors and local Poisson brackets

We first define more general, i.e. non-local \( k \)-vectors as elements of \( (\Lambda^1)^k \). They will be written as infinite sums of expressions of the form

\[
\alpha = \frac{1}{k!} \sum \frac{\partial}{\partial u^{i_1 s_1} (x_1)} \wedge \ldots \wedge \frac{\partial}{\partial u^{i_k s_k} (x_k)} \beta (x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), \ldots)
\]

(2.2.1)

(in this subsection we will consider only multivectors not containing \( \partial/\partial x \)). The coefficients must satisfy the antisymmetry condition w.r.t. simultaneous permutations

\( i_p, s_p, x_p \leftrightarrow i_q, s_q, x_q \).

The exterior algebra structure on multivectors is introduced in a usual way: the product of a \( k \)-vector \( \alpha \) by a \( l \)-vector \( \beta \) is a \((k+l)\)-vector

\[
(\alpha \wedge \beta)^{i_1 s_1; \ldots; i_k s_k; i_{k+1} s_{k+1}; \ldots; i_{k+l} s_{k+l}} (x_1, \ldots, x_{k+l}; u(x_1), \ldots, u(x_{k+l}), \ldots)
\]

\[
= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn} \sigma \alpha^{i_{\sigma(1)} s_{\sigma(1)}; \ldots; i_{\sigma(k)} s_{\sigma(k)}} (x_{\sigma(1)}, \ldots, x_{\sigma(k)}; u(x_{\sigma(1)}), \ldots, u(x_{\sigma(k)}), \ldots)
\]

\[
\times \beta^{i_{\sigma(k+1)} s_{\sigma(k+1)}; \ldots; i_{\sigma(k+l)} s_{\sigma(k+l)}} (x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)}; u(x_{\sigma(k+1)}), \ldots, u(x_{\sigma(k+l)}), \ldots)
\]

(2.2.2)

**Example 2.2.1** Lie derivative of a \( k \)-vector \( \alpha \) (2.2.1) along a vector field (2.1.20) reads

\[
\text{Lie}_\xi \alpha^{i_1 s_1; \ldots; i_k s_k} = \sum_{p=1}^k \left[ \xi^0 (x_p; u(x_p); \ldots) \frac{\partial}{\partial x_p} \alpha^{i_1 s_1; \ldots; i_k s_k} + \xi^{j_p t_p} (x_p; \ldots) \frac{\partial}{\partial u^{j_p t_p} (x_p)} \alpha^{i_1 s_1; \ldots; i_k s_k} \right]
\]

\[
- \sum_{p=1}^k \frac{\partial \xi^{j_p t_p}}{\partial u^{j_p t_p} (x_p)} \alpha^{i_1 s_1; \ldots; i_{k-1} s_{k-1}; j_p t_p; \ldots; i_k s_k}.
\]

(2.2.3)

Here we assume that \( \xi^0 \) does not depend on \( u^{j t} \).

**Definition.** A \( k \)-vector \( \alpha \) is called translation invariant if

\[
\text{Lie}_{\partial_x} \alpha = 0
\]
and also the sum of partial derivatives of the coefficients $\alpha^{i_1\ldots i_k s_k}$

\[ \alpha^{i_1\ldots i_k s_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); u_x(x_1), \ldots, u_x(x_k)) \text{ equals zero} \]

\[ \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_k} \alpha^{i_1\ldots i_k s_k} = 0. \]

**Lemma 2.2.2** Every translation invariant $k$-vector $\alpha$ has coefficients of the form

\[ \alpha^{i_1\ldots i_k s_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots) \]

\[ = \partial_{x_1}^{s_1} \ldots \partial_{x_k}^{s_k} A^{i_1\ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots) \]

(2.2.4)

where the differential polynomials $A^{i_1\ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots)$ are antisymmetric w.r.t. simultaneous permutations

\[ i_p, x_p \leftrightarrow i_q, x_q \]

and also they satisfy

\[ A^{i_1\ldots i_k}(x_1 + t, \ldots, x_k + t; u(x_1), \ldots, u(x_k); \ldots) = A^{i_1\ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots) \]

for any $t$.

The functions $A^{i_1\ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots)$ will be called components of the translation invariant $k$-vector $\alpha$.

Translation invariant multivectors form a graded Lie subalgebra of the full graded Lie algebra of multivectors closed w.r.t. Schouten - Nijenhuis bracket.

**Example 2.2.3** The Lie derivative of a bivector $\alpha$ with the components $A^{ij}(x - y; u(x), u(y); \ldots)$ along a translation invariant vector field $a$ with the components $a^i(u, u_x, \ldots)$ has the components

\[ \text{Lie}_a \alpha^{ij} = \partial_a^i a^k(u(x); \ldots) \frac{\partial A^{ij}}{\partial u^{k,l}(x)} + \partial_a^j a^k(u(y); \ldots) \frac{\partial A^{ij}}{\partial u^{k,l}(y)} \]

\[ - \frac{\partial a^i(u(x); \ldots)}{\partial u^{k,l}(x)} \partial_x^{k,l} A^{ij} - \frac{\partial a^j(u(y); \ldots)}{\partial u^{k,l}(y)} \partial_y^{k,l} A^{ij}. \]

(2.2.5)

**Example 2.2.4** Let $\alpha$, $\beta$ be two translation invariant bivectors with the components $A^{ij}(x - y; u(x), u(y); u_x(x), u_x(y), \ldots)$ and $B^{ij}(x - y; u(x), u(y); u_x(x), u_x(y), \ldots)$ that we redenote resp. $A^{ij}_{x,y}$ and $B^{ij}_{x,y}$ for brevity. The Schouten - Nijenhuis bracket $[\alpha, \beta]$ is a translation invariant trivector with the components

\[ [\alpha, \beta]^{ijk}_{x,y,z} = \partial A^{ij}_{x,y} \frac{\partial B^{kl}_{x,z}}{\partial u^{l,s}(x)} + \partial B^{ij}_{x,y} \frac{\partial A^{kl}_{x,z}}{\partial u^{l,s}(x)} + \partial A^{ij}_{x,y} \frac{\partial B^{kl}_{x,z}}{\partial u^{l,s}(y)} + \partial B^{ij}_{x,y} \frac{\partial A^{kl}_{x,z}}{\partial u^{l,s}(y)} \]

\[ + \frac{\partial A^{ij}_{x,y}}{\partial u^{l,s}(z)} \partial_z B^{kl}_{x,y} + \frac{\partial B^{ij}_{x,y}}{\partial u^{l,s}(z)} \partial_z A^{kl}_{x,y} \]

\[ + \frac{\partial A^{jk}_{y,z}}{\partial u^{l,s}(y)} \partial_y B^{ij}_{x,z} + \frac{\partial B^{jk}_{y,z}}{\partial u^{l,s}(y)} \partial_y A^{ij}_{x,z} \]

\[ + \frac{\partial A^{ij}_{y,z}}{\partial u^{l,s}(y)} \partial_y B^{kl}_{x,z} + \frac{\partial B^{ij}_{y,z}}{\partial u^{l,s}(y)} \partial_y A^{kl}_{x,z}. \]

(2.2.6)
For a translation invariant $k$-vector $\alpha$ and $k$ 1-forms $\omega^1, \ldots, \omega^k$,

$$\omega^j = \omega^j_{is} \delta u^i_s \wedge dx \in A_{1,1}, \ j = 1, \ldots, k$$

the contraction

$$< \alpha, \omega^1 \wedge \ldots \wedge \omega^k >$$

is well defined on $A_0^{\otimes k}$.

**Example 2.2.5** The value of a translation invariant $k$-vector $\alpha$ with the components $A^{i_1 \ldots i_k}$ on the 1-forms $\delta f^1, \ldots, \delta f^k$ equals

$$< \alpha, \delta f^1 \wedge \ldots \wedge \delta f^k > = \int \frac{\delta f^1}{\delta u^{i_1}(x_1)} \ldots \frac{\delta f^k}{\delta u^{i_k}(x_k)} A^{i_1 \ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); \ldots)dx_1 \ldots dx_k \in A_0^{\otimes k}. \quad (2.2.8)$$

The transformation law of components of translation invariant multivectors w.r.t. changes of coordinates on the intersection of two coordinate charts $(U, u^1, \ldots, u^n)$ and $(V, v^1, \ldots, v^n)$ is analogous to the transformation law of components of multivectors on a finite-dimensional manifolds:

$$A^{a_1 \ldots a_k}_{U}(x_1, \ldots, x_k; v(u(x_1)), \ldots, v(u(x_k)); \frac{\partial v}{\partial u}(x_1), \ldots, \frac{\partial v}{\partial u}(x_k), \ldots) = \frac{\partial v^{a_1}}{\partial u^{i_1}(x_k)} A^{i_1 \ldots i_k}_{U}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k); u(x_1), \ldots, u(x_k), \ldots). \quad (2.2.9)$$

