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Hamiltonian and quantum mechanics

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1 Historical and preliminary remarks

The summary of Hamiltonian formulation of the classical mechanics could be presented as follows. The state of an isolated physical system is described by its position q and momentum p. Any physical quantity is represented by a smooth function $f \in C^\infty(\mathbb{R}^{2N})$ of the canonical variables $(q_1, \dots, q_N, p_1, \dots, p_N)$. The appropriate physical law describing the time evolution of f is expressed by the ordinary differential equation

$$\frac{d}{dt} f = \{h, f\}, \tag{1.1}$$

where

$$\{h, f\} := \sum_{k=1}^{N} \left(\frac{\partial h}{\partial q^k} \frac{\partial f}{\partial p^k} - \frac{\partial f}{\partial q^k} \frac{\partial h}{\partial p^k} \right). \tag{1.2}$$

and $h \in C^\infty(\mathbb{R}^{2N})$ is Hamiltonian, i.e. the function describing the total energy of the system. The Poisson bracket defined by (1.2) has the crucial meaning for the integration of the Hamiltonian equations (1.1). It is bilinear operation on $C^\infty(\mathbb{R}^{2N})$ satisfying Leibniz

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{1.3}$$

and Jacobi

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{1.4}$$

identities. Hence, the space $\mathcal{I} \subset C^\infty(\mathbb{R}^{2N})$ of integrals of motion ($f \in \mathcal{I}$ iff $\{h, f\} = 0$) is closed:

i) with respect to function operations, i.e. if $f_1, \dots, f_K \in \mathcal{I}$ and $F \in C^\infty(\mathbb{R}^K)$ then $F(f_1, \dots, f_K) \in \mathcal{I}$;

ii) with respect to Poisson bracket, i.e. $f, g \in \mathcal{I}$ then $\{f, g\} \in \mathcal{I}$.

Such structure was called by Lie [26] (see also [64]) the **function group**. Assuming that $f_1, \dots, f_K$ are functionally independent and $\mathcal{I}$ is functionally generated by them one obtains relation

$$\{f_k, f_l\} = \pi_{kl}(f_1, \dots, f_K), \tag{1.5}$$

where $\pi_{kl} \in C^\infty(\mathbb{R}^K)$ for $k, l = 1, \dots, K$. The antisymmetry of Poisson bracket and Jacobi identity imply the following conditions

$$\pi_{kl} = -\pi_{lk} \tag{1.6}$$

and

$$\pi_{kl} \frac{\partial \pi_{rs}}{\partial f^k} + \pi_{ks} \frac{\partial \pi_{lr}}{\partial f^k} + \pi_{kr} \frac{\partial \pi_{sl}}{\partial f^k} = 0 \tag{1.7}$$

on the smooth functions $\pi_{kl} \in C^\infty(\mathbb{R}^K)$. Fixing the generating integrals of motion $f_1, \dots, f_K$ one can identify $\mathcal{I}$ with $C^\infty(\mathbb{R}^K)$. For $F, G \in C^\infty(\mathbb{R}^K)$ from Leibniz identity (1.3) one has

$$\{F(f_1, \dots, f_K), G(f_1, \dots, f_K)\} = \pi_{kl}(f_1, \dots, f_K) \frac{\partial F}{\partial f_k} \frac{\partial G}{\partial f_l}. \tag{1.8}$$

From the conditions (1.6) and (1.7) it follows that the bilinear operation

$$[F,G] := \pi_{kl} \frac{\partial F}{\partial f_k} \frac{\partial G}{\partial f_l} \tag{1.9}$$

defines a Poisson bracket on $C^\infty(\mathbb{R}^K)$. The map $\mathcal{J} : \mathbb{R}^{2N} \to \mathbb{R}^K$ defined by

$$\mathcal{J}(q,p) := \begin{pmatrix} f_1(q,p) \\ \vdots \\ f_K(q,p) \end{pmatrix} \tag{1.10}$$

is a Poisson map, i.e.

$$\{F \circ \mathcal{J}, G \circ \mathcal{J}\} = [F,G] \circ \mathcal{J}. \tag{1.11}$$

In the particular case when the Poisson tensor $\pi = (\pi_{kl})$ depends on the variables $f_1, \ldots, f_K$ linearly

$$\pi_{kl}(f_1, \ldots, f_K) = c_{klm} f_m, \tag{1.12}$$

where

$$c_{klm} = -c_{lkm} \tag{1.13}$$

and

$$c_{rnm} c_{klr} + c_{rln} c_{mkr} + c_{rkn} c_{lmr} = 0 \tag{1.14}$$

the vector subspace of linear functions $(\mathbb{R}^K)^* \subset C^\infty(\mathbb{R}^K)$ is preserved $[(\mathbb{R}^K)^*, (\mathbb{R}^K)^*] \subset (\mathbb{R}^K)^*$ by the Poisson bracket $[\cdot,\cdot]$ operation. The above explains how Sophus Lie came to the Lie algebra $\mathfrak{g} = (\mathbb{R}^K)^*$ with the bracket $[\cdot,\cdot]$ defined by the structural constants c_{klm}, i.e. one has

$$[e_k^*, e_l^*] = c_{klm} e_m^* \tag{1.15}$$

for the basis $\langle e_1^*, \ldots, e_K^* \rangle = \mathfrak{g}$ dual to the canonical basis $(e_1, \ldots, e_K)$ of $\mathbb{R}^K$. The vector space $\mathfrak{g}_* := \mathbb{R}^K$ predual to $\mathfrak{g}$ with linear Poisson bracket

$$[F,G] := c_{klm} f_m \frac{\partial F}{\partial f_k} \frac{\partial G}{\partial f_l} \tag{1.16}$$

defined by the Lie algebra structure of $\mathfrak{g}$ is called **Lie-Poisson space**. Since in the finite dimensions the predual $\mathfrak{g}_*$ is canonically isomorphic with the dual $\mathfrak{g}^*$ of Lie algebra $\mathfrak{g}$, one takes $\mathfrak{g}^*$ as the Lie-Poisson space related to $\mathfrak{g}$.

The integrals motion map $\mathcal{J} : \mathbb{R}^{2N} \to \mathfrak{g}^*$ defined by (1.10) in the case of linear Poisson tensor (1.12) is custommatory called the **momentum map**, see [52].

Contemporary Poisson geometry investigates the Lie's ideas [26] in the context of global differential geometry replacing $\mathbb{R}^{2N}$ by the symplectic manifold and $\mathbb{R}^K$ by the Poisson manifold. The notions of Lie-Poisson space and momentum map were rediscovered many years later, when the theory of Lie algebras and Lie groups as well as differential geometry have been already well founded mathematical disciplines, see [29, 60, 64, 67, 51, 4].

2 The Banach Lie-Poisson space of trace class operators

Now, we shall extend Lie ideas to the infinite dimensional case. In the first step of this effort we replace the elementary phase space $\mathbb{R}^{2N}$ by the space $\mathbb{CP}(\mathcal{H})$ of pure states of the quantum physical system. By the definition $\mathbb{CP}(\mathcal{H})$ is infinite dimensional complex projective separable Hilbert space. We fix in $\mathcal{H}$ an orthonormal basis using Dirac notation $\{|n\rangle\}_{n=0}^{\infty}$, i.e. $\langle n|m\rangle = \delta_{nm}$ and define the covering $\bigcup_{k\in\mathbb{N}\cup\{0\}} \Omega_k = \mathbb{CP}(\mathcal{H})$ of $\mathbb{CP}(\mathcal{H})$ by the open domains

$$\Omega_k := \{[\psi] : \psi_k \neq 0\}, \quad \text{where } [\psi] := \mathbb{C}|\psi\rangle \tag{2.1}$$

and $|\psi\rangle = \sum_{n=0}^{\infty} \psi_k |n\rangle$. Maps $\varphi_k : \Omega_k \to l^2$ defined by

$$\varphi_k([\psi]) := \frac{1}{\psi_k}(\psi_0, \psi_1, \ldots, \psi_{k-1}, \psi_{k+1}, \ldots) \tag{2.2}$$

similarly as in the finite dimensions case give the complex analytic atlas on $\mathbb{CP}(\mathcal{H})$.

The projective space $\mathbb{CP}(\mathcal{H})$ is an infinite dimensional Kähler manifold with Kähler structure given by the Fubini-Study form

$$\omega_{FS} := i\partial\bar{\partial}\log\langle\psi|\psi\rangle. \tag{2.3}$$

In the coordinates $(z_1, z_2, \ldots) = (\frac{\psi_1}{\psi_0}, \frac{\psi_2}{\psi_0}, \ldots) = \varphi_0([\psi])$ it is given by

$$\omega_{FS} = i\partial\bar{\partial}\log(1 + z^+z) = i(1 + z^+z)^{-2} \sum_{k,l=1}^{\infty} \left((1 + z^+z)\delta_{kl} - z_k\bar{z}_l\right) dz_l \wedge d\bar{z}_k \tag{2.4}$$

and the corresponding Poisson bracket for $f, g \in C^\infty(\mathbb{CP}(\mathcal{H}))$ by

$$\{f, g\}_{FS} = -i(1 + z^+z) \sum_{k,l=1}^{\infty} (\delta_{kl} + z_k\overline{z_l}) \left(\frac{\partial f}{\partial z_k}\frac{\partial g}{\partial \overline{z_l}} - \frac{\partial g}{\partial z_k}\frac{\partial f}{\partial \overline{z_l}}\right), \tag{2.5}$$

where we assumed notation

$$z^+z := \sum_{k=1}^{\infty} \bar{z}_k z_k. \tag{2.6}$$

In order to recognize the Lie Poisson space suitable for the predual space $\mathbb{R}^K = \mathfrak{g}_*$ of Lie algebra we will consider the functionally independent functions $f_{nm} = \overline{f_{mn}}$ defined by

$$f_{nm}(z) := \frac{z_n\bar{z}_m}{1 + z^+z}, \quad m, n \in \mathbb{N} \tag{2.7}$$

as an equivalent of the generating functions $f_1, \ldots, f_K$ from the previous section. The family of functions (2.7) is closed with respect to Poisson bracket (2.5), i.e.

$$\{f_{kl}, f_{mn}\}_{FS} = f_{ml}\delta_{kn} - f_{kn}\delta_{lm}. \tag{2.8}$$

Now, let us take the C^*-algebra $L^\infty(\mathcal{H})$ of the bounded operators acting in $\mathcal{H}$. It can be considered as the Banach space dual

$$L^\infty(\mathcal{H}) = (L^1(\mathcal{H}))^* \tag{2.9}$$

to the Banach space of the trace-class operators:

$$L^1(\mathcal{H}) := \left\{ \rho \in L^\infty(\mathcal{H}) : \|\rho\|_1 := \operatorname{Tr} \sqrt{\rho^* \rho} < \infty \right\}. \tag{2.10}$$

The duality is given by

$$\langle X; \rho \rangle := \operatorname{Tr}(X\rho), \tag{2.11}$$

where $X \in L^\infty(\mathcal{H})$ and $\rho \in L^1(\mathcal{H})$. Let us remark here that $L^1(\mathcal{H})$ is an ideal in $L^\infty(\mathcal{H})$ but not Banach subspace. The closure of $L^1(\mathcal{H})$ in the norm $\|X\|_\infty := \sup_{\psi \neq 0} \frac{\|X\psi\|}{\|\psi\|}$ gives the ideal $L^0(\mathcal{H}) \subset L^\infty(\mathcal{H})$ of compact operators.

Since $L^1(\mathcal{H}) \subset L^2(\mathcal{H})$, where

$$L^2(\mathcal{H}) := \{ \rho \in L^\infty(\mathcal{H}) : \|\rho\|_2 := \sqrt{\operatorname{Tr} \rho^* \rho} < \infty \} \tag{2.12}$$

is the ideal of Hilbert-Schmidt operators in $\mathcal{H}$, one can consider

$$\left\{ |m\rangle\langle n| \right\}_{n,m=0}^{\infty} \tag{2.13}$$

as Schauder basis [66] of $L^1(\mathcal{H})$. The functionals

$$\left\{ \operatorname{Tr}(|k\rangle\langle l| \,\cdot\,) \right\}_{k,l=0}^{\infty} \tag{2.14}$$

are biorthogonal with respect to the basis (2.13). Thus they form the basis of $L^\infty(\mathcal{H})$ in sense of the weak*-topology on $L^\infty(\mathcal{H})$.

The associative Banach algebra $L^\infty(\mathcal{H})$ can be considered as the Banach Lie algebra of the complex Banach Lie group $GL^\infty(\mathcal{H})$ of the invertible elements in $L^\infty(\mathcal{H})$. The real Banach Lie algebra

$$U^\infty(\mathcal{H}) := \{ X \in L^\infty(\mathcal{H}) : X^* + X = 0 \} \tag{2.15}$$

of the anti-hermitian operators is related to the real Banach Lie group $GU^\infty(\mathcal{H})$ of the unitary operators.

The predual Banach space of $U^\infty(\mathcal{H})$ is

$$U^1(\mathcal{H}) := \{ \rho \in L^1(\mathcal{H}) : \rho^* = \rho \} \tag{2.16}$$

and the isomorphism $U^1(\mathcal{H})^* \cong U^\infty(\mathcal{H})$ is given by

$$\langle X; \rho \rangle := i \operatorname{Tr}(X\rho). \tag{2.17}$$

Using (2.17) it is easy to check that

$$\operatorname{ad}^*_X \rho = [\rho, X], \tag{2.18}$$

what shows that Banach subspace $U^1(\mathcal{H}) \subset U^\infty(\mathcal{H})^*$ is invariant with respect to the coadjoint action of $U^\infty(\mathcal{H})$ on $U^\infty(\mathcal{H})^*$. The above allows us to define Poisson bracket

$$\{F, G\}_{U^1}(\rho) := i \operatorname{Tr}\left(\rho[DF(\rho, DG(\rho)]\right) \tag{2.19}$$

for $F, G \in C^\infty(U^1(\mathcal{H}))$, see paper by Bona [7].

From (2.17) we have

$$X_F(G)(\rho) = \operatorname{Tr}(\rho DF(\rho)DG(\rho) - \rho DG(\rho)DF(\rho)) = \operatorname{Tr}([\rho, DF(\rho)]DG(\rho) \tag{2.20}$$

for any $F, G \in C^\infty(U^1(\mathcal{H}))$. So,

$$X_F(\rho) = [\rho, DF(\rho)] = -\operatorname{ad}^*_{DF(\rho)} \rho \tag{2.21}$$

and then the Hamilton equations with Hamiltonian $H \in C^\infty(U^1(\mathcal{H}))$ assume for all $F \in C^\infty(U^1(\mathcal{H}))$ the form

$$\frac{d}{dt}F(\rho(t)) = \{H, \rho\}(\rho(t)) = i \operatorname{Tr}(\rho(t)[DH(\rho(t)), DF(\rho(t))]) = \tag{2.22}$$

$$= i \operatorname{Tr}([\rho(t), DH(\rho(t)]DF(\rho(t)))$$

or equivalently

$$-i\frac{d}{dt}\rho(t) = [\rho(t), DH(\rho(t))], \tag{2.23}$$

where we used the identity

$$\frac{d}{dt}F(\rho(t)) = \operatorname{Tr}\left(DF(\rho(t))\frac{d}{dt}\rho(t)\right). \tag{2.24}$$

The equation (2.23) is the non-linear version of the Liouville-von Neumann equation. One obtains the Liouville-von Neumann equation taking in (2.23) the Hamiltonian $H(\rho) = \operatorname{Tr}(\rho\hat{H})$, where $\hat{H} \in iU^\infty(\mathcal{H})$.

The **characteristic distribution**

$$S_\rho = \left\{X_F(\rho) \ : \ F \in C^\infty(U^1(\mathcal{H}))\right\} \qquad \rho \in U^1(\mathcal{H}) \tag{2.25}$$

for $U^1(\mathcal{H})$ is given by

$$S_\rho = \left\{[\rho, DF(\rho)] \ : \ F \in C^\infty(U^1(\mathcal{H}))\right\} = \{[\rho, X] \ : \ X \in U^\infty(\mathcal{H})\}. \tag{2.26}$$

Later we shall come back to it and will consider the symplectic leaves for $U^1(\mathcal{H})$.

Examples of **Casimirs**, i.e. the functions

$$K \in C^\infty(U^1(\mathcal{H})) \text{ such that } \{K, F\} = 0 \quad \forall F \in C^\infty(U^1(\mathcal{H})), \tag{2.27}$$

one obtains by

$$K_l(\rho) := \frac{1}{l+1} \operatorname{Tr} \rho^{l+1}, \qquad l = 0, 1, 2, \ldots \tag{2.28}$$

and one has

$$\{K_l, F\}(\rho) = \mathrm{Tr}(\rho[DK_l(\rho), DF(\rho)]) = \mathrm{Tr}([DK_l(\rho), \rho]DF(\rho)) = \mathrm{Tr}([\rho^l, \rho]DF(\rho)) = 0, \tag{2.29}$$

where we applied the formula

$$DK_l(\rho) = \rho^l. \tag{2.30}$$

For the case $l = 1$ let us prove (2.30) directly

$$\mathrm{Tr}(\rho + \Delta\rho)^2 - \mathrm{Tr}\,\rho^2 = Tr(2\rho\Delta\rho) + \mathrm{Tr}(\Delta\rho)^2 \tag{2.31}$$

$$\frac{\left|\mathrm{Tr}(\Delta\rho)^2\right|}{\|\Delta\rho\|_1} \leqslant \frac{\|\Delta\rho\|_1^2}{\|\Delta\rho\|_1} = \|\Delta\rho\|_1 \to 0, \tag{2.32}$$

when $\|\Delta\rho\|_1 \to 0$. Now using the identification $U^1(\mathcal{H})^* \cong U^\infty(\mathcal{H})$ by the trace we obtain (2.30).

Passing to the coordinate description

$$\rho = \sum_{n,m=0}^{\infty} \rho_{nm}\, |n\rangle\langle m|, \tag{2.33}$$

$$DF(\rho) = i \sum_{n,m=0}^{\infty} \frac{\partial F}{\partial \rho_{nm}}(\rho)\, |n\rangle\langle m|, \tag{2.34}$$

where $\overline{\rho_{nm}} = \rho_{mn}$, we obtain explicit formulas for:

i) Poisson bracket

$$\{F, G\}_{U^1}(\rho) = \sum_{k,l,m=0}^{\infty} \rho_{kl} \left(\frac{\partial F}{\partial \rho_{lm}} \frac{\partial G}{\partial \rho_{mk}} - \frac{\partial G}{\partial \rho_{lm}} \frac{\partial F}{\partial \rho_{mk}} \right) \tag{2.35}$$

ii) Hamiltonian vector field

$$X_F(\rho) = \sum_{k,m=0}^{\infty} \left(\sum_{l=0}^{\infty} \left(\rho_{kl} \frac{\partial F}{\partial \rho_{lm}} - \frac{\partial F}{\partial \rho_{kl}} \rho_{lm} \right) \right) |k\rangle\langle m| \tag{2.36}$$

iii) Hamilton equations

$$\frac{d}{dt}\rho_{km}(t) = \sum_{l=0}^{\infty} \left(\rho_{kl}(t) \frac{\partial H}{\partial \rho_{lm}(t)} - \frac{\partial H}{\partial \rho_{kl}(t)} \rho_{lm}(t) \right). \tag{2.37}$$

From (2.8) and (2.35) we see that the map $\iota : \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by

$$\iota([\psi]) := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \sum_{k,l=0}^{\infty} \frac{1}{1 + z^+ z} z_k \bar{z}_l |k\rangle\langle l|, \tag{2.38}$$

where we assume $z_0 = \bar{z}_0 = 1$, preserves Poisson bracket

$$\{F \circ \iota, G \circ \iota\}_{FS} = \{F, G\}_{U^1} \circ \iota \tag{2.39}$$

and in coordinates (2.33) has form $\rho_{kl} \circ \iota = f_{kl}$.

Therefore, we conclude:

Proposition 2.1. *The map $\iota : \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by* (2.38) *is the momentum map of the symplectic manifold $\mathbb{CP}(\mathcal{H})$ into the Banach Lie-Poisson space $U^1(\mathcal{H})$ predual to the Banach Lie algebra $U^\infty(\mathcal{H})$.*

In order to have a link with some physical models let us present the formulas given above in the **Schrödinger representation**.

We assume in that case that $\mathcal{H} = L^2(\mathbb{R}^N, d^N x)$ and represent $\rho \in U^1(\mathcal{H})$

$$(\rho\psi)(x) = \int \rho(x,y)\psi(y) d^N y, \tag{2.40}$$

where $\psi \in L^2(\mathbb{R}^N, d^N x)$, by the kernel $\rho(x,y) = \overline{\rho(y,x)}$, such that its diagonal $\rho(x,x)$ belongs to $L^1(\mathbb{R}^N, d^N x)$. For the derivative $DF(\rho) \in L^\infty(\mathcal{H})$ the kernel is given by $\frac{\delta F}{\delta \rho(x,y)}$, where we assumed the notation of functional derivative $\frac{\delta}{\delta\rho(x,y)}$, which is more familiar for physicists. Namely

$$DF(\rho)\psi(x) = \int \frac{\delta F}{\delta \rho(x,y)} \psi(y) d^N y. \tag{2.41}$$

Using (2.40) and (2.41) we obtain expressions for:

i) Poisson bracket

$$\{F,G\}(\rho) = i \iiint \rho(x,y) \left(\frac{\delta F}{\delta\rho(y,z)} \frac{\delta G}{\delta\rho(z,x)} - \frac{\delta G}{\delta\rho(y,z)} \frac{\delta F}{\delta\rho(z,y)} \right) d^N x d^N y d^N z \tag{2.42}$$

ii) Hamiltonian vector field

$$X_F(\rho) = \int d^N x \int d^N y \int d^N z \left(\rho(x,z) \frac{\delta F}{\delta\rho(z,y)} - \frac{\delta F}{\delta\rho(x,z)} \rho(z,y) \right) |\psi(x)\rangle\langle\psi(y)|, \tag{2.43}$$

where $\langle\psi(y)|\psi(x)\rangle = \delta(x-y)$ (Dirac notation for $L^2(\mathbb{R}^N, d^N x)$.

iii) Hamilton equations

$$-i\frac{d}{dt}\rho_t(x,y) = \int d^N z \left(\rho(x,z) \frac{\delta H}{\delta\rho_t(z,y)} - \frac{\delta H}{\delta\rho_t(x,z)} \rho_t(z,y) \right). \tag{2.44}$$

In the "basis" $\{|\psi(x)\rangle\langle\psi(y)|\}_{x,y\in\mathbb{R}^N}$ the mixed state $\rho \in U^1(\mathcal{H})$ is given by

$$\rho = \int d^N x d^N y \rho(x,y) |\psi(x)\rangle\langle\psi(y)| \tag{2.45}$$

and $DH(\rho) \in U^\infty(\mathcal{H})$ by

$$DH(\rho) = i \int d^N x d^N y \frac{\delta H}{\delta\rho(x,y)} |\psi(x)\rangle\langle\psi(y)|. \tag{2.46}$$

Let us end this section by applying the theory presented here to the cases of two well known dynamical systems.

Example 2.1 (*Unitary Shrödinger*)**.**

$$H(\rho) = \mathrm{Tr}(\rho\hat{H}), \qquad \text{where } \hat{H} \in iU^{\infty}(\mathcal{H}). \tag{2.47}$$

In this case one has

$$D\hat{H}(\rho) = \hat{H}, \tag{2.48}$$

$$-i\frac{d}{dt}\rho(t) = [\hat{H}, \rho], \tag{2.49}$$

This is Liouville-von Neumann equation for the dynamics of mixed states. The equation generates unitary (anti-unitary) flow, i.e.

$$\rho(t) = U_H(t)\rho_0 U_H^*(t), \tag{2.50}$$

where $\mathbb{R} \ni t \longrightarrow U_H(t) \in GU^{\infty}(\mathcal{H})$ is one–parameter unitary group

$$U_H(t) = e^{it\hat{H}} \tag{2.51}$$

generated by the self-adjoint operator $\hat{H}$.

In general quantum mechanical Hamiltonians $\hat{H}$ are unbounded self-adjoint operators. Hence, for the typical case the Hamilton function (2.47) is defined only on $\rho \in U^1(\mathcal{H})$ given by

$$\rho = \sum_{k=1}^{\infty} \rho_{kl} |\psi_k\rangle\langle\psi_l|, \tag{2.52}$$

where vectors ψ_k belong to the domain $D(\hat{H})$ of $\hat{H}$. In other words the domain of $\mathrm{ad}^*_{\hat{H}} = [\hat{H}, \cdot]$ is $U^1(\mathcal{H}) \cap (D(\hat{H}) \otimes D(\hat{H})^*) \subset U^1(\mathcal{H})$. In the following we will propose the way of avoiding this unpleasant on the first sight situation. Let us remark however that Hamiltonian (unitary) flow $U_H(t)$ generated by $H(\rho) = \mathrm{Tr}(\rho\hat{H})$ is well defined on all $U^1(\mathcal{H})$.

Ending the example we observe that unitary flow $U_H(t)$ preserves $\iota(\mathbb{CP}(\mathcal{H}))$ and in $\mathcal{H}$ it is given by

$$|\psi(t)\rangle = U_H(t)|\psi(0)\rangle \tag{2.53}$$

and $|\psi(t)\rangle \in \mathcal{H}$ fulfill the Schrödinger equation

$$-i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{2.54}$$

♢

Example 2.2 (*Non-linear Schrödinger*)**.**

For the investigation of that case we will use Schrödinger representation, i.e. the Hilbert space $\mathcal{H}$ will be realized as $L^2(\mathbb{R}^N, dx)$. The non-linear Schrödinger dynamics is given on $U^1(\mathcal{H})$ by the following Hamilton function

$$H(\rho) := \mathrm{Tr}(\hat{H}\rho) + \frac{1}{2}\kappa \int_{\mathbb{R}^N} (\rho(x,x))^2 d^N x, \tag{2.55}$$

where $\hat{H}$ is a self-adjoint operator with the kernel $H(x, y)$ and $\kappa > 0$ is the coupling constant.

The functional derivative of (2.55) is

$$\frac{\delta H}{\delta \rho(x,y)}(\rho) = H(x,y) + \kappa \delta(x-y)\rho(x,y). \tag{2.56}$$

Thus and from Hamilton equation in Schrödinger representation (2.44) one finds

$$-i\frac{d}{dt}\rho_t(x,y) = \int d^N z\,(\rho_t(x,y)H(z,y) - H(x,z)\rho(z,y)) +$$

$$+\kappa \int d^N z\big(\rho_t(x,z)\delta(z-y)\rho_t(z,y) - \delta(x-z)\rho_t(x,z)\rho_t(z,y)\big) = \tag{2.57}$$

$$= \int d^N z\,(\rho_t(x,z)H(z,y) - H(x,z)\rho_t(z,y)) + \kappa(\rho_t(x,y)\rho_t(y,y) - \rho_t(x,x)\rho_t(x,y)).$$

For the decomposable kernels

$$\rho_t(x,y) = \psi_t(x)\overline{\psi_t(y)} \tag{2.58}$$

i.e. after restriction to $\iota(\mathbb{CP}(\mathcal{H}))$ the equation (2.2) reduces to

$$-i\frac{d}{dt}\psi_t(x) = \int\limits_{\mathbb{R}^N} H(x,z)\psi(z)d^N z + \kappa\,|\psi_t(x)|^2\,\psi_t(x) \tag{2.59}$$

and for

$$H(x,z) = -\Delta_x \delta(x-z) + \delta(z-x)V(x) \tag{2.60}$$

gives the non-linear Schrödinger equation

$$-i\frac{d}{dt}\psi_t(x) = (-\Delta + V(x))\psi_t(x) + \kappa\,|\psi_t(x)|^2\,\psi_t(x). \tag{2.61}$$

Let us remark that the kernel (2.60) gives unbounded symmetric operator. So for that case one has Hamiltonian $H(\rho)$ defined on a dense subset of $U^1(\mathcal{H})$ only.

$\diamondsuit$

3 Banach Poisson Manifolds

Let us recall that topological space P locally isomorphic to Banach space $\mathfrak{b}$ with the fixed maximal smooth atlas is called Banach manifold modelled on $\mathfrak{b}$, see [8]. For any $p \in P$ one has canonical isomorphisms $T_pP \cong \mathfrak{b}$, $T_p^*P \cong \mathfrak{b}^*$ and $T_p^{**}P \cong \mathfrak{b}^{**}$ of Banach spaces. Since in general case $\mathfrak{b} \subsetneq \mathfrak{b}^{**}$ the tangent bundle TP is not isomorphic with twice-dual bundle $T^{**}P$. Hence one has only the canonical inclusion $TP \subset T^{**}P$ isometric on fibers. The isomorphism $TP \cong T^{**}P$ has place only if $\mathfrak{b}$ is reflexive. Particularly, when $\mathfrak{b}$ is finite dimensional.

Like in the finite dimensional case one defines the Poisson bracket on the space $C^\infty(P)$ as a bilinear smooth antisymmetric map

$$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P) \tag{3.1}$$

satisfying Leibniz and Jacobi identities. Due to the Leibniz property there exists antisymmetric 2-tensor field $\pi \in \Gamma^\infty(\bigwedge^2 T^{**}P)$ satisfying

$$\{f, g\} = \pi(df, dg) \tag{3.2}$$

for each $f, g \in C^\infty(P)$. In addition from Jacobi property and from

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = [\pi, \pi]_S(df \wedge dg \wedge dh), \tag{3.3}$$

see [29], one has that the 3-tensor field $[\pi, \pi]_S \in \Gamma^\infty(\bigwedge^3 T^{**}P)$, called the Skouten bracket of π, satisfies the condition

$$[\pi, \pi]_S = 0. \tag{3.4}$$

Hence the Poisson bracket can be equivalently described by the antisymmetric 2-tensor field satisfying the differential equation (3.4). One calls π the Poisson tensor.

Let us define by

$$\sharp df := \pi(\cdot, df) \tag{3.5}$$

the map $\sharp : T^*P \to T^{**}P$ covering the identity map $\mathrm{id} : P \to P$, for any locally defined smooth function f. One has $\sharp df \in \Gamma^\infty(T^{**}P)$, so, opposite to the finite dimensional case, it is not vector field in general. Thus according to [40] we give the following definition

Definition 3.1. A **Banach Poisson manifold** is a pair $(P, \{\cdot,\cdot\})$ consisting of a smooth Banach manifold and a bilinear operation $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ satisfying the following conditions:

i) $(C^\infty(P), \{\cdot,\cdot\})$ is a Lie algebra;

ii) $\{\cdot,\cdot\}$ satisfies the Leibniz property on each factor;

iii) the vector bundle map $\sharp : T^*P \to T^{**}P$ covering the identity satisfies $\sharp(T^*P) \subset TP$.

As we see, the condition iii) allows one to introduce for any function $f \in C^\infty(P)$ the **Hamiltonian vector field** X_f by

$$X_f := \sharp df. \tag{3.6}$$

In consequence after fixing Hamiltonian $h \in C^\infty(P)$ at the above one can consider Banach Hamiltonian system $(P, \{\cdot,\cdot\}, h)$ with equation of motion

$$\frac{d}{dt} f = -X_h(f) = \{h, f\}. \tag{3.7}$$

Definition 3.1 allows us to define the characteristic distribution $S \subset TP$ with fibers $S_p \subset T_pP$ defined by

$$S_p := \{X_f(p) : f \in C^\infty(P)\}. \tag{3.8}$$

The dependence of the characteristic subspace S_p on $p \in P$ is smooth, i.e. for every $v_p \in S_p \subset T_pP$ there is local Hamiltonian vector field X_f such that $v_p = X_f(p)$. The Hamiltonian vector fields X_f and X_g, $f, g \in C^\infty(P)$, are smooth sections of the characteristic distribution $S := \bigsqcup_p S_p$ and $[X_f, X_g] = X_{\{f,g\}}$ also belong to $\Gamma^\infty(S)$. So the vector space $\Gamma^\infty(S)$ of smooth sections of S is involutive.

By a leaf L of the characteristic distribution we will mean:

i) a connected Banach manifold L;

ii) a **weak injective immersion** $\iota : L \hookrightarrow P$, i.e. for every $q \in L$ the tangent map $T_q\iota : T_qL \to T_{\iota(q)}P$ is injective;

iii) $T_q\iota(T_qL) = S_q$ for each $q \in L$;

iv) L is maximal, i.e. if the $\iota' : L' \hookrightarrow P$ satisfies the above three conditions and $L \subset L'$ then $L = L'$.

Let us remark here that we did not assume in ii) that $\iota : L \hookrightarrow P$ is an **injective immersion**, i.e. for every $q \in L$ the tangent mao $T_q\iota : T_qL \to T_{\iota(q)}P$ is injective with the closed split range. In the finite dimensional case the concepts of weak injective immersion and injective immersion coincide. However in general Banach Poisson geometry context the weak injective immersion appeared in the generic case.

The leaf $\iota : L \to P$ is called **symplectic leaf** if:

i) there is a weak symplectic form ω_L on L;

ii) ω_L is consistent with the Poisson structure π of P, i.e.

$$\omega_L(v_q, u_q) = \pi(\iota(q))([\sharp_{\iota(q)}]^{-1} \circ T_q\iota(v_q), [\sharp_{\iota(q)}]^{-1} \circ T_q\iota(u_q)), \tag{3.9}$$

where $[\sharp_{\iota(q)}]^{-1}$ is inverse to the bijective map $[\sharp_p] : T_p^*P/\ker \sharp_p \to S_p$ generated by $\sharp_p(df) := \pi(df, \cdot)$.