We now proceed to the main definition of local multivectors. They are translation invariant multivectors $\alpha$ such that their dependence on $x_1, \ldots, x_k$ is given by a finite order distribution with the support on the diagonal $x_1 = x_2 = \ldots = x_k$

$$A^{i_1 \ldots i_k} = \sum_{p_2, p_3, \ldots, p_k \geq 0} B^{i_1 \ldots i_k}_{p_2 \ldots p_k}(u(x_1); u(x_1), \ldots) \delta(p_2)(x_1-x_2) \delta(p_3)(x_1-x_3) \ldots \delta(p_k)(x_1-x_k). \quad (2.2.10)$$

The coefficients $B^{i_1 \ldots i_k}_{p_2 \ldots p_k}(u(x_1); u(x_1), \ldots)$ are differential polynomials in $A$ not depending explicitly on $x$. All the sums at the moment are assumed to be finite. In the next section we will relax this condition. Delta functions and their derivatives and products are defined by the formulae

$$\int f(y)\delta(x-y)dy = f(x), \quad \int f(y)\delta^{(p)}(x-y)dy = f^{(p)}(x) \quad (2.2.11)$$
\[ \int f(x_1, \ldots, x_k) \delta^{(p_1)}(x_1 - x_2) \delta^{(p_3)}(x_1 - x_3) \ldots \delta^{(p_k)}(x_1 - x_k) \, dx_2 \ldots dx_k = \partial_{x_2}^{p_2} \ldots \partial_{x_k}^{p_k} f(x_1, \ldots, x_k)|_{x_1 = x_2 = \ldots = x_k}. \]

**Lemma 2.2.6** The value (2.2.7) of a local \( k \)-vector \( \alpha \) on \( k \) 1-forms \( \omega^1, \ldots, \omega^k \) is given by

\[ < \alpha, \omega^1 \wedge \ldots \wedge \omega^k > = \int B_{i_1 \ldots i_k}^{j_1 \ldots j_k} (u; u_x, u_{xx}, \ldots) \omega^1_{i_1} (x; u; u_x, \ldots) \partial_{x}^{j_2} \omega^2_{j_2} (x; u; u_x, \ldots) \ldots \partial_{x}^{j_k} \omega^k_{i_k} (x; u; u_x, \ldots) \, dx \in \Lambda_0. \]  

(2.2.12)

It gives a well-defined polylinear map

\[ \alpha : \Lambda^\otimes k \to \Lambda_0. \]

In calculations with local multivectors various simple identities for delta-functions will be useful. All of them are simple consequences of the definition (2.2.11). First,

\[ f(y) \delta^{(p)}(x - y) = \sum_{q=0}^{p} \binom{p}{q} f^{(q)}(x) \delta^{(p-q)}(x - y). \]  

(2.2.13)

Next,

\[ \delta(x_1 - x_2) \ldots \delta(x_1 - x_k) = \delta(x_2 - x_1) \delta(x_2 - x_3) \ldots \delta(x_2 - x_k) = \ldots = \delta(x_k - x_1) \ldots \delta(x_k - x_{k-1}). \]  

(2.2.14)

Differentiating (2.2.14) w.r.t. \( x_1, \ldots, x_k \) we will obtain relations between products of derivatives of delta-functions.

We leave as a simple exercise for the reader to prove that the space of local multivectors that we denote

\[ \Lambda^\ast_{\text{loc}} = \oplus \Lambda^k_{\text{loc}} \]

is closed w.r.t. the Schouten - Nijenhuis bracket. Warning: this space is not closed w.r.t. the exterior product! Because of this we were to introduce a wider algebra of multivectors to introduce the definition of the Schouten - Nijenhuis bracket according to the rules one uses in the finite dimensional case.

**Example 2.2.7** The component of a local bivector \( \varpi \) has the form

\[ \varpi^{ij} = \sum_{s \geq 0} A^{ij}_s (u(x); u_x(x), \ldots) \delta^{(s)}(x - y). \]  

(2.2.15)

The value of the bivector on two 1-forms \( \phi = \phi_i \delta u^i \wedge dx \) and \( \psi = \psi_i \delta u^i \wedge dx \) equals

\[ \int \phi_i A^{ij}_s \partial_x^j \psi_j \, dx. \]  

(2.2.16)
The conditions of antisymmetry of the bivector reads

\[ A^i_j = \sum_{t \geq s} (-1)^{t+1} \binom{t}{s} \partial_x^{t-s} A_t^i_j. \]  

**Proof** Let us explain how to prove the antisymmetry condition (2.2.17). We must have

\[ \sum_s A^i_j(u(y); \ldots)\delta^{(s)}(y - x) = - \sum_A^i_j(u(x); \ldots)\delta^{(s)}(x - y). \]

Using \( \delta^{(s)}(y - x) = (-1)^s\delta^{(s)}(x - y) \) and (2.2.13) we obtain (2.2.17). \( \Box \)

**Remark 2.2.8** One can represent the bivector as

\[ A^i_j(u(x); u_x(x), \ldots; \frac{d}{dx})\delta(x - y). \]  

Here the differential operators \( A^i_j \) are

\[ A^i_j(x; u(x); u_x(x), \ldots; \frac{d}{dx}) = \sum_s A^i_j d^s. \]

For local multivectors of higher rank the language of differential operators was used by Olver [129].

**Example 2.2.9** The Schouten - Nijenhuis bracket of the bivector

\[ \varpi = \pi^{ij}\delta(x - y), \]

where \( \pi^{ij} \) is a constant antisymmetric matrix, with \( \alpha \) of the form (2.2.15) reads

\[ ([\varpi, \alpha])_{x,y,z}^{ijk} = \left[ \frac{\partial A^{ij}_k}{\partial u^{l,s}} \pi^{lk} + \sum (-1)^{q+r+s} \binom{q + r + s}{q r} \left( \frac{\partial A^{ij}_k}{\partial u^{l,t-q}} \right)^{(r)} \pi^{lj} \right. \]

\[ + \sum (-1)^{q+r+t} \binom{q + r + t}{q r} \left( \frac{\partial A^{ij}_k}{\partial u^{l,q+r+t}} \right)^{(r)} \pi^{li} \delta^{(v)}(x - y)\delta^{(s)}(x - z). \]  

**Proof** Substituting into (2.2.6) we obtain

\[ ([\varpi, \alpha])_{x,y,z}^{ijk} = \frac{\partial A^{ij}_k(x)}{\partial u^{l,s}(x)} \pi^{lk}\delta^{(v)}(x - y)\delta^{(s)}(x - z) \]

\[ + \frac{\partial A^{ij}_k(z)}{\partial u^{l,s}(z)} \pi^{ij}\delta^{(v)}(z - x)\delta^{(s)}(z - y) + \frac{\partial A^{ij}_k(y)}{\partial u^{l,s}(y)} \pi^{ij}\delta^{(v)}(y - z)\delta^{(s)}(y - x). \]  

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Here $A_{ij}^l(x), A_{jk}^l(y), A_{ki}^l(z)$ stand for $A_{ij}^l(u(x); \ldots), A_{jk}^l(u(y); \ldots), A_{ki}^l(u(z); \ldots)$ respectively. Use the identities (2.2.14)

$$\delta^{(v)}(z - x)\delta^{(s)}(z - y) = (-\partial_y)^{t}(\partial_y)^{q}\delta^{(r)}(z - x)\delta^{(s)}(z - y)$$

$$= (-\partial_y)^{t}(\partial_y)^{q}\delta^{(r)}(x - y)\delta^{(s)}(x - z),$$

$$\delta^{(v)}(y - z)\delta^{(s)}(y - x) = (-\partial_x)^{t}(\partial_x)^{q}\delta^{(r)}(y - z)\delta^{(s)}(y - x)$$

$$= (-\partial_x)^{s}(\partial_x)^{t}\delta^{(r)}(x - y)\delta^{(s)}(x - z)$$

and also (2.2.13) to arrive at (2.2.19). \hfill \Box

**Definition.** A local Poisson structure on the formal loop space is a local bivector $\varpi \in \Lambda^2_{\text{loc}}$ (2.2.21) satisfying $[\varpi, \varpi] = 0$.

Adopting notation common in the physics literature we will represent the Poisson structure in the form

$$\{u^i(x), u^j(y)\} = \sum_s A_{ij}^s(u(x); u_x(x), u_{xx}(x), \ldots)\delta^{(s)}(x - y). \tag{2.2.21}$$

The Poisson bracket of two local functionals $\bar{f} = \int f(x; u; u_x, \ldots)dx$ and $\bar{g} = \int g(x; u; u_x, \ldots)dx$ can be written in the following equivalent forms (see above the general theory of multivectors)

$$\{\bar{f}, \bar{g}\} = \langle \varpi, \delta \bar{f} \wedge \delta \bar{g} \rangle = \int dx dy \frac{\delta \bar{f}}{\delta u^i(x)}\frac{\delta \bar{g}}{\delta u^j(y)}\left\{u^i(x), u^j(y)\right\} \tag{2.2.22}$$

$$= \sum_s \int dx \frac{\delta \bar{f}}{\delta u^i(x)} A_{ij}^s(u; u_x, u_{xx}, \ldots)\left(\frac{\delta \bar{g}}{\delta u^j(x)}\right)^{(s)} \in \Lambda_0. \tag{2.2.22}$$

Therefore it is again a local functional.

The crucial property of local Poisson brackets is that, the Hamiltonian systems

$$u^i_t = -\delta_{\bar{H}} \varpi = \{u^i(x), \bar{H}\} = A_{ij}^s(u; u_x, u_{xx}, \ldots)\delta^s \frac{\delta \bar{H}}{\delta u^j(x)} \tag{2.2.23}$$

with local translation invariant Hamiltonians

$$\bar{H} = \int H(u; u_x, \ldots)dx$$

are translation invariant evolutionary PDEs (2.1.23).

Living in the infinite dimensional loop space we will not impose conditions on the rank of the Poisson bracket. However, in the main examples the corank of the bivector will be finite.
Example 2.2.10 For a constant antisymmetric matrix $\pi^{ij}$ the bivector
\[
\{u^i(x), u^j(y)\} = \pi^{ij} \delta(x - y) \tag{2.2.24}
\]
is a local Poisson structure. It is called ultralocal Poisson bracket. This is a symplectic structure on the loop space if $\det \pi^{ij} \neq 0$. The Hamiltonian evolutionary PDEs read
\[
u^i_t = \pi^{ij} \frac{\delta H}{\delta u^j(x)}.
\]
Reducing the nodegenerate matrix $\pi^{ij}$ to the canonical form we arrive at the Hamiltonian formulation of 1+1 dimensional variational problems
\[
q^i_t = \frac{\delta H}{\delta p_i(x)}, \quad p^i_t = -\frac{\delta H}{\delta q^i(x)}.
\]

Example 2.2.11 For a constant symmetric matrix $\eta^{ij}$ the bivector
\[
\{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x - y) \tag{2.2.25}
\]
is a local Poisson structure. Under the assumption $\det \eta^{ij} \neq 0$ this Poisson bracket has $n$ independent Casimirs
\[
\bar{u}^1 = \int u^1 \, dx, \ldots, \bar{u}^n = \int u^n \, dx. \tag{2.2.26}
\]
The annihilator of (2.2.25) is generated by the above Casimirs.

The Hamiltonian evolutionary PDEs read
\[
u^i_t = \eta^{ij} \frac{\partial}{\partial u^j(x)} \frac{\delta H}{\delta u^i(x)}. \tag{2.2.27}
\]

We finish this section with spelling out the transformation law of coefficients of local bivectors imposed by the general formula (2.2.9). If $A^a_i(u; u, \ldots)$ and $A^a_b(v; v, \ldots)$ are the coefficients of a local bivector in two coordinate charts $(U, u^1, \ldots, u^n)$ and $(V, v^1, \ldots, v^n)$ resp. then, on $U \cap V$ one has
\[
A^a_b(v; \frac{\partial v}{\partial u} u_x, \ldots) = \sum_{s \geq t} \binom{s}{t} \frac{\partial v^a}{\partial u^t} \left( \frac{\partial v^b}{\partial u^s} \right)^{(s-t)} A^a_b(u; u_x, \ldots). \tag{2.2.28}
\]

Example 2.2.12 Applying the transformation
\[
u = \frac{1}{4} v^2 \tag{2.2.29}
\]
to the bivector
\[
\{u(x), u(y)\} = u(x) \delta'(x - y) + \frac{1}{2} u'(x) \delta(x - y) \tag{2.2.30}
\]
we obtain a constant Poisson bracket of the form (2.2.25)
\[
\{v(x), v(y)\} = \delta'(x - y). \tag{2.2.31}
\]
Hence (2.2.30) is itself a Poisson structure. This is the Lie - Poisson bracket on the space dual to the Lie algebra of vector fields on the circle [46].
3 Lecture 3. Problem of classification of local Poisson brackets

The last example of the previous section is the simplest issue of the problem of reduction of local Poisson brackets to the simplest (possibly, to the constant one) form. In this example the reduction to the constant form was achieved by a change of coordinates in the target space \( M \) (\( M \) was one-dimensional). We give now another well-known example: to transform the bivector (the *Magri bracket* for the KdV equation)

\[
\{u(x), u(y)\} = u(x)\delta'(x - y) + \frac{1}{2}u'(x)\delta(x - y) - \delta'''(x - y)
\]  

(3.0.1)

to the constant form (2.2.31) one is to use the celebrated *Miura transformation* [120]

\[
u = \frac{1}{4}v^2 + v'.
\]  

(3.0.2)

Our strategy will be to classify local Poisson brackets on the loop space \( \mathcal{L}(M) \) (the target space \( M \) will be a ball in this section) with respect to the action of the group of Miura-type transformations.

The problem of reduction of certain classes of Poisson brackets to a canonical form by coordinate transformations was first investigated in [44] for the Poisson brackets of hydrodynamic type and in [45] for the so-called differential geometric Poisson brackets (see also [46] and the references therein). Some results regarding reduction of the local Poisson brackets to the canonical form by using Miura - Bäcklund transformations were obtained in [2], [69], [130], [134] (in the latter non translation invariant Poisson brackets were studied).

We want to classify local Poisson brackets w.r.t. general Miura type transformations of the form

\[
u^i \rightarrow \tilde{\nu}^i = F^i(u; u_x, u_{xx}, \ldots).
\]  

(3.0.3)

The problem is that these transformations do not form a group. The main trouble is with inverting such a transformation. E.g., to invert the Miura transformation one is to solve Riccati equation (3.0.2) w.r.t. \( v \). To resolve this problem we will extend the class of Miura-type transformations. Simultaneously we will also be extending the class of local functionals, vector fields, and Poisson brackets.

3.1 Extended formal loop space

Let us introduce gradation on the ring \( \mathcal{A} \) of differential polynomials putting

\[
\deg u^{i,k} = k, \quad k \geq 1, \quad \deg f(x; u) = 0.
\]  

(3.1.4)

We extend the gradation onto the spaces \( \mathcal{A}_{k,l} \) of differential forms by

\[
\deg dx = -1, \quad \deg \delta u^{i,s} = s, \quad s \geq 0.
\]
The differentials \( d \) and \( \delta \) preserve the gradation. Introduce a formal indeterminate \( \epsilon \) of the degree
\[
\deg \epsilon = -1.
\]
Let us define a subcomplex
\[
\hat{A}_{k,l} \subset A_{k,l} \otimes \mathbb{C}[\epsilon], \epsilon^{-1}
\]
collecting all the elements of the total degree \( k - l \). The corresponding subcomplex
\[
\hat{\Lambda}_k \subset \Lambda_k \otimes \mathbb{C}[\epsilon], \epsilon^{-1}
\]
spanned by the elements of the total degree \( k - 1 \). In particular, the space of local functionals \( \hat{\Lambda}_0 \) consists of integrals of the form
\[
\bar{f} = \int f(u; u_x, u_{xx}, \ldots ; \epsilon) dx,
\]
\[
f(u; u_x, u_{xx}, \ldots) = \sum_{k=0}^{\infty} \epsilon^k f_k(u; u_x, \ldots, u^{(k)}), \quad f_k \in A, \quad \deg f_k = k.
\]
We will still call such a series differential polynomials when it will not cause confusions. Taking the tensor algebra of \( \hat{\Lambda}_0 \) we obtain the ring of functionals on the extended formal loop space that we will denote \( \hat{L}(M) \). The vertical differential \( \delta \) of the bicomplex must be renormalized
\[
\delta \mapsto \hat{\delta} = \frac{1}{\epsilon} \delta.
\]
It follows from Theorem 2.1.4 the bicomplex \( (\hat{A}_{k,l}, d, \hat{\delta}) \) is exact
The gradation on the vector and multivector fields is defined by
\[
\deg \frac{\partial}{\partial x} = 1, \quad \deg \frac{\partial}{\partial u^{i,s}} = -s, \quad s \geq 0.
\]
Observe that \( \partial_x \) increases degrees by one:
\[
\deg \partial_x f = \deg f + 1.
\]
The space \( \hat{\Lambda}^1 \) of vector fields on \( \hat{L}(M) \) is obtained by collecting all the elements in \( \Lambda^1 \otimes \mathbb{C}[\epsilon], \epsilon^{-1} \) of the total degree 1. In particular, the translation invariant evolutionary vector fields are
\[
a^i = \sum_{k=0}^{\infty} \epsilon^{k-1} a^i_k(u; u_x, \ldots, u^{(k)}), \quad a^i_k \in A, \quad \deg a^i_k = k.
\]
The corresponding evolutionary system of PDEs reads
\[
u^i_t = \epsilon^{-1} a^i_0(u) + a^i_1(u; u_x) + \epsilon a^i_2(u; u_x, u_{xx}) + O(\epsilon^2)
\]
\[
a^i_1(u; u_x) = v^i_j(u) u^j_x,
\]
\[
a^i_2(u; u_x, u_{xx}) = b^i_j(u) u^j_{xx} + \frac{1}{2} c^i_{jk}(u) u^j_x u^k_x
\]
(3.1.7)
Proceeding in a similar way we introduce the subspace
\[ \hat{\Lambda}^k \subset \Lambda^k \otimes \mathbb{C}[[\epsilon], \epsilon^{-1}] \]
of \(k\)-vectors of the total degree \(k\).