If $\iota : L \hookrightarrow P$ is a symplectic leaf of the characteristic distribution S, then

i) for each $f, g \in C^\infty(P)$ one has

$$\{f \circ \iota, g \circ \iota\}_L = \{f, g\} \circ \iota, \tag{3.10}$$

where

$$\{f \circ \iota, g \circ \iota\}_L(q) := \omega_L(q)((T_q\iota)^{-1}X_f(\iota(q)), (T_q\iota)^{-1}X_g(\iota(q))). \tag{3.11}$$

Let us recall that the closed differential 2-form ω is a **weak symplectic** form if for each $q \in L$ the map $\flat_q : T_qL \ni v_q \to \omega(p)(v_q, \cdot) \in T_q^*L$ is an injective continuous map of Banach spaces. The 2-form $\omega \in \Gamma^\infty(\bigwedge^2 T^*L)$ is **strong symplectic** if the maps $\flat_q$, $q \in L$, are continuous bijections.

For finite dimensional case the problem of finding symplectic leaves for the characteristic distribution S (i.e. the integration of S) is solved by the Stefan-Susman or Viflyantsev theorems (eg. see [60, 63]). For the infinite dimensional case one has not the corresponding theorems and the problem is open in general case. The answer is only known for some special subcases, see e.g. for this next section.

The Banach Poisson manifolds form the category with the morphisms between $(P_1, \{\cdot,\cdot\}_1)$ and $(P_2, \{\cdot,\cdot\}_2)$ being a smooth map $\varphi : P_1 \to P_2$ preserving Poisson structure, i.e.

$$\{f, g\}_2 \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_1 \tag{3.12}$$

for locally defined smooth functions f and g on P_2. Equivalently $X_f^2 \circ \varphi = T\varphi \circ X^1_{f\circ\varphi}$, therefore the flow of a Hamiltonian vector field is a Poisson map.

Returning to Definition 3.1, it should be noted that the condition $\sharp(T^*P) \subset TP$ is automatically satisfied in certain cases:

- if P is a smooth manifold modelled on a reflexive Banach space, that is $\mathfrak{b}^{**} = \mathfrak{b}$, or

- P is a strong symplectic manifold with symplectic form ω.

In particular, the first condition holds if P is a Hilbert (and, in particular, a finite dimensional) manifold.

Any strong symplectic manifold (P, ω) is a Poisson manifold in the sense of Definition 3.1. Recall that **strong** means that for each $p \in P$ the map

$$v_p \in T_pP \mapsto \omega(p)(v_p, \cdot) \in T_p^*P \tag{3.13}$$

is a bijective continuous linear map. Therefore, given a smooth function $f : P \to \mathbb{R}$ there exists a vector field X_f such that $df = \omega(X_f, \cdot)$. The Poisson bracket is defined by $\{f, g\} = \omega(X_f, X_g) = \langle df, X_g\rangle$, thus $\sharp df = X_f$, so $\sharp(T^*P) \subset TP$.

On the other hand, a weak symplectic manifold is not a Poisson manifold in the sense of Definition 3.1. Recall that **weak** means that the map defined by (3.13) is an injective continuous linear map that is, in general, not surjective. Therefore, one cannot construct the map that associates to every differential df of a smooth function $f : P \to \mathbb{R}$ the Hamiltonian vector field X_f. Since the definition of the Poisson bracket should be $\{f, g\} = \omega(X_f, X_g)$, one cannot define this operation on functions and hence weak symplectic manifold structures do not define, in general, Poisson manifold structures in the sense of Definition 3.1. There are various ways to deal with this problem. One of them is to restrict the space of functions on which one is working, as is often done in field theory. Another is to deal with densely defined vector fields and invoke the theory of (nonlinear) semigroups; see [9] for this approach. A simple example illustrating the importance of the underlying topology is given by the canonical symplectic structure on $\mathfrak{b} \times \mathfrak{b}^*$, where $\mathfrak{b}$ is a Banach space. This canonical symplectic structure is in general weak; if $\mathfrak{b}$ is reflexive then it is strong.

Similarly to the finite dimensional case (see eg. [60]) the product $P_1 \times P_2$ of the Banach Poisson manifolds and the reduction in the sense of Marsden-Ratiu [29] the Poisson structure of P to the submanifolds $\iota : N \hookrightarrow P$ have the functional character.

Theorem 3.2. *Given the Banach Poisson manifolds* $(P_1, \{\cdot,\cdot\}_1)$ *and* $(P_2, \{\cdot,\cdot\}_2)$ *there is a unique Banach Poisson structure* $\{\cdot,\cdot\}_{12}$ *on the product manifold* $P_1 \times P_2$ *such that:*

i) the canonical projections $\pi_1 : P_1 \times P_2 \to P_1$ *and* $\pi_2 : P_1 \times P_2 \to P_2$ *are Poisson maps;*

ii) $\pi_1^*(C^\infty(P_1))$ *and* $\pi_2^*(C^\infty(P_2))$ *are Poisson commuting subalgebras of* $C^\infty(P_1 \times P_2)$*.*

This unique Poisson structure on $P_1 \times P_2$ *is called the* **product** *Poisson structure and its bracket is given by the formula*

$$\{f, g\}_{12}(p_1, p_2) = \{f_{p_2}, g_{p_2}\}_1(p_1) + \{f_{p_1}, g_{p_1}\}_2(p_2), \tag{3.14}$$

where $f_{p_1}, g_{p_1} \in C^\infty(P_2)$ *and* $f_{p_2}, g_{p_2} \in C^\infty(P_1)$ *are the partial functions given by* $f_{p_1}(p_2) = f_{p_2}(p_1) = f(p_1, p_2)$ *and similarly for* g*.*

Proof of this theorem one finds in [40]. The functional character of the product follows from the formula (3.14).

One shall follow [40] to introduce oneself to Poisson reduction for Banach Poisson manifolds. Let $(P, \{\cdot,\cdot\}_P)$ be a real Banach Poisson manifold (in the sense of Definition 3.1), $i : N \hookrightarrow P$ be a (locally closed) submanifold, and $E \subset (TP)|_N$ be a subbundle of the tangent bundle of P restricted to N. For simplicity we make the following topological regularity assumption throughout this section: $E \cap TN$ is the tangent bundle to a foliation $\mathcal{F}$ whose leaves are the fibers of a submersion $\pi : N \to M := N/\mathcal{F}$, that is, one assumes that the quotient topological space $N/\mathcal{F}$ admits the quotient manifold structure. The subbundle E is said to be **compatible with the Poisson structure** provided the following condition holds: if $U \subset P$ is any open subset and $f, g \in C^\infty(U)$ are two arbitrary functions whose differentials df and dg vanish on E, then $d\{f, g\}_P$ also vanishes on E. The triple (P, N, E) is said to be **reducible**, if E is compatible with the Poisson structure on P and the manifold $M := N/\mathcal{F}$ carries a Poisson bracket $\{\cdot,\cdot\}_M$ (in the sense of Definition 3.1) such that for any smooth local functions $\bar{f}, \bar{g}$ on M and any smooth local extensions f, g of $\bar{f} \circ \pi$, $\bar{g} \circ \pi$ respectively, satisfying $df|_E = 0$, $dg|_E = 0$, the following relation on the common domain of definition of f and g holds:

$$\{f, g\}_P \circ i = \{\bar{f}, \bar{g}\}_M \circ \pi. \tag{3.15}$$

If (P, N, E) is a reducible triple then $(M = N/\mathcal{F}, \{\cdot, \cdot\}_M)$ is called the **reduced manifold** of P via (N, E). Note that (3.15) guarantees that if the reduced Poisson bracket $\{\cdot, \cdot\}_M$ on M exists, it is necessarily unique.

Given a subbundle $E \subset TP$, its **annihilator** is defined as the subbundle of T^*P given by $E^\circ := \{\alpha \in T^*P \mid \langle \alpha, v \rangle = 0 \text{ for all } v \in E\}$.

The following statement generalizes the finite dimensional Poisson reduction theorem of [29].

Theorem 3.3. *Let P, N, E be as above and assume that E is compatible with the Poisson structure on P. The triple (P, N, E) is reducible if and only if $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$ for every $n \in N$.*

Proof is given in [40]. Also there you find

Theorem 3.4. *Let (P_1, N_1, E_1) and (P_2, N_2, E_2) be Poisson reducible triples and assume that $\varphi : P_1 \to P_2$ is a Poisson map satisfying $\varphi(N_1) \subset N_2$ and $T\varphi(E_1) \subset E_2$. Let $\mathcal{F}_i$ be the regular foliation on N_i defined by the subbundle E_i and denote by $\pi_i : N_i \to M_i := N_i/\mathcal{F}_i$, $i = 1, 2$, the reduced Poisson manifolds. Then there is a unique induced Poisson map $\overline{\varphi} : M_1 \to M_2$, called the* **reduction** *of φ, such that $\pi_2 \circ \varphi = \overline{\varphi} \circ \pi_1$.*

It shows the functional character of the proposed Poisson reduction procedure.

If the Banach Poisson manifold $(P, \{\cdot, \cdot\})$ has an almost complex structure, that is, there is a smooth vector bundle map $I : TP \to TP$ covering the identity which satisfies $I^2 = -id$. The question then arises what does it mean for the Poisson and almost complex structures to be compatible. The Poisson structure π is said to be **compatible with the almost complex structure** I if the following diagram commutes:

$$\begin{array}{ccc} T^*P & \xrightarrow{\ \sharp\ } & TP \\ {\scriptstyle I^*}\uparrow & & \downarrow{\scriptstyle I} \\ T^*P & \xrightarrow[\ \sharp\ ]{} & TP \end{array} ,$$

that is,

$$I \circ \sharp + \sharp \circ I^* = 0. \tag{3.16}$$

The decomposition

$$\pi = \pi_{(2,0)} + \pi_{(1,1)} + \pi_{(0,2)} \tag{3.17}$$

induced by the almost complex structure I and the reality of π, implies that the compatibility condition (3.16) is equivalent to

$$\pi_{(1,1)} = 0 \qquad \text{and} \qquad \overline{\pi}_{(2,0)} = \pi_{(0,2)}. \tag{3.18}$$

In view of (3.18), $[\pi, \pi]_S = 0$ is equivalent to

$$[\pi_{(2,0)}, \pi_{(2,0)}]_S = 0 \qquad \text{and} \qquad [\pi_{(2,0)}, \overline{\pi}_{(2,0)}]_S = 0. \tag{3.19}$$

If (3.16) holds, the triple $(P, \{\cdot, \cdot\}, I)$ is called an **almost complex Banach Poisson manifold**. If I is given by a complex analytic structure $P_{\mathbb{C}}$ on P it will be called a **complex Banach Poisson manifold**. For finite dimensional complex manifolds these structures were introduced and studied by [25].

Denote by $\mathcal{O}\Omega^{(k,0)}(P_{\mathbb{C}})$ and $\mathcal{O}\Omega_{(k,0)}(P_{\mathbb{C}})$ the space of holomorphic k-forms and k-vector fields respectively. If

$$\sharp\left(\mathcal{O}\Omega^{(1,0)}(P_{\mathbb{C}})\right) \subset \mathcal{O}\Omega_{(1,0)}(P_{\mathbb{C}}), \tag{3.20}$$

that is, the Hamiltonian vector field X_f is holomorphic if f is a holomorphic function, then, in addition to (3.18) and (3.19), one has $\pi_{(2,0)} \in \mathcal{O}\Omega_{(2,0)}(P_{\mathbb{C}})$. As expected, the compatibility condition (3.20) is stronger than (3.16). Note that (3.20) implies the second condition in (3.19). Thus the compatibility condition (3.20) induces on the underlying complex Banach manifold $P_{\mathbb{C}}$ a holomorphic Poisson tensor $\pi_{\mathbb{C}} := \pi_{(2,0)}$. A pair $(P_{\mathbb{C}}, \pi_{\mathbb{C}})$ consisting of an analytic complex manifold $P_{\mathbb{C}}$ and a holomorphic skew symmetric contravariant two-tensor field $\pi_{\mathbb{C}}$ such that $[\pi_{\mathbb{C}}, \pi_{\mathbb{C}}]_S = 0$ and (3.20) holds will be called a **holomorphic Banach Poisson manifold**.

Consider now a holomorphic Poisson manifold (P, π). Denote by $P_{\mathbb{R}}$ the underlying real Banach manifold and define the real two-vector field $\pi_{\mathbb{R}} := \operatorname{Re} \pi$. It is easy to see that $(P_{\mathbb{R}}, \pi_{\mathbb{R}})$ is a real Poisson manifold compatible with the complex Banach manifold structure of P and $(\pi_{\mathbb{R}})_{\mathbb{C}} = \pi$. Summarizing, we have shown that there are two procedures that are inverses of each other: a holomorphic Poisson manifold corresponds in a bijective manner to a real Poisson manifold whose Poisson tensor is compatible with the underlying complex manifold structure. One can call these constructions the **complexification** and **realification** of Poisson structures on complex manifolds.

4 Banach Lie-Poisson spaces

Now we shall consider a subcategory of Banach Poisson manifolds consisting of the linear Banach Poisson manifolds, i.e. $P = \mathfrak{b}$ and the Poisson tensor π is also linear. In order to give the formal definition let us recall that the **Banach Lie algebra** $(\mathfrak{g}, [\cdot, \cdot])$ is a Banach space imposed in the continuous Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. For $x \in \mathfrak{g}$ one defines the adjoint $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\operatorname{ad}_x g := [x, y]$, and coadjoint $\operatorname{ad}_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$ map which are also continuous.

According to [40] we assume the following definition, which formalizes the concept of Lie-Poisson space discussed in the Section 1.

Definition 4.1. A Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot, \cdot\})$ is a real or complex Poisson manifold such that $\mathfrak{b}$ is a Banach space and its dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation.

The relation between the category of Banach Lie-Poisson spaces and the category of Banach Lie algebras is described by

Theorem 4.2. *The Banach space* $\mathfrak{b}$ *is a Banach Lie-Poisson space* $(\mathfrak{b}, \{\cdot,\cdot\})$ *if and only if it is predual* $\mathfrak{b}^* = \mathfrak{g}$ *of some Banach Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *satisfying* $\mathrm{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}^*$ *for all* $x \in \mathfrak{g}$*. The Poisson bracket of* $f, g \in C^\infty(\mathfrak{b})$ *is given by*

$$\{f, g\}(b) = \langle [Df(b), Dg(b)]; b\rangle, \tag{4.1}$$

where $b \in \mathfrak{b}$*.*

For proof of the theorem see [40]. One can see from (4.1) that the Poisson tensor $\pi \in \Gamma^\infty(\bigwedge^2 T^{**}\mathfrak{b})$ of Banach Lie-Poisson space is given by

$$\pi(b) = \langle [\cdot,\cdot]; b\rangle. \tag{4.2}$$

Here we used identification $T\mathfrak{b} \cong \mathfrak{b} \times \mathfrak{b}$, $T^*\mathfrak{b} \cong \mathfrak{g} \times \mathfrak{b}$ and $T^{**}\mathfrak{b} \cong \mathfrak{g}^* \times \mathfrak{b}$. So π linearly depends on $b \in \mathfrak{b}$. Therefore, as a **morphism** between two Banach Lie-Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ we assume a continuous linear map $\Phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ that preserves the linear Poisson structure, i.e.

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi \tag{4.3}$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$. It will be called a **linear Poisson map**. Therefore Banach Lie-Poisson spaces form a category, which we will denote by $\mathcal{P}$.

Let us denote by $\mathcal{L}$ the category of Banach Lie algebras. Let $\mathcal{L}_0$ be subcategory of $\mathcal{L}$ which consists Banach Lie algebras $\mathfrak{g}$ admitting preduals $\mathfrak{b}^* = \mathfrak{g}$, and $\mathrm{ad}_{\mathfrak{g}}^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}^*$. A morphism $\Psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ in the category $\mathcal{L}_0$ is a Banach Lie algebras homomorphism such that its dual map $\Psi^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ preserves preduals, i.e. $\Psi^*\mathfrak{b}_2 \subset \mathfrak{b}_1$. In general it could happen that the same Banach algebra $\mathfrak{g}$ has more than one non-isomorphic preduals. Therefore, let us define the category $\mathcal{PL}_0$ which has as objects the pairs $(\mathfrak{b}, \mathfrak{g})$ such that $\mathfrak{b}^* = \mathfrak{g}$ and morphisms are defined as for $\mathcal{L}_0$.

Proposition 4.3. *The contravariant functor* $\mathcal{F} : \mathcal{P} \to \mathcal{PL}_0$ *defined by* $\mathcal{F}(\mathfrak{b}) = (\mathfrak{b}, \mathfrak{b}^*)$ *and* $\mathcal{F}(\Phi) = \Phi^*$ *gives categories isomorphism. The inverse of* $\mathcal{F}$ *is given by* $\mathcal{F}^{-1}(\mathfrak{b}, \mathfrak{g}) = \mathfrak{b}$ *and* $\mathcal{F}^{-1}(\Psi) = \Psi^*_{|\mathfrak{b}_2}$*, where* $\Psi : \mathfrak{g}_1 \to \mathfrak{g}_2$*.*

The proof of the theorem is the direct consequence of Theorem 4.2.

The linearity of Poisson tensor π allows us to present Hamilton equation (3.7) in the form

$$\frac{d}{dt} b = -\,\mathrm{ad}^*_{dh(b)}\, b, \tag{4.4}$$

which, as we will see later, is natural generalization on the case of general Banach Lie-Poisson space of the rigid body equation of motion as well as von Neumann-Liouville equation.

For the same reasons the fiber S_b of the characteristic distribution at $b \in \mathfrak{b}$ is given by

$$S_b = \{-ad_x^* b : x \in \mathfrak{g}\}. \tag{4.5}$$

We recall here that $T\mathfrak{b} \cong b \times \mathfrak{b}$ and $T_b\mathfrak{b} \cong \mathfrak{b}$.

Now, let us discuss the question of integrability of the characteristic distribution S. Following of [40] we shall assume that:

(i) $\mathfrak{b}$ is a predual $\mathfrak{g}_*$ of $\mathfrak{g}$ which is Banach Lie algebra of a connected Banach Lie group G;

(ii) the coadjoint action of G on the dual $\mathfrak{g}^*$ preserve $\mathfrak{g}_* \subset \mathfrak{g}^*$, i.e. $\mathrm{Ad}_g^* \mathfrak{g}_* \subset \mathfrak{g}_*$ for any $g \in G$;

(iii) for any $b \in \mathfrak{b}$ the coadjoint isotropy subgroup $G_b := \{g \in G : \mathrm{Ad}_g^* b = b\}$ is a Lie subgroup of G that is a submanifold of G.

It was shown, see Theorem 7.3 and Theorem 7.4 in [40] that under these assumptions one has:

(i) the quotient space G/G_b is a connected Banach weak symplectic manifold with the weak symplectic form ω_b given by

$$\omega_b([g])(T_g\pi(T_eL_g\xi), T_g\pi(T_eL_g\eta)) := \langle b; [\xi, \eta]\rangle, \tag{4.6}$$

where $\xi, \eta \in \mathfrak{g}$, $g \in G$, $[g] := \pi(g)$ and $\pi : G \to G/G_b$ is quotient submersion, $L_g : G \to G$ is left action map;

(ii) the map

$$\iota_b : [g] \in G/G_b \longrightarrow \mathrm{Ad}_{g^{-1}}^* b \in \mathfrak{g}_* = \mathfrak{b} \tag{4.7}$$

is an injective weak immersion of the quotient manifold G/G_b into $\mathfrak{b}$;

(iii) $T_{[g]}\iota_b(T_{[g]}(G/G_b)) = S_{\mathrm{Ad}_{g^{-1}}^* b}$ for each $[g] \in G/G_b$;

(iv) the weak immersion $b_b : G/G_b \to \mathfrak{b}$ is maximal;

(v) the form ω_b is consistent with the Banach Lie-Poisson structure of $\mathfrak{b}$ defined by (4.1).

Summing up the above facts we conclude that $\iota_b : G/G_b \to \mathfrak{b}$ is a symplectic leaf of the characteristic distribution (3.8).

Endowing the coadjoint orbit

$$\mathcal{O}_b := \{\mathrm{Ad}_{g^{-1}}^* b : g \in G\} \tag{4.8}$$

with the smooth manifold structure of the quotient space G/G_b one makes $\iota_b : G/G_b \to \mathcal{O}_b$ into a diffeomorphism. The weak symplectic form $(\iota_b^{-1})^*\omega_b$ is, given like in the finite dimensional case, by the Kiryllov formula

$$(\iota_b^{-1})^*\omega_b(\mathrm{Ad}_{g^{-1}}^* b)(\mathrm{ad}_{\mathrm{Ad}_g \xi}^* \mathrm{Ad}_{g^{-1}}^* b, \mathrm{ad}_{\mathrm{Ad}_g \eta}^* \mathrm{Ad}_{g^{-1}}^* b) = \langle b; [\xi, \eta]\rangle \tag{4.9}$$

for $g \in G$, $\xi, \eta \in \mathfrak{g}$ and $b \in \mathfrak{b} = \mathfrak{g}_*$.

The following theorem gives equivalent conditions on $b \in \mathfrak{b}$ which provided that $\iota_b : G/G_b \to \mathfrak{g}_*$ is an injective immersion.

Theorem 4.4. *Let the Banach Lie group G and $b \in \mathfrak{g}_*$ be such that $\mathrm{Ad}^*_g \mathfrak{g}_* \subset \mathfrak{g}_*$, for any $g \in G$, and the isotropy subgroup G_b is a Lie subgroup of G. Then the following conditions are equivalent*

(i) $\iota_b : G/G_b \to \mathfrak{g}_*$ *is an injective immersion;*

(ii) the characteristic subspace $S_\rho = \{\mathrm{ad}^*_\xi b : \xi \in \mathfrak{g}\}$ *is closed in* $\mathfrak{g}_*$*;*

(iii) $S_\rho = \mathfrak{g}^0_\rho$*, where* $\mathfrak{g}^0_\rho$ *is the annihilator of* $\mathfrak{g}_\rho$ *in* $\mathfrak{g}_*$.

Moreover the coadjoint orbit $\mathcal{O}_b$ with the manifold structure making $\iota_b : G/G_b \to \mathcal{O}_b$ a diffeomorphism. Then, under any of the hypotesis (i), (ii) and (iii), the two-form defined by (4.9) *is a strong symplectic form.*

For proof see Theorem 7.5 in [40].

There is the concept of quasi immersion $\iota : N \to M$ between the two Banach manifolds, see [1] and [8] for example. By the definition $\iota : N \to M$ is **quasi immersion** if for every $n \in N$ the tangent map $T_n\iota : T_nN \to T_{\iota(n)}M$ is injective with the closed range. From Theorem 4.4 we conclude that $\iota_b : G/G_b \to \mathfrak{g}_*$ is a quasi immersion if and only if it is an immersion.

The another important question is which conditions on $b \in \mathfrak{b}$ guarantee that $\iota_b : G/G_b \to \mathfrak{g}_*$ is an embedding, i.e. when $\mathcal{O}_b$ is submanifold of the Banach Lie-Poisson space $\mathfrak{g}_*$. Even there are examples of finite dimension of G and $b \in \mathfrak{g}_*$ such that $\iota_b : G/G_b \to \mathfrak{g}_*$ is not embedding. For the general Banach case this problem is evidently more complicated. Here, opposite to the finite dimensional case, we will be looking for the examples of an embedded symplectic leaves $\iota_b : G/G_b \to \mathfrak{g}_*$.

Example 4.1. The Lie algebra $(L^\infty(\mathcal{H}), [\cdot,\cdot])$ is the one of the Banach group $GL^\infty(\mathcal{H})$ which is open in $L^\infty(\mathcal{H})$. The same has place for $(U^\infty(\mathcal{H}), [\cdot,\cdot])$ which is Lie algebra of the Banach Lie group $GU^\infty(\mathcal{H})$ of the unitary operators in $\mathcal{H}$. So the group $GL^\infty(\mathcal{H})$ ($GU^\infty(\mathcal{H})$ respectively) acts on $L^1(\mathcal{H})$ (on $U^1(\mathcal{H})$) by the coadjoint representation

$$\mathrm{Ad}^*_g : L^1(\mathcal{H}) \to L^1(\mathcal{H}) \quad \text{for } g \in GL^\infty(\mathcal{H}) \tag{4.10}$$

$$\mathrm{Ad}^*_g(\rho) = g\rho g^{-1}. \tag{4.11}$$

For $g \in GU^\infty(\mathcal{H})$ and $\rho \in U^1(\mathcal{H})$ one has

$$\mathrm{Ad}^*_g \rho = g\rho g^* \quad . \tag{4.12}$$

In [40] it is proved that orbits

$$\mathcal{O}_{\rho_0} = \mathrm{Ad}^*_G \rho_0, \tag{4.13}$$

where $G = GL^\infty(\mathcal{H})$ or $G = GU^\infty(\mathcal{H})$, are symplectic leaves. But in general case the Kiryllov symplectic form $\omega_{\mathcal{O}}$ is only weak symplectic and in consequence the quotient manifold G/G_ρ is weak symplectic manifold and the map

$$\iota : G/G_{\rho_0} \ni [g] \to \mathrm{Ad}^*_{\mathfrak{g}} \rho_0 \in \mathcal{O}_{\rho_0} \subset \mathfrak{g}_* = L^1(\mathcal{H}) \text{ or } U^1(\mathcal{H}) \tag{4.14}$$

is an injective weak immersion. The weak symplectic structure $\omega_{\mathcal{O}}$ is consistent with Banach Lie–Poisson structure of $\mathfrak{g}_*$ It means that

$$\{f, g\}_{\mathfrak{g}_*} \circ \iota = \{f \circ \iota, g \circ \iota\}_{\mathcal{O}}, \tag{4.15}$$

where $f, g \in C^\infty(\mathfrak{g}_*)$ and Poisson bracket $\{\cdot, \cdot\}_{\mathcal{O}}$ is defined for the function $f \circ \iota$ and $g \circ \iota$ only. The situation looks better for orbits $\mathcal{O}_{\rho_0}$ generated from finite rank $(\dim(\operatorname{im}\rho_0) < \infty)$ elements ρ_0. In that case hermitian element $\rho_0 = \rho_0^*$ can be decomposed on the finite sum of orthonormal projectors

$$\rho_0 = \sum_{k=1}^{N} \lambda_k P_k \quad , \quad \sum_{k=0}^{N} P_k = \mathbb{I} \quad , \quad P_k P_l = \delta_{kl} P_l, \tag{4.16}$$

where $\dim(\ker P_0)^\perp = \infty$, $\dim(\ker P_k)^\perp < \infty$ $\lambda_k \neq \lambda_l \in \mathbb{R}$ and $\lambda_k \neq 0$ for $N \geqslant k \geqslant 1$ and $\lambda_0 = 0$. In that case one has **splitting** (see [40])

$$T_{\rho_0} U^1(\mathcal{H}) = \left\{ \sum_{k \neq l = 0}^{N} P_k \rho P_l : \rho \in U^1(\mathcal{H}) \right\} \oplus \left\{ \sum_{k=0}^{N} P_k \rho P_k : \rho \in U^1(\mathcal{H}) \right\} \tag{4.17}$$

in which the first component is

$$S_{\rho_0} \cong T_{\rho_0}\mathcal{O} = i[\rho_0, U^\infty(\mathcal{H})] \tag{4.18}$$

and the second one is the intersection

$$U^\infty_{\rho_0}(\mathcal{H}) \cap U^1(\mathcal{H}) \tag{4.19}$$

of the stabilizer Lie–Banach subalgebra $U^\infty_{\rho_0}(\mathcal{H})$ with $U^1(\mathcal{H})$. One can conclude from this (see [40]) that the map

$$\iota : GU^\infty(\mathcal{H})/GU^\infty_{\rho_0}(\mathcal{H}) \xrightarrow{\sim} \mathcal{O}_{\rho_0} \subset U^1(\mathcal{H}) \tag{4.20}$$

is an injective smooth immersion and $(\mathcal{O}_{\rho_0}, \omega_{\mathcal{O}})$ is strong symplectic manifold.

The orbit $\mathcal{O}_{\rho_0}$ had two naturally defined topologies:

$\mathcal{T}_R$) the relative topology: Ω is open iff there exists $\tilde{\Omega}$ open in $U^1(\mathcal{H})$ such that $\Omega = \tilde{\Omega} \cap \mathcal{O}_{\rho_0}$

$\mathcal{T}_Q$) the quotient topology: Ω is open iff $(\iota \circ \pi)^{-1}(\Omega)$ is open in $GU^\infty(\mathcal{H})$.

The map π is the quotient projection

$$\pi : GU^\infty(\mathcal{H}) \longrightarrow GU^\infty(\mathcal{H})/GU^\infty_{\rho_0}(\mathcal{H}) \tag{4.21}$$

of the Banach–Lie group $GU^\infty(\mathcal{H})$ onto the quotient space $GU^\infty(\mathcal{H})/GU^\infty_{\rho_0}(\mathcal{H})$.

The coadjoint action map

$$\mathrm{Ad}^* : GU^\infty(\mathcal{H}) \times U^1(\mathcal{H}) \longrightarrow U^1(\mathcal{H}) \tag{4.22}$$

is continuous and thus the map

$$\mathrm{Ad}^*_{\rho_0} : GU^\infty(\mathcal{H}) \longrightarrow U^1(\mathcal{H}) \tag{4.23}$$

defined by $\mathrm{Ad}^*_{\rho_0} g = g\rho_0 g^*$ is also continuous. Because of this the set

$$\left(\mathrm{Ad}^*_{\rho_0}\right)^{-1}(\Omega) = \pi^{-1} \circ \iota^{-1}(\Omega) = (\iota \circ \pi)^{-1}(\omega) = \{g \in GU^\infty(\mathcal{H}) : g\rho_0 g^* \in \Omega\} \tag{4.24}$$

is open in $\|\cdot\|_\infty$ topology of the unitary group $GU^\infty(\mathcal{H})$ if $\mathcal{O}_{\rho_0} \supset \Omega$ is open in relative topology $\mathcal{T}_R$. The above proves that if $\Omega \in \mathcal{T}_R$ than $\Omega \in \mathcal{T}_Q$.

We would have shown that the injective smooth immersion is an **embedding** if we have constructed a section

$$S : \Omega \longrightarrow GU^\infty(\mathcal{H}) \tag{4.25}$$

continuous with respect to the relative topology $\mathcal{T}_R$.

Indeed, then assuming that ι is continuous in quotient topology we find that $(\iota \circ \pi)^{-1}(\Omega)$ is open in $GU^\infty(\mathcal{H})$. Thus $S^{-1}\left((\iota \circ \pi)^{-1}(\Omega)\right) = (\iota \circ \pi \circ S)^{-1} = \mathrm{id}^{-1}(\Omega) = \Omega$ is open in topology $\mathcal{T}_R$.

In particular we have the above situation if ρ_0 has finite rank. Therefore, for example, the map $\iota : \mathbb{CP}(\mathcal{H}) \to U^1(\mathcal{H})$ defined by (2.38) is an embedding.

♢

Now, following of [40] we will describe of the internal structure of morphisms of Banach Lie-Poisson spaces.

Proposition 4.5. *Let $\Phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ be a linear Poisson map between Banach Lie-Poisson spaces and assume that $\mathrm{im}\,\Phi$ is closed in $\mathfrak{b}_2$. Then the Banach space $\mathfrak{b}_1/\ker\Phi$ is predual to $\mathfrak{b}_2^*/\ker\Phi^*$, that is $(\mathfrak{b}_1/\ker\Phi)^* \cong \mathfrak{b}_2^*/\ker\Phi^*$. In addition, $\mathfrak{b}_2^*/\ker\Phi^*$ is a Banach Lie-Poisson algebra satisfying the condition $\mathrm{ad}^*_{[x]}(\mathfrak{b}_1/\ker\Phi) \subset \mathfrak{b}_1/\ker\Phi$ for all $x \in b_2^*$ and $\mathfrak{b}_1/\ker\Phi$ is a Banach Lie-Poisson space. Moreover, the following properties hold*

(i) the quotient map $\pi : b_1 \to \mathfrak{b}_1/\ker\Phi$ is a surjective linear Poisson map;

(ii) the map $\iota : \mathfrak{b}_1/\ker\Phi \to \mathfrak{b}_2$ defined by $\iota([b]) = \Phi(b)$ is an injective linear Poisson map;

(iii) the decomposition $\Phi = \iota \circ \pi$ into the surjective and injective linear Poisson map is valid.

For proof of the proposition see [40].

So, as in linear algebra, one can reduce the investigation of linear Poisson maps with closed range to the surjective and injective subcases. Since of Theorem 4.2 and Proposition 4.3 one can characterize linear Poisson maps using Banach Lie algebraic terminology.

Let us consider firstly the surjective linear continuous map $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$ of a Banach Lie-Poisson space $(b_1, \{\cdot,\cdot\}_1)$ just only on a Banach space. It is easy to see that the dual map $\pi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a continuous injective linear map and $\operatorname{im}\pi^*$ is closed in $\mathfrak{b}_1^*$. So, one can identify $\operatorname{im}\pi^*$ with the dual $\mathfrak{b}_2^*$ of Banach space $\mathfrak{b}_2$.

Assuming additionally that $\operatorname{im}\pi^*$ is Banach Lie subalgebra one shows that $\operatorname{im}\pi^* \cong \mathfrak{b}_2^*$ satisfies conditions of Theorem 4.2 (see Section 4 of [40]) and thus conclude that the following proposition is valid.

Proposition 4.6. *Let $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ be a Banach Lie-Poisson space and let $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$ be a continuous linear surjective map onto $\mathfrak{b}_2$. Then $\mathfrak{b}_2$ carries the unique Banach Lie-Poisson structure $\{\cdot,\cdot\}_2$ if and only if $\operatorname{im}\pi^* \subset \mathfrak{b}_1^*$ is closed under the Lie bracket $[\cdot,\cdot]_1$ of b_1^*. The map $\pi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a Banach Lie algebra morphism whose dual $\pi^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ maps b_1 into b_2. The uniquely defined Banach Poisson-Lie structure $\{\cdot,\cdot\}_2$ following, e.g. [60] we shall call* **coinduced** *by π from Banach Lie-Poisson space $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$.*

We shall illustrate the importance of the coinduction procedure presenting the following example, see [40].