**Lemma 3.1.1** The Schouten - Nijenhuis bracket gives a well-defined map
\[ \epsilon[\ , \ ] : \hat{\Lambda}^k \times \hat{\Lambda}^l \rightarrow \hat{\Lambda}^{k+l-1}. \]

There is an important subtlety with the grading of the local multivectors. Indeed, a local \(k\)-vector is a map
\[ \Lambda_1^k \rightarrow \Lambda_0 \]
not
\[ \Lambda_1^k \rightarrow \hat{\Lambda}_0^k \]
(cf. the formulae (2.2.7), (2.2.10) and (2.2.12)). Such a map does not respect the grading. So we must assign a nonzero degree to delta-function
\[ \deg \delta(x - y) = 1, \quad \deg \delta^{(s)}(x - y) = s + 1 \quad (3.1.8) \]
Therefore a general local bivector in \(\hat{\Lambda}_{\text{loc}}\) will be represented by an infinite sum
\[ \{u^i(x), u^j(y)\} = \sum_{k=-1}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]} \quad (3.1.9) \]
\[ \{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A_{k,s}^{ij}(u; u_x, \ldots, u^{(s)}) \delta^{(k-s+1)}(x - y), \]
\[ A_{k,s}^{ij} \in \mathcal{A}, \quad \deg A_{k,s}^{ij} = s, \quad s = 0, 1, \ldots, k + 1. \]

More explicitly, the first three terms in the expansion (3.1.9) read
\[ \{u^i(x), u^j(y)\}^{[-1]} = \pi^{ij}(u(x))\delta(x - y) \]
\[ \{u^i(x), u^j(y)\}^{[0]} = g^{ij}(u(x))\delta'(x - y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x - y) \]
\[ \{u^i(x), u^j(y)\}^{[1]} = a^{ij}(u(x))\delta''(x - y) + b_k^{ij}(u(x))u_x^k\delta'(x - y) \]
\[ +[c_k^{ij}(u(x))u_x^k u_x^l + \frac{1}{2}d_{kl}^{ij}(u(x))u_x^k u_x^l]\delta(x - y) \quad (3.1.12) \]
where \(\pi^{ij}, g^{ij}(u), \Gamma^{ij}(u), a^{ij}(u), b_k^{ij}(u), c_k^{ij}(u), d_{kl}^{ij}(u)\) are some functions on the manifold \(M\).
Remark 3.1.2 Our rules of introducing the gradation can be memorized using the following simple trick. Do a rescaling of the independent variable \( x \)

\[
x \mapsto \epsilon x. \tag{3.1.13}
\]

The \( x \)-derivatives \( u^{i,k} = d^k u^i/dx^k \) will change

\[
u^{i,k} \mapsto \epsilon^k u^{i,k}. \tag{3.1.14}
\]

We also have

\[
dx \mapsto \epsilon^{-1} dx. \tag{3.1.15}
\]

The delta-function, according to the definition (2.2.11) must be rescaled as

\[
\delta(x) \mapsto \epsilon \delta(x). \tag{3.1.16}
\]

In other words, “delta-function” is not a function but a density. Simultaneously with the rescaling we will also redefine the integrals

\[
\int . \, dx_1 \ldots dx_k \mapsto \epsilon^k \int . \, dx_1 \ldots dx_k.
\]

After such a rescaling we expand all the formulae of the previous two sections in a power series in \( \epsilon \) to arrive at our grading conventions.

Example 3.1.3 Rescaling (3.1.13) the KdV equation \( u_t = uu_x + u_{xxx} \) one obtains

\[
u_t = \epsilon(uu_x + \epsilon^2 u_{xxx}).
\]

One usually introduces slow time variable \( t \mapsto \epsilon t \) to recast the last equation into the form

\[
u_t = uu_x + \epsilon^2 u_{xxx}. \tag{3.1.17}
\]

This is the small dispersion expansion of the KdV equation. The smooth solutions of (3.1.17) describe solutions to KdV slow varying in space and time.

Remark 3.1.4 Besides the rescaling procedure, one can arrive at the above series (3.1.5), (3.1.6), (3.1.9) considering the continuous limits of differential-difference systems. E.g., for the well-known example of Toda lattice

\[
\dot{u}_n = v_n - v_{n-1}, \quad \dot{v}_n = e^{u_{n+1}} - e^{u_n}, \quad n \in \mathbb{Z} \tag{3.1.18}
\]

the continuous limit \( u_n = u(en) = u(x), \quad v_n = v(en) = v(x), \quad t \mapsto \epsilon t \) gives an evolutionary system of the form (3.1.7) with an infinite series in the r.h.s.

\[
u_t = \frac{1}{\epsilon} (v(x) - v(x - \epsilon)) = [v' - \frac{1}{2} \epsilon v'' + O(\epsilon^2)],
\]

\[
v_t = \frac{1}{\epsilon} [e^{u(x+\epsilon)} - e^{u(x)}] = [(e^u)'+ \frac{1}{2} \epsilon (e^u)'' + O(\epsilon^2)]. \tag{3.1.19}
\]
Replacing in the Hamiltonian structure
\[ \{ u_m, u_n \} = \{ v_m, v_n \} = 0, \quad \{ u_m, v_n \} = \delta_{mn} - \delta_{m,n+1} \]
the Kronecker symbols \( \delta_{mn} \) by \( \epsilon^{-1}\delta(x-y) \), \( \delta_{m,n+1} \) by \( \epsilon^{-1}\delta(x-y-\epsilon) \), we obtain a Poisson bracket of the (3.1.9) form
\[
\{ u(x), u(y) \} = \{ v(x), v(y) \} = 0, \quad \{ u(x), v(y) \} = \frac{1}{\epsilon} [\delta(x-y) - \delta(x-y-\epsilon)] = \delta'(x-y) - \frac{\epsilon}{2}\delta''(x-y) + O(\epsilon^2).
\]

The Hamiltonian of the “interpolated” Toda lattice (3.1.19) is a local functional of the form
\[
H = \int \left[ \frac{1}{2} v^2 + \epsilon u \right] dx.
\]

3.2 Miura group

The next definition will be that to extend the class of Miura-type transformations (3.0.3). Let us consider the transformations
\[
u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n
\]
\[F_k^i \in \mathcal{A}, \quad \deg F_k^i = k, \quad \det \left( \frac{\partial F_i^0(u)}{\partial u^j} \right) \neq 0.
\]

Lemma 3.2.5 The transformations of the form (3.2.22) form a group. The Lie algebra of the group is isomorphic to the subalgebra \( \hat{\Lambda}_1^{ev} \) of all translation invariant evolutionary vector fields in \( \hat{\Lambda}_1 \) with the Lie bracket operation.

Example 3.2.6 Let us invert the classical Miura transformation
\[ u = \frac{1}{4} v^2 + \epsilon v' \]
using successive approximations. Rewriting the equation in the form
\[ v = 2\sqrt{u - \epsilon v'} = 2\sqrt{u} - \epsilon \frac{v'}{\sqrt{u}} + O(\epsilon^2) = 2\sqrt{u} - \epsilon \frac{u'}{u} + O(\epsilon^2) \]
we obtain first two terms of the solution \( v = F(u; u', \ldots; \epsilon) \).
**Remark 3.2.7** This way of solving the Riccati equation is essentially equivalent to the classical WKB method of solving the related linear second order ODE

\[ \epsilon^2 y'' = \frac{1}{4} u y, \quad v = 4 \epsilon \frac{y'}{y}. \]

Substituting the above series solution to Riccati we obtain the WKB asymptotic solution to the second order ODE with the small parameter \( \epsilon \to 0 \)

\[ y = u^{-1/4} \exp \frac{1}{2 \epsilon} \int \sqrt{u} dx \left( 1 + O(\epsilon) \right). \]

**Definition.** The group \( G \) of all the transformations of the form (3.2.22) is called Miura group.

The Miura group \( G \) looks to be a natural candidate for the role of the group of “local diffeomorphisms” of the extended formal loop space \( \hat{L}(M) \) (recall that at the moment \( M \) is a ball). \( G \) contains the group of diffeomorphisms \( Diff(M) \) of the manifold \( M \) as a subgroup. It coincides with the semidirect product of \( Diff(M) \) and the pro-unipotent subgroup \( G_0 \) consisting of Miura-type transformations close to identity,

\[ u^i \mapsto u^i + \epsilon A^i_j(u)u^j_x + \epsilon^2 \left( B^i_j(u)u^j_{xx} + \frac{1}{2} C^i_{jk}(u)u^j_x u^k_x \right) + \ldots \]  

(3.2.23)

The product in the group \( G_0 \) reads

\[
A^j_i = A^j_{1i} + A^j_{2i}, \\
B^j_i = B^j_{1i} + B^j_{2i} + A^j_{2k} A^{k} \!_{1i}, \\
C^i_{jk} = C^i_{1jk} + C^i_{2jk} + \frac{1}{2} \left[ \partial_s A^i_{2j} A^j_1 s + \partial_s A^i_{2k} A^j_1 s + A^i_{2k} \partial_k A^j_1 s + A^i_{2j} \partial_j A^j_1 s \right] \\
\ldots
\]

(3.2.24)

The following simple statements immediately follow from Lemma 3.2.5.