Example 4.2. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a complex Banach Lie algebra admitting a predual $\mathfrak{g}_*$ satisfying $\operatorname{ad}_x^* \mathfrak{g}_* \subset \mathfrak{g}_*$ for every $x \in \mathfrak{g}$. Then, by Theorem 4.2, the predual $\mathfrak{g}_*$ admits a holomorphic Banach Lie-Poisson structure, whose holomorphic Poisson tensor π is given by (4.1). We shall work with the realification $(\mathfrak{g}_{*\mathbb{R}}, \pi_{\mathbb{R}})$ of $(\mathfrak{g}_*, \pi)$ in the sense of Section 3. We want to construct a real Banach space $\mathfrak{g}_*^\sigma$ with a real Banach Lie-Poisson structure π_σ such that $\mathfrak{g}_*^\sigma \otimes \mathbb{C} = \mathfrak{g}_*$ and π_σ is coinduced from $\pi_{\mathbb{R}}$ in the sense of Proposition 4.6. To achieve this, introduce a continuous $\mathbb{R}$-linear map $\sigma : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_{*\mathbb{R}}$ satisfying the properties:

(i) $\sigma^2 = id$;

(ii) the dual map $\sigma^* : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ defined by

$$\langle \sigma^* z, b\rangle = \overline{\langle z, \sigma b\rangle} \tag{4.26}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$ and where $\langle\cdot,\cdot\rangle$ is the pairing between the complex Banach spaces $\mathfrak{g}$ and $\mathfrak{g}_*$, is a homomorphism of the Lie algebra $(\mathfrak{g}_{\mathbb{R}}, [\cdot,\cdot])$;

(iii) $\sigma \circ I + I \circ \sigma = 0$, where $I : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ is defined by

$$\langle z, Ib\rangle := \langle I^* z, b\rangle := i\langle z, b\rangle \tag{4.27}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$.

Consider the projectors

$$R := \frac{1}{2}(id + \sigma) \qquad R^* := \frac{1}{2}(id + \sigma^*) \tag{4.28}$$

and define $\mathfrak{g}_*^\sigma := \operatorname{im} R$, $\mathfrak{g}^\sigma := \operatorname{im} R^*$. Then one has the splittings

$$\mathfrak{g}_{*\mathbb{R}} = \mathfrak{g}_*^\sigma \oplus I\mathfrak{g}_*^\sigma \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^\sigma \oplus I\mathfrak{g}^\sigma \tag{4.29}$$

into real Banach subspaces. One can identify canonically the splittings (4.29) with the splittings

$$\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} = (\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}) \oplus (\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}i)\,. \tag{4.30}$$

Thus one obtains isomorphisms $\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_*$ and $\mathfrak{g}^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$ of complex Banach spaces.

For any $x, y \in \mathfrak{g}_{\mathbb{R}}$ one has

$$[R^*x, R^*y] = R^*[x, R^*y] \tag{4.31}$$

and thus $\mathfrak{g}^\sigma$ is a real Banach Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$. From

$$\begin{aligned}\operatorname{Re}\langle z, b\rangle &= \langle R^*zRb\rangle + \langle I^*R^*I^*z, IRIb\rangle \\ &= \langle R^*zRb\rangle + \langle (1-R^*)z, (1-R)b\rangle \end{aligned}\tag{4.32}$$

for all $z \in \mathfrak{g}_{\mathbb{R}}$ and all $b \in \mathfrak{g}_{*\mathbb{R}}$, where for the last equality we used $R = 1 + IRI$ and $R^* = 1 + I^*R^*I^*$, one concludes that the annihilator $(\mathfrak{g}_*^\sigma)^\circ$ of $\mathfrak{g}_*^\sigma$ in $\mathfrak{g}_{\mathbb{R}}$ equals $I^*\mathfrak{g}^*$. Therefore $\mathfrak{g}_*^\sigma$ is the predual of $\mathfrak{g}^\sigma$.

Taking into account all of the above facts we conclude from Proposition 4.6 that $\mathfrak{g}_*^\sigma$ carries a real Banach Lie-Poisson structure $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ coinduced by $R : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$. According to (4.32), the bracket $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ is given by

$$\{f, g\}_{\mathfrak{g}_*^\sigma}(\rho) = \langle [df(\rho), dg(\rho)], \rho\rangle, \tag{4.33}$$

where $\rho \in \mathfrak{g}_*^\sigma$ and the pairing on the right is between $\mathfrak{g}_*^\sigma$ and $\mathfrak{g}^\sigma$. In addition, for any real valued functions $f, g \in C^\infty(\mathfrak{g}_*^\sigma)$ and any $b \in \mathfrak{g}_{*\mathbb{R}}$ we have

$$\begin{aligned}&\{f\circ R, g\circ R\}_{\mathfrak{g}_{\mathbb{R}}}(b) = \operatorname{Re}\langle [d(f\circ R)(b), d(g\circ R)(b)], b\rangle \\ &= \langle R^*[d(f\circ R)(b), d(g\circ R)(b)], R(b)\rangle + \langle (1-R^*)[d(f\circ R)(b), d(g\circ R)(b)], (1-R)b\rangle \\ &= \langle R^*[R^*df(R(b)), R^*dg(R(b))], R(b)\rangle + \langle (1-R^*)[R^*df(R(b)), R^*dg(R(b))], (1-R)b\rangle \\ &= \langle [df(R(b)), dg(R(b))], R(b)\rangle = \{f, g\}_{\mathfrak{g}_*^\sigma}(R(b)),\end{aligned}$$

where we have used (4.31). The above computation proves, independently of Proposition 4.6, that $R : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$ is a linear Poisson map.

$\diamondsuit$

The injective ingredient of the linear Poisson map is described as follows.

Proposition 4.7. *Let $\mathfrak{b}_1$ be a Banach space, $(b_2, \{\cdot,\cdot\}_2)$ be a Banach Lie-Poisson space, and $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ be an injective continuous linear map with closed range. Then $\mathfrak{b}_1$ carries a unique Banach Lie-Poisson structure $\{\cdot,\cdot\}_1$ such that ι is a linear Poisson map if and only if $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$.*

In analogy to the previous case, we will call the Banach Lie-Poisson structure $\{\cdot,\cdot\}_1$ **induced** from $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$ by the map ι.

Example 4.3 (*Banach Lie-Poisson spaces related to infinite Toda latice*, see [41]).

We begin by defining the family of Banach subspaces of $L^1(\mathcal{H})$ and $L^\infty(\mathcal{H})$ where $\mathcal{H}$ is real separable Hilbert space. Given the Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of $L^1(\mathcal{H})$, we define the Banach subspaces of $L^1(\mathcal{H})$:

- $L^1_-(\mathcal{H}) := \{\rho \in L^1(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } m > n\}$ (lower triangular trace class)
- $L^1_{-,k}(\mathcal{H}) := \{\rho \in L^1_-(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } n > m + k\}$ (lower k-diagonal trace class)
- $I^1_{-,k}(\mathcal{H}) := \{\rho \in L^1_-(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } n \leqslant m + k\}$ (lower triangular trace class with zero first k-diagonals)
- $I^1_{+,k}(\mathcal{H}) := \{\rho \in L^1_{+,k}(\mathcal{H}) \mid \rho_{nm} = 0 \text{ for } m \geqslant n + k\}$ (upper triangular trace class with zero first k-diagonals).

Similarly, using the biorhogonal family of functionals $\{|l\rangle\langle k|\}_{l,k=0}^\infty$ in $L^\infty(\mathcal{H}) \cong L^1(\mathcal{H})^*$ we define Banach subspaces of $L^\infty(\mathcal{H})$:

- $L^\infty_+(\mathcal{H}) := \{x \in L^\infty(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m < n\}$ (upper triangular bounded)
- $L^\infty_{+,k}(\mathcal{H}) := \{x \in L^\infty_+(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m > n + k\}$ (upper k-diagonal bounded)
- $I^\infty_{-,k}(\mathcal{H}) := \{x \in L^\infty_-(\mathcal{H}) \mid x_{nm} = 0 \text{ for } n \leqslant m + k\}$ (lower triangular bounded with zero first k-diagonals)
- $I^\infty_{+,k}(\mathcal{H}) := \{x \in L^\infty_+(\mathcal{H}) \mid x_{nm} = 0 \text{ for } m \geqslant n + k\}$ (upper triangular bounded with zero first k-diagonals)

With these subspaces we have the following splittings

$$L^1(\mathcal{H}) = L^1_-(\mathcal{H}) \oplus I^1_{+,1}(\mathcal{H}) \tag{4.34}$$

$$L^1_-(\mathcal{H}) = L^1_{-,k}(\mathcal{H}) \oplus I^1_{-,k}(\mathcal{H}) \tag{4.35}$$

$$L^\infty(\mathcal{H}) = L^\infty_+(\mathcal{H}) \oplus I^\infty_{-,1}(\mathcal{H}) \tag{4.36}$$

$$L^\infty_+(\mathcal{H}) = L^\infty_{+,k}(\mathcal{H}) \oplus I^\infty_{+,k}(\mathcal{H}) \tag{4.37}$$

Non-degenerate pairing (2.11) relates the above splitting by

$$(L^1_-(\mathcal{H}))^* \cong (I^1_{+,1}(\mathcal{H}))^\circ = L^\infty_{+,1}(\mathcal{H}) \tag{4.38}$$

$$(L^1_{-,k}(\mathcal{H}))^* \cong (I^1_{-,k}(\mathcal{H}))^\circ = L^\infty_{+,k}(\mathcal{H}), \tag{4.39}$$

where $^\circ$ denotes the annihilator of the Banach subspace in the dual of the ambient space.

The $L^\infty_+(\mathcal{H})$ is the associative Banach subalgebra of $L^\infty(\mathcal{H})$ and $I^\infty_{+,k}(\mathcal{H})$ is the Banach ideal of $L^\infty_+(\mathcal{H})$. Then they are Banach Lie subalgebra and Banach Lie ideal of $(L^\infty(\mathcal{H}), [\cdot,\cdot])$ respectively.

The associative Banach Lie groups are

$$GL^{\infty}_{+}(\mathcal{H}) := GL^{\infty}(\mathcal{H}) \cap L^{\infty}_{+}(\mathcal{H}) \tag{4.40}$$

and

$$GI^{\infty}_{+,k}(\mathcal{H}) := (\mathbb{I} + I^{\infty}_{+,k}(\mathcal{H}) \cap GL^{\infty}_{+}(\mathcal{H}). \tag{4.41}$$

Now, let us take the Banach spaces map $\iota_k : L^1_{-,k}(\mathcal{H}) \hookrightarrow L^1_{-}(\mathcal{H})$ defined by the splitting (4.35). It is clear that it satisfies the conditions of the Proposition 4.7. Therefore $L^1_{-,k}$ is the Banach Lie-Poisson space predual to the Banach Lie algebra $L^{\infty}_{+}(\mathcal{H})/I^{\infty}_{+,k}(\mathcal{H}) \cong L^{\infty}_{+,k}(\mathcal{H})$ with the bracket

$$[X,Y]_k = \sum_{l=0}^{k-1}\sum_{i=0}^{l}\left(x_i s^i(y_{l-i}) - y_i s^i(x_{l-i})\right) S^l \tag{4.42}$$

of $X = \sum_{l=0}^{k-1} x_l S^l$, $Y = \sum_{l=0}^{k-1} y_l S^l$, where

$$S := \sum_{n=0}^{\infty} |n\rangle\langle n+1| \in L^{\infty}(\mathcal{H}) \tag{4.43}$$

and $x_l, y_l \in L^{\infty}_{0}(\mathcal{H})$, where $L^{\infty}_{0}(\mathcal{H})$ by definition is subalgebra of diagonal elements in $L^{\infty}_{+}(\mathcal{H})$. We defined the map $s : L^{\infty}_{0}(\mathcal{H}) \to L^{\infty}_{0}(\mathcal{H})$ by

$$Sx = s(x)S. \tag{4.44}$$

One has isomorphism of $GL^{\infty}_{+}(\mathcal{H})/GI^{\infty}_{+,k}$ with the group

$$GL^{\infty}_{+,k} = \left\{ g = \sum_{i=0}^{k-1} g_i S^i \;\middle|\; g_i \in L^{\infty}_{0}, |g_0| \geqslant \varepsilon(g_0)\mathbb{I} \quad \text{for some} \quad \varepsilon(g_0) > 0 \right\}, \tag{4.45}$$

of invertible elements in the Banach associative algebra $(L^{\infty}_{+,k}(\mathcal{H}), \circ_k)$ with the product of elements given by

$$X \circ_k Y := \sum_{l=0}^{k-1}\left(\sum_{i=0}^{l} x_i s^i(y_{l-i})\right) S^l. \tag{4.46}$$

Finally the induced Poisson bracket on $L^1_{-,k}(\mathcal{H})$ is given by

$$\{f,g\}_k(\rho) = \\ = \sum_{l=0}^{k-1}\sum_{i=0}^{l} \mathrm{Tr}\left[\rho_l \left(\frac{\delta f}{\delta \rho_i}(\rho) s^i\left(\frac{\delta g}{\delta \rho_{l-i}}(\rho)\right) - \frac{\delta g}{\delta \rho_i}(\rho) s^i\left(\frac{\delta f}{\delta \rho_{l-i}}(\rho)\right)\right)\right], \tag{4.47}$$

where $\rho = \sum_{l=0}^{k-1} (S^T)^l \rho_l$, here ρ_i are diagonal trace-class operators and S^T is conjugation of S.

♢

Example 4.4 (*Flaschka map as a momentum map*, see [41]). Let us recall that by definition l^∞ and l^1 are

$$l^\infty := \left\{ q = \{q_k\}_{k=0}^\infty : \|q\|_\infty := \sup_{k=0,1,\dots} |q_k| < \infty \right\} \tag{4.48}$$

$$l^1 := \left\{ p = \{p_k\}_{k=0}^\infty : \|p\|_1 := \sum_{k=0}^\infty |p_k| < \infty \right\} \tag{4.49}$$

The spaces l^∞ and l^1 are in duality, that is, $(l^1)^* = l^\infty$ relative to the strongly nondegenerate duality pairing

$$\langle q, p \rangle = \sum_{k=0}^\infty q_k p_k. \tag{4.50}$$

Thus the space $l^\infty \times l^1$ is a weak symplectic Banach space relative to the canonical weak symplectic form

$$\omega((q,p),(q',p')) = \langle q, p' \rangle - \langle q', p \rangle, \tag{4.51}$$

for $q, q' \in l^\infty$ and $p, p' \in l^1$.

Let us define the map

$$\mathcal{J}_\nu(q,p) := p + S^T \nu e^{s(q)-q} \tag{4.52}$$

of the canonical weak symplectic Banach space $(l^\infty \times l^1, \omega)$ into the Banach Lie-Poisson space $L^1_{-,2}(\mathcal{H})$, where $S^T\nu$ is a generic lower diagonal element of $L^1_{-,2}(\mathcal{H})$. We identify l^1 with $L^1_0(\mathcal{H})$ and l^∞ with $L^\infty_0(\mathcal{H})$. Having fixed $S^T\nu \in L^1_{-,2}(\mathcal{H})$, we define the action

$$\sigma^\nu_g(q,p) := \left(q + \log g_0, p + g_1 g_0^{-1} \nu e^{s(q)-q} + \tilde{s}\left(g_1 g_0^{-1} \nu e^{s(q)-q} \right) (\mathbb{I} - p_0) \right), \tag{4.53}$$

where $g_0 + g_1 S \in GL^\infty_{+,2}(\mathcal{H})$ and $(q,p) \in l^\infty \times l^1$.

One can prove that

i) $\mathcal{J}_\nu$ is an Poisson map, that is, $\{f \circ \mathcal{J}_\nu, g \circ \mathcal{J}_\nu\}_\omega = \{f, g\}_2 \circ \mathcal{J}_\nu$, for all $f, g \in C^\infty(L^1_{-,2}(\mathcal{H}))$;

ii) $\mathcal{J}_\nu$ is $GL^\infty_{+,2}(\mathcal{H})$-equivariant, that is, $\mathcal{J}_\nu \circ \sigma^\nu_g = \left(\mathrm{Ad}^{-,2}\right)^*_{g^{-1}} \circ \mathcal{J}_\nu$ for any $g \in GL^\infty_{+,2}(\mathcal{H})$.

Reassuming above statements we can say that Flaschka map (4.52) is a momentum map of the weak symplectic Banach space $(l^\infty \times l^1, \omega)$ into the Banach Lie-Poisson space $L^1_{-,2}(\mathcal{H})$.

♢

In order to summarize the above considerations let us formulate the following theorem, see [40].

Theorem 4.8. *The linear continuous closed range map* $\Phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ *between the Banach Lie-Poisson spaces* $\mathfrak{b}_1$ *and* $\mathfrak{b}_2$ *is a linear Poisson map if and only if it has a decomposition* $\Phi = \iota \circ \pi$*, where*

(i) $\pi : \mathfrak{b}_1 \to \mathfrak{b}$ *is a linear continuous surjective map of Banach spaces such that* $\operatorname{im} \pi^* \subset \mathfrak{b}_1^*$ *is closed with respect to a Lie bracket of* $\mathfrak{b}_1^*$*;*

(ii) $\iota : \mathfrak{b} \to \mathfrak{b}_2$ *is a continuous injective linear map of Banach spaces with closed range such that* $\ker \iota^*$ *is an ideal in the Banach Lie algebra* $\mathfrak{b}_2^*$*.*

Let $\mathfrak{b}$ be a Banach Lie-Poisson space and let $\mathfrak{g}$ be Banach Lie algebra defined by $\mathfrak{b}^* = \mathfrak{g}$. From Proposition 4.6 we see that there exists a bijective correspondence between the coinduced Banach Lie-Poisson structures from $\mathfrak{b}$ and the Banach Lie subalgebras of $\mathfrak{g}$. If the surjective continuous linear map $\pi : \mathfrak{b} \to \mathfrak{c}$ coinduces a Banach Lie-Poisson structure on $\mathfrak{c}$, the Banach Lie subalgebra of $\mathfrak{g}$ given by this correspondence is $\pi^*(\mathfrak{c}^*)$.

Conversely, if $\mathfrak{k} \subset \mathfrak{g}$ is a Banach Lie subalgebra then the Banach Lie-Poisson space given by this correspondence is $\mathfrak{b}/\mathfrak{k}^\circ$, where $\mathfrak{k}^\circ$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{b}$, and $\pi : \mathfrak{b} \to \mathfrak{b}/\mathfrak{k}^\circ$ is the quotient projection.

Similarly from Proposition 4.7 we conclude that there exists a bijective correspondence between the induced Banach Lie-Poisson structures in $\mathfrak{b}$ (i.e., the Banach Lie-Poisson subspaces of $\mathfrak{b}$) and the Banach ideals of $\mathfrak{g}$. If the injection $\iota : \mathfrak{c} \to \mathfrak{b}$ with closed range induces a Banach Lie-Poisson structure on $\mathfrak{c}$, then the ideal in $\mathfrak{g}$ given by this correspondence is $\ker \iota^*$.

Conversely, if $\mathfrak{i} \subset \mathfrak{g}$ is a Banach ideal, then the Banach Lie- Poisson subspace of $\mathfrak{b}$ given by this correspondence is $\mathfrak{i}^\circ$, where $\mathfrak{i}^\circ$ is the annihilator of $\mathfrak{i}$ in $\mathfrak{b}$ and $\iota : \mathfrak{i}^\circ \to \mathfrak{b}$ is the inclusion.

The product $\mathfrak{b}_1 \times \mathfrak{b}_2$ of the Banach Lie-Poisson spaces $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ and $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$ has the Banach Poisson manifold structure $\{\cdot,\cdot\}_{12}$ defined by Theorem 3.2. Since the Banach spaces isomorphism $(\mathfrak{b}_1 \times \mathfrak{b}_2)^* \cong \mathfrak{b}_1^* \times \mathfrak{b}_2^*$ the dual Banach space $(\mathfrak{b}_1 \times \mathfrak{b}_2)^*$ is closed with respect to the Poisson bracket $\{\cdot,\cdot\}_{12}$. The inclusions $i_k : \mathfrak{b}_k \to \mathfrak{b}_1 \times \mathfrak{b}_2$ and projections $\pi_k : \mathfrak{b}_1 \times \mathfrak{b}_2 \to \mathfrak{b}_k$, $k = 1, 2$. The inverse procedure to the product is the splitting $b = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ of a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$.

Definition 4.9. Let $(\mathfrak{b}, \{\cdot,\cdot\})$ be a Banach Lie-Poisson space. The splitting $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ into two Banach subspaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ is called a **Poisson splitting** if

(i) $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Banach Lie Poisson spaces whose brackets shall be denoted by $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ respectively;

(ii) the projections $\pi_k : \mathfrak{b} \to \mathfrak{b}_k$ and the inclusions $i_k : \mathfrak{b}_k \to \mathfrak{b}$, $k = 1, 2$, consistent with the above splitting, are Poisson maps;

(iii) if $f \in \pi_1^*(C^\infty(P_1))$ and $g \in \pi_2^*(C^\infty(P_2))$, then $\{f, g\} = 0$.

The following conditions are equivalent:

(i) the Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ admits a Poisson splitting into the two Banach Lie-Poisson subspaces $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ and $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$;

(ii) the Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ is isomorphic to the product Banach Lie-Poisson space $(\mathfrak{b}_1 \times \mathfrak{b}_2, \{\cdot,\cdot\}_{12})$;

(iii) the components $\mathfrak{b}_1^*$ and $\mathfrak{b}_2^*$ of the dual splitting $\mathfrak{b}^* = \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*$ are ideals of the Banach Lie algebra $\mathfrak{b}^*$, where one identifies $\mathfrak{b}_1^*$ and $\mathfrak{b}_2^*$ with the annihilators of $\mathfrak{b}_2$ and $\mathfrak{b}_1$ in $\mathfrak{b}^*$ respectively.

5 Preduals of W^*-algebras and the conditional expectation

The physically important and mathematically interesting subcategory of Banach Lie-Poisson spaces is given by the preduals of W^*-algebras. Let us recall that W^*-algebra is a C^*-algebra $\mathfrak{M}$, which allows a predual Banach space $\mathfrak{M}_*$. For given $\mathfrak{M}$ the predual $\mathfrak{M}_*$ is defined in the unique way, see eg. [48, 53]. By Sakai theorem the W^*-algebra is abstract presentation of von Neumann algebra. The existence of $\mathfrak{M}_*$ defines $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ topology on the $\mathfrak{M}$. Below we will use term σ-topology. A net $\{x_\alpha\}_{\alpha\in A} \subset \mathfrak{M}$ converges to $x \in \mathfrak{M}$ in σ-topology if, by definition, $\lim_{\alpha\in A}\langle x_\alpha; b\rangle = \langle x; b\rangle$ for any $b \in \mathfrak{M}_*$. One can characterize the predual space $\mathfrak{M}_*$ as the Banach subspace of $\mathfrak{M}^*$ consisting of all σ-continuous linear functionals, eg. see [48]. The left

$$L_a : \mathfrak{M} \ni x \longrightarrow ax \in \mathfrak{M} \tag{5.1}$$

and right

$$R_a : \mathfrak{M} \ni c \longrightarrow xa \in M \tag{5.2}$$

multiplication by $a \in M$ are norm and σ-continuous maps. Thus their duals $L_a^* : \mathfrak{M}^* \to \mathfrak{M}^*$ and $R_a^* : \mathfrak{M}^* \to \mathfrak{M}^*$ preserve $\mathfrak{M}_*$ which is canonically embedded Banach subspace of $\mathfrak{M}^*$.

The W^*-algebra is a Banach Lie algebra with the commutator $[\cdot,\cdot]$ as Lie bracket. One has $\mathrm{ad}_a = [a,\cdot] = L_a - R_a$ and $\mathrm{ad}_a^* = L_a^* - R_a^*$. Therefore $\mathrm{ad}_a^* \mathfrak{M}_* \subset \mathfrak{M}_*$ for each $a \in \mathfrak{M}$. The above proves that the conditions of Theorem 4.2 are satisfied. Thus one has

Proposition 5.1. *The predual $\mathfrak{M}_*$ of W^*-algebra $\mathfrak{M}$ is a Banach Lie-Poisson space with the Poisson bracket $\{f, g\}$ of $f, g \in C^\infty(\mathfrak{M}_*)$ given by* (4.1)*.*

The above statement is remarkable, since it says that the space of quantum states $\mathfrak{M}_*$ can be considered as an infinite dimensional classical phase space.

Now, let us introduce the concept of **quantum reduction** physical meaning of which will be elucidated subsequently.

Definition 5.2. A **quantum reduction** is the linear map $R : \mathfrak{M}_* \to \mathfrak{M}_*$ of the predual of W^*-algebra $\mathfrak{M}$ such that

(i) $R^2 = R$ and $\|R\| = 1$

(ii) the range $\operatorname{im} R^*$ of the dual map $\mathbb{R}^* : \mathfrak{M} \to \mathfrak{M}$ is a C^*-subalgebra of $\mathfrak{M}$.

The properties of $R^* : \mathfrak{M} \to \mathfrak{M}$ we present in the following proposition.

Proposition 5.3. *One has*

i) $R^{*2} = R^*$ *and* $\|R^*\| = 1$

ii) $R^* : \mathfrak{M} \to \mathfrak{M}$ *is* σ*-continuous*

iii) $\operatorname{im} R^*$ *is* σ*-closed*

iv) $\operatorname{im} R^*$ *is* W^**-subalgebra of* $\mathfrak{M}$.

Proof. i) For any $x \in \mathfrak{M}$ and $b \in \mathfrak{M}_*$ one has

$$\langle R^{*2} x; b\rangle = \langle x; R^2 b\rangle = \langle x; Rb\rangle = \langle R^* x; b\rangle, \tag{5.3}$$

which gives $R^{*2} = R^*$ and $1 \leqslant \|R^*\|$. On the other side

$$\|R^* x\| = \sup_{b\neq 0} \frac{|\langle R^* x; b\rangle|}{\|b\|} = \sup_{b\neq 0} \frac{\langle x; Rb\rangle}{\|b\|} \leqslant \sup_{b\neq 0} \|x\| \frac{\|Rb\|}{\|b\|} = \|x\|, \tag{5.4}$$

so, $\|R^*\| \leqslant 1$.

ii) Let a net $\{x_\alpha\}_{\alpha\in A} \subset \mathfrak{M}$ converges $x_\alpha \xrightarrow{\sigma} x$ to $x \in \mathfrak{M}$ in σ-topology. Thus

$$\forall b \in \mathfrak{M}_* \quad \langle R^* x_\alpha; b\rangle = \langle x_\alpha; Rb\rangle \xrightarrow{\sigma} \langle x; Rb\rangle = \langle R^* x; b\rangle, \tag{5.5}$$

what means $R^* x_\alpha \xrightarrow{\sigma} R^* x$.

iii) If $R^* x_\alpha \xrightarrow{\sigma} y$ from ii) one has $R^* x_\alpha = R^* R^* x_\alpha \xrightarrow{\sigma} R^* y$. Thus $y = R^* y \in \operatorname{im} R^*$.

iv) From iii) $\operatorname{im} R^*$ is σ-closed then it is a W^*-subalgebra.

□

We see from the point iv) of Proposition 5.3 that in the condition ii) of Definition 5.2 one can equivalently assume that $\operatorname{im} R^*$ is W^*-subalgebra of $\mathfrak{M}$.

In the probability theory there is the concept of conditional expectation. It can be extended to the non-commutative probability theory which forms mathematical language of quantum statistical physics and the theory of quantum measurement, see [53, 54, 55, 48, 17]. By the definition, see e.g. [48, 56] the **normal conditional expectation** is a σ-continuous, norm one idempotent map $\mathfrak{E} : \mathfrak{M} \to \mathfrak{M}$ which maps $\mathfrak{M}$ onto a C^*-subalgebra $\mathfrak{N}$.

Proposition 5.4. *Let* $R : \mathfrak{M}_* \to \mathfrak{M}_*$ *be a quantum reduction then* $R^* : \mathfrak{M} \to \mathfrak{M}$ *is the normal conditional expectation.*

Conversely if $\mathfrak{E} : \mathfrak{M} \to \mathfrak{M}$ *is a normal conditional expectation then* $\mathfrak{E}^* : \mathfrak{M}^* \to \mathfrak{M}^*$ *preserves* $\mathfrak{M}_* \subset \mathfrak{M}^*$ *and* $R := \mathfrak{E}^*_{|\mathfrak{M}_*}$ *is the quantum reduction.*

Proof. It follows from Proposition 5.3 that $R^* : \mathfrak{M} \to \mathfrak{M}$ is normal conditional expectation. Since $\mathfrak{E} : \mathfrak{M} \to \mathfrak{M}$ is σ-continuous one has

$$\langle \mathfrak{E}^* b; x_\alpha \rangle = \langle b; \mathfrak{E} x_\alpha \rangle \to \langle b; \mathfrak{E} x \rangle = \langle \mathfrak{E}^* b; x \rangle \tag{5.6}$$

for any $x_\alpha \xrightarrow{\sigma} x$ and $b \in \mathfrak{M}_*$, so $\mathfrak{E}^* b \in \mathfrak{M}_*$. It is clear that ${\mathfrak{E}^*_{|\mathfrak{M}_*}}^2 = \mathfrak{E}^*_{|\mathfrak{M}_*}$ and $\left\| \mathfrak{E}^*_{|\mathfrak{M}_*} \right\| = 1$. For any $x \in \mathfrak{M}$ and $b \in \mathfrak{M}_*$ one has

$$\langle \mathfrak{E} x; b \rangle = \langle x; \mathfrak{E}^*_{|\mathfrak{M}_*} b \rangle = \langle (\mathfrak{E}^*_{|\mathfrak{M}_*})^* x; b \rangle, \tag{5.7}$$

which is equivalent to $(\mathfrak{E}^*_{|\mathfrak{M}_*})^* = \mathfrak{E}$. The above shows the last statement of the proposition. □

Concluding, we see that any quantum reduction R is the predual $\mathfrak{E}_*$ of a normal conditional expectation and vice versa any normal conditional expectation $\mathfrak{E}$ is the dual R^* of some quantum reduction. From σ-continuity of $\mathfrak{E}$ follows that the C^*-subalgebra $\mathfrak{N} = \operatorname{im} \mathfrak{E}$ is σ-closed, i.e. it is W^*-subalgebra. Its predual Banach space $\mathfrak{N}_*$ is isomorphic to $\operatorname{im} \mathfrak{E}_*$. Thus and from Proposition 4.6 we obtain:

Proposition 5.5. *The predual $\mathfrak{E}_* : \mathfrak{M}_* \to \mathfrak{N}_*$ of a normal conditional expectation $\mathfrak{E} : \mathfrak{M} \to \mathfrak{N} \subset \mathfrak{M}$ is the surjective linear Poisson map of Banach Lie-Poisson spaces. The Lie-Poisson structure of $\mathfrak{N}_*$ is coinduced by $\mathfrak{E}_*$ from Banach Lie-Poisson space $\mathfrak{M}_*$.*

We will see from the examples presented below that $\mathfrak{E}_* : \mathfrak{M}_* \to \mathfrak{M}_*$ could be considered as the mathematical realization of the measurement operation. Therefore in virtue of Proposition 5.5 one can consider the measurement as a linear Poisson morphism.

Example 5.1. If $p \in \mathfrak{M}$ is self-adjoint projector, i.e. $p^2 = p = p^*$, then the map

$$\mathfrak{E}_p(x) := pxp, \tag{5.8}$$

as it easily to see, satisfies the all defining properties of the normal conditional expectation. The $\operatorname{im} \mathfrak{E}_p$ is a hereditary W^*-subalgebra of $\mathfrak{M}$. Any hereditary W^*-subalgebra of $\mathfrak{M}$ is the range of the normal conditional expectation $\mathfrak{E}_p$ for some self-adjoint projector $p \in \mathfrak{M}$, see for example [48] for the proof of the above facts.

♢

Example 5.2. Let the family $\{p_\alpha\}_{\alpha \in I} \subset \mathfrak{M}$ of self-adjoint mutually orthogonal $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$ projectors gives the orthogonal resolution $\sum_\alpha p_\alpha = 1$ of unit $1 \in \mathfrak{M}$. It defined the normal conditional expectation $\mathfrak{E} : \mathfrak{M} \to \mathfrak{M}$ by

$$\mathfrak{E}(x) := \sum_{\alpha \in I} p_\alpha x p_\alpha, \tag{5.9}$$

where the summation in (5.9) is taken in sense of the σ-topology.

In order to see that let us consider $\mathfrak{M}$ as a von Neumann algebra of operators on the Hilbert space $\mathcal{H}$. Then for $v \in \mathcal{H}$ one has

$$\|\mathfrak{E}(x)v\|^2 = \left\|\sum_\alpha p_\alpha x p_\alpha v\right\|^2 = \sum_\alpha \|p_\alpha x p_\alpha v\|^2 \leqslant \sum_\alpha \|x\|^2 \|p_\alpha v\|^2 = \|x\|^2 \|v\|^2 , \tag{5.10}$$

which gives $\|\mathfrak{E}(x)\| \leqslant 1$. Thus since $\{p_\alpha\}_{\alpha\in I}$ is orthogonal resolution of unit the map $\mathfrak{E}$ is an idempotent, i.e. $\mathfrak{E}^2 = \mathfrak{E}$, of the norm $\|\mathfrak{E}\| = 1$. The direct computation shows that

$$\mathfrak{E}(x)^* = \mathfrak{E}(x^*) \tag{5.11}$$

$$\mathfrak{E}(x)\mathfrak{E}(y) = \mathfrak{E}(\mathfrak{E}(x)\mathfrak{E}(y)) \tag{5.12}$$

for any $x, y \in \mathfrak{M}$. Let $x_i \xrightarrow{\sigma} x$ and $\rho \in L^1(\mathcal{H})$ be such that$\langle x; b\rangle = \mathrm{Tr}(x\rho)$. Thus

$$\langle \mathfrak{E}(x); b\rangle = \mathrm{Tr}(\rho \sum_\alpha p_\alpha x_i p_\alpha) = \mathrm{Tr}(x_i \sum_{\alpha\in I} p_\alpha \rho p_\alpha) \xrightarrow{\sigma} \mathrm{Tr}(x \sum_\alpha p_\alpha \rho p_\alpha) = \mathrm{Tr}(\mathfrak{E}(x)\rho) = \langle \mathfrak{E}(x); b\rangle \tag{5.13}$$

for any $b \in \mathfrak{M}_*$. This shows that $\mathfrak{E}$ is σ-continuous.

The range $\mathrm{im}\,\mathfrak{E}$ of the normal conditional expectation (5.9) can be characterized as the commutant of the set $\{p_\alpha\}_{\alpha\in I}$ of self-adjoint projectors.

$\diamondsuit$

Example 5.3. The W^*-tensor product $\mathfrak{M}\otimes\mathfrak{N}$ of the W^*-algebras $\mathfrak{M}$ and $\mathfrak{N}$ by definition is $(\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*)^* = (\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*)^{**}/\mathcal{I}$, where the two-side ideal $\mathcal{I}$ is the polar (annihilator) of $\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*$ in the second dual of $\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*$. ($\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*$ is a closed subspace of $(\mathfrak{M} \otimes_{\alpha_0} \mathfrak{N})^*$). In order to explain the above definition in the details we will follows of [48]. The cross norm α_0 is a least C^*-norm among all norms α on the algebraic tensor product $\mathfrak{M}\otimes\mathfrak{N}$ satisfying $\alpha(x^*x) = \alpha(x)^2$ and $\alpha(xy) \leqslant \alpha(x)\alpha(y)$ for $x, y, \in \mathfrak{M}\otimes\mathfrak{N}$. Existence of α_0 is proved in Theorem 1.2.2 of [48]. The C^*-algebra $\mathfrak{M} \otimes_{\alpha_0} \mathfrak{N}$ (called C^*-tensor product of $\mathfrak{M}$ and $\mathfrak{N}$) denotes the completion of $\mathfrak{M}\otimes\mathfrak{N}$ with respect to α_0. The predual Banach space $\mathfrak{M}_* \otimes_{\alpha_0} \mathfrak{N}_*$ is completion of algebraic tensor product $\mathfrak{M}_* \otimes \mathfrak{N}_*$ with respect to the dual form α_0^*. Finally let us recall (e.g. see Theorem 1.17.2 in [48]) that the second dual $\mathcal{A}^{**}$ of C^*-algebra $\mathcal{A}$ is a W^*-algebra and $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{A}^{**}$.

After these preliminary definitions let us define the linear map $\mathfrak{E}_{m_0} : \mathfrak{M}\bar{\otimes}\mathfrak{N} \to \mathfrak{M}\bar{\otimes}\mathfrak{N}$ indexed by a positive $m_0 \in \mathfrak{M}_*$ which satisfies $\langle 1; m_0\rangle = 1$ and $\|m_0\| = 1$. It is sufficient to fix the values of $\mathfrak{E}_{m_0}$ on the decomposable elements:

$$\mathfrak{E}_{m_0}(x \otimes y) := 1 \otimes \langle x; m_0\rangle y, \tag{5.14}$$

where $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$.

Proposition 5.6. *If $m_0 \in \mathfrak{M}_*$ is positive, $\|m_0\| = 1$ and $\langle 1; m_0\rangle = 1$ then $\mathfrak{E}_{m_0} : \mathfrak{M}\bar{\otimes}\mathfrak{N} \to \mathfrak{M}\bar{\otimes}\mathfrak{N}$ defined by* (5.14) *is a normal conditional expectation. Moreover* *(i)* $\operatorname{im}\mathfrak{E}_{m_0} = 1\bar{\otimes}\mathfrak{N}$

(ii) $\mathfrak{E}_{m_0}(1 \otimes 1) = 1$

(iii) *predual* $R_{m_0} = (\mathfrak{E}_{m_0})_*$ *of* $\mathfrak{E}_{m_0}$ *is given by*

$$R_{m_0}(n \otimes m) = \langle 1; m\rangle m_0 \otimes n \tag{5.15}$$

for $m \in \mathfrak{M}_*$ *and* $n \in \mathfrak{N}_*$*;*

(iv) $\mathfrak{E}_{m_0}(axb) = a\mathfrak{E}_{m_0}(x)b$ *for* $a, b \in \mathfrak{E}_{m_0}$ *and* $x \in \mathfrak{M}\bar{\otimes}\mathfrak{N}$*;*

(v) $\mathfrak{E}_{m_0}(x)^*\mathfrak{E}_{m_0}(x) \leqslant \mathfrak{E}_{m_0}(x^*x)$ *for* $x \in \mathfrak{M}\bar{\otimes}\mathfrak{N}$*;*

(vi) *if* $\mathfrak{E}_{m_0}(x^*x) = 0$ *then* $x = 0$*;*

(vii) *if* $x \geqslant 0$ *then* $\mathfrak{E}_{m_0}(x) \geqslant 0$.

For the proof of this proposition see Theorem 2.6.4 in [48].

♢

Subsequently we shall discuss those three examples in detail when W^*-algebra $\mathfrak{M}$ be the algebra of all bounded operators $L^\infty(\mathcal{H})$ on Hilbert space $\mathcal{H}$. As we will see the normal conditional expectations and quantum reduction in this case have concrete physical meaning.

6 Statistical models of physical systems

Any investigation of the physical system always establishes the existence of the system states set $\mathcal{S}$ and the set $\mathcal{O}$ of the observables related to the system. The choice of $\mathcal{S}$ and $\mathcal{O}$ depends on the our actual knowledge, experimental as well as theoretical, concerning the system under considerations.

The observable $X \in \mathcal{O}$ which describes measurement procedure is realized by an experimental device which after application to the system prepared in the state $s \in \mathcal{S}$ gives some real number $x \in \mathbb{R}$. Repetition of the X observable measurement on the ansamble of systems in the same state gives a sequence

$$\{x_1, \ldots, x_N\} \tag{6.1}$$

of the real numbers. The limit of the relative frequencies

$$\lim_{N\to\infty} \frac{\#\{x_i : x_i \in \Omega\}}{N} =: \mu_s^X(\Omega), \tag{6.2}$$

where $\Omega \in \mathcal{B}(\mathbb{R})$ is the Borel subset of $\mathbb{R}$, defines a probabilistic measure μ_s^X on the σ-algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of the real line $\mathbb{R}$.

Thereby the measurement procedure gives the prescription

$$\mu : \mathcal{O} \times \mathcal{S} \ni (X, s) \longrightarrow \mu_s^X \in \mathcal{P}(\mathbb{R}), \tag{6.3}$$

which maps $\mathcal{O}\times\mathcal{S}$ into the space $\mathcal{P}(\mathbb{R})$ of probabilistic measures on the σ-algebra of Borel subsets $\mathcal{B}(\mathbb{R})$.

Let us remark here that the experimental construction of the map (6.3) is based on the confidence that one can repeat individual measurement and the limit (6.2) is stable under independent repetitions.

The pairing $\langle X; s\rangle$ defined by the integral

$$\langle X; s\rangle := \int_{\mathbb{R}} y\mu_s^X(dy) \tag{6.4}$$

has the physical interpretation of the **mean value** of the observable X in the state s, i.e. the $\langle X; s\rangle$ could be cosidered as the "value" of the observable X in the state s.

This, what was presented above, is in some sense the shortest and most abstract description of the statistical structure of the physical measurement applied to the system. It is obviously not complete, since it does not yield any information concerning structures of the spaces $\mathcal{S}$ and $\mathcal{O}$. In order to recognize these structures one postulates additionally certain system of axioms, see [27].

Axiom 1. *From the fact that $\mu_{s_1}^X = \mu_{s_2}^X$ for all $X \in \mathcal{O}$ it follows that $s_1 = s_2$ and from $\mu_s^{X_1} = \mu_s^{X_2}$ for all $s \in \mathcal{S}$ it follows that $X_1 = X_2$.*

This **separability axiom** means that one can separate states of the investigated system in the experimental way; also observables are distinguished by their experimentally obtained probability distributions in all states $s \in \mathcal{S}$ of the system.

The rejection of Axiom 1 leads to the possibility of non-experimental recognition of states and observables, what is in contradiction with rational point of view on physical phenomena. So, the necessity of Axiom 1 follows from Okham razor principle.

The space of probabilistic measures $\mathcal{P}(\mathbb{R})$ has two properties important for the statistical approach to the physical systems:

i) $\mathcal{P}(\mathbb{R})$ is a convex set, i.e. for any $\mu_1, \mu_2 \in (\mathbb{R})$ and $p \in [0, 1]$ one has

$$p\mu_1 + (1-p)\mu_2 \in \mathcal{P}(\mathbb{R}). \tag{6.5}$$

ii) Let $\mathcal{M}(\mathbb{R})$ denote the set of measurable functions from $\mathbb{R}$ to $\mathbb{R}$. It is semigroup with respect to superposition operation and it acts on $\mathcal{P}(\mathbb{R})$ from the left side by

$$f^*\mu(\Omega) := \mu(f^{-1}(\Omega)) \tag{6.6}$$

i.e. $f^*\mu \in \mathcal{M}(\mathbb{R})$ and $(g \circ f)^* = g^* \circ f^*$ for $f, g \in \mathcal{M}(\mathbb{R})$.

The following axiom allows the possibility of mixing of the states. Expressing that in the precise manner, it allows to define the convex structure on $\mathcal{S}$.

Axiom 2. *For arbitrary $s_1, s_2 \in \mathcal{S}$ and any $p \in [0, 1]$ there exists $s \in \mathcal{S}$ such that $\mu_s^X = p\mu_{s_1}^X + (1-p)\mu_{s_2}^X$ for all $X \in \mathcal{O}$.*

It follows from Axiom 1 that s is defined by Axiom 2 in the unique way.

Let us denote by $\mathcal{F}(\mathcal{S})$ the vector space of real valued functions $\varphi : \mathcal{S} \to \mathbb{R}$ which have the property

$$\varphi(s) = p\varphi(s_1) + (1-p)\varphi(s_2) \tag{6.7}$$

for any $s_1, s_2 \in \mathcal{S}$ defined by Axiom 2. For example the mean values function $\langle X; \cdot \rangle$, $X \in \mathcal{O}$ given by (6.4) fulfills the property (6.7). It is natural to assume, what we will do, that $\mathcal{F}(\mathcal{S})$ is spanned by mean values functions. Additionally we assume that $\mathcal{F}(\mathcal{S})$ separates elements of $\mathcal{S}$, i.e. for any $s_1, s_2 \in \mathcal{S}$ there exist $\varphi \in \mathcal{F}(\mathcal{S})$ that $\varphi(s_1) \neq \varphi(s_2)$. Under such assumptions the evaluation map $\mathcal{E} : \mathcal{S} \to \mathcal{F}(\mathcal{S})$, defined by

$$\mathcal{E}(\mathcal{S})(\varphi) := \varphi(s) \qquad \varphi \in \mathcal{F}(\mathcal{S}), \tag{6.8}$$

is one-to-one. Therefore, in the considered case, states space $\mathcal{S}$ can be identified with the convex subset of the vector space $\mathcal{F}(\mathcal{S})'$ dual to $\mathcal{F}(\mathcal{S})$. Usually $\mathcal{S}$ is always considered as a convex subset of some topological vector space, e.g. see Examples 6.1 and 6.2 presented below. So, summing up the above considerations, we see that Axiom 2 allows to take the mixture

$$s := ps_1 + (1-p)s_2 \tag{6.9}$$

of the states s_1 and s_2.

The extremal element of $\mathcal{S}$, i.e. one which does not have the decomposition (6.9) with $p \in]0, 1[$ is called a **pure state**.

One also postulates the axiom which permits to define the semigroup $\mathcal{M}(\mathbb{R})$ action on the set of observables.

Axiom 3. *For any $X \in \mathcal{O}$ and any $f \in \mathcal{M}(\mathbb{R})$ there exists $Y \in \mathcal{O}$ such that*

$$\mu_s^Y = f^* \mu_s^X \tag{6.10}$$

for all $s \in \mathcal{S}$.

By the Axiom 1 the observable Y is defined univocally. One calls Y **functionally subordinated** to the observable X. We shall use the commonly accepted notation $Y = f(X)$ subsequently. The functional subordination gives a partial ordering on $\mathcal{O}$ defined by

$$X \prec Y \quad \text{iff} \quad \text{exists } f \in \mathcal{M}(\mathbb{R}) \text{ such that } Y = f(X). \tag{6.11}$$

Since the antisymmetry property, i.e. if $X \prec Y$ and $Y \prec X$ then $X = Y$, is not satisfied, the relation $\prec$ is not the ordering in general.

Observables $Y_1, \ldots Y_n$ are called **compatible** if they are functionally subordinated to some observable X: $X \prec Y_1, \ldots, X \prec Y_n$. One can measure compatible observables by measuring observable X, it means that they can be measured simultaneously, what is not true for the arbitrary set of observables.

Therefore, by postulated axioms, spaces of states $\mathcal{S}$ and of observables $\mathcal{O}$ inherit from $\mathcal{P}(\mathbb{R})$ the convex geometry and the partial ordering respectively.

According to Mackey [27] we introduce the notion of the experimentally verifiable **proposition (question)**. By definition it is an observable $q \in \mathcal{O}$ such that

$$\forall\, s \in \mathcal{S} \quad \mu_s^q(\{0,1\}) = 1. \tag{6.12}$$

Let us denote by $\mathcal{L}$ the set of all propositions. Since for any $X \in \mathcal{O}$ and $A \in \mathcal{B}(\mathbb{R})$ the observable $\chi_\Delta(X)$, where χ_Δ is the indicator function of Δ, belongs to $\mathcal{L}$, one has lots of propositions.

For any proposition $q \in \mathcal{L}$ one defines its negation $q^\perp \in \mathcal{L}$ by

$$\forall\, s \in \mathcal{S} \quad \mu_s^q(\{1\}) + \mu_s^{q^\perp}(\{1\}) = 1. \tag{6.13}$$

It is easy to see that $\perp: \mathcal{L} \to \mathcal{L}$ is an involution.

One assumes the following formal definition of the logic, eg. see [61].

Definition 6.1. The **logic** is an orthomodular lattice $\mathcal{L}$ such that $\bigvee_n a_n$ and $\bigwedge_n a_n$ exist in $\mathcal{L}$ for any countable subset $\{a_1, a_2, \ldots\} \subset \mathcal{L}$.

For the sake of self-completeness of the text let us recall that partially ordered set $\mathcal{L}$ is a lattice iff for any two $a, b \in \mathcal{L}$ there is $a \wedge b \in \mathcal{L}$ ($a \vee b \in \mathcal{L}$) such that $a \wedge b \prec a$ and $a \wedge b \prec b$ ($a \prec a \vee b$ and $b \prec a \vee b$) and $c \prec a \wedge b$ ($a \vee b \prec c$) for any $c \prec a$ and $c \prec b$ ($a \prec c$ and $b \prec c$). Binary operations $\wedge$ and $\vee$ define the algebra structure on $\mathcal{L}$ and conditions $\bigvee_n a_n$, $\bigwedge_n a_n \in \mathcal{L}$ mean that $\mathcal{L}$ is closed with respect to the countable application of $\wedge$ and $\vee$ operations. If $\mathcal{L}$ has zero 0 and unit 1 and there exists map $\perp: \mathcal{L} \to \mathcal{L}$ such that

$$a \wedge a^\perp = 0\,, \qquad a \vee a^\perp = 1 \tag{6.14}$$

$$a^{\perp\perp} = a \tag{6.15}$$

$$a \prec b \Rightarrow b^\perp \prec a^\perp \tag{6.16}$$

$$a \prec b \Rightarrow b = a \vee (b \wedge a^\perp) \tag{6.17}$$

one says that $\mathcal{L}$ is **orthomodular lattice**, see [61].

The element $b \wedge a^\perp$ from (6.17) is written as $b - a$. The proposition $a^\perp$, which is the negation of the proposition a, is called the orthogonal complement of a in $\mathcal{L}$. One says that propositions a and b are orthogonal and writes $a \perp b$ iff $a \prec b^\perp$ and $b \prec a^\perp$. If $a \perp b$ then proposition (question) a excludes the proposition b.

Moreover one has

$$\bigvee_n a_n^\perp = \left(\bigwedge_n a_n\right)^\perp \tag{6.18}$$

$$\bigwedge_n a_n^\perp = \left(\bigvee_n a_n\right)^\perp \tag{6.19}$$

and

$$(a \vee b) \wedge c = (a \wedge c) \vee b \tag{6.20}$$

for $a \perp b$ and $b \prec c$.

The condition (6.20) is called **orthomodularity property**. Stronger condition that if $b \prec c$ then (6.20) is called **modularity** property.

The element $a \in \mathcal{L}$ is called an atom of $\mathcal{L}$ if $a \neq 0$ and if $b \prec a$ then $b = 0$, i.e. a is minimal non-zero element of $\mathcal{L}$. The logic $\mathcal{L}$ is **atomic** if for any element $0 \neq b \in \mathcal{L}$ there exists atom $a \prec b$.

By **morphism** of two logics we will understand the map $\Phi : \mathcal{L}_1 \to \mathcal{L}_2$ which preserves their operations $\vee$, $\wedge$, involutions $\perp$, zeros and units.

Axiom 4. *The set of propositions* $\mathcal{L}$ *with* $\perp$ *defined by* (6.13) *is logic and for any* $X \in \mathcal{O}$ *the map*

$$\mathcal{B}(\mathbb{R}) \ni \Delta \to \chi_\Delta(X) \in \mathcal{L} \tag{6.21}$$

is a morphism of the logic $\mathcal{B}(\mathbb{R})$ *of Borel subsets of* $\mathbb{R}$ *into* $\mathcal{L}$.

For any logic $\mathcal{L}$ of Definition 6.1 one can define the space $\mathcal{P}(\mathcal{L})$ of σ-additive measures and the space $\mathcal{E}(\mathcal{L})$ of proposition valued measures, e.g. see [61]. The space $\mathcal{P}(\mathcal{L})$ by definition will consist of measures on $\mathcal{L}$, i.e. functions

$$\pi : \mathcal{L} \to [0, 1] \tag{6.22}$$

such that

i) $\pi(0) = 0$ and $\pi(1) = 1$

ii) if $a_1, a_2, \ldots$ is a countable or finite sequence of elements of $\mathcal{L}$ then

$$\pi(\bigvee_n a_n) = \sum_n \pi(a_n) \tag{6.23}$$

if $a_n \perp a_m$ for $n \neq m$.

It follows from properties (6.17) and (6.23) that

$$\pi(a) \leqslant \pi(b) \tag{6.24}$$

if $a \prec b$. It is also clear that $\mathcal{P}(\mathcal{L})$ has naturally defined convex structure.

The space $\mathcal{E}(\mathcal{L})$ of proposition valued measures is defined in some sense as a dual object to $\mathcal{P}(\mathcal{L})$. Namely, the proposition valued measure $E \in \mathcal{E}(\mathcal{L})$ associated to $\mathcal{L}$ is a map

$$E : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L} \tag{6.25}$$

such that

i) $E(\emptyset) = 0$ and $E(\mathbb{R}) = 1$

ii) if $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$ and $\Delta_1 \cap \Delta_2 = \emptyset$ then $E(\Delta_1) \perp E(\Delta_2)$.

iii) if $\Delta_1, \Delta_2, \ldots \in \mathcal{B}(\mathbb{R})$ and $\Delta_k \cap \Delta_l = \emptyset$ for $k \neq l$ then

$$E(\cup_k \Delta_k) = \bigvee_k E(\Delta_k), \tag{6.26}$$

i.e. it is a logic morphism.

If $f \in \mathcal{M}(\mathbb{R})$ is measurable real valued function, then $f(E)$ defined by

$$f(E)(\Delta) := E(f^{-1}(\Delta)) \tag{6.27}$$

belongs to $\mathcal{E}(\mathcal{L})$ if $E \in \mathcal{E}(\mathcal{L})$. The above defines subordination relation in $\mathcal{E}(\mathcal{L})$. One defines the map $\nu : \mathcal{E}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathbb{R})$ by

$$\nu^E_\pi(\Delta) := \pi(E(\Delta)). \tag{6.28}$$

From the definition (6.28) one has

$$f^*\nu^E_\pi = \nu^{f(E)}_\pi \tag{6.29}$$

for any $\pi \in \mathcal{P}(\mathcal{L})$ and

$$\nu^E_{p\pi_1+(1-p)\pi_2} = p\nu^E_{\pi_1} + (1-p)\nu^E_{\pi_2} \tag{6.30}$$

for any $E \in \mathcal{E}(\mathcal{L})$.

Let us now define the maps $\chi : \mathcal{O} \to \mathcal{E}(\mathcal{L})$ and $\iota : \mathcal{S} \to \mathcal{P}(\mathcal{L})$ in the following way

$$\chi(X)(\Delta) := \chi_\Delta(X) \tag{6.31}$$

and

$$\iota(s)(q) := \mu^q_s(\{1\}) \tag{6.32}$$

for any $q \in \mathcal{L}$.

Proposition 6.2.

i) One has

$$\chi(f(X)) = f(\chi(X)) \tag{6.33}$$

for $f \in \mathcal{M}(\mathbb{R})$, and

$$\iota(ps_1 + (1-p)s_2) = p\,\iota(s_1) + (1-p)\,\iota(s_2), \tag{6.34}$$

i.e. $\chi : \mathcal{O} \to \mathcal{E}(\mathcal{L})$ is equivariant with respect to the action of the semigroup $\mathcal{M}(\mathbb{R})$ and $\iota : \mathcal{S} \to \mathcal{P}(\mathcal{L})$ preserves the convex structure.

ii) For $X \in \mathcal{O}$ and $s \in \mathcal{S}$ the equality

$$\nu^{\chi(X)}_{\iota(s)} = \mu^X_s \tag{6.35}$$

is valid.

Proof.

i) The formula (6.33) follows from the definition (6.31) and from

$$\chi_\Delta \circ f = \chi_{f^{-1}(\Delta)} \tag{6.36}$$

and (6.34) follows in the trivial way from the definition (6.32)

ii) In order to prove (6.35) let us observe that

$$\nu_{\iota(s)}^{\chi(X)}(\Delta) = \iota(s)(\chi(X)(\Delta)) = \iota(s)(\chi_\Delta(X)) = \mu_s^{\chi_\Delta(X)}(\{1\}) = \mu_s^X(\Delta). \tag{6.37}$$

□

Finally we assume the following

Axiom 5. *The maps* $\chi : \mathcal{O} \to \mathcal{E}(\mathcal{L})$ *and* $\iota : \mathcal{S} \to \mathcal{P}(\mathcal{L})$ *are bijective.*

In order to elucidate physical as well as mathematical substance of the formulated above separable statistical model let us present few examples.

Example 6.1 (*Kolmogorov model*)**.** In the Kolmogorov model the space of states $\mathcal{S}$ is given by the convex set $\mathcal{P}(M)$ of all probability measures on a Borel space $(M, \mathcal{B}(M))$. The space of observables $\mathcal{O}$ is the set $\mathcal{M}(M)$ of measurable real functions (real random variables). The subordination relation for $X, Y \in \mathcal{M}(M)$ is given canonically by

$$X \prec Y \quad \text{iff} \quad \text{exists } f \in \mathcal{M}(\mathbb{R}) \text{ such that } Y = f \circ X. \tag{6.38}$$

One defines $\mu : \mathcal{M}(M) \times \mathcal{P}(M) \to \mathcal{P}(\mathbb{R})$ by

$$\mu_s^X(\Delta) := s(X^{-1}(\Delta)), \tag{6.39}$$

where $\Delta \in \mathcal{B}(\mathbb{R})$.

Now let us check that separability axiom is fulfilled. If $s_1(X^{-1}(\Delta)) = s_2(X^{-1}(\Delta))$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$ and arbitrary $X \in \mathcal{M}(M)$. Then since one can take as X any indicator function it follows that $s_1(\Omega) = s_2(\Omega)$ for arbitrary $\Omega \in \mathcal{B}(M)$. This gives $s_1 = s_2$. If $s(X_1^{-1}(\Delta)) = s(X_2^{-1}(\Delta))$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$ and $s \in \mathcal{P}(M)$ then $X_1^{-1}(\Delta) = X_2^{-1}(\Delta)$ for arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$. This gives $X_1 = X_2$.

From (6.39) it follows

$$p\mu_{s_1}^X + (1-p)\mu_{s_2}^X = \mu_{ps_1+(1-p)s_2}^X. \tag{6.40}$$

Thus Axiom 2 is fulfilled.

Also from (6.39) one has

$$\mu_s^{f\circ X}(\Delta) = s(X^{-1}(f^{-1}(\Delta))) = (f^*\mu_s^X)(\Delta), \tag{6.41}$$

what shows that Axiom 3 is also fulfilled.

Logic of propositions $\mathcal{L}$ in this case is equal to the Boolean algebra $\mathcal{B}(M)$ of Borel subsets of M. We identify here $A \in \mathcal{B}(\mathcal{L})$ with its indicator function χ_A. The partial order $\prec$ in $\mathcal{B}(M)$ is given by the inclusion $\subset$. The orthocomplement operation is defined by

$$A^\perp := M \setminus A \qquad \text{for } A \in \mathcal{B}(M). \tag{6.42}$$

The lattice $\mathcal{B}(M)$ is distributive

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \tag{6.43}$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \tag{6.44}$$

and $\bigwedge\limits_{\alpha \in F} A_\alpha$ and $\bigvee\limits_{\alpha \in F} A_\alpha$ belongs to $\mathcal{B}(M)$ for any countable subset F. So $\mathcal{B}(M)$ is a Boolean σ-algebra. The map (6.21) in this case assumes the form

$$\mathcal{B}(\mathbb{R}) \ni \Delta \to \chi_\Delta \circ X = \chi_{X^{-1}(\Delta)} \in \mathcal{L} \cong \mathcal{B}(M). \tag{6.45}$$

So it is logic morphism. One can show that any logic morphism of $\mathcal{B}(\mathbb{R})$ into $\mathcal{B}(M)$ is given in this way. Thus Axioms 4 and 5 are fulfilled.

Finally let us remark that in Kolmogorov model all observables are compatible.

$\diamondsuit$

The other example of the statistical model related to a logic is the standard statistical model of quantum mechanics.

Example 6.2 (*standard statistical model of quantum mechanics*)**.** For the standard statistical model of quantum mechanics the logic $\mathcal{L}(\mathcal{H})$ is given by the orthomodular lattice of Hilbert subspaces of the Hilbert space $\mathcal{H}$. Any element $\mathcal{M} \in \mathcal{L}(\mathcal{H})$ can be identified with the orthogonal projector $E : \mathcal{H} \to \mathcal{M}$, i.e. $E^2 = E = E^*$. The logic $\mathcal{L}(\mathcal{H})$ is non-distributive and for the infinite-dimensional Hilbert space $\mathcal{H}$ non-modular, see [61].

For this model as the state space $\mathcal{S}(\mathcal{H})$ one assumes the set of all positive trace class operators satisfying additional condition $\|\rho\|_1 = \operatorname{Tr}\rho = 1$. The set $\mathcal{S}(\mathcal{H})$ is convex and extreme points (pure states) of it are rank one orthogonal projectors

$$E_{[\psi]} := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}. \tag{6.46}$$

By spectral theorem the arbitrary state $\rho \in \mathcal{S}(\mathcal{H})$ has convex decomposition

$$\rho = \sum_{k=1}^{\infty} p_k E_{[\psi_k]}, \tag{6.47}$$

on the pure states $E_{[\psi_k]}$, where ψ_k are eigenvectors of ρ and $p_k \geqslant 0$ are the corresponding eigenvalues. One has $\operatorname{Tr}\rho = \sum_{k=1}^{\infty} p_k = 1$.

The set of observables $\mathcal{O}(\mathcal{H})$ consists of selfadjoint operators, unbounded in general. Taking the spectral decomposition

$$X = \int xE(dx), \tag{6.48}$$

where $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ denotes the spectral measure of $X \in \mathcal{O}(\mathcal{H})$, one defines the probability distribution of the observable X in the state ρ by

$$\mu^X_\rho(\Delta) := \mathrm{Tr}(\rho E(\Delta)). \tag{6.49}$$

The map $\mu : \mathcal{O}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathcal{P}(\mathbb{R})$ defined by (6.49) satisfies all axioms postulated above. Let us check it.

If $\mu^{X_1}_\rho = \mu^{X_2}_\rho$ for any $\rho \in \mathcal{S}(\mathcal{H})$ then

$$\mathrm{Tr}\, \rho(E_1(\Delta) - E_2(\Delta)) = 0 \tag{6.50}$$

for arbitrary ρ and $\Delta \in \mathcal{B}(\mathbb{R})$. Since $E_1(\Delta) - E_2(\Delta) \in iU^\infty(\mathcal{H})$ and $U^\infty(\mathcal{H}) \cong U^1(\mathcal{H})^*$ one obtains $E_1(\Delta) = E_2(\Delta)$ for any $\Delta \in \mathcal{B}(\mathbb{R})$, what means $X_1 = X_2$.

If $\mu^X_{\rho_1} = \mu^X_{\rho_1}$ for any $X \in \mathcal{O}(\mathcal{H})$ then

$$\mathrm{Tr}(\rho_1 - \rho_2)E = 0 \tag{6.51}$$

for an arbitrary orthogonal projector $E \in \mathcal{O}(\mathcal{H})$. Now since $U^\infty(\mathcal{H})$ is dual to $U^1(\mathcal{H})$ and the lattice $\mathcal{L}(\mathcal{H})$ of orthogonal projections is linearly dense in $U^\infty(\mathcal{H})$ with respect to $\sigma(U^\infty(\mathcal{H}), U^1(\mathcal{H}))$-topology one obtains $\rho_1 = \rho_2$.

From the definition (6.49) one has

$$\mu^X_{p\rho_1+(1-p)\rho_2}(\Delta) = p\mu^X_{\rho_1}(\Delta) + (1-p)\mu^X_{\rho_2}(\Delta) \tag{6.52}$$

and

$$f^*\mu^X_\rho(\Delta) = \mathrm{Tr}(\rho E(f^{-1}(\Delta))) = \mu^{f(X)}_\rho(\Delta), \tag{6.53}$$

where

$$f(X) = \int f(x)E(dx), \tag{6.54}$$

for an arbitrary $\Delta \in \mathcal{B}(\mathbb{R})$. This shows that Axiom 2 and Axiom 3 are fulfilled.

The Axiom 4 is the consequence of the spectral theorem. The Axiom 5 is the statement of the Gleason theorem, see [14]

♢

Example 6.3 (*models related to W^*-algebras*)**.**

In that case the logic $\mathcal{L}(\mathfrak{M})$ of the physical system under consideration is given by the lattice of all self-adjoint idempotents of W^*-algebra $\mathfrak{M}$. The space of observables consists of $\mathcal{L}(\mathfrak{M})$-valued spectral measures or equivalently the self-adjoint operators affiliated to the faithful representations of $\mathfrak{M}$ in the Hilbert space $\mathcal{H}$. The state space $\mathcal{S}(\mathfrak{M}) \subset \mathfrak{M}_*$ given by the positive $0 \leqslant b \in \mathfrak{M}_*$ normalized $\|b\| = 1$ σ-continuous linear functionals.

This class of physical systems contains the standard statistical model of quantum mechanics, which is given by W^*-algebra $\mathfrak{M} = L^\infty(\mathcal{H})$. Also Kolmogorov models ca be considered as models related to the subcategory of commutative W^*-algebras $\mathfrak{M} = \mathcal{L}^\infty(M, d\mu)$.

$\diamondsuit$

Let us now explain what we shall mean by the quantization of the classical phase space M which, according to the classical statistical mechanics, is naturally considered as a Kolmogorov model $(M, \mathcal{B}(M), \mathcal{M}(M), \mathcal{P}(M))$. Our approach will be done by fixing two transforms:

(i) The morphism

$$E : \mathcal{B}(M) \to \mathcal{L}(\mathfrak{M}) \tag{6.55}$$

of the Borel logic $\mathcal{B}(M)$ into the logic $\mathcal{L}(\mathfrak{M})$ of all self-adjoint idempotents of the W^*-algebra $\mathfrak{M}$.

(ii) The normal conditional expectation map

$$\mathfrak{E} : \mathfrak{M} \longrightarrow \mathfrak{M}, \tag{6.56}$$

which maps $\mathfrak{M}$ on the C^*-subalgebra $\mathfrak{N} \subset \mathfrak{M}$.

Definition 6.3. The **quantum phase space** $\mathcal{A}_{M,\mathfrak{E},E}$ related to E and $\mathfrak{E}$ is the C^*-subalgebra of $\mathfrak{N}$ generated by $\mathfrak{E}(E(\mathcal{B}(M)))$.

Many known procedures of quantization are included in this general scheme. For example, one obtains in this way the Toeplitz C^*-algebra related to the symmetric domain, see [57, 58], in the case if one takes conditional expectation $\mathfrak{E} : L^\infty(\mathcal{H}) \to L^\infty(\mathcal{H})$ defined by the coherent state map, see Section 11.

7 The coherent state map

The idea which we want to present is based on the conviction that all experimentally achievable quantum states $s \in \mathcal{S}(\mathcal{H})$ of any considered physical system are parametrized by a finite number of continuous or discrete parameters. One can prove it by the experiment ad absurdum method. Because, assuming the contrary, i.e. an infinite number of parameters, one will need infinite time for the measurement. Mathematical correctness suggests the idea that the space of parameters be a smooth finite dimensional manifold M (the discrete parameter case will not be discussed here), and that the parametrizing map

$$\mathcal{K} : M \longrightarrow \mathcal{S}(\mathcal{H}) \subset U^1(\mathcal{H}) \tag{7.1}$$

be a smooth map. However in order to preserve generality of our considerations we will also admit the possibility that M is infinite dimensional Banach manifold. Even having such general assumptions one can investigate which models are physically interesting and mathematically fruitful. However, since we are within the framework of mechanics, we restrict the generality, assuming the following definitions

Definition 7.1. The **coherent state map** is a map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ such that:

i) the differential form $\mathcal{K}^*\omega_{FS} =: \omega$ is a symplectic form;

ii) the rank $\mathcal{K}(M)$ of $\mathcal{K}$ is linearly dense in $\mathcal{H}$.