**Lemma 3.2.8** The class of local functionals (3.1.5), evolutionary PDEs (3.1.7), and local translation invariant multivectors (see the formula (3.1.9) for the bivectors) on the extended formal loop space \( \hat{L}(M) \) is invariant w.r.t. the action of the Miura group.

**Lemma 3.2.9** An arbitrary vector field \( a \) of the form (3.1.6) with \( a_0 \neq 0 \) can be reduced, by a transformation of the Miura group, to a constant form

\[ a_0 = \text{const}, \quad a_i = 0 \text{ for } i > 0. \]
This is an infinite dimensional analogue of the theorem of “rectifying of a vector field”.

Proof By using the theorem of “rectifying of a vector field” on a finite dimensional manifold, we can reduce the vector field \( (a_0^1, \ldots, a_0^n) \) to a constant one by performing a Miura transformation of the form (3.2.22) with \( F_k = 0 \), \( k \geq 1 \). We prove the lemma by induction. Let us assume the vector field \( a \) to be of the form

\[
a^i = a_0^i + \varepsilon^k a_k^i(u, u_x, \ldots, u^{(k)}) + \mathcal{O}(\varepsilon^{k+1})
\]

with \( \deg a_k = k \). Since \( a_0 \neq 0 \), we can find differential polynomials \( F_k^i(u, \ldots, u^{(k)}) \) of degree \( k \) such that

\[
a_0^i \frac{\partial F_k^i}{\partial u^i} + a_k^i = 0.
\]

Then the Miura transformation

\[
\bar{u}^i = u^i + \varepsilon^k F_k^i(u, \ldots, u^{(k)})
\]

reduces the vector field \( a \) to the form

\[
a^i = a_0^i + \mathcal{O}(\varepsilon^{k+1}).
\]

\[\square\]

3.3 \((p, q)\)-brackets on the extended formal loop space

Let us write explicitly down the transformation law of the coefficients of a local Poisson bracket w.r.t. transformations from the Miura group (cf. [129]). Let \( A^{kl} \) be the differential operator of the Poisson bracket given in (2.2.18). In the new “coordinates” \( \bar{u}^i \) of the form (3.2.22) the Poisson bracket will be given by the operator

\[
\bar{A}_{ij} = L_{kj}^s A^{kl} L_{lj}^s
\]  

(3.3.25)

where the matrix-valued operator \( L_{kj}^s \) and the adjoint one \( L_{kj}^s \) are given by

\[
L_{kj}^s = \sum_s (-\partial_x)^s \circ \frac{\partial \bar{u}^i}{\partial u^{k,j}}, \quad L_{kj}^s = \sum_s \frac{\partial \bar{u}^i}{\partial u^{k,j}} \partial_{x,s}.
\]

Main Problem. To describe the orbits of the action of the Miura group \( \mathcal{G} \) on \( \hat{\Lambda}^2 \).

In our opinion this problem is the natural setup of the problem of classification of Poisson structures of one-dimensional evolutionary PDEs with respect to Miura-type transformations (they are called also Darboux, or Bianchi, or Bäcklund transformations. Our transformations do not involve a change of the independent variable \( x \) since we consider only translation invariant PDEs).
Our conjecture is that, for a reasonable class of Poisson brackets to be defined below, the orbits are labelled by certain finite-dimensional geometrical structures on the underlying manifold $M$. Below we will illustrate this claim describing two orbits being, in a certain sense, generic. The full problem remains open.

We first explain how a local Poisson bracket from $\hat{\Lambda}_{\text{loc}}^2$ induces certain finite-dimensional geometrical structures on $M$.

**Lemma 3.3.10** The subgroup $\text{Diff}(M) \subset G$ acts independently on every term $\{ , \}^k$ of the expansion (3.1.9), $k \geq -1$. In particular, the leading term $A^ij_{k,0}(u)$ is a $(2,0)$-tensor field on $M$, symmetric/antisymmetric for even/odd $k$.

*Proof* This follows from the transformation law (2.2.28). The symmetry/antisymmetry of the coefficients follows from the general antisymmetry condition (2.2.17). The lemma is proved. □

The transformations of the Miura group mix the terms of the expansion (3.1.9). However, the following simple statement holds true.

**Lemma 3.3.11** After a transformation from the Miura group $G$ the $k$-th term of the transformed bracket is expressed via $\{ , \}^{-1}, \{ , \}^0, \ldots, \{ , \}^k$ for any $k \geq -1$.

This follows from the explicit formula (3.3.25).

**Lemma 3.3.12** The first non-zero term in the expansion (3.1.9) is itself a local Poisson bracket.

This is obvious.

**Corollary 3.3.13** The coefficient $\pi^{ij}(u)$ in (3.1.10) is a Poisson structure on $M$. This Poisson structure is invariant w.r.t. the action of $G$ on $\hat{\Lambda}_{\text{loc}}^2$.

We obtain a map

$$\hat{\Lambda}_{\text{loc}}^2 / G \rightarrow \text{Poisson structures on } M.$$ (3.3.26)

Let us assume that the Poisson structure $\pi^{ij}(u)$ on $M$ has constant rank $p = 2p_1$. Denote $q := n - p$ the corank of $\pi^{ij}(u)$.

**Definition.** (3.1.9) is called $(p,q)$-bracket if the coefficient $g^{ij}(u)$ in (3.1.11) does not degenerate on $\operatorname{Ker} \pi^{ij}(u) \subset T_u^* M$ on an open dense subset in $M \ni u$.

**Example 3.3.14** The ultralocal Poisson bracket (2.2.24) with a non-degenerate matrix $\pi^{ij}$ is a $(n,0)$-bracket.
Example 3.3.15 The Poisson bracket (2.2.25) with a non-degenerate matrix $\eta^{ij}$ is a $(0,n)$-bracket.

Example 3.3.16 Let $c^i_j$ be the structure constants of a semisimple $n$-dimensional Lie algebra $\mathfrak{g}$. The Killing form $\eta^{ij}$ on $\mathfrak{g}$ defines a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ called Kac-Moody Lie algebra [85]. The Lie-Poisson bracket (1.1.8) on the dual space $\hat{\mathfrak{g}}^*$

$$\frac{1}{\epsilon} \{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x - y) + \frac{1}{\epsilon} c^i_j u^k \delta(x - y) \quad (3.3.27)$$

can be considered as a Poisson bracket of the form (3.1.9) on the loop space $\hat{\mathcal{L}}(\mathfrak{g}^*)$. Here $\epsilon$ is the central charge. This is a $(p,q)$-bracket with

$$q = \text{rk} \mathfrak{g}, \quad p = \text{dim} \mathfrak{g} - \text{rk} \mathfrak{g}.$$

Let a $(p,q)$-Poisson bracket on $\hat{\mathcal{L}}(M)$ of the form (3.1.9) - (3.1.11) be given. We will now construct a flat metric on the base of the symplectic foliation of $M$ defined by the finite-dimensional Poisson bracket $\pi^{ij}(u)$. Let us assume that $M$ is a small ball such that the symplectic foliation defines a fibration

$$M \to N, \quad \text{dim } N = q. \quad (3.3.28)$$

Functions on $N$ are Casimirs of the finite-dimensional Poisson bracket $\pi^{ij}(u)$. Define first a symmetric bilinear form $(\ , \ )^*$ on $T^*N$ putting

$$(df_1, df_2)^* := \frac{\partial f_1}{\partial u^i} \frac{\partial f_2}{\partial u^j} \eta^{ij}(u) \quad (3.3.29)$$

for any two Casimirs of $\pi^{ij}(u)$. By the assumption this bilinear form does not degenerate. Define the non-degenerate symmetric tensor $(\ , \ )$ on $TN$ by

$$(\ , \ ) := [(\ , \ )^*]^{-1}. \quad (3.3.30)$$

Theorem 3.3.17 The metric (3.3.30) on $TN$ is well-defined and flat.