We shall call the states $\mathcal{K}(m)$, where $m \in M$, the **coherent states**.

Definition 7.2. The **mechanical system** will be the triple: M, $\mathcal{H}$, $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$, where

i) M is a smooth Banach manifold;

ii) $\mathcal{H}$ is a complex separable Hilbert space;

iii) $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ is a coherent state map.

In order to illustrate the introduced notions let us present the example important from physical point of view.

Example 7.1 (*Gauss coherent state map*)**.** Historically this coherent state map can be addressed to E. Schrödinger who in the paper [49] considered the wave packets minimalizing Heisenberg uncertainty principle.

In our presentation we will use Fock representation. The classical phase space of the system will be assumed to be $M = \mathbb{R}^{2N}$ with the symplectic form ω given by

$$\omega_\hbar = \hbar^{-1} d\left(\sum_{k=1}^{N} p_k dq_k\right) , \tag{7.2}$$

where $(q_1, \ldots, q_N, p_1, \ldots, p_N)$ are the canonical coordinates describing position and momentum respectively. By $z_k = q_k + ip_k$, $k = 1, \ldots, N$, we will identify $\mathbb{R}^{2N}$ with $\mathbb{C}^N$ and thus $\omega_\hbar$ will be given by

$$\omega_\hbar = \frac{\hbar^{-1}}{2i} \sum_{k=1}^{N} dz_k \wedge d\bar{z}_k . \tag{7.3}$$

In the Hilbert space $\mathcal{H}$ we fix Fock basis

$$\Big\{|n_1, \ldots, n_N\rangle\Big\}_{n_1,\ldots,n_N \in \mathbb{N}\cup\{0\}} \tag{7.4}$$

and define the complex analytic map $K_\hbar : \mathbb{C}^N \longrightarrow \mathcal{H}$ by

$$K_\hbar(z_1, \ldots, z_N) := \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\frac{1}{\hbar}\right)^{\frac{n_1+\ldots+n_N}{2}} \frac{z_1^{n_1} \cdot \ldots \cdot z_N^{n_N}}{\sqrt{n_1! \cdot \ldots \cdot n_N!}} |n_1 \ldots n_N\rangle , \tag{7.5}$$

where $\hbar$ is some positive parameter interpreted as the Planck constant. Since

$$\langle K_\hbar(z_1, \ldots, z_N) | K_\hbar(z_1, \ldots, z_N)\rangle = \exp\left(\hbar^{-1} \sum_{k=1}^{N} |z_k|^2\right) < +\infty , \tag{7.6}$$

the map K is well defined on $\mathbb{C}^N$ and $K_\hbar(z_1,\dots,z_N)\neq 0$. Thus one can define $\mathcal{K}_\hbar:\mathbb{C}^N\longrightarrow\mathbb{CP}(\mathcal{H})$ by

$$\mathcal{K}_\hbar(z):=[K_\hbar(z)]\ , \tag{7.7}$$

where $[K_\hbar(z)]=\mathbb{C}K_\hbar(z)$ and $z=(z_1,\dots,z_N)$. Simple computation shows that

$$\mathcal{K}_\hbar^*\omega_{FS}=-\frac{i}{2}\partial\bar\partial\log\langle K_\hbar(z)|K_\hbar(z)\rangle=\frac{1}{2i}\hbar^{-1}\partial\bar\partial\left(\sum_{k=1}^{N}z_k\bar z_k\right)=\omega_\hbar\ , \tag{7.8}$$

i.e. $\mathcal{K}_\hbar$ is complex analytic immersion intertwining Kähler structure of $\mathbb{CP}(\mathcal{H})$ and $\mathbb{C}^N$. Taking derivatives of $\mathcal{K}_\hbar(z)$ in the point $z=0$ one reconstructs the Fock basis of $\mathcal{H}$. Thus we conclude that vectors $K_\hbar(z)$, where $z\in\mathbb{C}^N$, form linearly dense subset of $\mathcal{H}$. Summing up we see that $\mathcal{K}_\hbar:\mathbb{C}^N\longrightarrow\mathbb{CP}(\mathcal{H})$, given by (7.7), is the coherent state map.

Let us consider the operators $A_1,\dots,A_N$ defined by

$$A_kK_\hbar(z)=z_kK_\hbar(z)\ , \tag{7.9}$$

i.e. we assume that the coherent states $K_\hbar(z)$, $z\in\mathbb{C}^N$, are the eigenstates of A_k with eigenvalues equal to the kth coordinate z_k of z.

One can check by the direct computation that

$$A_k|n_1\dots n_k\dots n_N\rangle=\sqrt{\hbar}\sqrt{n_k}|n_1\dots n_k-1\dots n_N\rangle \tag{7.10}$$

for $n_k\geqslant 1$ and $A_k|n_1\dots n_k\dots n_N\rangle=0$ for $n_k=0$. It follows from (7.10) that A_k is an unbounded operator with dense domain given by finite linear combinations of elements of the Fock basis.

Operator A_k^* conjugated to A_k acts on the elements of Fock basis by

$$A_k^*|n_1\dots n_k\dots n_N\rangle=\sqrt{\hbar}\sqrt{n_k+1}|n_1\dots n_k+1\dots n_N\rangle. \tag{7.11}$$

From (7.10) and (7.11) one obtain the Heisenberg canonical commutation relations

$$[A_k,A_l^*]=A_kA_l^*-A_l^*A_k=\hbar\delta_{kl}\mathbb{I} \tag{7.12}$$
$$[A_k,A_l]=[A_k^*,A_l^*]=0$$

for annihilation A_k and creation A_l^* operators. Taking self-adjoint operators

$$Q_k=\frac{1}{2}(A_k+A_k^*) \tag{7.13}$$

$$P_k=\frac{1}{2i}(A_k-A_k^*)\ ,$$

which have the physical interpretation of position an momentum operators one obtains the more familiar form of the Heisenberg commutation relations

$$[Q_k,P_l]=\frac{\hbar}{2}i\delta_{kl}\mathbb{I}\ . \tag{7.14}$$

The mean values of Q_l and P_l in the coherent states $[K_\hbar(z)]$ are given by

$$\langle Q_l \rangle = \frac{\langle K_\hbar(z)|Q_l|K_\hbar(z)\rangle}{\langle K_\hbar(z)|K_\hbar(z)\rangle} = q_l \tag{7.15}$$

$$\langle P_l \rangle = \frac{\langle K_\hbar(z)|P_l|K_\hbar(z)\rangle}{\langle K_\hbar(z)|K_\hbar(z)\rangle} = p_l$$

and their dispersions minimalize Heisenberg uncertainty inequalities, i.e.

$$\Delta Q_l \Delta P_l = \frac{\hbar}{2} \, . \tag{7.16}$$

In conclusion let us remark that the above facts show that coherent states given by (7.7) are pure quantum states with the properties which qualify them to be ones the moat similar to the classical pure states. After long period since 1926, when paper of Schrödinger appeared, it was Glauber who discovered, see [13], that the Gauss coherent states $K_\hbar(z)$ have fundamental meaning for the quantum optical phenomena.

Afterwards we will come back to the Gaussian coherent state map. It will play in our considerations the role similar to the role of Euclidean geometry in Riemannian geometry.

$\diamondsuit$

The notion of the mechanical system given by Definition 7.2 is too restrictive from the point of view of the measurement procedures. For this reason let us modify this definition as follows.

Definition 7.3. The **physical system** will be a triple: $M, \mathcal{H}, \mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{H})$, where

i) M is a smooth Banach manifold;

ii) $\mathcal{H}$ is a complex separable Hilbert space;

iii) $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ is a smooth map.

Let us remark here that, since 7.3 one neglects the symplectic structure, in the class of physical systems the mechanical systems form a subclass.

Now we come back to the statistical interpretation of quantum mechanics and discuss the metric structure of $\mathbb{CP}(\mathcal{H})$ in this context. To this end, let us fix two pure states

$$\iota([\psi]) = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \quad \& \quad \iota([\varphi]) = \frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle} \, , \tag{7.17}$$

where $\varphi, \psi \in \mathcal{H}$. Since $U^1(\mathcal{H}) \subset iU^\infty(\mathcal{H}) \subset \mathcal{O}(\mathcal{H})$ one can consider, for example the state $\iota([\varphi])$ as an observable. Thus, according to standard statistical model of quantum mechanics, one fixes that the probability of finding the system in the state $\iota([\varphi])$, when one knows that it is in the state $\iota([\psi])$, is given by

$$\mathrm{Tr}(\iota([\psi])\iota([\varphi])) = \frac{|\langle\psi|\varphi\rangle|^2}{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle} \ . \tag{7.18}$$

The complex valued quantity

$$a([\psi],[\varphi]) := \frac{\langle\psi|\varphi\rangle}{\sqrt{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}} \ , \tag{7.19}$$

called the **transition amplitude** between the pure states $\iota([\psi])$ and $\iota([\varphi])$, plays the fundamental role in quantum mechanical considerations, see for example R. Feynman book [12]. The following formula

$$\|\iota([\psi]) - \iota([\varphi])\|_1 = 2(1 - |a([\psi],[\varphi])|^2)^{\frac{1}{2}} \tag{7.20}$$

explains the relation between $\|\cdot\|_1$-distance and transition probability $|a([\psi],[\varphi])|^2$. One sees from (7.20) that transition probability from $\iota([\psi])$ to $\iota([\varphi])$ is nearly one if theses states are close in sense of $\|\cdot\|_1$-metric. The sequence of states $\{\iota([\psi_n])\}_{n=0}^{\infty}$ of the physical system is a Cauchy sequence if starting from some state $\iota([\psi_{\mathcal{N}}])$ the probability $|a([\psi],[\varphi])|^2$ of successive transitions $\iota([\psi]) \to \iota([\varphi])$ is arbitrarily close to one for $n > \mathcal{N}$.

The transition probability $|a([\psi],[\varphi])|^2$ is a quantity measurable in the direct way. So, it is natural to assume that the set $\mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$ of morphisms between $\mathbb{CP}(\mathcal{H}_1)$ and $\mathbb{CP}(\mathcal{H}_2))$ consists of the maps $\Sigma : \mathbb{CP}(\mathcal{H}_1) \longrightarrow \mathbb{CP}(\mathcal{H}_2))$ which preserve corresponding transition probabilities, i.e. $\Sigma \in \mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$ if

$$|a_2(\Sigma([\psi]), \Sigma([\varphi]))|^2 = |a_1([\psi],[\varphi])|^2 \tag{7.21}$$

for any $[\psi],[\varphi] \in \mathbb{CP}(\mathcal{H}_1)$, or equivalently the maps which preserve $\|\cdot\|_1$ metric.

For two physical systems $(M_1, \mathcal{H}_1, \mathcal{K}_1 : M_1 \to \mathbb{CP}(\mathcal{H}_1))$ and $(M_2, \mathcal{H}_2, \mathcal{K}_2 : M_2 \to \mathbb{CP}(\mathcal{H}_2))$ we shall define morphisms by the following commutative diagrams

$$\begin{array}{ccc} M_1 & \xrightarrow{\mathcal{K}_1} & \mathbb{CP}(\mathcal{H}_1) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ M_2 & \xrightarrow[\mathcal{K}_2]{} & \mathbb{CP}(\mathcal{H}_2) \end{array} \ , \tag{7.22}$$

where $\sigma \in C^\infty(M_1, M_2)$ and $\Sigma \in \mathrm{Mor}(\mathbb{CP}(\mathcal{H}_1), \mathbb{CP}(\mathcal{H}_2))$. The morphism Σ is univocally defined by σ. It is so by Wigner theorem [65] and the assumption that $\mathcal{K}(M)$ is linearly dense in $\mathcal{H}$.

Therefore physical systems form the category. We shall denote it by $\mathcal{P}$. Category of mechanical systems can be distinguished as the subcategory of $\mathcal{P}$ by the conditions that M_1 and M_2 are symplectic manifolds and maps $\mathcal{K}_1$, $\mathcal{K}_2$ and σ are symplectic maps.

We shall now present the coordinate description of the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$. In order to do this let us fix an atlas $\{\Omega_\alpha, \Phi_\alpha\}_{\alpha\in I}$, where Ω_i is the open domain of the chart $\Phi_\alpha : \Omega_\alpha \to \mathbb{R}^n$, with the property that for any $\alpha \in I$ there exists smooth map

$$K_\alpha : \Omega_\alpha \to \mathcal{H} \tag{7.23}$$

such that $K_\alpha(q) \neq 0$ for $q \in \Omega_\alpha$. One has consistency condition

$$K_\beta(q) = g_{\beta\gamma}(q) K_\gamma(q) \tag{7.24}$$

for $q \in \Omega_\beta \cap \Omega_\gamma$, where maps

$$g_{\beta\gamma} : \Omega_\beta \cap \Omega_\gamma \longrightarrow \mathbb{C} \setminus \{0\} \tag{7.25}$$

form smooth cocycle, i.e.

$$g_{\beta\gamma}(p) = g_{\beta\delta}(p) g_{\delta\gamma}(p) \tag{7.26}$$

for $p \in \Omega_\beta \cap \Omega_\gamma \cap \Omega_\delta$. The system of maps $\{\mathcal{K}_\alpha\}_{\alpha\in I}$ we shall call the trivialization of coherent state map $\mathcal{K}$ if one has

$$\mathcal{K}(q) = [K_\alpha(q)] = \mathbb{C} K_\alpha(q) \tag{7.27}$$

for $q \in \Omega_\alpha$.

Let us recall that tautological complex line bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{E} \\ & & \big\downarrow \pi \\ & & \mathbb{CP}(\mathcal{H}) \end{array}$$

over $\mathbb{CP}(\mathcal{H})$ is defined by

$$\mathbb{E} := \{(\psi, l) \in \mathcal{H} \times \mathbb{CP}(\mathcal{H}) : \psi \in l\} \tag{7.28}$$

and the bundle projection π is by definition the projection on the second component of the product $\mathcal{H} \times \mathbb{CP}(\mathcal{H})$. The bundle fibre $\pi^{-1}(l) =: \mathbb{E}_l$ is given by the complex line $l \subset \mathcal{H}$. With the use of projection $\mu : \mathbb{E} \to \mathcal{H}$ on the first factor of the product $\mathcal{H} \times \mathbb{CP}(\mathcal{H})$ we obtain the Hermitian kernel $K_\mathbb{E}(l, k) : \pi^{-1}(l) \times \pi^{-1}(k) \to \mathbb{C}$ given by

$$K_\mathbb{E}(l, k)(\xi, \eta) := \langle \mu(\xi) | \mu(\eta) \rangle, \tag{7.29}$$

where $\xi \in \pi^{-1}(l)$ and $\eta \in \pi^{-1}(k)$. It follows directly from definition that $K_\mathbb{E}$ is a smooth section of the bundle

$$\mathrm{pr}_1^* \overline{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \overline{\mathbb{E}}^* \longrightarrow \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H}), \tag{7.30}$$

where $\mathrm{pr}_1^* \overline{\mathbb{E}}^* \to \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H})$ is the pull back of the complex conjugate bundle dual to $\mathbb{E}$

$$\overline{\mathbb{E}} \longrightarrow \mathbb{CP}(\mathcal{H}) \tag{7.31}$$

given by the projection $\mathrm{pr}_1 : \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H}) \to \mathbb{CP}(\mathcal{H})$ on the first factor of the product and $\mathrm{pr}_2^* \mathbb{E}^* \to \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathcal{H})$ is the pull back by the projector on the second factor.

Therefore, the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ has canonically defined Hermitian kernel $K_{\mathbb{E}} \in \Gamma^\infty(\mathrm{pr}_1^* \overline{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \overline{\mathbb{E}}^*, \mathbb{CP}(\mathcal{H}) \times \mathbb{CP}(\mathfrak{M}))$.

Another remarkable property of $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ is that the map

$$I : \mathcal{H} \ni \psi \longrightarrow \langle \mu(\cdot)|\psi\rangle =: I(\psi) \in \Gamma(\mathbb{CP}(\mathcal{H}), \overline{\mathbb{E}}^*) \tag{7.32}$$

defines monomorphism of vector spaces. Its image $\mathcal{H}_{\mathbb{E}} := I(\mathcal{H}) \subset \Gamma(\mathbb{CP}(\mathfrak{M}), \overline{\mathbb{E}}^*)$ can be considered as a Hilbert space with the scalar product defined by

$$\langle I(\psi)|I(\varphi)\rangle_{\mathbb{E}} := \langle \psi|\varphi\rangle. \tag{7.33}$$

Then, after fixing the frame sections $S_\alpha : \Omega_\alpha \to \mathbb{E}$ one finds

$$I(\psi)(l) = \langle \mu(S_\alpha(l))|\psi\rangle \overline{S_\alpha^*}(l) =: \psi_\alpha(l)\overline{S_\alpha^*(l)}. \tag{7.34}$$

Here, $\{\Omega_\alpha\}_{\alpha\in I}$ stands for the covering of $\mathbb{CP}(\mathcal{H})$ by open subset Ω_α such that $\pi^{-1}(\Omega_\alpha) \cong \Omega_\alpha \times \mathbb{C}$. Due to Schwartz inequality one gets

$$|\psi_\alpha(l)| = |\langle \mu(S_\alpha(l))|\psi\rangle| \leqslant \|\mu(S_\alpha(l))\| \, \|\psi\| \tag{7.35}$$

which shows that the evaluation functional $e_{\alpha,l} : \mathcal{H}_{\mathbb{E}} \to \mathbb{C}$ defined by

$$e_{\alpha,l}(I(\psi)) := \psi_\alpha(l) \tag{7.36}$$

is continuous and smoothly depends on $l \in \Omega_\alpha$, what follows from the smoothness of the frame section $S_\alpha : \Omega_\alpha \to \mathbb{E}$.

Reassuming, we see that to the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ one has canonically related Hilbert space $\mathcal{H}_{\mathbb{E}} \subset \Gamma^\infty(\overline{\mathbb{E}^*}, \mathbb{CP}(\mathcal{H}))$ with continuous smoothly dependent on $l \in \Omega_\alpha$ evaluation functionals $e_{\alpha,l}$.

We will discuss later other properties of the bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ important for the theory investigated here.

Finally, let us mention that the coordinate representation of the Hermitian kernel $K_{\mathbb{E}}$ is given by

$$K_{\mathbb{E}} = \langle \mu(S_\alpha)|\mu(S_\beta)\rangle \, \mathrm{pr}_1^* \, \overline{S}_\alpha^* \overline{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \, S_\beta^*. \tag{7.37}$$

After passing to the unitary frame $u_\alpha : \Omega_\alpha \to \mathbb{C}$, given by

$$u_\alpha(l) := \|S_\alpha(l)\|^{-1} \, S_\alpha(l) \tag{7.38}$$

one obtains

$$K_{\mathbb{E}} = a_{\overline{\alpha}\beta}([\mu(u_\alpha)], [\mu(u_\beta)]) \, \mathrm{pr}_1^* \, u_\alpha^* \overline{\mathbb{E}}^* \otimes \mathrm{pr}_2^* \, \overline{u}_\beta^*, \tag{7.39}$$

where $a_{\overline{\alpha}\beta}([\mu(u_\alpha)], [\mu(u_\beta)])$ is transition amplitude between the states $[\mu(u_\alpha)] \in \Omega_\alpha$ and $[\mu(u_\beta)] \in \Omega_\beta$. Therefore, canonical Hermitian kernel $K_{\mathbb{E}}$ is the geometric realization of quantum mechanical transition amplitude.

8 Three realizations of the physical systems

In this section we shall present additionally to the standard representation the geometric and analytic representation of the category of physical systems $\mathcal{C}$ introduced in Section 7. We will show that all representations are equivalent. The geometric one is directly related to the straightforward construction of coherent state map in the experimental way. In order to describe it let us choose an atlas $(\Omega_\alpha, \varphi_\alpha)_{\alpha \in I}$ of the parametrizing manifold M consistent with the definition of the coherent state map given by (7.23). For two fixed points $q \in \Omega_\alpha$ and $p \in \Omega_\beta$ the transition amplitude $a_{\overline{\alpha}\beta}(q,p)$ from the coherent state $\iota([K_\alpha(q)])$ to the coherent state $\iota([K_\beta(p)])$ according to (7.19) is given by

$$a_{\overline{\alpha}\beta}(q,p) = \frac{\langle K_\alpha(q) | K_\beta(p) \rangle}{\|K_\alpha(q)\| \, \|K_\beta(p)\|}. \tag{8.1}$$

From (7.24) one has

$$\begin{aligned} a_{\overline{\alpha}\beta}(q,p) &= u_{\beta\gamma}(p) a_{\overline{\alpha}\gamma}(q,p) \\ a_{\overline{\beta}\alpha}(p,q) &= \overline{u_{\beta\gamma}}(p) a_{\overline{\gamma}\alpha}(p,q) \end{aligned} \tag{8.2}$$

for $p \in \Omega_\beta \cap \Omega_\gamma$, where $u_{\alpha\beta} : \Omega_\beta \cap \Omega_\gamma \to U(q)$ is the unitary cocycle defined by

$$u_{\beta\gamma} := |g_{\beta\gamma}|^{-1} g_{\beta\gamma} \tag{8.3}$$

Additionally to the transformation property (8.2) the transition amplitude satisfies two other

$$\overline{a_{\overline{\alpha}\beta}(q,p)} = a_{\overline{\beta}\alpha}(p,q) \tag{8.4}$$

$$a_{\overline{\alpha}\alpha}(q,q) = 1 \tag{8.5}$$

for $q \in \Omega_\alpha$ and

$$\det \begin{pmatrix} a_{\overline{\alpha}_1\alpha_1}(q_1,q_1) & \dots & a_{\overline{\alpha}_1\alpha_N}(q_1,q_N) \\ \vdots & & \vdots \\ a_{\overline{\alpha}_N\alpha_1}(q_N,q_1) & \dots & a_{\overline{\alpha}_N\alpha_N}(q_N,q_N) \end{pmatrix} \geqslant 0 \tag{8.6}$$

for all $N \in \mathbb{N}$ and $q_1 \in \Omega_{\alpha_1}, \dots, q_N \in \Omega_{\alpha_N}$.

The transition amplitude $\{a_{\overline{\alpha}\beta}(q,p)\}$ is the quantity which can be directly obtained by the measurement procedure. Let us recall for this reason that $|a_{\overline{\alpha}\beta}(q,p)|^2$ is the transition probability and phase $|a_{\overline{\alpha}\beta}(q,p)|^{-1} a_{\overline{\alpha}\beta}(q,p)$ is responsible for the quantum interference effects.

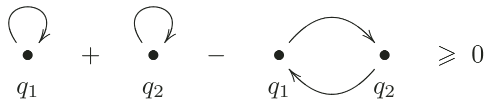
The property (8.5) means that transition amplitude for the process

$= 1$

is equal to 1. We shall illustate the physical meaning of the property (8.6) considering it for $N = 2$ and $N = 3$. In the case $N = 2$ one has inequality

$$\det \begin{pmatrix} 1 & a_{\overline{\alpha}_1\alpha_2}(q_1,q_2) \\ a_{\overline{\alpha}_2\alpha_1}(q_2,q_1) & 1 \end{pmatrix} = 1 - |a_{\overline{\alpha}_1\alpha_2}(q_1,q_2)|^2 \geqslant 0, \tag{8.7}$$

which states that transition probability between two coherent states is not greater than 1. One can express (8.7) graphically in the following way

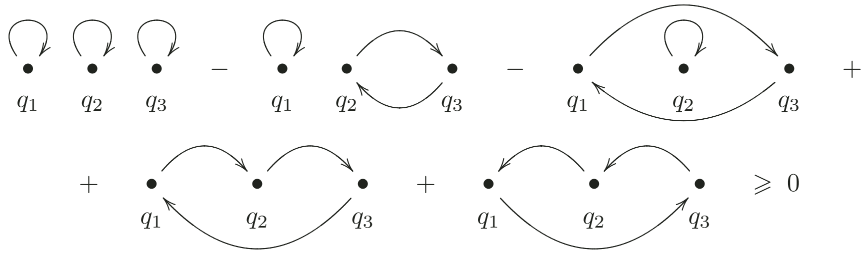


For the case $N = 3$ one obtains

$$1 - |a_{\overline{\alpha}_2\alpha_3}(q_2, q_3)|^2 - |a_{\overline{\alpha}_3\alpha_1}(q_3, q_1)|^2 - |a_{\overline{\alpha}_1\alpha_2}(q_1, q_2)|^2 + \tag{8.8}$$

$$+a_{\overline{\alpha}_2\alpha_1}(q_2, q_1)a_{\overline{\alpha}_1\alpha_3}(q_1, q_3)a_{\overline{\alpha}_3\alpha_2}(q_3, q_2) + a_{\overline{\alpha}_2\alpha_3}(q_2, q_3)a_{\overline{\alpha}_3\alpha_1}(q_3, q_1)a_{\overline{\alpha}_1\alpha_2}(q_1, q_2) \geqslant 0,$$

which corresponds to the positive probability of the following alternating sum of the virtual processes



The property (8.4) says that the transition amplitudes of the processes

are complex conjugated.

In order to understand the geometric sens of the transition amplitude we shall introduce the notion of the positive Herimitian kernel. Therefore let us consider the complex line bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{L} \\ & & \downarrow \\ & & M \end{array}$$

over manifold M with the fixed local trivialization

$$S_\alpha : \Omega_\alpha \longrightarrow \mathbb{L} \tag{8.9}$$

$$g_{\alpha\beta} : \Omega_\alpha \cap \Omega_\beta \longrightarrow \mathbb{C} \setminus \{0\},$$

i.e. $S_\alpha(m) \neq 0$ for $m \in \Omega_\alpha$ and

$$S_\alpha = g_{\alpha\beta} S_\beta \quad \text{on } \Omega_\alpha \cap \Omega_\beta \tag{8.10}$$

and

$$g_{\alpha_\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma, \tag{8.11}$$

where $(\Omega_\alpha, \varphi_\alpha)_{\alpha\in I})$ forms an atlas of M.

Using the projections

$$\begin{array}{ccc} & M \times M & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ M & & M \end{array}$$

on the first and second components of the product $M \times M$ one can define the line bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathrm{pr}_1^* \overline{\mathbb{L}^*} \otimes \mathrm{pr}_2^* \mathbb{L}^* \\ & & \downarrow \\ & & M \times M \end{array} \tag{8.12}$$

with the local trivialization defined by the tensor product

$$\mathrm{pr}_1^* \overline{S}^*_\alpha \otimes \mathrm{pr}_2^* \overline{S}^*_\beta : \Omega_\alpha \times \Omega_\beta \to \mathrm{pr}_1^* \overline{\mathbb{L}^*} \otimes \mathrm{pr}_2^* \mathbb{L}^* \tag{8.13}$$

of the pullbacks of the local frames given by (8.9).

Let us explain here that $\mathbb{L}^*$ is dual to $\mathbb{L}$ and $\overline{\mathbb{L}^*}$ is complex conjugation of $\mathbb{L}^*$. The line bundle (8.12) by definition is the tensor product of the pullbacks $\mathrm{pr}_1^* \overline{\mathbb{L}^*}$ and $\mathrm{pr}_2^* \mathbb{L}^*$ of $\overline{\mathbb{L}^*}$ and $\mathbb{L}^*$ respectively.

Definition 8.1. The section $K_\mathbb{L} \in C^\infty(M \times M, \mathrm{pr}_1^* \overline{\mathbb{L}^*} \otimes \mathrm{pr}_2^* \overline{\mathbb{L}^*})$ we shall call the positive Hermitian kernel iff

$$\begin{aligned} \overline{K_{\overline{\alpha}\beta(q,p)}} &= K_{\overline{\beta}\alpha}(p,q), \\ K_{\overline{\alpha}\alpha}(q,q) &> 0, \\ \sum_{k,j=1}^{N} K_{\overline{\alpha_j}\alpha_k}(q_j, q_k)\overline{v^j} v^k &\geqslant 0 \end{aligned} \tag{8.14}$$

for any $q \in \Omega_\alpha$, $p \in \Omega_\beta$, $q_k \in \Omega_{\alpha_k}$, $v^1, \ldots, v^N \in \mathbb{C}$ and any set of indices $\alpha, \beta, \alpha_1, \ldots, \alpha_N$ resulting from covering M by open sets Ω_α, $\alpha \in I$, where

$$K_{\overline{\alpha}\beta} : \Omega_\alpha \times \Omega_\beta \longrightarrow \mathbb{C} \tag{8.15}$$

are the coordinate functions of $K_\mathbb{L}$ defined by

$$K_\mathbb{L} = K_{\overline{\alpha}\beta}(q,p)\, \mathrm{pr}_1^* \overline{S}^*_\alpha(q) \otimes \mathrm{pr}_2^* \overline{S}^*_\beta(p) \tag{8.16}$$

on $\Omega_\alpha \times \Omega_\beta$.

It follows immediately from the transformation rule

$$K_{\overline{\alpha}\beta}(q,p) = \overline{g_{\alpha\gamma}(q)}g_{\beta\delta}(p)K_{\overline{\gamma}\delta}(q,p) \tag{8.17}$$

for $q \in \Omega_\alpha \cap \Omega_\gamma$ and $p \in \Omega_\beta \cap \Omega_\delta$ that the conditions (8.14) are independent with respect ot the choice of frame.

The relation of $K_{\mathbb{L}}$ to the transition amplitude on M is recognized by noticing that

$$\frac{K_{\overline{\alpha}\beta}(q,p)}{K_{\overline{\alpha}\alpha}(q,q)^{\frac{1}{2}}K_{\overline{\beta}\beta}(p,p)^{\frac{1}{2}}} \tag{8.18}$$

fulfills properties (8.2)-(8.6) of $\{a_{\overline{\alpha}\beta(q,p)}\}$ defined by (8.1).

The line bundles with the distinguished positive Hermitian kernel $(\mathbb{L} \to M, K_{\mathbb{L}})$ form the category $\mathfrak{K}$ for which the morphisms set $\mathrm{Mor}[(\mathbb{L}_1 \to M_1, K_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, K_{\mathbb{L}_2})]$ is given by $f : M_2 \to M_1$ such that

$$\mathbb{L}_2 = f^*\mathbb{L}_1 = \{(m,\xi) \in M_2 \times \mathbb{L}_1 : f(m) = \pi_1(\xi)\} \tag{8.19}$$

and

$$K_{\mathbb{L}_2} = f^*K_{\mathbb{L}_1} = K^1_{\overline{\alpha}\beta}(f(q), f(p))\, \mathrm{pr}_1^* \,\overline{S^*}^1_\alpha(f(q)) \otimes \mathrm{pr}_2^* \,\overline{S^*}^1_\beta(f(p)) \tag{8.20}$$

i.e.

$$K^2_{\overline{\alpha}\beta}(q,p) = K^1_{\overline{\alpha}\beta}(f(q), f(p)) \tag{8.21}$$

for $q \in f^{-1}(\Omega_\alpha)$ and $p \in f^{-1}(\Omega_\beta)$.

The above expresses the covariant character of the transition amplitude and its independence of choice of the coordinates.

There is a covariant functor

$$\mathcal{F}_{\mathcal{KP}} : \mathcal{O}b(\mathcal{P}) \longrightarrow \mathcal{O}b(\mathcal{K}) \tag{8.22}$$

between the category of physical systems $\mathcal{P}$ and the category of the positive Hermitian kernels, naturally defined by

$$\mathbb{L} = \mathfrak{K}^*\mathbb{E} = \{(m,\xi) \in M \times \mathbb{E} : \mathfrak{K}(m) = \pi(\xi)\} \tag{8.23}$$

and by

$$K_{\overline{\alpha}\beta}(q,p) = \langle K_\alpha(q)|K_\beta(p)\rangle, \tag{8.24}$$

where $K_\alpha : \Omega_\alpha \to \mathbb{C} \setminus \{0\}$ is given by (7.23)-(7.27). The functor $\mathcal{F}_{\mathfrak{K}\mathcal{P}}$ maps $(\sigma, \Sigma) \in \mathrm{Mor}(M_1 \xrightarrow{\mathfrak{K}_1} \mathbb{CP}(\mathcal{H}_1), M_2 \xrightarrow{\mathfrak{K}_2} \mathbb{CP}(\mathcal{H}_2))$ on $f \in \mathrm{Mor}[(\mathbb{L}_1 \to M_1, K_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, K_{\mathbb{L}_2})]$ by $f = \sigma$.

The positive Hermitian kernel $K_{\mathbb{L}}$ canonically defines the complex separable Hilbert space $\mathcal{H}_{\mathbb{L}}$ realized as a vector subspace of the space $\Gamma(M, \overline{\mathbb{L}^*})$ of the sections of the bundle $\overline{\mathbb{L}}^* \to M$. One obtains $\mathcal{H}_{\mathbb{L}}$ in the following way. Let us take the vector space $V_{K,\mathbb{L}}$ of finite linear combinations

$$v = \sum_{i=1}^{N} v_i K_{\beta_i}(q_i) = \sum_{i=1}^{N} v_i K_{\overline{\alpha}\beta_i}(p, q_i)\overline{S^*_\alpha}(p) \tag{8.25}$$

of sections

$$K_{\beta_i}(q_i) = K_{\overline{\alpha}\beta_i}(p, q_i)\overline{S^*_\alpha}(p) \in \Gamma(M, \overline{\mathbb{L}^*}), \tag{8.26}$$

where $q \in \Omega_\beta$ with the scalar product defined by

$$\langle v|w\rangle := \sum_{i,j=1}^{N} \overline{v_i} w_j K_{\overline{\beta}_i \beta_j}(q_i, q_j). \tag{8.27}$$

It follows from the properties (8.14) that the pairing (8.26) is $1\frac{1}{2}$-linear and that

$$\left|\sum_{i=1}^{N} \overline{v_i} K_{\beta_i \beta}(q_i, p)\right|^2 = |\langle v|K_\beta(p)\rangle|^2 \leqslant \langle v|v\rangle K_{\overline{\beta}\beta}(p, p), \tag{8.28}$$

from which one has $v = 0$ iff $\langle v|v\rangle = 0$. Therefore (8.26) defines positive scalar product on $V_{K,\mathbb{L}}$.

Proposition 8.2. *The unitary space $V_{K,\mathbb{L}}$ extends in the canonical and unique way to the Hilbert space $\mathcal{H}_{K,\mathbb{L}}$, which is a vector subspace of* $\Gamma(M, \overline{L^*})$.

Proof. Let $\{v_n\}$ be the Cauchy sequence in $V_{K,\mathbb{L}}$. Then

$$v_n = v_{n\alpha}(p)\overline{S^*_\alpha}(p), \tag{8.29}$$

where

$$v_{n\alpha}(p) = \langle K_\alpha(p)|v_n\rangle. \tag{8.30}$$

From Schwartz inequality (8.28) one obtains that $\{v_{n\alpha}(p)\}$ is also Cauchy sequence. So one can define by

$$v(p) = \lim_{n\to\infty} v_{n\alpha}(p)\overline{S^*_\alpha}(p) = v_\alpha(p)\overline{S^*_\alpha}(p) \tag{8.31}$$

the section $v \in \Gamma(M, \overline{\mathbb{L}^*})$. Let $\overline{V_{K,\mathbb{L}}}$ be the abstract completion of $V_{K,\mathbb{L}}$. Using (8.29) one defines one-to-one linear map of $\overline{V_{K,\mathbb{L}}}$ into $\Gamma(M, \overline{\mathbb{L}^*})$ by

$$I([\{v_n\}])(p) := v(p), \tag{8.32}$$

where $[\{v_n\}]$ is equivalence class of Cauchy sequences. Let us now define Hilbert space $\mathcal{H}_{K,\mathbb{L}}$ as $I(\overline{V_{K,\mathbb{L}}})$ with the scalar product given by

$$\langle t|s\rangle := \langle I^{-1}(t)|I^{-1}(s)\rangle \quad \text{for } s, t \in \mathcal{H}_{K,\mathbb{L}}. \tag{8.33}$$

The Hilbert space $H_{K,\mathbb{L}}$ is realized by sections of $\overline{\mathbb{L}^*}$ and extends uniquely the unitary space $V_{K,\mathbb{L}}$. □

Obviously for $v \in \mathcal{H}_{K,\mathbb{L}}$ one has

$$v = v_\alpha(p)\overline{S^*_\alpha} = \langle K_\alpha(p)|v\rangle\overline{S^*_\alpha}, \tag{8.34}$$

which shows that the evaluation functional $e_\alpha(p) : \mathcal{H}_{K,\mathbb{L}} \to \mathbb{C}$ defined by

$$e_\alpha(p)(v) := v_\alpha(p) \tag{8.35}$$

is a continuous linear functional and $e_\alpha(p)$ depends smoothly on $p \in \Omega_\alpha$. Hence, we see that Hilbert space $\mathcal{H}_{K,\mathbb{L}} \subset \Gamma(M, \overline{\mathbb{L}^*})$ possesses the property that evaluation functionals $e_\alpha(p) : \mathcal{H}_{K,\mathbb{L}} \to \mathbb{C}$ are continuous and define smooth maps

$$e_\alpha : \Omega_\alpha \to \mathcal{H}^*_{K,\mathbb{L}} \setminus \{0\} \tag{8.36}$$

for $\alpha \in I$. Since $e_\alpha(p)(K_\alpha(p)) = K_{\overline{\alpha}\alpha}(p,p) > 0$ e_α does not assume zero value in $\mathcal{H}_{K,\mathbb{L}}$.

Motivated by the preceding construction let us introduce the category $\mathfrak{H}$ of line bundles $\mathbb{L} \to M$ with distinguished Hilbert space $H_\mathbb{L}$ which is realized as a vector subspace of $\Gamma(M, \overline{L^*})$ and has the property that evaluation functionals $e_\alpha(p)$ are continuous, i.e. $\|e_\alpha(p)(v)\| \leqslant M_{\alpha,p} \|v\|$ for $v \in \mathcal{H}_\mathbb{L}$, $M_{\alpha,p} > 0$ and define smooth maps $e_\alpha : \Omega_\alpha \to \mathcal{H}^*_\mathbb{L} \setminus \{0\}$.

By the definition the morphisms set

$$\mathrm{Mor}[(\mathbb{L}_1 \to M_1, \mathcal{H}_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, \mathcal{H}_{\mathbb{L}_2})] \tag{8.37}$$

will consist of maps $f : M_2 \to M_1$ which satisfy $f^*\mathbb{L}_1 = \mathbb{L}_2$ and $f^*\mathcal{H}_{\mathbb{L}_1} = \mathcal{H}_{\mathbb{L}_2}$. In order to prove the correctness of the definition let us show that the vector space

$$f^*\mathcal{H}_{\mathbb{L}_1} = \{f^*v | v \in \mathcal{H}_{\mathbb{L}_1}\} \tag{8.38}$$

of the inverse image sections has a canonically defined Hilbert space structure with continuous evaluation functionals smoothly dependent on the argument.

It is easy to see that

$$\ker f^* = \{v \in \mathcal{H}_{\mathbb{L}_1} | f^*v = 0\} \tag{8.39}$$

is the Hilbert subspace of $\mathcal{H}_{\mathbb{L}_1}$. We define the Hilbert space structure on $f^*\mathcal{H}_{\mathbb{L}_1}$ by the vector spaces identifications

$$f^*\mathcal{H}_{\mathbb{L}_1} \cong \mathcal{H}_{\mathbb{L}_1} / \ker f^* \cong (\ker f^*)^\perp \tag{8.40}$$

i.e. $f^*\mathcal{H}_{\mathbb{L}_1}$ inherits the Hilbert space structure from the Hilbert subspace $(\ker f^*)^\perp$. In order to prove the property (8.36) for $f^*e_\alpha(p) = e_\alpha(f(p))$ we notice that

$$|(f^*v)_\alpha(p)| = |v_\alpha(f(p))| \leqslant M_{\alpha,f(p)}(\|\psi^0\| + \|\psi^\perp\|) \tag{8.41}$$

for $p \in f^{-1}(\Omega_\alpha)$. Because of $\psi^0_\alpha(f(p)) = 0$, the left hand side of the inequality (8.36) does not depend on $\psi^0 \in \ker f^*$. This results in

$$|(f^*(v)_\alpha(p))| \leqslant M_{\alpha,f(p)} \min_{\psi^0 \in \ker f^*} (\|\psi^0\| + \|\psi^\perp\|) = M_{\alpha,f(p)} \|\psi^\perp\| = M_{\alpha,f(p)} \|f^*\psi\| . \tag{8.42}$$

The above proves the continuity of the evaluation functionals f^*e_α. The smooth dependence of $f^*e_\alpha(p) = e_\alpha(f(p))$ on p follows from the smoothness of f.

In such a way the category $\mathfrak{h}$ is defined correctly.

Proposition 8.3. *Let $f : M_2 \to M_1$ be such that $f^*\mathbb{L}_1 = \mathbb{L}_2$ and $f^*K_1 = K_2$ then $f^*\mathcal{H}_{\mathbb{L}_1,K_1} = \mathcal{H}_{\mathbb{L}_2,K_2}$.*

Proof. The equality $f^*K_1 = K_2$ means $K_{1\alpha}(f(p)) = K_{2\alpha}(p)$ for $p \in f^{-1}(\Omega_{1\alpha})$ and $S_{2\alpha} = f^*S_{1\alpha} : f^{-1}(\Omega_{1\alpha}) \to \mathbb{L}_2$. Because $K_{1\alpha}(f(p))$ are linearly dense if $f^*\mathcal{H}_{K_1,\mathbb{L}_1}$ and $K_{2\alpha}(p)$ are linearly dense in $\mathcal{H}_{K_2,\mathbb{L}_2}$, this shows that $f^*\mathcal{H}_{\mathbb{L}_1,K_1} = \mathcal{H}_{\mathbb{L}_2,K_2}$. □

Now we conclude from Proposition 8.2 that there is canonically defined covariant functor $\mathcal{F}_{\mathfrak{h},\mathfrak{K}} : \mathfrak{K} \to \mathfrak{h}$:

$$\mathcal{F}_{\mathfrak{h}\mathfrak{K}}(\mathbb{L} \to M, K) = (\mathbb{L} \to M, \mathcal{H}_{\mathbb{L},K}) \tag{8.43}$$

from the category $\mathfrak{K}$ of line bundles with distinguished positive Hermitian kernel to the category $\mathfrak{h}$ of $K_{\mathbb{L}} \in \Gamma(M \times M, \mathrm{pr}_1^*\overline{\mathbb{L}^*} \otimes \mathrm{pr}_2^*\mathbb{L}^*)$ line bundles with distinguished Hilbert space $\mathcal{H}_{\mathbb{L}} \subset \Gamma^\infty(M, \overline{\mathbb{L}^*})$ with some additional conditions on the evaluation functionals.

Now let us discuss the relation between the category $\mathfrak{h}$ and the category of physical systems. Let $(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})$ be an object of the category $\mathfrak{h}$. Taking the smooth maps

$$K_\alpha : \Omega_\alpha \to \mathcal{H}_{\mathbb{L}} \setminus \{0\}, \tag{8.44}$$

which represent the evaluation functional maps $e_\alpha : \Omega_\alpha \to \mathcal{H}_{\mathbb{L}} \setminus \{0\}$

$$e_\alpha(p) = \langle K_\alpha(p)|\cdot\rangle \tag{8.45}$$

on Ω_α, we construct the smooth map $K^{\mathbb{L}} : M \to \mathbb{CP}(\mathcal{H}_{\mathbb{L}})$ given by

$$K^{\mathbb{L}}(q) := [K_\alpha^{\mathbb{L}}(q)]. \tag{8.46}$$

Because of $K_\alpha(q) = g_{\alpha\beta}(q)K_\beta(q)$ the definition (8.46) of $K^{\mathbb{L}}$ is independent on the choice of frame. The smoothness of $K^{\mathbb{L}}$ is ensured by the one of $e_\alpha : \Omega_\alpha \to \mathcal{H}_{\mathbb{L}}^*$.

Proposition 8.4. *The correspondence*

$$\mathcal{F}_{\mathcal{P}\mathfrak{h}}[(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})] := (M, \mathcal{H}_{\mathbb{L}}, \mathfrak{K}^{\mathbb{L}} : M \to \mathbb{CP}(\mathcal{H}^*)) \tag{8.47}$$

$$\mathcal{F}_{\mathcal{P}\mathfrak{h}}(f^*) := (f[\varphi_f]),$$

where $(\mathbb{L} \to \Gamma, \mathcal{H}_{\mathbb{L}}) \in \mathcal{O}b(\mathfrak{h})$, $f^* \in \mathrm{Mor}[(\mathbb{L}_1 \to M_1, \mathcal{H}_{\mathbb{L}_1}), (\mathbb{L}_2 \to M_2, \mathcal{H}_{\mathbb{L}_2})]$ and $\varphi_f : \mathcal{H}_{\mathbb{L}_2} \to \mathcal{H}_{\mathbb{L}_1}$ is the monomorphism given by

$$K_{1\alpha}(f(p)) = K_{1\alpha}^{\perp}(f(p)) =: \varphi(K_{2\alpha}(p)), \tag{8.48}$$

defines the contravariant functor

$$\mathcal{F}_{\mathcal{P}\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathcal{P}. \tag{8.49}$$

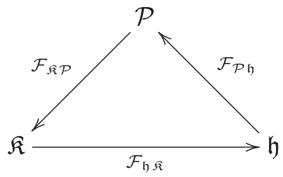
Proof. It is enough to mention that relation (8.48) implies the commutativity of the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\mathcal{K}_1^{\mathbb{L}_1}} & \mathbb{CP}(\mathcal{H}^{\mathbb{L}_1}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [\varphi_f]} \\ M_2 & \xrightarrow[\mathcal{K}_2^{\mathbb{L}_2}]{} & \mathbb{CP}(\mathcal{H}^{\mathbb{L}_2}) \end{array}, \tag{8.50}$$

□

Summing up statements discussed in above one has

Proposition 8.5. *The categories $\mathfrak{K}$, $\mathfrak{h}$ and $\mathcal{P}$ satisfy the following relation*



(8.51)

i.e. functors defined by $\mathcal{F}_{\mathcal{P}\mathfrak{h}} \circ \mathcal{F}_{\mathfrak{h}\mathfrak{K}} =: \mathcal{F}_{\mathcal{P}\mathfrak{K}}$, $\mathcal{F}_{\mathfrak{K}\mathcal{P}} \circ \mathcal{F}_{\mathcal{P}\mathfrak{h}} =: \mathcal{F}_{\mathfrak{K}\mathfrak{h}}$ *and* $\mathcal{F}_{\mathfrak{h}\mathfrak{K}} \circ \mathcal{F}_{\mathfrak{K}\mathcal{P}} =: \mathcal{F}_{\mathfrak{h}\mathcal{P}}$ *are inverse to* $\mathcal{F}_{\mathfrak{K}\mathcal{P}}$, $\mathcal{F}_{\mathfrak{h}\mathfrak{K}}$ *and* $\mathcal{F}_{\mathcal{P}\mathfrak{h}}$ *respectively. Moreover, functors* $\mathcal{F}_{\mathfrak{K}\mathfrak{h}}$ *and* $\mathcal{F}_{\mathfrak{K}\mathcal{P}}$ *are given explicitly by*

$$\mathcal{F}_{\mathfrak{K}\mathfrak{h}}[(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}})] = (\mathbb{L} \to M, K = \langle K_\alpha | K_\beta \rangle \operatorname{pr}_1^* \overline{\mathbb{L}}^* \otimes \operatorname{pr}_2^* \mathbb{L}) \tag{8.52}$$

$$\mathcal{F}_{\mathfrak{K}\mathfrak{h}}(f^*) = f^*,$$

where $K_\alpha : \Omega_\alpha \to \mathcal{H}_m athbbL \setminus \{0\}$ *is given by* (8.45), *and*

$$\mathcal{F}_{\mathfrak{h}\mathcal{P}}[(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}))] = (\mathcal{K}^* \mathbb{E} \to M, \mathcal{K}^* \mathcal{H}_E) \tag{8.53}$$

$$\mathcal{F}_{\mathfrak{h}\mathcal{P}}(\sigma, \Sigma) = \sigma.$$

The following definition introduces an equivalence among the objects taken into consideration.

Definition 8.6.

i) The objects $(\mathbb{L} \to M, K_{\mathbb{L}}), (\mathbb{L}' \to M', K'_{\mathbb{L}'}) \in Ob(\mathfrak{K})$ are equivalent iff $M = M'$ and there exists a bundle isomorphism $\kappa : \mathbb{L} \to \mathbb{L}'$ such that $\kappa^* K'_{\mathbb{L}'} = K_{\mathbb{L}}$.

ii) The objects $(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}}), (\mathbb{L}' \to M', \mathcal{H}'_{\mathbb{L}'}) \in Ob(\mathfrak{h})$ are equivalent iff $M = M'$ and there is a bundle isomorphism $\kappa : \mathbb{L} \to \mathbb{L}'$ such that $\kappa^* H'_{\mathbb{L}'} = \mathcal{H}_{\mathbb{L}}$.

iii) The objects $(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(M)), (M', \mathcal{H}', \mathcal{K}' : M' \to \mathbb{CP}(M')) \in Ob(\mathcal{P})$ are equivalent iff $M = M'$ and there is automorphism $\Sigma : \mathbb{CP}(\mathcal{H}) \to \mathbb{CP}(\mathcal{H}')$ such that $\mathcal{K}' = \Sigma \circ \mathcal{K}$.

These equivalences are presented by morphisms of between all three categories. This allows us to define the categories $\tilde{\mathfrak{h}}$, $\tilde{\mathfrak{K}}$ and $\tilde{\mathcal{P}}$ whose object classes consists described above equivalence classes and morphisms are canonically generated by morphisms of the categories $\mathfrak{h}$, $\mathfrak{K}$ and $\mathcal{P}$ respectively.

The main result of considerations given above, which shows that are three independent ways of presentation of the physical systems, is expressed below.

Theorem 8.7. *The categories $\tilde{\mathfrak{h}}$, $\tilde{\mathfrak{K}}$ and $\tilde{\mathcal{P}}$ are isomorphic.*

Proof. It follows by a straightforward verification. Let us recall only that two categories $\mathcal{X}$ and $\mathcal{Y}$ are said to be isomorphic if there exists a functor $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$ such that

i) for any object Y of $\mathcal{Y}$ there exists a unique object X of $\mathcal{X}$ such that $\mathcal{F}(X)=Y$;

ii) for any pair X_1, X_2 of objects of $\mathcal{X}$ the map which associates to each morphism $f^*: X_1\to X_2$ the morphism $\mathcal{F}(f^*):\mathcal{F}(X_1)\to\mathcal{F}(X_2)$ is a bijection of the sets of morphisms.

□

9 The Kostant-Souriau prequantization and positive hermitian kernels

We shall present indispensable for the investigated theory of the physical systems, elements of the geometric quantization in sense of B. Kostant [24] and J. M. Souriau [52]. It is based on the notion of the complex line bundle $\mathbb{L}\to M$ with the fixed Hermitian metric $H\in C^\infty(M,\overline{\mathbb{L}}^*\otimes\mathbb{L}^*)$ and metrical connection $\nabla: C^\infty(\Omega,\mathbb{L})\to C^\infty(\Omega,\mathbb{L}\otimes T^*M)$, i.e.

i)
$$\nabla(fs)=df\otimes s+f\nabla s \tag{9.1}$$

ii)
$$dH(s,t)=H(\nabla s,t)+H(s,\nabla t) \tag{9.2}$$

for any local smooth sections $s,t\in C^\infty(\Omega,\mathbb{L})$ and $f\in C^\infty(\Omega)$, where Ω is the open subset of M. Let $s_\alpha:\Omega_\alpha\to\mathbb{L}$, $\alpha\in I$ be a local trivialization of $\mathbb{L}\to M$, see (8.9). According to the property of (9.1) one gives ∇ and H univocally by defining them on the local frames

$$\nabla s_\alpha=k_\alpha\otimes s_\alpha \tag{9.3}$$

$$H(s_\alpha,s_\alpha)=H_{\overline{\alpha}\alpha}, \tag{9.4}$$

where $k_\alpha\in C^\infty(\Omega_\alpha,T^*M)$ and $0<H_{\overline{\alpha}\alpha}\in C^\infty(\Omega_\alpha)$ and assuming the transformation rules

$$k_\alpha(m)=k_\beta(m)+g_{\alpha\beta}^{-1}(m)dg_{\alpha\beta}(m) \tag{9.5}$$

$$H_{\overline{\alpha}\beta}(m) = |g_{\alpha\beta}(m)|^2 H_{\overline{\beta}\beta}(m) \tag{9.6}$$

for $m \in \Omega_\alpha \cap \Omega_\beta$, where cocycle $g_{\alpha\beta} : \Omega_\alpha \cap \Omega_\beta \to \mathbb{C}\setminus\{0\}$ is defined by $s_\alpha = g_{\alpha\beta} s_\beta$. Let us remark here since $\mathbb{L} \to M$ is complex line bundle, that the connection 1-form

$$k_\alpha(x) = k_{\alpha\mu}(x) dx^\mu, \tag{9.7}$$

where $(x^1, \ldots, x^n)$ are real coordinates on Ω_α, assume complex values, i.e.

$$k_{\alpha\mu} : \Omega_\alpha \longrightarrow \mathbb{C}. \tag{9.8}$$

The consistency condition (9.4) locally assumes the form

$$d \log H_{\overline{\alpha}\beta} = \overline{k}_\alpha + k_\alpha. \tag{9.9}$$

Thus and from the gauge transformation (9.5) one obtains that

$$\operatorname{curv} \nabla := dk_\alpha \qquad \text{on } \Omega_\alpha \tag{9.10}$$

is globally defined $i\mathbb{R}$-valued 2-form, i.e. curvature form for the hermitian connection defined on $U(1)$-principal bundle $U\mathbb{L} \to M$. By definition we will consider $U\mathbb{L} \to M$ as the subbundle of $\mathbb{L} \to M$ consisting of elements $\xi \in \pi^{-1}(m)$ of the norm $H(m)(\xi, \xi) = 1$.

If one assumes

$$g_{\alpha\beta} = e^{2\pi i c_{\alpha\beta}} \tag{9.11}$$

then

$$c_{\alpha\beta\gamma} := c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} \tag{9.12}$$

is $\mathbb{Z}$-valued cocycle on M related to the covering $\{\Omega_\alpha\}_{\alpha\in I}$ and defines the element $c_1(\mathbb{L}) \in H^2(M, \mathbb{Z})$ called the Chern class of the bundle $\mathbb{L} \to M$. $c_1(\mathbb{L})$ defines the last one up to bundle isomorphism, see for example [24]. Because of

$$2\pi i d c_{\alpha\beta} = k_\alpha - k_\beta \tag{9.13}$$

the real-valued form

$$\omega := \frac{1}{2\pi i} \operatorname{curv} \nabla \tag{9.14}$$

satisfies

$$[\omega] = c_1(\mathbb{L}) \in H^2(M, \mathbb{Z}). \tag{9.15}$$

Therefore it has integer cohomology class is. So (9.15) is the necessary condition the closed form $2\pi i\omega$ to be the curvature form of a Hermitian connection on the complex line bundle. It follows from Narisimhan and Ramanan paper [31] that this is also the sufficient condition. We will come back in the below to the question.

One has the identity

$$[\nabla_x, \nabla_y] - \nabla_{[x,y]} = 2\pi i \omega(x, y) \tag{9.16}$$

which can be proved by direct computation.

Now, let us assume that curvature 2-form is non-singular. Thus, since $d\omega = 0$, ω is symplectic form and one can define the Poisson bracket for $f, g \in C^\infty(M, \mathbb{R})$ as usually by

$$\{f, g\} = \omega(X_f, X_g) = -X_f(g), \tag{9.17}$$

where X_f is hamiltonian vector field defined by

$$\omega(X_f, \cdot) = -df. \tag{9.18}$$

That was idea of Souriau (and Kostant) to consider the differential operator $Q_f : C^\infty(M, \mathbb{L}) \to C^\infty(M, \mathbb{L})$ defined by

$$Q_f : \nabla_{X_f} + 2\pi i f \tag{9.19}$$

for $f \in C^\infty(M, \mathbb{R})$. It is easy to see from (9.1) and (9.16) that

$$Q_{\{f,g\}} = [Q_f, Q_g] \tag{9.20}$$

i.e. the map Q called **Kostant-Souriau prequantization** is a homomorphism of the Poisson-Lie algebra $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ into the Lie algebra of the first-order differential operators acting in the space $C^\infty(M, \mathbb{L})$ of the smooth sections of the line bundle $\mathbb{L} \to M$.

In this moment we are far from the quantization of the classical mechanical physical quantity $f \in C^\infty(M, \mathbb{R})$. Since this reasons it is necessary to construct the Hilbert space $\mathcal{H}_\mathbb{L}$ related to $C^\infty(M, \mathbb{L})$ in which the differential operator Q_f can be extended to self-adjoint operator $\overline{Q_f}$ being the quantum counterpart of f. An effort in this direction was done by using the notion of polarization, see for example [51, 67]. In the sequel we shall explain how one can obtain the polarization from the coherent state map, which as we will see is most physically fundamental object.

After this short review of the Kostant-Souriau geometric prequantization, we shall describe as it is related to our model of the mechanical (physical) system. In order of this let us fix the line bundle $\mathbb{L} \to M$ with distinguished positive Hermitian kernel $K_\mathbb{L}$, which as it was shown, equivalently describes the fixed physical system. We define the differential 2-forms $\omega_{1,2}$ and $\omega_{2,1}$ on the product $M \times M$ by

$$\omega_{12} = i d_1 d_2 \log K_{\overline{\alpha}_1 \alpha_2} \tag{9.21}$$

$$\omega_{22} = i d_2 d_1 \log K_{\overline{\alpha}_2 \alpha_1}, \tag{9.22}$$

where $K_{\overline{\alpha}_1 \alpha_2}$ are coordinates of $K_\mathbb{L}$ in the local frames

$$\mathrm{pr}_1^* \, \overline{s}^*_{\alpha_1} \otimes \mathrm{pr}_2^* \, s^*_{\alpha_2} : \Omega_{\alpha_1} \times \Omega_{\alpha_2} \to \mathrm{pr}_1^* \, \overline{\mathbb{L}}^* \otimes \mathrm{pr}_2 \, \mathbb{L}^*. \tag{9.23}$$

Operations d_1 and d_2 are differentials with respect to the first and the second component of the product $M \times M$, respectively. The complete differential on $M \times M$ is their sum $d = d_1 + d_2$.

From the transformation rule (8.17) and from the hermicity of $K_\mathbb{L}$ we get the following properties of ω_{12} and ω_{21}.

Proposition 9.1.

i) $\omega_{12} = -\omega_{21}$ *does not depend on the choice of trivialization;*

ii) $\overline{\omega_{12}} = \omega_{21}$

iii) $d\omega_{12} = 0$.

Let us also consider 1-forms

$$k_{2\alpha_2} := d_2 \log K_{\overline{\alpha}_1\alpha_2} \tag{9.24}$$

$$k_{1\overline{\alpha}_1} := d_1 \log K_{\overline{\alpha}_1\alpha_2} \tag{9.25}$$

which are independent on indices $\overline{\alpha}_1$ and $\overline{\alpha}_2$, respectively, and satisfy the transformation rules

$$k_{2\alpha_2} = k_{2\beta_2} + d_2 \log g_{\alpha_2\beta_2} \tag{9.26}$$

$$k_{1\overline{\alpha}_1} = k_{1\overline{\beta}_2} + d_1 \log \overline{g_{\alpha_1\beta_1}}. \tag{9.27}$$

Let $\Delta : M \to M \times M$ be the diagonal embedding i.e. $\Delta(m) = (m, m)$ for $m \in M$. We introduce the following notation

$$\Delta^* K = H \qquad \frac{1}{2\pi i}\Delta^*\omega_{12} = \omega \quad \text{and} \quad \Delta^* \mathrm{k}_{2\alpha} = \mathrm{k}_\alpha. \tag{9.28}$$

Now, it is easy to see that the following proposition is valid.

Proposition 9.2.

i) H *defined by* (9.28) *is a positive hermitian metric on* $\mathbb{L}$.

ii) The 1-form $k_\alpha \in C^\infty(\Omega_\alpha, \mathbb{L} \otimes T^*M)$ $(\overline{k}_\alpha \in C^\infty(\Omega_\alpha, \overline{\mathbb{L}} \otimes T^*M)$ *defined by* (9.28) *gives local representation of a connection* ∇ $(\overline{\nabla})$ *on the bundle* $\mathbb{L}$ $(\overline{\mathbb{L}})$.

iii) One has $\operatorname{curv}\nabla = 2\pi i\omega$ *for* ω *defined by* (9.28).

iv) The connection ∇ *is metric with respect to* H.

According to [24] we assume the following terminology.

Definition 9.3. The line bundle $\mathbb{L} \to M$ with distinguished hermitian metric H and the connection ∇ satisfying the consistency condition (9.2) we shall call pre-quantum bundle and denote by $(\mathbb{L} \to M, H, \nabla)$.

The pre-quantum line bundles form the category with the morphisms defined in the standard way. We shall call $\mathcal{L}$ the category of pre-quantum bundles.

From [31] one can obtain

Proposition 9.4. *For any pre-quantum bundle* $(\mathbb{L} \to M, H, \nabla)$ *there exists smooth map* $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ *such that*

$$(\mathbb{L} \to M, H, \nabla) = (\mathcal{K}\mathbb{E} \to M, \mathcal{K}^* H_{FS}, \mathcal{K}^* \nabla_{FS}) \tag{9.29}$$

i.e. the line bundle $\mathbb{L} \to M$*, the hermitian metric* H *and the metric connections* ∇ *can be obtained as the respective pullbacks of their counterparts* $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$, H_{FS} *and* ∇_{FS} *on the complex projective Hilbert space* $\mathbb{CP}(\mathcal{H})$

From the Proposition 9.4 and Theorem 8.7 one concludes that construction given by formula (9.24)-(9.28) define covariant functor from the category of positive hermitian kernels $\mathfrak{K}$ on the category of pre-quantum line bundles $\mathcal{L}$.

Taking the above remarks into account that metric structure H, the connection ∇ and curvature form ω related to the positive hermitian kernel $K_{\mathbb{L}}$ by the (9.28) are given equivalently by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ as follows

$$H_{\overline{\alpha}\alpha}(q, q) = K_{\overline{\alpha\alpha}}(q, q) = \langle K_\alpha(q)|K_\alpha(q)\rangle \tag{9.30}$$

$$k_\alpha(q) = \frac{\langle K_\alpha(q)|dK_\alpha(q)\rangle}{\langle K_\alpha(q)|K_\alpha(q)\rangle} \tag{9.31}$$

$$\omega = \frac{1}{2\pi i} d\left(\frac{\langle K_\alpha(q)|dK_\alpha(q)\rangle}{\langle K_\alpha(q)|K_\alpha(q)\rangle}\right)(q) \tag{9.32}$$

for $q \in \Omega_\alpha$.

In order to find the quantum mechanical interpretation of the connection ∇ and its curvature form $2\pi i\omega$ let us take the sequence $q = q_1, \dots, q_{N-1}, q_N = p$ of the points $q_i \in \Omega_{\alpha_i}$, for which we assumed $\Omega_{\alpha_1} = \Omega_\alpha$ and $\Omega_{\alpha_N} = \Omega_\beta$. According to the multiplication property of the transition amplitude, the following expression

$$a_{\alpha\beta}(q, q_2, \dots, q_{N-1}, p) := a_{\overline{\alpha}_1\alpha_2}(q, q_2) \cdots a_{\overline{\alpha}_{N-1}\beta}(q_{N-1}, p) \tag{9.33}$$

gives the transition amplitude from the state $\iota([K_\alpha(q)])$ to the state $\iota([K_\beta(p)])$ under the condition that the system has gone through all the intermediate coherent states $\iota([K_{\alpha_2}(q_2)]), \dots, \iota([K_{\alpha_{N-1}}(q_{N-1})])$. We shall call the sequence

$$\iota([K_{\alpha_1}(q_1)]), \dots, \iota([K_{\alpha_N}(q_N)]) \tag{9.34}$$

of coherent states a process starting at q and ending at p. Consequently $a_{\alpha\beta}(q, q_2, \dots, q_{N-1}, p)$ will be called the transition amplitude for that process.

Let us investigate further the process in $\iota(\mathcal{K}(M))$ parametrized by a piecewise smooth curve $\gamma : [\tau_i, \tau_f] \to M$ such that $\gamma(\tau_k) = q_k$ for $\tau_k \in [\tau_i, \tau_f]$ defined by $\tau_{k+1} - \tau_k = \frac{1}{N-1}(\tau_f - \tau_i)$. Then in the limit $N \to \infty$ this γ-process may be viewed as a process approximately described by the discrete one $(q, q_2, \dots, q_{N-1}, p)$. The transition amplitude for the process γ is obtained from (9.33) by the limit $N \to \infty$

$$a_{\overline{\alpha}\beta}(q, \gamma, p) = \lim_{N\to\infty} \prod_{k=1}^{N-1} a_{\overline{\alpha}_k,\alpha_{k+1}}(\gamma(\tau_k), \gamma(\tau_{k+1})). \tag{9.35}$$

Taking into account the smoothness of $K_\alpha : \Omega_\alpha \to \mathcal{H}$ and piecewise smoothness of γ we define

$$\Delta K_{\alpha_k}(\gamma(\tau_k)) := K_{\alpha_k}(\gamma(\tau_{k+1})) - K_{\alpha_k}(\gamma(\tau_k)), \tag{9.36}$$

where we assumed in (9.36) that $\gamma(\tau_k), \gamma(\tau_{k+1}) \in \Omega_{\alpha_k}$. Then, using (9.26) and assuming that $\gamma([\tau_i, \tau_f]) \subset \Omega_{\alpha_k}$ one has

$$a_{\overline{\alpha}_k \alpha_k}(q, \gamma, p) = \lim_{N\to\infty} \left(\frac{\langle K_{\alpha_k}(q)|K_{\alpha_k}(q)\rangle}{\langle K_{\alpha_k}(p)|K_{\alpha_k}(p)\rangle} \right)^{\frac{1}{2}} \cdot \tag{9.37}$$

$$\cdot \prod_{l=1}^{N-1} \left(1 - \frac{\langle K_{\alpha_k}(\gamma(\tau_l))|\Delta K_{\alpha_k}(\gamma(\tau_l))\rangle}{\langle K_{\alpha_k}(\gamma(\tau_l))|K_{\alpha_k}(\gamma(\tau_l))\rangle} \right) =$$

$$= \lim_{N\to\infty} \left(\frac{\langle K_{\alpha_k}(q)|K_{\alpha_k}(q)\rangle}{\langle K_{\alpha_k}(p)|K_{\alpha_k}(p)\rangle} \right)^{\frac{1}{2}} \exp \sum_{l=1}^{N-1} \frac{\langle K_{\alpha_k}(\gamma(\tau_l))|\Delta K_{\alpha_k}(\gamma(\tau_l))\rangle}{\langle K_{\alpha_k}(\gamma(\tau_l))|K_{\alpha_k}(\gamma(\tau_l))\rangle} =$$

$$= \left(\frac{\langle K_{\alpha_k}(q)|K_{\alpha_k}(q)\rangle}{\langle K_{\alpha_k}(p)|K_{\alpha_k}(p)\rangle} \right)^{\frac{1}{2}} e^{{}_{\tau_i}^{\tau_f} K_{\alpha_k} \llcorner \frac{d\gamma}{d\tau} d\tau} =$$

$$\exp i \int_{\tau_i}^{\tau_f} Im K_{\alpha_k} \llcorner \frac{d\gamma}{d\tau} d\tau = \exp i \int_{\tau_i}^{\tau_f} Im \frac{\langle K_{\alpha_k}|dK_{\alpha_k}\rangle}{\langle K_{\alpha_k}|K_{\alpha_k}\rangle} \llcorner \frac{d\gamma}{d\tau} d\tau.$$

After expressing the connection $\nabla = \mathcal{K}^* \nabla^{FS}_K$ in the unitary gauge frame

$$u_\alpha := \frac{1}{H(s_\alpha, s_\alpha)}^{\frac{1}{2}} s_\alpha, \tag{9.38}$$

i.e.

$$\nabla u_\alpha = i Im \frac{\langle K_{\alpha_k}|dK_{\alpha_k}\rangle}{\langle K_{\alpha_k}|K_{\alpha_k}\rangle} \otimes u_\alpha, \tag{9.39}$$

we obtain that transition for the piecewise process $\gamma([\tau_i, \tau_f])$ starting at q and ending at p is given by the parallel transport

$$a_{\overline{\alpha}\beta}(q, \gamma, p) = \exp i \int_{\gamma([\tau_i, \tau_f])} Im \frac{\langle K|dK\rangle}{\langle K|K\rangle} \tag{9.40}$$

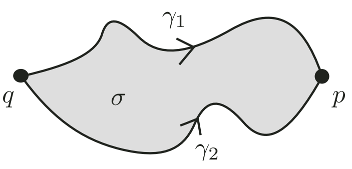
from $\mathbb{L}_q$ to $\mathbb{L}_p$ along γ with respect to the connection ∇. In (9.40) we applied the notation $\frac{\langle K|dK\rangle}{\langle K_{\alpha_k}|K\rangle} := \frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}$ on Ω_α and by the integral $\int_{\gamma([\tau_i,\tau_f])} Im \frac{\langle K|dK\rangle}{\langle K|K\rangle}$ we mean the sum of integrals over the pieces of the curve $\gamma([\tau_i, \tau_f])$ which are contained in Ω_α.