Proof The simplest way to prove that the metric (3.3.30) is constant along symplectic leaves and also prove vanishing of the curvature of the metric is the following one. Choose local coordinates $u = (w^a, v^\alpha)$ on $M$, $a = 1, \ldots, p = 2p_1$, $\alpha = 1, \ldots, q$, $p+q = n$, such that $w^1, \ldots, w^p$ are canonical coordinates on the symplectic leaves and $v^1, \ldots, v^q$ is a system of independent Casimirs of $\pi^{ij}(u)$. The $v$’s can be considered as coordinates on $N$. In the local coordinates the Poisson bracket (3.1.9) reads

$$\{w^a(x), w^b(y)\} = \frac{1}{\epsilon} \eta^{ab} \delta(x - y) + A^{ab}_{00}(u) \delta'(x - y) + A^{ab}_{01}(u, u_\alpha) \delta(x - y) + O(\epsilon)$$

$$\{v^\alpha(x), w^a(y)\} = A^{a\alpha}_{00}(u) \delta'(x - y) + A^{a\alpha}_{01}(u, u_\alpha) \delta(x - y) + O(\epsilon)$$

$$\{v^\alpha(x), v^\beta(y)\} = g^{\alpha\beta}(u) \delta'(x - y) + \left( \Gamma^{\alpha\beta}_a(u) w^a_\alpha + \Gamma^{\alpha\beta}_a(u) v^a_\alpha \right) \delta(x - y) + O(\epsilon) \quad (3.3.31)$$
Here $\pi^{ab}$ is a constant antisymmetric nondegenerate matrix, the $q \times q$ matrix $g^{\alpha\beta}(u)$ coincides with the Gram matrix of the bilinear form (3.3.29) in the basis $dv^1, \ldots, dv^q$.

Let us consider the following foliation on the loop space $\hat{L}(M)$

$$w^a(x) \equiv w^a_0, \quad a = 1, \ldots, p$$

(3.3.32)

for arbitrary given numbers $w^1_0, \ldots, w^p_0$. Due to nondegeneracy of $\pi^{ab}$ this foliation is cosymplectic. The corresponding Dirac bracket $\{ , \}_D$ can be considered as a Poisson bracket on $\hat{L}(N)$ since, by definition

$$\{ v^\alpha(x), w^a(y) \}_D = \{ w^a(x), w^b(y) \}_D = 0.$$

Let us show that the Dirac bracket is a $(0, q)$-bracket on $\hat{L}(N)$ with the same leading term

$$\{ v^\alpha(x), v^\beta(y) \}_D = \{ v^\alpha(x), v^\beta(y) \} + O(\epsilon).$$

Introduce the differential operator

$$\Pi_{ab} = \pi_{ab} - \epsilon \pi_{aa'} \left( A_{00}^{\alpha\gamma}(u) \frac{d}{dx} + A_{01}^{\alpha\gamma}(u, u_x) \right) \pi_{b' b} + O(\epsilon^2)$$

inverse to the operator $\epsilon A_{ab}$. Then the Dirac bracket has the form

$$\{ v^\alpha(x), v^\beta(y) \}_D = \{ v^\alpha(x), v^\beta(y) \} - \epsilon \sum_{a,b=1}^p A_{ab} \Pi_{ab} A_{b\beta} \delta(x-y).$$

(3.3.33)

We obtain a $(0, q)$ Poisson bracket on $\hat{L}(N)$ eventually depending on the parameters $w^1_0, \ldots, w^p_0$. The leading term

$$\{ v^\alpha(x), v^\beta(y) \}_D \equiv \{ v^\alpha(x), v^\beta(y) \} = g^{\alpha\beta}(v, w_0)\delta(x-y) + \Gamma^{\alpha\beta}(v, w_0)v_x^\gamma \delta(x-y)$$

is itself a Poisson bracket (the so-called Poisson bracket of hydrodynamic type). According to the theory of such brackets [44] one can choose local coordinates $v^\alpha$ on $N$ in such a way that

$$g^{\alpha\beta} = \text{const}, \quad \Gamma^{\alpha\beta} = 0.$$

This proves the theorem. \hfill \Box

An alternative way to prove the Theorem is to write explicitly down the terms of the order $\epsilon^{-1}$ in the Jacobi identity

$$\{ \{ v^\alpha(x), v^\beta(y) \}, w^\alpha(z) \} + (\text{cyclic}) = 0$$

in order to prove that the leading term in $\{ v^\alpha(x), v^\beta(y) \}$ does not depend on $w, w_x$. Then from the leading term in the Jacobi identity

$$\{ \{ v^\alpha(x), v^\beta(y) \}, v^\gamma(z) \} + (\text{cyclic}) = 0$$
it follows, as in [44], vanishing of the curvature of the metric $g^{\alpha\beta}(v)$.

We believe that the problem of classification of $(p, q)$-brackets (and also classification of pencils of $(p, q)$-brackets) is very important in the Hamiltonian theory of integrable PDEs. In this paper we will mainly consider $(0, n)$-brackets leaving the general case for a subsequent publication.

To illustrate our technique we will begin with a more simple example of $(n, 0)$-brackets.

3.3.1 Classification of $(n, 0)$-brackets

Our first result is

**Theorem 3.3.18** If $M$ is a ball then all $(n, 0)$ Poisson brackets in $\hat{\Lambda}^2_{loc}$ are equivalent w.r.t. the action (3.3.25) of the Miura group $\mathcal{G}$.

**Proof** First we choose the Darboux coordinates for the symplectic structure on $M$. The Poisson bracket in question will read

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon} \pi^{ij} \delta(x-y) + \sum_{k=0}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]}.$$  (3.3.34)

Next we will try to kill all the terms of the expansion (3.3.34) by transformations of the form (3.2.22) with $F^i_0(u) = u^i, i = 1, \ldots, n$. To this end an appropriate version of Poisson cohomology will be useful. We define the Poisson cohomology $H^*(\hat{\Lambda}(M), \varpi)$ for a Poisson structure $\varpi \in \hat{\Lambda}^2_{loc}$ as the cohomology of the complex

$$0 \to \hat{\Lambda}^0_{loc} \xrightarrow{\partial} \hat{\Lambda}^1_{loc} \xrightarrow{\partial} \hat{\Lambda}^2_{loc} \to \cdots$$  (3.3.35)

with the differential $\partial \beta := [\varpi, \beta]$. In the present proof $\varpi = \pi^{ij} \delta(x-y)$. The cohomology is naturally decomposed into the direct sum

$$H^k = \bigoplus_{m \geq -1} H^{k,m}$$  (3.3.36)

with respect to monomials in $\epsilon$, where $H^{k,m}$ consists of the cocycles proportional to $\epsilon^m$. Denote

$$\hat{H}^k := \bigoplus_{m \geq 0} H^{k,m}.$$  (3.3.37)

The following obvious statement holds true.

**Lemma 3.3.19** The first non-zero term in the expansion (3.3.34) is a 2-cocycle in the Poisson cohomology $H^2$ of the ultralocal Poisson bracket (2.2.24).
So, we will be able to kill this first nonzero term of the expansion if we prove that, for any 2-cocycle $\epsilon \in \hat{\Lambda}^2_{\text{loc}}$ of the ultralocal bracket $\varpi := \epsilon \{, \}$, there exists a vector field $a$ of the form (3.1.6) such that the Lie derivative $\text{Lie}_a \varpi$ gives the cocycle and $a|_{\epsilon=0} = 0$. This will follow from the following general statement about triviality of the ultralocal Poisson bracket.

**Lemma 3.3.20** For $M = \text{ball}$ and the ultralocal Poisson bracket $\varpi$ (2.2.24) with $\det(\pi^{ij}) \neq 0$ the Poisson cohomology $\tilde{H}^1(\hat{\mathcal{L}}(M), \varpi)$, $\tilde{H}^2(\hat{\mathcal{L}}(M), \varpi)$ vanish.

The first proof of the lemma (and of the lemma 3.3.22 below) was obtained by E. Getzler [73] (also triviality of the higher cohomology has been proved). Independently, L. Degiovanni, F. Magri, V. Sciacca obtained another proof [18]. We have decided to present here our own proofs of triviality of cohomologies that closely follow the finite-dimensional case. Our proof will also be useful in the study of bihamiltonian structures below.