Since the connection ∇ is metric, one has

$$|a_{\overline{\alpha}\beta}(q, \gamma, p)|^2 = 1 \tag{9.41}$$

for the transition probability of the considered γ-process. This is a consequence of the continuity of the coherent state map $\iota \circ \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}) \subset U^1(\mathcal{H})$ with respect of $\|\cdot\|_1$-metric, which causes that $a_{\overline{\alpha}\beta}(q(\tau), q(\tau + \Delta\tau)) \approx 1$ for $\Delta\tau \approx 0$. Therefore, for the classical process, i.e. continuous ones, the interference effects disappear between the infinitely close $q(\tau) \approx q(\tau + \Delta\tau)$ classical pure states. It remains only as a global effect given by the parallel transport (9.40) with respect to ∇.

For two piecewise smooth processes starting from q and ending in p



one has the following relation

$$a_{\overline{\alpha}\beta}(q, \gamma_2, p) = a_{\overline{\alpha}\beta}(q, \gamma_1, p) e^{2\pi i \ {}_{\sigma} \omega} \tag{9.42}$$

between the transition amplitudes, where the boundary $\partial\sigma = \gamma_1 - \gamma_2$. The factor $e^{2\pi i \ {}_{\sigma} \omega}$ does not depend on the choice of σ. Hence, one concludes that the curvature 2-form ω measures the phase change of transition amplitude for the cyclic piecewise smooth process.

According to path-integral approach the quantum probability amplitude one can define the path integral over the processes starting from q and ending in p

$$a_{\overline{\alpha}\beta}(q, p) := \int \mathcal{D}[\gamma] e^{i \ {}_{\gamma} Im \frac{\langle K|dK\rangle}{\langle K|K\rangle}} == \int \prod_{\tau \in [\tau_i, \tau_f]} d_k\gamma(t) \exp\left(i \int_{\tau_i}^{\tau_f} Im \frac{\langle K|dK\rangle}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{d\tau} d\tau \right), \tag{9.43}$$

where

$$\int \prod_{\tau \in [\tau_i, \tau_f]} d_k\gamma(t) := \tag{9.44}$$

$$:= \lim_{N\to\infty} \int_M \sum_{\delta_2} h_{\delta_2}(\gamma(\tau_2)) d\mu_L(\gamma(\tau_2)) \cdot \ldots \times \int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(\gamma(\tau_{N-1})) d\mu_L(\gamma(\tau_{N-1}))$$

and $\mu_L = \bigwedge^n \omega$ is the Liouville measure on (M, ω), as the transition amplitude $a_{\overline{\alpha}\beta}$. This point of view on the transition amplitude we will use to find the Lagrangian description of the system.

Having in the mind the energy conservation law we will admit in (9.44) only those trajectories which are confined to the equienergy surface $H^{-1}(E)$, where $H \in C^\infty(M)$ is the function of total energy of the considered system. Let then $a_{\overline{\alpha}\beta}(q, p; H = E = \text{const})$ denotes the transition amplitude from $\mathcal{K}(q)$ to $\mathcal{K}(p)$

which is the result of the superposition of the equienergy processes. In order to find $a_{\overline{\alpha}\beta}(q,p;H=E=\text{const})$ one should insert the δ-factor

$$\delta(H(\gamma(\tau_k))-E)d\mu_L(\gamma(\tau_k)) = \int^{+\infty} e^{-i(H(\gamma(\tau_k))-E)\lambda(\tau_k)}d\lambda(\tau_k)d\mu_L(\gamma(\tau_k)) \quad (9.45)$$

into (9.43). Thus we obtain

$$a_{\overline{\alpha}\beta}(q,p;H=E=\text{const}) = \int \prod_{\tau\in[\tau_i,\tau_f]} d_k\gamma(t)d\lambda(\tau)\cdot \quad (9.46)$$

$$\cdot \exp\left(i \int_{\tau_i}^{\tau_f} \left(Im\frac{\langle K|dK\rangle}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{d\tau}d\tau - (h(\gamma(\tau_k))-E)\lambda(\tau)\right) d\tau \right).$$

Now according to Feynman the Lagrangian L of the system is given by

$$\frac{dL}{dt} = Im\frac{\langle K|dK\rangle}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{dt} - H(\gamma(t)), \quad (9.47)$$

where the summand $Im\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{dt}$ is responsible for the interaction of the system with the effective external field resulting from the way the coherent state map $\mathcal{K}: M \to \mathbb{CP}(\mathcal{H})$ has been realized.

10 Relation between classical and quantum observables (quantization)

The fundamental problem in the theory of physical systems is to explain how to construct the quantum observables if one has their classical counterparts. Traditionally one calls this procedure the quantization. Let us now explain what we mean by quantization in the framework of our model of the mechanical system. In order to do this let us take two mechanical systems $(M_i, \omega_i, \mathcal{K}_i : M_i \to \mathbb{CP}(\mathcal{H}_i))$, $i = 1, 2$, and consider the symplectomorphism $\sigma : M_1 \to M_2$. By the **quantization** of σ we shall mean the morphism

$$\Sigma : \mathrm{Sp}C^\infty(M_1, M_2) \ni \sigma \to \Sigma(\sigma) \in \mathrm{Mor}(\mathbb{CP}(\mathcal{H}), \mathbb{CP}(\mathcal{H})))$$

defined for such σ for which the diagram (7.22) commutes. One has

$$\Sigma(\sigma_1 \circ \sigma_2) = \Sigma(\sigma_2) \circ \Sigma(\sigma_1) \quad (10.1)$$

for $\sigma_1 : M_1 \to M_2$ and $\sigma_2 : M_2 \to M_3$. It is clear that not all elements of $\mathrm{Sp}C^\infty(M_1, M_2)$ are quantizable in this way. If $M_1 = M_2$, $\mathcal{H}_1 = \mathcal{H}_2$ and $\mathcal{K}_1 = \mathcal{K}_2$ the quantizable symplectic diffeomorphisms $\sigma : M \to M$ form the subgroup $\mathrm{SpDiff}_{\mathcal{K}}(M, \omega)$ of the group $\mathrm{SpDiff}(M, \omega)$ of all symplectic diffeomorphism of M. Since $\Sigma(\sigma) : \mathbb{CP}(\mathcal{H}) \to \mathbb{CP}(\mathcal{H})$ preserve the transition probability it follows from Wigner theorem, see [61],that there exists a unitary or anti-unitary map $U(\sigma) : \mathcal{H} \to \mathcal{H}$ such that

$$\Sigma(\sigma) = [U(\sigma)]\,. \tag{10.2}$$

The phase ambiguity in the choice of $U(\sigma)$ in (10.2) one removes by the passing to the lifting

$$\begin{CD} \mathbb{L}' @>\mathcal{K}'>> \mathbb{E}' \\ @VVV @VVV \\ M @>>\mathcal{K}> \mathbb{CP}(\mathcal{H}) \end{CD}\quad , \tag{10.3}$$

of the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$, where the $\mathbb{C}^*$–principal bundles $\mathbb{L}'$ and $\mathbb{E}'$ are obtained from $\mathbb{L}$ and $\mathbb{E}$ by the cutting off zeros sections. Fixing the unitary (anti-unitary) representative $U(\sigma)$ one obtains σ' from (12.112) and from $\mathbb{E}' \backsimeq \mathcal{H} \setminus \{0\}$

$$\begin{CD} \mathbb{L}' @>\mathcal{K}'>> \mathbb{E}' \\ @V\sigma'VV @VVU(\sigma)V \\ \mathbb{L}' @>>\mathcal{K}'> \mathbb{E}' \end{CD}\;, \tag{10.4}$$

where the lifting

$$\begin{CD} \mathbb{L}' @>\sigma'>> \mathbb{L}' \\ @VVV @VVV \\ M @>>\sigma> M \end{CD}\quad , \tag{10.5}$$

is defined by $U(\sigma)$ in the unique way. The map σ' defines the principal bundles authomorphism and preserves the positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{E}}$, i.e.

$$\sigma'(c\xi) = c\sigma'(\xi) \tag{10.6}$$

for $c \in \mathbb{C} \setminus \{0\}$ and $\xi \in \mathbb{L}'$ and

$$K_{\mathbb{L}}\left(\sigma'(\xi_1), \sigma'(\xi_2)\right) = K_{\mathbb{L}}(\xi_1, \xi_2) \tag{10.7}$$

for $\xi_1, \xi_2 \in \mathbb{L}'$. The inverse statement is also valid.

Proposition 10.1. *Let $\mathbb{L} \to M$ be the complex line bundle with distinguished positive Hermitian Kernel $K_{\mathbb{L}}$ and the diffeomorpism $\sigma : M \to M$ that has a lifting $\sigma' : \mathbb{L}' \to \mathbb{L}'$ which satisfies (10.6) and (10.7). Then there are uniquely defined coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ and the unitary (anti-unitary) operator $U(\sigma)$ with the property (10.4).*

Definition 10.2. The one–parameter subgroup $\sigma(t) \subset \mathrm{Diff}M, t \in \mathbb{R}$, we call the **prequantum flow** if and only if it admits the lifting $\sigma'(t) \in Diff\mathbb{L},\ \ t \in \mathbb{R}$, which preserves the structure of the prequatum bundle $(\mathbb{L} \to \mathbb{M}, \bigtriangledown, \mathcal{H})$.

It was shown by Kostant [24] that the Lie algebra $Lie(\mathbb{L}^*, \bigtriangledown, \mathcal{H})$ of the vector fields tangent to the prequantum flows is isomorphic with the Poisson algebra $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ where the Poisson bracket $\{\cdot,\cdot\}$ is defined by ω. It follows form Proposition 12.3 and Proposition 9.4 that the prequantum bundle structure is always defined by a coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ or, equivalently, by a positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{E}}$.

Definition 10.3. The one–parameter subgroup $\sigma(t) \in \mathrm{SpDiff}M, t \in \mathbb{R}$, we call the **quantum flow** if and only if it preserve the structure of the physical system $(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}))$, i.e. there is one parameter subgroup $\Sigma(t),\ t \in \mathbb{R}$ such that

$$\begin{array}{ccc} M & \xrightarrow{\ \mathcal{K}\ } & \mathbb{CP}(\mathcal{H}) \\ {\scriptstyle \sigma(t)}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma(t)} \\ M & \xrightarrow[\ \mathcal{K}\ ]{} & \mathbb{CP}(\mathcal{H}) \end{array}, \tag{10.8}$$

for any $t \in \mathbb{R}$.

Theorem 10.4. *The following statements are equivalent:*

(i) The one–parameter subgroup $\sigma(t) \in \mathrm{Diff}M,\ \ t \in \mathbb{R}$, is a quantum flow of the physical system $(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}))$.

(ii) The one–parameter subgroup $\sigma(t) \in \mathrm{Diff}M,\ \ t \in \mathbb{R}$, has the lifting $\sigma'(t) : \mathbb{L}' \to \mathbb{L}',\ t \in \mathbb{R}$, which preserves the bundle structure of $\mathbb{L}'$ and the positive Hermitian kernel $K_{\mathbb{L}} = \mathcal{K}^* K_{\mathbb{L}}$.

(iii) There are the lifting $\sigma'(t) \in \mathrm{Diff}\mathbb{L}',\ \ t \in \mathbb{R}$ and the strong unitary (anti–unitary) one parameter subgroup $U(t) \in Aut\mathcal{H},\ t \in \mathbb{R}$, such that

$$\begin{array}{ccc} \mathbb{L}' & \xrightarrow{\ \mathcal{K}'\ } & \mathbb{E}' \\ {\scriptstyle \sigma'(t)}\big\downarrow & & \big\downarrow{\scriptstyle U(t)} \\ \mathbb{L}' & \xrightarrow[\ \mathcal{K}'\ ]{} & \mathbb{E}' \end{array}, \tag{10.9}$$

for any $t \in \mathbb{R}$, where $\mathbb{E}' \cong \mathcal{H} \setminus \{0\}$.

The vector field tangent to the quantum flow $\sigma'(t),\ t \in \mathbb{R}$, is the lifting of the Hamiltonian field $X_f \in \Gamma^\infty(TM)$ generated by $f \in C^\infty(M,\mathbb{R})$, see [24]. So, the strong unitary one–parameter subgroup $U^f(t),\ t \in \mathbb{R}$ given by (10.9) depends univocally on f. The Stone-von Neumann theorem states that there is the self–adjoint operator F on $\mathcal{H}$ such that

$$U^f(t) = e^{-itF}. \tag{10.10}$$

The domain $D(F)$ of F is the linear span $l.s.(\mathcal{K}(M))$ of the set of coherent states. Representing F in the $\mathcal{H}_{\mathbb{K},\mathbb{L}} \subset \Gamma^\infty(M, \mathbb{L}^*)$ we obtain

$$-iF\Psi = \lim_{t\to 0} \frac{(U^f(t) - 1)\Psi}{t} = \lim_{t\to 0} \frac{1}{t}(\sigma'(-t)\Psi - \Psi) = \left(\nabla_{X_f} + 2\pi i f\right)\Psi \tag{10.11}$$

for $\Psi \in D(F)$. The second equality in (10.11) is possible since $l.s.(\mathcal{K}(M))$ is $U^f(t)$ invariant , $t \in \mathbb{R}$. Hence, Kostant–Souriau operator $-iQ_f$ is essentially self-adjoint on $l.s.(\mathcal{K}(M))$ and the infinitesimal generator of $U^f(t)$ is its closure.

Let us denote by $C^\infty_{\mathcal{K}}(M, \mathbb{R})$ the space of function which generate the quantum flows on $(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}))$. It follows from (ii) of the Theorem 10.4 that it is the Lie subalgebra of Poisson algebra $C^\infty(M, \mathbb{R})$. One also has

$$[Q_f, Q_g] = iQ_{\{f,g\}} \tag{10.12}$$

what means that $-iQ$ defines Lie algebras homomorphism, i.e. it is quantization in Kostant-Souriau sense. We remark that one does not use the notion of polarization, which plays the crucial role in the Kostant–Souriau geometric quantization [67]. In the theory developed here the polarization does not have the crucial meaning. It could be reconstructed from the coherent state map or from the positive Hermitian kernel [39].

11 Quantum phase spaces defined by the coherent state map

This section is based on the paper [39]. We will begin by explaining how coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ defines the polarization $P \subset T^{\mathbb{C}}M$ in the sense of geometric quantization.

Therefore let us consider the complex distribution $P \subset T^{\mathbb{C}}M$ spanned by smooth complex vector fields $X \in \Gamma^\infty(T^{\mathbb{C}}M)$ which annihilate the Hilbert space $I(\mathcal{H}) \subset \Gamma^\infty(M, \overline{\mathcal{L}}^*)$, i.e.

$$P := \bigsqcup_{m\in M} P_m, \tag{11.1}$$

where

$$P_m := \{X(m) : X \in \Gamma^\infty(T^{\mathbb{C}}M) \text{ and } \overline{\nabla}^*_X \psi = 0 \text{ for any } \psi \in I(\mathcal{H})\}. \tag{11.2}$$

To summarize the properties of P we formulate

Proposition 11.1.

i) The necessary and sufficient condition for X to belong to $\Gamma^\infty(P)$ is

$$\overline{X}(K_\alpha) = k_\alpha(\overline{X})K_\alpha. \tag{11.3}$$

ii) The distribution P is involutive and isotropic, i.e. for $X, Y \in \Gamma^\infty(P)$ one has

$$[X,Y] \in \Gamma^\infty(P) \quad \textit{and}\ \omega(X,Y) = 0. \tag{11.4}$$

iii) If $X \in \Gamma^\infty(P \cap \overline{P})$ then

$$X \llcorner \omega = 0 \tag{11.5}$$

iv) Positivity condition:

$$i\omega(X, \overline{X}) \geqslant 0 \tag{11.6}$$

for all $X \in \Gamma^\infty(P)$.

Proof.

i) By the definition one has that $X \in \Gamma^\infty(P)$ iff $\overline{\nabla}^*_X I(v) = 0$ for any $v \in \mathcal{H}$. From (12.95) and (9.31) we get

$$\overline{\nabla}^*_X I(v) = X \llcorner \left(\langle dK_\alpha|v\rangle - \frac{\langle dK_\alpha|K_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}\langle K_\alpha|v\rangle \right) \otimes \overline{s}^*_\alpha =$$

$$= \langle \overline{X}(K_\alpha)| \left(\mathbb{I} - \frac{|K_\alpha\rangle\langle K_\alpha|}{\langle K_\alpha|K_\alpha\rangle} \right) v > \overline{s}^*_\alpha = \langle \overline{X}(K_\alpha) - k_\alpha(\overline{X})K_\alpha|v > \overline{s}^*_\alpha$$

Thus we have proven (11.3).

ii) From

$$dk_\alpha = d\frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} = \frac{\langle dK_\alpha \wedge |dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} - \frac{\langle dK_\alpha|K_\alpha\rangle \wedge \langle K_\alpha|dK_\alpha\rangle}{(\langle K_\alpha|K_\alpha\rangle)^2} \tag{11.7}$$

and (11.3) we obtain

$$dk_\alpha(X,Y) = \frac{1}{2}\left(\frac{\langle \overline{X}(K_\alpha)|Y(K_\alpha)\rangle - \langle \overline{Y}(K_\alpha)|X(K_\alpha)\rangle}{\langle K_\alpha|K_\alpha\rangle} - \right.$$

$$\left. - \frac{\langle \overline{X}(K_\alpha)|K_\alpha\rangle\langle K_\alpha|Y(K_\alpha)\rangle - \langle \overline{Y}(K_\alpha)|K_\alpha\rangle\langle K_\alpha|X(K_\alpha)\rangle}{(\langle K_\alpha|K_\alpha\rangle)^2} \right) =$$

$$= \frac{1}{2}\left(\overline{k_\alpha(\overline{X})}k_\alpha(Y) - \overline{k_\alpha(\overline{Y})}k_\alpha(X) - \overline{k_\alpha(\overline{X})}k_\alpha(Y) + \overline{k_\alpha(\overline{Y})}k_\alpha(X) \right) = 0 \tag{11.8}$$

for $X, Y \in \Gamma^\infty(P)$. Using the identity

$$dk_\alpha(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla[X,Y] \tag{11.9}$$

and (11.8) we conclude that P is involutive isotropic distribution.

iii) Let $X \in \Gamma^\infty(P \cap \overline{P})$, then

$$\pounds_X k_\alpha = \pounds_X \frac{\langle K_\alpha | dK_\alpha \rangle}{\langle K_\alpha | K_\alpha \rangle} = -\frac{1}{\langle K_\alpha | K_\alpha \rangle} \left(\langle \overline{X}(K_\alpha) | K_\alpha \rangle + \right.$$

$$\left. + \langle K_\alpha | X(K_\alpha) \rangle \right) k_\alpha + \frac{1}{\langle K_\alpha | K_\alpha \rangle} \left(\langle \overline{X}(K_\alpha) | dK_\alpha \rangle + \langle K_\alpha | dX(K_\alpha) \rangle \right) =$$

$$= -\left(\overline{k_\alpha(\overline{X})} + k_\alpha(X) \right) k_\alpha + \overline{k_\alpha(\overline{X})} k_\alpha + k_\alpha(X) k_\alpha + d\left(k_\alpha(X)\right) =$$

$$d\left(k_\alpha(X)\right) + X \llcorner dk_\alpha - X \llcorner dk_\alpha = \pounds_X k_\alpha - X \llcorner dk_\alpha \tag{11.10}$$

Hence one has (11.5).

iv) For $X \in \Gamma^\infty(P)$ one can write

$$dk_\alpha(X, \overline{X}) = \frac{1}{2} \left(\overline{k_\alpha(\overline{X})} k_\alpha(\overline{X}) - \right.$$

$$\left. - \frac{\langle X(K_\alpha) | X(K_\alpha) \rangle}{\langle K_\alpha | K_\alpha \rangle} - \overline{k_\alpha(\overline{X})} k_\alpha(\overline{X}) + \frac{\langle X(K_\alpha) | K_\alpha \rangle \langle K_\alpha | X(K_\alpha) \rangle}{(\langle K_\alpha | K_\alpha \rangle)^2} \right) =$$

$$= -\frac{1}{2\,\|K_\alpha\|^2} \left(\|X(K_\alpha)\|^2 \|K_\alpha\|^2 - |\langle K_\alpha | X(K_\alpha) \rangle|^2 \right). \tag{11.11}$$

Now from Schwartz inequality one gets (11.8).

□

Let $\mathcal{O}_\mathcal{K}$ denote the algebra of functions $\lambda \in C^\infty(M)$ such that $\lambda\psi \in I(\mathcal{H})$ if $\psi \in I(M)$.

In all further considerations we shall restrict ourselves to coherent state maps $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{H})$ which do satisfy the following conditions:

a) *The curvature 2-form*

$$\omega = i\,\mathrm{curv}\,\nabla = \mathcal{K}^* \omega_{FS}$$

is non-degenerate, i.e. ω is symplectic.

b) *The distribution P is maximal. i.e.*

$$\dim_\mathbb{C} P = \frac{1}{2} \dim M =: N. \tag{11.12}$$

c) *For every $m \in M$ there exists an open neighborhood $\Omega \ni m$ and functions $\lambda_1, \ldots, \lambda_N \in \mathcal{O}_\mathcal{K}$ such that $d\lambda_1, \ldots, d\lambda_N$ are linearly independent on Ω.*

Proposition 11.2.

i) *The manifold M is Kähler manifold and $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{H})$ is a Kähler immersion of M into $\mathbb{CP}(\mathcal{H})$.*

ii) The distribution P is Kähler polarization of symplectic manifold (M, ω). Moreover P is spanned by the Hamiltonian vector fields X_λ generated by $\lambda \in \mathcal{O}_\mathcal{K}$.

Proof.

i) The condition $a)$ and $b)$ imply that P define almost complex structure on M. The condition $c)$ guarantees its integrability. The property of M being Kähler follows from the fact that ω is symplectic and from positivity property (11.6) of Proposition 11.1. The immersion property of $\mathcal{K}$ follows from $\omega = \mathcal{K}^*\omega_{FS}$ and ω is symplectic.

ii) Let us take $X \in \Gamma^\infty(P)$ and $\lambda \in \mathcal{O}_\mathcal{K}$. Then from

$$\overline{\nabla}^*_X \psi = 0 \qquad \text{and} \qquad \overline{\nabla}^*_X(\lambda\psi) = 0$$

for any $\psi \in I(\mathcal{H})$, it follows $X(\lambda) = 0$. Let X_λ be the Hamiltonian vector field corresponding to λ

$$X_\lambda \llcorner \omega = d\lambda \tag{11.13}$$

Then

$$\omega(X_\lambda, X) = d\lambda(X) = X(\lambda) = 0.$$

Since P is maximal isotropic one gets $X_\lambda \in \Gamma^\infty(P)$. The condition $c)$ implies now that P is spanned by X_λ where $\lambda \in \mathcal{O}_\mathcal{K}$. In this way we have shown that P is integrable Kähler polarization on (M, ω).

□

We conclude this section by making the following comment. In the symplectic case the Lie subalgebra $(\mathcal{O}_\mathcal{K}, \{\cdot,\cdot\})$ is a maximal commutative subalgebra of the algebra of classical observables $(C^\infty(M), \{\cdot,\cdot\})$. The corresponding Hamiltonian vector fields $X_{\lambda_1}, \dots X_{\lambda_N} \in \Gamma^\infty(P)$ span the **Kähler polarization** P in the sense of Kostant-Souriau geometric quantization.

Now let us define the quantum Kähler polarization corresponding to the classical polarization P defined above.

Let $\mathcal{D}$ be the vector subspace of the Hilbert space $\mathcal{H}$ generated by finite combinations of the vectors $K_\alpha(m)$, where $\alpha \in I$ and $m \in \Omega_\alpha$. The linear operator $a : \mathcal{D} \to \mathcal{H}$ such that

$$aK_\alpha(m) = \lambda(m)K_\alpha(m) \tag{11.14}$$

for any $\alpha \in I$ and $m \in \Omega_\alpha$, will be called the **annihilation operator**. On the other hand the operator a^* conjugated to a we shall call the **creation operator**. The eigenvalue function $\lambda : M \to \mathbb{C}$ is well defined on M since $K_\alpha(m) \neq 0$ and the condition (11.14) does not depend on the choice of gauge.

In general the annihilation operators are not bounded as it is in the case of the Gaussian coherent states map (see Example 7.1). In this paper we restrict ourselves to the case when the annihilation operators are bounded.

Proposition 11.3. *The bounded annihilation operators form a commutative unital Banach subalgebra $\overline{\mathcal{P}}_{\mathcal{K}}$ in the algebra $L^\infty(\mathcal{H})$ of all bounded operators in the Hilbert space $\mathcal{H}$.*

Proof. It follows directly from the definition (11.14) that for any elements $a_1, a_2 \in \overline{\mathcal{P}}_{\mathcal{K}}$ their product $a_1 a_2$ and linear combination $c_1 a_1 + c_2 a_2$ belong to $\overline{\mathcal{P}}_{\mathcal{K}}$. It is also clear that identity operator $\mathbb{I} \in \overline{\mathcal{P}}_{\mathcal{K}}$. We shall show completeness of the subalgebra $\overline{\mathcal{P}}_{\mathcal{K}} \subset L^\infty(\mathcal{H})$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of annihilation operators and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of their eigenfunctions. From the condition (11.14)

$$|\lambda_k(m) - \lambda_n(m)| = \frac{\|(a_k - a_n)K_\alpha(m)\|}{\|K_\alpha(m)\|} \leqslant \|a_k - a_n\|$$

for all $\alpha \in I$ and $m \in \Omega_\alpha$. Hence the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ converges pointwise to some function $\lambda : \mathcal{H} \to \mathbb{C}$. Since $a_n \xrightarrow{n\to\infty} a$ converges in the operator norm to some bounded operator $a \in L^\infty(\mathcal{H})$ one has

$$\|(a - \lambda(m)\mathbb{I})K_\alpha(m)\| = \lim_{n\to\infty} \|(a_n - \lambda_k(m)K_\alpha(m)\| = 0$$

Consequently λ is the eigenvalues function for a and $a \in \overline{\mathcal{P}}_{\mathcal{K}}$. The annihilation operators a_1, $a_2 \in \overline{\mathcal{P}}_{\mathcal{K}}$ commute on a dense domain $\mathcal{D} \subset \mathcal{H}$ implying the commutativity of the subalgebra $\overline{\mathcal{P}}_{\mathcal{K}}$. □

The eigenvalues function is the covariant symbol

$$\lambda(m) = \frac{\langle K_\alpha(m)|aK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle} =: \langle a\rangle(m) \tag{11.15}$$

of the annihilation operator. It is thus a bounded complex analytic function on the complex manifold M.

We shall describe now the algebra of annihilation operator covariant symbols in terms of the vector space $I(\mathcal{H})$. Let $\Lambda : \mathcal{H} \to \mathcal{H}$ be the linear operator defined by the condition one has

$$\lambda I(v) = I(\Lambda v)$$

for all $v \in \mathcal{H}$. The operator defined above has the following properties.

Proposition 11.4.

i) If $\lambda \in \mathcal{O}_{\mathcal{K}}$ then Λ is a bounded operator on $\mathcal{H}$.

ii) The operator Λ^* adjoint to Λ is an annihilation operator with the covariant symbol given by bounded function $\overline{\lambda}$.

Proof.

i) From the sequence of equalities

$$\langle\overline{\lambda(m)}K_\alpha(m)|v\rangle\overline{s}^*_\alpha(m) = \lambda(m)\langle K_\alpha(m)|v\rangle\overline{s}^*_\alpha =$$

$$= \langle K_\alpha(m)|\Lambda v\rangle \overline{s}^*_\alpha(m) \tag{11.16}$$

where $v \in \mathcal{H}$, $\alpha \in I$ and $m \in \Omega_\alpha$ it follows that $\mathcal{D}$ is the domain of the conjugated operator Λ^*. Since $\mathcal{D}$ is dense in $\mathcal{H}$ the operator Λ admits the closure $\overline{\Lambda} = \Lambda^{**}$, see [2]. We have $\mathcal{H} = D(\Lambda) \subset D(\overline{\Lambda})$ which implies the boundedness of Λ.

ii) Let us notice that from (11.16) it follows

$$\Lambda^* K_\alpha(m) = \overline{\lambda}(m) K_\alpha(m). \tag{11.17}$$

Thus Λ^* is the annihilation operator with $\overline{\lambda}$ as its covariant symbol.

□

From this two propositions above one can deduce the following.

Theorem 11.5. *The mean value map* $\langle\cdot\rangle$ *defined by* (11.15) *gives the continuous*

$$\|\langle b\rangle\|_\infty = \sup_{m\in M} |\langle b\rangle(m)| \leqslant \|b\| \tag{11.18}$$

isomorphism of the operator commutative Banach algebra $\mathcal{P}_\mathcal{K} := \{a^* : a \in \overline{\mathcal{P}}_\mathcal{K}\}$ *of creation operators with the function Banach algebra* $(\mathcal{O}_\mathcal{K}, \|\cdot\|_\infty)$.

Let us assume that for some measure μ one has the resolution of the identity operator

$$\mathbb{I} = \int_M P(m) d\mu(m), \tag{11.19}$$

where

$$P(m) := \frac{|K_\alpha(m)\rangle\langle K_\alpha(m)|}{\langle K_\alpha(m)|K_\alpha(m)\rangle} \tag{11.20}$$

is the orthogonal projection operator $P(m)$ on the coherent state $\mathcal{K}(m)$, $m \in M$. In such case the scalar product of the functions $\psi = I(v)$ and $\varphi = I(w)$ can be expressed in terms of the integral

$$\langle\psi|\varphi\rangle = \langle v|w\rangle = \int_M \overline{H}^*(\psi,\varphi) d\mu =$$

$$= \int_M \frac{\overline{\langle K_\alpha(m)|v\rangle}\langle K_\alpha(m)|w\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle} d\mu(m). \tag{11.21}$$

Moreover one has

$$\|\Lambda v\|^2 = \int_M |\lambda|^2\, \overline{H}^*(\psi,\psi) d\mu \leqslant \|\lambda\|^2_\infty \|v\|^2 \tag{11.22}$$

for $v \in \mathcal{H}$ and thus it follows that

$$\|\Lambda\| \leqslant \|\lambda\|_\infty . \tag{11.23}$$

Taking into account the inequalities (11.18) and (11.23) we obtain

Theorem 11.6. *If the coherent states map admits the measure μ defining the resolution of identity* (11.19) *then the mean value map $\langle\cdot\rangle$ is the isomorphism of the Banach algebra $(\overline{\mathcal{P}}_{\mathcal{K}}, \|\cdot\|)$ onto Banach algebra $(\mathcal{O}_{\mathcal{K}}, \|\cdot\|_\infty)$.*

From the theorem above one may draw the conclusion that the necessary condition for the existence of the identity decomposition for the coherent states map $\mathcal{K}$ is the uniformity of the algebra $\overline{\mathcal{P}}_{\mathcal{K}}$, i.e.

$$\left\|a^2\right\| = \|a\|^2 \quad \text{for } a \in \overline{\mathcal{P}}_{\mathcal{K}}.$$

We shall show some facts allowing deeper understanding of the covariant symbols algebra $\mathcal{O}_{\mathcal{K}}$ in the context of the geometric quantization and Hamiltonian mechanics.

According to Theorem 11.6 the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ of annihilation operators is isomorphic to Banach algebra $\mathcal{O}_{\mathcal{K}}$. It is easy to notice that Kostant-Souriau quantization

$$\mathcal{O}_{\mathcal{K}} \ni \lambda \longrightarrow Q_\lambda = i\nabla_{X_\lambda} + \lambda \tag{11.24}$$

restricted to $I(\mathcal{H})$ gives inverse of the mean value isomorphism $\langle\cdot\rangle$ defined by (11.15).

In the light of the remarks above it is strongly justified to call the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ a **quantum Kähler polarization** of the mechanical system defined by Kähler coherent immersion $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{H})$.

Now we will concentrate on the purely quantum description of the mechanical system in C^*-algebra approach.