Let us prove first triviality of $H^1$. Let an evolutionary vector field $a$ with the components $a^1, \ldots, a^n$ be a cocycle. Denote

$$\omega_i = \pi_{ij} a^j$$

where the constant matrix $\pi_{ij}$ is inverse to $\pi^{ij}$. The condition $\partial a = \text{Lie}_a \varpi = 0$ reads

$$\frac{\partial \omega_i}{\partial u^{j,s}} = \sum_{t \geq s} (-1)^t \binom{t}{s} \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{i,t}}.$$ 

Using (2.1.27) we conclude that there exists a local functional $\tilde{f} = \int f \, dx$ such that

$$\omega_i = \frac{\delta \tilde{f}}{\delta u^i(x)}.$$ 

Therefore the vector field is a Hamiltonian one,

$$a^i = \pi^{ij} \frac{\delta \tilde{f}}{\delta u^j(x)}.$$ 

Let us now proceed to the proof of triviality of $H^2$. The idea is very simple: the bivector

$$\varpi + \varepsilon \alpha = \pi^{ij} \delta(x-y) + \varepsilon \sum_s A_s^{ij} \delta(s)(x-y)$$

satisfies the Jacobi identity

$$[\varpi + \varepsilon \alpha, \varpi + \varepsilon \alpha] = 0(\mod \varepsilon^2)$$

iff the inverse matrix is a closed differential form

$$\frac{1}{2} \pi_{ij} dx \wedge \delta u^i \wedge \delta u^j + \frac{1}{2} \varepsilon \omega_{i,j,s} dx \wedge \delta u^i \wedge \delta u^{i,s} (\mod \varepsilon^2)$$
where
\[ \omega_{ijs} := \pi_{ip} \pi_{jq} A_{s}^{pq}. \] (3.3.38)

Denote
\[ \omega = \frac{1}{2} \omega_{ijs} \delta u^i \wedge \delta u^j. \]

From the condition of closedness \( \delta(dx \wedge \omega) = 0 \in \Lambda_3 \) we derive, due to Corollary 2.1.14, existence of a one-form \( dx \wedge \phi, \phi = \phi_i \delta u^i \) such that \( \delta(dx \wedge \phi) = dx \wedge \omega. \) The vector field \( a \) with the components
\[ a^i = \pi^{ij} \phi_j \]
gives a solution to the equation
\[ [\omega, a] = \alpha. \]

To be on the safe side we will now show, by straightforward calculations, that, indeed, the above geometrical arguments work. First, from the antisymmetry condition (2.2.17) for the bivector \( \alpha \) it readily follows the antisymmetry condition (2.1.30) for the 2-form \( \omega \) with the reduced components (3.3.38). Next, we are to verify that from the cocycle condition \([\omega, \alpha] = 0\) where the Schouten - Nijenhuis bracket \([\omega, \alpha]\) is written in (2.2.19), it follows closedness (2.1.32) of \( dx \wedge \omega \in \Lambda_2. \) First we will rewrite the formula for the bracket in a slightly modified form. Differentiating the antisymmetry condition
\[ A_{s}^{ik} = - \sum (-1)^{m} \binom{m}{s} \partial_{x}^{m-s} A_{m}^{ki} \]
w.r.t. \( u^{l,t} \) and using the commutators (2.1.5) we obtain
\[ \frac{\partial A_{s}^{ik}}{\partial u^{l,t}} = - \sum (-1)^{q+r+s} \binom{q+r+s}{q, r} \left( \frac{\partial A_{q+r+s}^{kj}}{\partial u^{l,t-q}} \right)^{(r)}. \]

So the coefficients of the Schouten - Nijenhuis bracket (2.2.19) can be rewritten as follows
\[ [\omega, \alpha]_{x,y,z}^{ij} = \left[ \frac{\partial A_{i}^{kj}}{\partial u^{l,s}} \pi_{jk} - \frac{\partial A_{i}^{jk}}{\partial u^{l,t}} \pi_{kj} \right] + \sum (-1)^{q+r+t} \binom{q+r+t}{q, r} \left( \frac{\partial A_{q+r+s}^{ij}}{\partial u^{l,q+r+t}} \right)^{(r)} \partial_{x}^{t} \delta^{(t)}(x-y) \delta^{(s)}(x-z). \] (3.3.39)

The coefficient of \( \delta^{(t)}(x-y) \delta^{(s)}(x-z) \) must vanish for every \( t \) and \( s. \) Multiplying this coefficient by \( \pi_{ia} \pi_{ja} \pi_{kc} \) we arrive at the condition of closedness (2.1.32) of the 2-form (3.3.38). Using Corollary 2.1.14 we establish existence of differential polynomials \( \phi_1, \ldots, \phi_n \) representing the 2-form as in (2.1.31). The translation invariant vector field
\[ a^i = \pi^{ij} \phi_j \]
will satisfy \( \partial a = \alpha. \) This proves the lemma, and also the theorem. \( \square \)
3.3.2 Classification of \((0, n)\)-brackets

Let us now consider the Poisson structures in \(\hat{\Lambda}^2\) with identically vanishing leading term \(\{\ ,\ \}\). As above, the first nonzero term (3.1.11) is itself a Poisson bracket. The leading coefficient \(g^{ij}(u)\) of it determines a symmetric tensor field on \(M\) invariant w.r.t. the action of the Miura group. This gives a map

\[\hat{\Lambda}^2/G \rightarrow \text{symmetric tensors on } M.\]

**Theorem 3.3.21** Let \(M\) be a ball. Then the only invariant of a \((0, n)\) Poisson bracket in \(\hat{\Lambda}^2\) with respect to the action of the Miura group is the signature of the quadratic form \(g^{ij}(u)\).

Proof. The symmetric nondegenerate tensor \(g^{ij}(u)\) defines a pseudoriemannian metric

\[g_{ij}(u)du^i du^j, \quad (g_{ij}) = (g^{ij})^{-1}\]

on the manifold \(M\). From the general theory of [44] of the Poisson brackets of the form (3.1.11) it follows that the Riemann curvature of the metric vanishes, and that the coefficient \(\Gamma^j_{ik}(u)\) in (3.1.11) is related to the Christoffel coefficients \(\Gamma^k_{ij}(u)\) of the Levi-Civita connection for the metric by

\[\Gamma^j_{ik} = -g^{ij} \Gamma^j_{sk}.\]

Using standard arguments of differential geometry we deduce that, locally coordinates \(v^1(u), \ldots, v^n(u)\) exist such that, in the new coordinates the metric becomes constant

\[\partial v^k/\partial u^i \partial v^l/\partial u^j g^{ij}(u) = \eta^{kl} = \text{const}.\]

The Christoffel coefficients in these coordinates vanish. Of course, all constant symmetric matrices \(\eta^{kl}\) of a given signature are equivalent w.r.t. linear changes of coordinates.

We have reduced the proof of the theorem to reducing to the normal form (2.2.25) the Poisson bracket

\[\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x - y) + \sum_{k=1}^{\infty} c_k \{u^i(x), u^j(y)\}^{[k]} \quad (3.3.40)\]

by the transformations of the form (3.2.22) with \(F^i_0 = \text{id}\). As in the proof of Theorem 3.3.18 the latter problem is reduced to proving triviality of the second Poisson cohomology of the Poisson bracket \(\alpha\) of the form (2.2.25). Triviality of the cohomology is somewhat surprising from the point of view of finite-dimensional Poisson geometry. Indeed, as we have seen above this Poisson bracket degenerates. So we will be to also prove that all the cocycles are tangent to the leaves of the symplectic foliation (see the end of Section 1.1).
Lemma 3.3.22 For $M = \text{ball}$ all the cocycles in $H^1(\hat{L}(M))$ and $H^2(\hat{L}(M))$ vanishing at $\epsilon = 0$ are trivial.

Denote, like in (3.3.37),

$$\hat{H}^k = \oplus_{m>0} H^{k,m}. \quad (3.3.41)$$

We are to prove that $\hat{H}^1 = \hat{H}^2 = 0$.

Let us begin with proving triviality of $\hat{H}^1$. Using (2.2.5) we obtain

$$\partial_a \varpi = \text{Lie}_a \varpi^{ij} = - \left( \partial_x \sum_{t \geq 0} (-1)^t \eta^{ip} \left( \frac{\partial a^j}{\partial u^{p,t}} \right)^{(t)} \right) \delta(x-y)$$

$$- \sum_{r \geq 0} \left[ \frac{\partial a^i}{\partial u^{p,r}} \eta^{pj} + \sum_{t \geq r} (-1)^t \left( \frac{t + 1}{r + 1} \right) \eta^{ip} \left( \frac{\partial a^j}{\partial u^{p,t}} \right)^{(t-r)} \right] \delta^{(r+1)}(x-y). \quad (3.3.42)$$

Since

$$\frac{\delta \tilde{a}^j}{\delta u^p(x)} = \sum_{t \geq 0} (-1)^t \left( \frac{\partial a^j}{\partial u^{p,t}} \right)^{(t)}$$

is a differential polynomial in $\Lambda_0 \otimes \mathbb{C}[[\epsilon]]$ of the degree 0 vanishing at $\epsilon = 0$, from vanishing of the coefficient in front of $\delta(x-y)$ we derive that

$$\frac{\delta \tilde{a}^j}{\delta u^p(x)} = 0, \quad j, p = 1, \ldots, n.$$

Using Example 2.1.5 we derive existence of differential polynomials $b^j$ s.t.

$$a^j = \partial_x b^j, \quad j = 1, \ldots, n.$$

This is the crucial point in the proof: we have shown that the vector field $a$ is tangent to the level surface of the Casimirs (2.2.26). The remaining part of the proof is rather straightforward. Using (2.1.5) and also the Pascal triangle identity

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

we rewrite the coefficient of $\delta^{(r+1)}(x-y)$ in the form

$$\partial_x \left[ \frac{\partial \omega_k}{\partial u^{r}} - \sum_{t \geq r} (-1)^t \left( \frac{t + 1}{r} \right) \left( \frac{\partial \omega_l}{\partial u^{p,t}} \right)^{(t-r)} \right]$$

$$+ \frac{\partial \omega_k}{\partial u^{r-1}} + (-1)^r \frac{\partial \omega_l}{\partial u^{k,r-1}} = 0.$$
the last two terms are not present for \( r = 0 \). As above, for \( r = 0 \) we derive that
\[
\frac{\partial \omega_k}{\partial u^l} = \sum_{t \geq 0} (-1)^t \left( \frac{\partial \omega_l}{\partial u^{k,t}} \right)^{(t)}.
\]
Proceeding by induction in \( r \) we prove that the 1-form \( \int dx \wedge \omega \delta u^i \) is closed. Using the Volterra criterion we derive existence of a differential polynomial \( f \) s.t. \( \omega = \delta \int f \, dx \). Hence
\[
a^i = \eta^{ij} \partial_x \frac{\delta \tilde{f}}{\delta u^j(x)}.
\]
We proved triviality of \( \tilde{H}^1 \).

Let us proceed to prove the triviality of \( \tilde{H}^2 \). The condition \( \partial \alpha = 0 \) for \( \alpha \) of the form (2.2.15) can be computed similarly to Example 2.2.9. We obtain a system of equations
\[
\frac{\partial A^{ij}_{k}}{\partial u^{s-1}} \eta^{jk} + \sum (-1)^{q+r+s} \left( \begin{array}{cc} q+r+s & q \cr r & r \end{array} \right) \left( \frac{\partial A^{k}_{q+r+s}}{\partial u^{q+r+t}} \right)^{(r)} \eta^{ij} + \sum (-1)^{q+r+t} \left( \begin{array}{cc} q+r+t & q \cr r & r \end{array} \right) \left( \frac{\partial A^{k}_{q-r}}{\partial u^{q+r+t}} \right)^{(r)} \eta^{li} = 0
\]
for any \( i, j, k, s, t \) (3.3.43)

(it is understood that the terms with \( s-1, t-q-1 \) or \( t+q+r-1 \) negative do not appear in the sum). Recall that the crucial point in the proof of triviality of the 2-cocycle is to establish validity of (1.1.18) for the Casimirs (2.2.26) of \( \omega \). Explicitly, we need to show that
\[
\alpha(\delta \bar{u}^i, \delta \bar{u}^j) = \int A^{ij}_{0} dx = 0 \text{ for any } i, j. \tag{3.3.44}
\]
We first use (3.3.43) for \( s = t = 0 \) to prove that
\[
\partial_x \sum_r (-1)^r \left( \frac{\partial A^{jk}_{0}}{\partial u^{q+r}} \right)^{(r)} \eta^{ij} = 0.
\]
Hence
\[
A^{jk}_{0} = \partial_x B^{jk}
\]
for some differential polynomial \( B^{jk} \). This implies (3.3.44). The rest of the proof is identical to the proof of Lemma 1.1.3. We first construct the vector field \( z \) (see the proof of the lemma 1.1.3). To this end we use the equation (3.3.43) for \( s = 0, t > 0 \):
\[
\sum_{q,r} (-1)^{q+r} \left( \begin{array}{c} q+r \\ r \end{array} \right) \left( \frac{\partial A^{ki}_{q+r}}{\partial u^{q+r+t}} \right)^{(r)} \eta^{ij} + \sum_{r} (-1)^{t+r} \left( \begin{array}{c} t+r \\ r \end{array} \right) \left( \frac{\partial A^{jk}_{0}}{\partial u^{t+r-1}} \right)^{(r)} \eta^{li} = 0.
\]
Differentiating the antisymmetry condition

\[ A_{0}^{ik} = \sum (-1)^{r+1} \left( A_{r}^{i(k)} \right) \quad (r) \]

w.r.t. \( u^{l,t-1} \) we identify the first term of the previous equation with

\[ - \frac{\partial A_{0}^{ik}}{\partial u^{l,t-1}} \eta^{ij}. \]

The resulting equation coincides with the condition \( \partial a^{k} = 0 \) of closedness of the 1-cocycle

\[ (a^{k})^{i} = A_{0}^{ik} \]

for every \( k = 1, \ldots, n \) (see (3.3.42) for the explicit form of this condition). Using the first part of Lemma we arrive at existence of \( n \) differential polynomials \( q^{1}, \ldots, q^{n} \) s.t.

\[ A_{0}^{ik} = \eta^{js} \frac{\delta q^{k}}{\delta u^{s}(x)}, \quad (3.3.45) \]

The last step, as in the proof of Lemma 1.1.3, is to change the cocycle \( \alpha \) to a cohomological one to obtain a closed 2-cocycle

\[ \alpha \mapsto \alpha + \partial z =: \alpha' \]

for

\[ z = q^{i} \frac{\partial}{\partial u^{i}}. \]

The new 2-cocycle \( \alpha' \) will have the same form as above with \( A_{0}^{ij} = 0 \). Denote

\[ g_{i;js} := n_{ip} \eta_{ij} A_{s}^{ij}, \quad s \geq 1. \]

We will now show existence of differential polynomials \( \omega_{i;0}, \omega_{i;1}, \ldots \) s.t.

\[ g_{i;js} = \partial^{2} \omega_{i,j,s-1} + \omega_{i,j,s-2} \text{ for } s \geq 2. \quad (3.3.46) \]

From (3.3.43) for \( s = 1, \ t = 0 \) we obtain

\[ \partial^{2} \sum_{r} (-1)^{r} \left( \frac{\partial A_{1}^{ik}}{\partial u^{l,r}} \right) \quad (r) = 0. \]

As we already did many times, from the last equation it follows that

\[ \sum_{r} (-1)^{r} \left( \frac{\partial A_{1}^{ik}}{\partial u^{l,r}} \right) \quad (r) = 0. \]
This shows existence of $\omega_{i;j0}$. Using (3.3.43) for $s = 1$ and $t > 0$ we inductively prove existence of the differential polynomials $\omega_{i;j,t-1}$. Actually, we can obtain

$$\omega_{i;j} = \sum_{s \geq t+2} \partial_x^{l-2} g_{i;js}. \quad (3.3.47)$$

From this it readily follows that the coefficients $\omega_{i;js}$ satisfy the antisymmetry conditions (2.1.30). Thus they determine a 2-form $\omega$.

Let us prove that the 2-form $\omega$ is closed. Denote

$$J_{ijk;st} := \left( \sum_{m=s}^{l+s} \sum_{r=0}^{m-s} \sum_{m\geq t+s+1}^{t} (-1)^m \binom{m}{r} \partial_x^{m-r-s} \frac{\partial \omega_{j;k,t-r}}{\partial u^{t,m}} \right)$$

the l.h.s. of the equation (2.1.32) of closedness of a 2-form. Let us show that the coefficient of $\delta^{(v)}(x-y)\delta^{(s)}(x-z)$ in (3.3.43) is equal to

$$\partial_x J_{ijk;1,s-1} + J_{ijk;1,s-2} + J_{ijk;2,t-1}. \quad (3.3.48)$$

To this end we replace the second sum in (3.3.43) by

$$- \frac{\partial A_{ik}^{jk} }{\partial u^{t,l-1}} \eta^{lj}. \quad \text{lowering the indices by means of } \eta_{ij} \text{ and using (3.3.46) we obtain (3.3.48).}$$

From vanishing of (3.3.48) we inductively deduce that $J_{ijk;st} = 0$ for all $i, j, k = 1, \ldots, n$ and all $s, t \geq 0$ (observe that the coefficients $J_{ijk;0} = J_{ijk;0s} = 0$ due to our assumption $A_{ij}^{0} = 0$. This proves that the 2-form $\omega$ is closed. So $\omega = \delta \int dx \wedge \phi$ for some 1-form $\phi = \phi_i \delta u^i$. Introducing the vector field

$$a^i = \eta^{ik} \phi_k$$

we finally obtain, for the original cocycle $\alpha$,

$$\alpha = \partial(a - z).$$

Theorem is proved.

**Example 3.3.23** The Poisson brackets (3.1.20) of the “interpolated” Toda lattice (3.1.19) can be reduced to the canonical form

$$\{u(x), w(y)\} = \delta'(x-y), \quad \{u(x), u(y)\} = \{w(x), w(y)\} = 0$$

by the Miura-type transformation $(u, v) \mapsto (u, w)$,

$$v'(x) = \frac{1}{\epsilon} [w(x + \epsilon) - w(x)] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\epsilon \partial_x)^n w(x). \quad (3.3.49)$$

The inverse transformation reads

$$w(x) = \epsilon \partial_x [e^{\epsilon \partial_x} - 1]^{-1} v(x) = \sum_{n=0}^{\infty} B_n \frac{\epsilon^n}{n!} v(x). \quad (3.3.50)$$

Here $B_n$ are Bernoulli numbers.
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