The function algebra $\mathcal{O}_{\mathcal{K}}$ defines the complex analytic coordinates of the classical phase space (M, ω), i.e. for any $m \in M$ there are open neighborhoods $\Omega \ni m_0$ and $z_1, \ldots, z_N \in \mathcal{O}_{\mathcal{K}}$ such that the map $\varphi : \Omega \to \mathbb{C}^N$ defined by $\varphi(m) := (z_1(m), \ldots, z_N(m))$ for $m \in \Omega$, is a holomorphic chart from the complex analytic atlas of M. The annihilation operators $a_1, \ldots, a_N \in \overline{\mathcal{P}}_{\mathcal{K}}$ correspond to $z_1, \ldots, z_N$ through the defining relation (11.14) is naturally to consider as a quantum coordinate system. The operators from $\mathcal{P}_{\mathcal{K}}$, for example such as $a_1^*, \ldots, a_N^*$, conjugated to those of $\overline{\mathcal{P}}_{\mathcal{K}}$ are the creation operators.

Definition 11.7. The unital C^*-algebra $\mathcal{A}_{\mathcal{K}}$ generated by the the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ will be called **quantum phase space** defined by coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$.

Let us define **Berezin covariant symbol**

$$\langle F\rangle(m) = \frac{\langle K_\alpha(m)|FK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle}, \qquad m \in M \tag{11.25}$$

of the operator F (unbounded in general) which domain $\mathcal{D}$ contains all finite linear combinations of coherent states. Since $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ is a complex analytic map, the Berezin covariant symbol $\langle F\rangle$ is a real analytic function of the coordinates $\overline{z}_1, \ldots, \overline{z}_N, z_1, \ldots, z_N$.

For $n \in \mathbb{N}$ let $F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \in \mathcal{A}_{\mathcal{K}}$ be a normally ordered polynomials of creation and annihilation operators. We say that

$$F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \xrightarrow[n\to\infty]{} F =: F(a_1^*, \dots, a_N^*, a_1, \dots, a_N) \qquad (11.26)$$

converges in **coherent state weak topology** if

$$\langle F_n(a_1^*, \dots, a_N^*, a_1, \dots, a_N)\rangle(m) \xrightarrow[n\to\infty]{} \langle F\rangle(m). \qquad (11.27)$$

Therefore thinking about observables of the considered system, i.e. self-adjoint operators, as the weak coherent state limits of normally ordered polynomials of annihilation and creation operators we are justified to consider $\mathcal{A}_{\mathcal{K}}$ as the quantum phase space of the physical system defined by $(M, \mathcal{H}, \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}))$.

Taking into account the properties of $\mathcal{A}_{\mathcal{K}}$ we define the abstract polarized C^*-algebra.

Definition 11.8. The **polarized C^*-algebra** is a pair $(\mathcal{A}, \overline{\mathcal{P}})$ consisting of the unital C^*-algebra $\mathcal{A}$ and its Banach commutative subalgebra $\overline{\mathcal{P}}$ such that

i) $\overline{\mathcal{P}}$ generates $\mathcal{A}$

ii) $\overline{\mathcal{P}} \cap \mathcal{P} = \mathbb{C}\mathbb{I}$

It is easy to see that $\mathcal{A}_{\mathcal{K}}$ is polarized C^*-algebra in the sense of this definition.

Also the notion of coherent state can be generalized to the case of abstract polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$, namely

Definition 11.9. A **coherent state** ω on polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$ is the positive linear functional of the norm equal to one satisfying the condition

$$\omega(xa) = \omega(x)\omega(a) \qquad (11.28)$$

for any $x \in \mathcal{A}$ and any $a \in \overline{\mathcal{P}}$.

Let us stress that in the case when $(\mathcal{A}, \overline{\mathcal{P}})$ is defined by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ then the state

$$\omega_m(x) := Tr\big(xP(m)\big), \qquad (11.29)$$

where $m \in M$ and $P(m)$ is given by (11.20), is coherent in the sense of Definition 11.9

Proceeding as in motivating remarks we shall introduce the notion of the norm normal ordering in polarized C^*-algebra $(\mathcal{A}, \mathcal{P})$.

Definition 11.10. The C^*-algebra $\mathcal{A}$ of quantum observables with fixed polarization $\overline{\mathcal{P}}$ admits the **norm normal ordering** if and only if the set of elements of the form

$$\sum_{k=1}^{N} b_k^* a_k$$

where $N \in \mathbb{N}$ and $a_1, \dots, a_N, b_1, \dots, b_N \in \overline{\mathcal{P}}$, is dense in $\mathcal{A}$ in C^*-algebra norm topology.

Since we assume that $\mathcal{A}$ is unital the coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ are positive continuous functionals satisfying the condition $\omega(\mathbb{I}) = 1$. The set of all coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ will be denoted by $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. The structure of $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ is investigated and described in the next section of this paper. Some properties of coherent states are however needed now for the description of algebra $\mathcal{A}_{\mathcal{K}}$ defined by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$.

Theorem 11.11. *Let $\rho \neq 0$ be a positive linear functional on $(\mathcal{A}, \overline{\mathcal{P}})$. Assume that $\rho \leqslant \omega$, where $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ is a coherent state. Then*

i) the functional $\frac{1}{\rho(\mathbb{I})}\rho$ is the coherent state and

$$\frac{1}{\rho(\mathbb{I})}\rho(a) = \omega(a)$$

for $a \in \overline{\mathcal{P}}$.

ii) If $(\mathcal{A}, \overline{\mathcal{P}})$ admits the norm normal ordering then

$$\frac{1}{\rho(\mathbb{I})}\rho = \omega,$$

i.e. the coherent state ω is pure.

Proof.

i) Let $\pi_\omega : \mathcal{A} \longrightarrow \mathcal{H}_\omega$ be the GNS representation of $\mathcal{A}$ and let $v_\omega \in \mathcal{H}_\omega$ be the cyclic vector of this reprezentation corresponding to ω. Then there exists an operator $T \in \pi_\omega(\mathcal{A})'$, $0 \leqslant T \leqslant 1$, such that

$$\rho(x) = \langle T v_\omega | \pi_\omega(x) T v_\omega \rangle \tag{11.30}$$

for any $x \in \mathcal{A}$, see [10, 30]. From the defining property (11.28) of the coherent state one gets

$$\langle v_\omega | \pi_\omega(x)(\pi_\omega(a) - \omega(a)) v_\omega \rangle = 0.$$

Since v_ω is cyclic for $\pi_\omega(\mathcal{A})$ it must be

$$\pi_\omega(a) v_\omega = \omega(a) v_\omega \tag{11.31}$$

for any $a \in \overline{\mathcal{P}}$. From (11.30) and (11.31) it follows that

$$\rho(xa) = \rho(x)\omega(a) \tag{11.32}$$

for an $x \in \mathcal{A}$ and $a \in \overline{\mathcal{P}}$. Taking $x = \mathbb{I}$ in (11.32) we get $\frac{1}{\rho(\mathbb{I})}\rho(a) = \omega(a)$. Substituting $\omega(a) = \frac{1}{\rho(\mathbb{I})}\rho(a)$ into (11.32) and dividing both sides of (11.32) by $\rho(\mathbb{I}) \neq 0$ we find that $\frac{1}{\rho(\mathbb{I})}\rho(a)$ belongs to $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$.

ii) Since $\frac{1}{\rho(\mathbb{I})}\rho(a)$ is equal to ω on $\overline{\mathcal{P}}$ we have

$$\frac{1}{\rho(\mathbb{I})}\rho\left(\sum_{k=1}^{N} b_k^* a_k\right)=\sum_{k=1}^{N} \overline{\frac{1}{\rho(\mathbb{I})}\rho(b_k)}\frac{1}{\rho(\mathbb{I})}\rho(a_k) =\sum_{k=1}^{N} \overline{\omega(b_k)}\omega(a_k) = \omega\left(\sum_{k=1}^{N} b_k^* a_k\right).$$

From the existence of the normal ordering on $(\mathcal{A}, \overline{\mathcal{P}})$ and continuity of ρ and ω it follows that $\frac{1}{\rho(\mathbb{I})}\rho = \omega$ on $\mathcal{A}$.

□

Let us remark that the norm normal ordering property of the polarized C^*-algebra $\mathcal{A}$ is stronger than the normal ordering in the Heisenberg quantum mechanics or quantum field theory where it is considered in the weak topology sense.

One of the commonly accepted principles of quantum theory is irreducibility of the algebra of quantum observables. For the Heisenberg-Weyl algebra case the irreducible representations are equivalent to Schrödinger representation due to the von Neumann theorem [45]. In the case of general coherent states map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ the irreducibility of the corresponding algebra $\mathcal{A}_{\mathcal{K}}$ of observables depends on the existence of the norm normal ordering.

Theorem 11.12. *Let $\mathcal{A}_{\mathcal{K}}$ be polarized algebra of observables defined by the coherent states map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$. If M is connected and there exists the norm normal ordering on $\mathcal{A}_{\mathcal{K}}$ then the auto-representation* $\mathrm{id} : \mathcal{A}_{\mathcal{K}} \longrightarrow L^\infty(\mathcal{H})$ *is irreducible.*

Proof. It was stated in Theorem 11.11 that the vector coherent state $\mathcal{K}(m_1)$ is pure one. This implies irreducibility of representation

$$\pi_{m_1} := \mathrm{id}_{|\mathcal{H}_{m_1}} : \mathcal{A}_{\mathcal{K}} \longrightarrow \mathrm{End}\,\mathcal{H}_{m_1}$$

of the algebra $\mathcal{A}_{\mathcal{K}}$ in the Hilbert subspace $\mathcal{H}_{m_1} = \mathcal{A}\mathcal{K}(m_1)$. There are two possibilities: either $\mathcal{K}(m_2) \subset \mathcal{H}_{m_1}$ for any $m_2 \in M$ or there exists $m_2 \in M$ such that $\mathcal{K}(m_2) \not\subseteq \mathcal{H}_{m_1}$. In the second case it follows from irreducibility of representation

$$\pi_{m_2} := \mathrm{id}_{|\mathcal{H}_{m_2}} : \mathcal{A}_{\mathcal{K}} \longrightarrow \mathrm{End}\,\mathcal{H}_{m_2}$$

that $\mathcal{H}_{m_2} \subset \mathcal{H}_{m_1}^{\perp}$. Applying this procedure step by step one obtain the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_{m_i}$$

of the Hilbert space $\mathcal{H}$. From assumed separability of $\mathcal{H}$ we find that I is at most countable.

Let

$$M_i := \{m \in M : \mathcal{K}(m) \subset \mathcal{H}_{m_i}\}$$

where $i \in I$. If $m \in M_i \cap \Omega_\alpha$ then

$$\langle K_\alpha(m)|K_\alpha(m)\rangle > 0.$$

Since $K_\alpha : \Omega_\alpha \to \mathbb{C}$ is continuous there exists a open neighborhood $m \in \mathcal{O} \subset \Omega_\alpha$ of m such that

$$\langle K_\alpha(m)|K_\alpha(m')\rangle \neq 0$$

for $m' \in \mathcal{O}$. The following inclusion must be valid $\mathcal{K}(\mathcal{O}) = \big[K_\alpha(\mathcal{O})\big] \subset \mathcal{H}_{m_i}$. Otherwise one would have

$$\langle K_\alpha(m)|K_\alpha(m')\rangle = 0$$

which contradicts the definition of the set $\mathcal{O}$. In this way we have shown that $\mathcal{O} \subset M_i$ and M_i is open in M. Thus M is disjoint union

$$M = \bigcup_{i \in I} M_i$$

of the open sets. Since, by assumption, M is connected it must be $M = M_i$ for some $i \in I$. The above means that $\mathcal{H} = \mathcal{A}_\mathcal{K}\mathcal{K}(m)$ for any $m \in M$ and consequently the representation

$$\mathrm{id} : \mathcal{A}_\mathcal{K} \longrightarrow L^\infty(\mathcal{H})$$

is irreducible. □

In general case one can decompose the Hilbert space $\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i$, where $N \in \mathbb{N}$ or $N = \infty$, on the invariant $\mathcal{A}_\mathcal{K}\mathcal{H}_i \subset \mathcal{H}_i$ orthogonal Hilbert subspaces. Superposing $\mathcal{K} : M \to \mathbb{CP}(\mathcal{H})$ with the orthogonal projectors $P_i : \mathcal{H} \to \mathcal{H}_i$ one obtains the family of coherent state maps $\mathcal{K}_i := P_i \circ \mathcal{K} : M \to \mathbb{CP}(\mathcal{H}_i)$, $i = 1, \dots N$. One has $\mathcal{A}_{\mathcal{K}_i} = P_i \mathcal{A}_\mathcal{K} P_i$ and the decomposition $\mathcal{A}_\mathcal{K} = \bigoplus_{i=1}^N A_{\mathcal{K}_i}$ is consistent with the decomposition

$$K_\alpha(m) = \sum_{i=1}^N (P_i \circ K_\alpha)(m), \quad m \in \Omega_\alpha \tag{11.33}$$

of the coherent state map.

Example 11.1 (*Toeplitz Algebra*)**.**

Fix an orthonormal basis $\{|n\rangle\}_{n=1}^\infty$ in the Hilbert space M. The coherent states map $\mathcal{K} : \mathbb{D} \to \mathbb{CP}(\mathcal{H})$ is defined by

$$\mathbb{D} \ni z \longrightarrow \mathcal{K}(z) := \sum_{n=1}^\infty z^n |n\rangle \tag{11.34}$$

where $\mathcal{K}(z) = [K(z)]$.

Quantum polarization $\overline{\mathcal{P}}_\mathcal{K}$ is generated in this case by the one-side shift operator

$$a|n\rangle = |n-1\rangle \tag{11.35}$$

which satisfies

$$aa^* = \mathbb{I} \tag{11.36}$$

From this relation it follows that the algebra $\mathcal{A}_{\mathcal{K}}$ of physical observables generated by the coherent states map (11.34) is Toeplitz C^*-algebra. The existence of normal ordering in $(\mathcal{A}_{\mathcal{K}}, \overline{\mathcal{P}}_{\mathcal{K}})$ is guaranteed by the property that monomial

$$a^{*k}a^l \qquad k, l \in \mathbb{N} \cup \{0\}$$

are linearly dense in $\mathcal{A}_{\mathcal{K}}$.

Let us finally remark that the space $I(\mathcal{H})$ is exactly the Hardy space $H^2(\mathbb{D})$, see [11, 46]. According to the Theorem 11.12 the auto-representation of Toeplitz algebra is irreducible as the unit disc $\mathbb{D}$ is connected and there exists the norm normal ordering in $\mathcal{A}_{\mathcal{K}}$.

Example 11.2 (quantum disc algebra)**.**

Following [38] one can generalize the construction presented in Example 11.1 taking

$$\mathbb{D}_{\mathcal{R}} \ni z \longrightarrow K_{\mathcal{R}}(z) := \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\mathcal{R}(q) \cdots \mathcal{R}(q^n)}} |n\rangle, \tag{11.37}$$

where $0 < q < 1$ and $\mathcal{R}$ is a meromorphic function on $\mathbb{C}$ such that $\mathcal{R}(q^n) > 0$ for $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R}(1) = 0$. For $z \in \mathbb{D}_{\mathcal{R}} := \{z \in \mathbb{C} : |z| < \sqrt{\mathcal{R}(0)}\}$ one has $K_{\mathcal{R}}(z) \in \mathcal{H}$ and the coherent state map $\mathcal{K} : \mathbb{D}_{\mathcal{R}} \to \mathbb{CP}(\mathcal{H})$ is defined by $\mathcal{K}_{\mathcal{R}}(z) = \mathbb{C}K_{\mathcal{R}}(z)$. The annihilation a and creation a^* operators defined by (11.37) satisfy the relations

$$\begin{aligned} a^*a &= \mathcal{R}(Q) \\ aa^* &= \mathcal{R}(qQ) \\ aQ &= qQa \\ Qa^* &= qa^*Q, \end{aligned} \tag{11.38}$$

where the compact self-adjoint operator Q is defined by $Q|n\rangle = q^n|n\rangle$. Hence one obtains the class of C^*-algebras $\mathcal{A}_{\mathcal{R}}$ parametrized by the meromorphic functions $\mathcal{R}$, which includes the q-Heisenberg-Weyl algebra of one degree of freedom and the quantum disc in sense of [23] if

$$\mathcal{R}(x) = \frac{1-x}{1-q} \tag{11.39}$$

and

$$\mathcal{R}(x) = r\frac{1-x}{1-\rho x}, \tag{11.40}$$

where $0 < r, \rho \in \mathbb{R}$, respectively. These algebras find the application for the integration of quantum optical models, see [18]. For the rational $\mathcal{R}$ they also can be considered as the symmetry algebras in the theory of the basic hypergeometric series, see [38]

Example 11.3 (*q-Heisenberg-Weyl Algebra*)**.**

Let M be the polydisc $\mathbb{D}_q\times\cdots\times\mathbb{D}_q$, where $\mathbb{D}_q\subset\mathbb{C}$ is the disc of radius $\frac{1}{\sqrt{1-q}}$, $0<q<1$. The orthonormal basis in the Hilbert space $\mathcal{H}$ will be parameterized in the following way

$$\{|n_1\dots n_N\rangle\}$$

where $n_1,\dots,n_N\in\mathbb{N}\cup\{0\}$, and

$$\langle n_1\dots n_N|k_1\dots k_N\rangle=\delta_{n_1k_1}\dots\delta_{n_Nk_N}$$

The coherent states map

$$\mathcal{K}:\mathbb{D}_q\times\cdots\times\mathbb{D}_q\longrightarrow\mathbb{CP}(\mathcal{H})$$

is defined by $\mathcal{K}(z_1,\dots,z_N)=[K(z_1,\dots,z_N)]$ where

$$K(z_1,\dots,z_N):=\sum_{k_1,\dots,k_N=0}^{\infty}\frac{z_1^{k_1}\cdots z_N^{k_N}}{\sqrt{[k_1]!_q\cdots[k_N]!_q}}|k_1\dots k_N\rangle \tag{11.41}$$

The standard notation

$$[n]:=1+\cdots+q^{n-1}$$
$$[n]!_q:=[1]\cdots[n]$$

was used in (11.41).

The quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}}$ is the algebra generated by the operators $a_1,\dots,a_N$ defined by

$$a_iK(z_1,\dots,z_N)=z_iK(z_1,\dots,z_N) \tag{11.42}$$

It is easy to show that $\|a_i\|=\frac{1}{\sqrt{1-q}}$. Hence $\overline{\mathcal{P}}_{\mathcal{K}}$ is commutative and algebra $\mathcal{A}_{\mathcal{K}}$ of all quantum observables is generated by $\mathbb{I},a_1,\dots,a_N,a_1^*,\dots,a_N^*$ satisfying the relations

$$[a_i,a_j]=[a_i^*,a_j^*]=0$$
$$a_ia_j^*-qa_j^*a_i=\delta_{ij}\mathbb{I}. \tag{11.43}$$

The C^*-algebra $\mathcal{A}_{\mathcal{K}}$ is then the q-deformation of Heisenberg-Weyl algebra, see [20]. The structural relations (11.43) imply that $a_i^{*k}a_j^l$, where $i,j=1,\dots,N$ and $k,l\in\mathbb{N}\cup\{0\}$ do form linearly dense subset in $\mathcal{A}_{\mathcal{K}}$. Consequently $\mathcal{A}_{\mathcal{K}}$ admits the norm normal ordering. Since the polydisc is connected the auto-representation of $\mathcal{A}_{\mathcal{K}}$ is irreducible.

In the limit $q\to1$ $\mathcal{A}_{\mathcal{K}}$ becomes the standard Heisenberg-Weyl algebra for which the creation and annihilation operators are unbounded.

The method of quantization of classical phase space which we have presented and illustrated by examples can be included in the general scheme of quantization given by Definition 6.3. In order to se this let us notice that because of the resolution of the identity (11.19) the coherent state map $\mathcal{K}:M\to\mathbb{CP}(\mathcal{H})$ defines the projection

$$\Pi:L^2(M,\Gamma(\overline{\mathbb{L}}^*),d\mu)\to I(\mathcal{H}) \tag{11.44}$$

of the Hilbert space of sections $\psi : M \to \overline{\mathbb{L}}^*$ square integrable with respect to $d\mu$, i.e. such that $\int_M \overline{H}^*(\psi, \psi) d\mu < \infty$ on its Hilbert subspace $I(\mathcal{H}) \subset L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)$.

Using the projector Π given above we define conditional expectation $\mathfrak{E}_\Pi : L^\infty(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)) \to L^\infty(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu))$ by (5.8).

On the other side one has naturally defined logic morphism E given by

$$E : \mathcal{B}(M) \ni \Omega \longrightarrow M_{\chi_\Omega} \in \mathcal{L}(L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)), \tag{11.45}$$

where the projector $M_{\chi_\Omega} : L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu) \to L^2(M, \Gamma(\overline{\mathbb{L}}^*), d\mu)$ is given as multiplication by indicator function χ_Ω of the Borel set Ω.

One can show that the quantum phase space $\mathcal{A}_\mathcal{K}$ given by Definition 11.7 coincides with $\mathcal{A}_{M,\mathfrak{E}_\Pi,E}$ related to conditional expectation $\mathfrak{E}_\Pi$ and logic morphism E defined above in sense of Definition 6.3.

12 Quantum complex Minkowski space

This section is based on [19].

Extending the Poincaré group by dilation and acceleration transformations, one obtains the conformal group $SU(2,2)/\mathbb{Z}_4$, which is the symmetry group of the conformal structure of compactified Minkowski space-time M, where $\mathbb{Z}_4 = \{i^k \,\mathrm{id} : k = 0,1,2,3\}$ is the centralizer of $SU(2,2)$. According to the prevailing point of view $SU(2,2)/\mathbb{Z}_4$ is the symmetry group for physical models which describe massless fields or particles, but has no application to the theory of massive objects. However, using the twistor description [42] of Minkowski space-time and the orbit method [22], the different orbits of $SU(2,2)/\mathbb{Z}_4$ in the conformally compactified complex Minkowski space $\mathfrak{M} := M^{\mathbb{C}}$ may be considered to be the classical phase spaces of massless and massive scalar conformal particles, antiparticles and tachyons, see [32, 34].

The motivation for various attempts to construct models of non-commutative Minkowski space-time is the belief that this is the proper way to avoid divergences in quantum field theory [28]. Here, on the other hand, our aim is to quantize the classical phase space $\mathfrak{M}^{++} \subset \mathfrak{M}$ of the massive particle by replacing it by the Toeplitz-like operator C^*-algebra $\mathcal{M}^{++}$. To this end we first quantize the classical states of the massive scalar conformal particle by constructing the coherent state map $\mathcal{K} : \mathfrak{M}^{++} \to \mathbb{CP}(\mathcal{H})$ of $\mathfrak{M}^{++}$ into the complex projective Hilbert space $\mathbb{CP}(\mathcal{H})$, i.e. the space of the pure states of the system. In the next step we define the Banach algebra $\overline{\mathcal{P}}^{++}$ of annihilation operators as the ones having the coherent states $\mathcal{K}(m)$, $m \in \mathfrak{M}^{++}$, as eigenvectors. Finally, the quantum phase space $\mathcal{M}^{++}$ will be the C^*-algebra generated by $\overline{\mathcal{P}}^{++}$.

12.1 Complex Minkowski space as the phase space of the conformal scalar massive particle

Following [32, 33, 34], we present the twistor description of phase spaces of the conformal scalar massive particles. Let us recall that twistor space $\mathbf{T}$ is $\mathbb{C}^4$

equipped with a Hermitian form η of signature $(++--)$. The symmetry group of $\mathbf{T}$ is the group $\mathrm{SU}(2,2)$, where $g \in \mathrm{SU}(2,2)$ iff $g^\dagger \eta g = \eta$ and $\det g = 1$.

In relativistic mechanics the elementary phase spaces are given by the coadjoint orbits of the Poincaré group, see [52], which are parametrized in this case by mass, spin, and signature of the energy of the relativistic particle. Similarly, elementary phase spaces for conformal group one identifies with its coadjoint orbits. Since conformal Lie algebra $su(2,2)$ is simple we will identify its dual $su(2,2)^*$ with $su(2,2)$ using Cartan-Killing form:

$$\langle X, Y\rangle = \frac{1}{2}\operatorname{Tr}(XY), \tag{12.1}$$

where $X, Y \in$ su(2,2). Thus the coadjoint representation $\mathrm{Ad}_g^* : su(2,2)^* \to su(2,2)^*$ is identified with the adjoint one

$$\mathrm{Ad}_g X = gXg^{-1}, \tag{12.2}$$

where $g \in \mathrm{SU}(2,2)$. For the complete description and physical interpretation of $\mathrm{Ad}^\star(\mathrm{SU}(2,2))$-orbits see [21, 33].

One defines the compactified complex Minkowski space $\mathbb{M}$ as the Grassmannian of 2-dimensional complex vector subspaces $w \subset \mathbf{T}$ of the twistor space and $\mathrm{SU}(2,2)$ acts on $\mathbb{M}$ by

$$\sigma_g : w \to gw. \tag{12.3}$$

The Grassmannian $\mathbb{M}$ splits into the orbits $\mathfrak{M}^{kl}$ indexed by the signatures of the restricted Hermitian forms $sign\ \eta|_z = (k,l)$, where $k, l = +, -, 0$.

The orbit $\mathbb{M}^{00}$ consisting of subspaces isotropic with respect to η is the conformal compactification M of real Minkowski space and $\mathbb{M}$ is the complexification of $M = \mathbb{M}^{00}$.

The cotangent bundle $T^\star\mathbb{M}^{00} \to \mathbb{M}^{00}$ is isomorphic with the vector bundle $\{(x,X) \in \mathbb{M}^{00} \times su(2,2) : \operatorname{im} X \subset x \subset \ker X\} =: \mathbb{N} \xrightarrow{pr_1} \mathbb{M}^{00}$, where pr_1 is the projection on the first component of the product $\mathbb{M}^{00} \times su(2,2)$. The vector bundle isomorphism $T^*\mathbb{M}^{00} \cong \mathbb{N}$ is defined by the following sequence $T^*_x\mathbb{M}^{00} \cong (su(2,2)\diagup su(2,2)_x)^* \cong \{X \in su(2,2) : \operatorname{Tr} YX = 0 \ \ \forall Y \in su(2,2)_x\} \cong \{X \in su(2,2) : \operatorname{im} X \subset x \subset \ker X\} = pr_1^{-1}(x)$ of the vector space isomorphisms.

There exists a conformal structure on $\mathbb{N}$ defined by the cones $C_x := \{X \in pr_1^{-1}(x) : \dim_{\mathbb{R}} \operatorname{im} X \leqslant 1\} \subset pr_1^{-1}(x) \cong T^*_x\mathbb{M}^{00}$, $x \in \mathbb{M}^{00}$. This conformal structure is invariant with respect to the action of $\mathrm{SU}(2,2)$ on $\mathbb{N}$ defined by

$$\alpha_g : (x, X) \mapsto (gx, gXg^{-1}) \tag{12.4}$$

for $g \in \mathrm{SU}(2,2)$.

The 8-dimensional orbits of the action (12.4) are: the bundle $\mathbb{N}^{++} \to \mathbb{M}^{00}$ of upper halves of the interiors of the cones, the bundle $\mathbb{N}^{--} \to \mathbb{M}^{00}$ of bottom halves of the interiors of the cones and the bundle $\mathbb{N}^{+-} \to \mathbb{M}^{00}$ of exteriors of the cones.

Similarly, the action (12.3) of $\mathrm{SU}(2,2)$ on $\mathfrak{M}$ generates three 8-dimensional orbits: $\mathbb{M}^{++}$, $\mathbb{M}^{--}$ and $\mathbb{M}^{+-}$.

One has maps $J_0 : \tilde{\mathbb{N}} \to su(2,2)$ and $J_\lambda : \tilde{\mathbb{M}} \to su(2,2)$ of $\tilde{\mathbb{N}} := \mathbb{N}^{++} \cup \mathbb{N}^{--} \cup \mathbb{N}^{+-}$ and $\tilde{\mathbb{M}} := \mathbb{M}^{++} \cup \mathbb{M}^{--} \cup \mathbb{M}^{+-}$ into $su(2,2)$ defined by:

$$J_0(x, X) := X \tag{12.5}$$

$$J_\lambda(w) := i\lambda(\pi_w - \pi_{w^\perp}), \tag{12.6}$$

where $\perp: \tilde{\mathbb{M}} \mapsto \tilde{\mathbb{M}}$ is the involution, which maps $w \in \tilde{\mathbb{M}}$ on its orthogonal complement $w^\perp$ (with respect to the twistor forms η) and $\pi_w : \mathbf{T} \mapsto \mathbf{T}$ and $\pi_{w^\perp} : \mathbf{T} \mapsto \mathbf{T}$ are the projections defined by the decomposition $\mathbf{T} = w \oplus w^\perp$.

The maps J_0 and J_λ are equivariant with respect to the actions α and σ respectively and Ad-action of the conformal group. Thus J_0 maps $\mathbb{N}^{++}$, $\mathbb{N}^{+-}$, $\mathbb{N}^{--}$ on the 8-dimensional nilpotent Ad-orbits and J_λ maps $\mathbb{M}^{++}, \mathbb{M}^{++}, \mathbb{M}^{++}$ on the 8-dimensional simple Ad-orbits which consist of $X \in su(2,2)$ with eigenvalues $i\lambda$ and $-i\lambda$. Using the Kirillov construction [22] we obtain the conformally invariant symplectic form ω_0 on $\tilde{\mathbb{N}}$ (identical with the canonical symplectic form of $T^*\mathbb{M}^{00}$) and the conformally invariant Kähler form ω_λ on $\tilde{\mathbb{M}}$. So $(\tilde{\mathbb{N}}, \omega_0)$ and $(\tilde{\mathbb{M}}, \omega_\lambda)$ are 8-dimensional conformal symplectic manifolds with momentum maps given by (12.5),(12.6).

In order to show that $\tilde{\mathbb{N}}$ and $\tilde{\mathbb{M}}$ have a physical interpretation of the phase spaces of the conformal scalar massive particles, let us take the coordinate description of the presented models. We fix an element $\infty \in \mathbb{M}^{00}$, called point at infinity. One defines the Minkowski space $\mathbb{M}^{00}_\infty$ as the affine space of elements $w \in \mathbb{M}^{00}$ which are transversal to ∞, i.e. $w \oplus \infty = \mathbf{T}$. The elements $w \in \mathbb{M}^{00}$ which intersect with ∞ in more than one-dimension, i.e. $\dim_{\mathbb{C}}(w \cap \infty) \geqslant 1$, form a cone C_∞ at infinity, so

$$\mathbb{M}^{00} = \mathbb{M}^{00}_\infty \cup C_\infty \cong \mathbb{S}^1 \times \mathbb{S}^3.$$

The cones $C_x = \{x' \in \mathbb{M}^{00} : \dim_{\mathbb{C}}(x \cap x') \geqslant 1\}$ define a conformal structure on $\mathbb{M}^{00}$, invariant with respect to the conformal group action given by (12.3).

The Poincaré group $\mathbf{P}_\infty$ extended by the dilations is defined as the stabilizer $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty$ of the element ∞. The intersections of the stabilizers $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty \cap (\mathrm{SU}(2,2)/\mathbb{Z}_4)_0$, where $0 \in \mathbb{M}^{00}_\infty$ is the origin of the inertial coordinates system, is the Lorentz group extended by dilations. One defines the Lorentz group $\mathbf{L}_{0,\infty}$ and the group of dilations $\mathbf{D}_{0,\infty}$ respectively as the commutator and the centralizer of $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty \cap (\mathrm{SU}(2,2)/\mathbb{Z}_4)_0$ respectively. Finally, the group of Minkowski space translations $\mathbf{T}_\infty$ consists of the elements $\exp X$, where $X \in su(2,2)$ satisfies $\operatorname{im} X \subset \infty \subset \ker X$, while the elements $\exp X$ fulfilling $\operatorname{im} X \subset 0 \subset \ker X$, define the commutative subgroup $\mathbf{A}_0$ of four-accelerations.

Let us assume in the following that

$$\eta = i\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \infty = \left\{ \begin{pmatrix} \zeta \\ 0 \end{pmatrix} : \zeta \in \mathbb{C}^2 \right\}, \mathbf{0} = \left\{ \begin{pmatrix} 0 \\ \zeta \end{pmatrix} : \zeta \in \mathbb{C}^2 \right\}, \tag{12.7}$$

where we use the 2×2 matrix representation with Pauli basis:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $Mat_{2\times 2}(\mathbb{C})$. This choice of $\eta, \infty, \mathbf{0}$ gives us the decomposition

$$su(2,2) = \mathcal{T}_\infty \oplus \mathcal{L}_{0,\infty} \oplus \mathcal{D}_{0,\infty} \oplus \mathcal{A}_0 \tag{12.8}$$

where the subalgebras of 4-translations, Lorentz, dilations and 4-accelerations are given respectively by

$$\mathcal{T}_\infty = \{\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : T = T^\dagger \in Mat_{2\times 2}(\mathbb{C}) \text{ and } T = t^\mu \sigma_\mu\} \tag{12.9a}$$

$$\mathcal{L}_{0,\infty} = \{\begin{pmatrix} L & 0 \\ 0 & -L^\dagger \end{pmatrix} : \operatorname{Tr} L = 0 \text{ and } L \in Mat_{2\times 2}(\mathbb{C})\} \tag{12.9b}$$

$$\mathcal{D}_{0,\infty} = \{d \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} : d \in \mathbb{R}\} \tag{12.9c}$$

$$\mathcal{A}_0 = \{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C = C^\dagger \in Mat_{2\times 2}(\mathbb{C}) \text{ and } C = c^\mu \sigma_\mu\} \tag{12.9d}$$

The basis of $su(2,2)^* \cong su(2,2)$ dual to the one defined by Pauli matrices in the Lie subalgebras $\mathcal{T}_\infty$, $\mathcal{L}_{0,\infty}$, $\mathcal{D}_{0,\infty}$, $\mathcal{A}_0$ is

$$\mathcal{T}^*_\infty \ni \mathcal{P}^*_\mu = \begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix} \tag{12.10a}$$

$$\mathcal{L}^*_{0,\infty} \ni \mathcal{L}^*_{kl} = \frac{1}{2}\epsilon_{klm} \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} \qquad \mathcal{L}^*_{0,\infty} \ni \mathcal{L}^*_{0k} = \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \tag{12.10b}$$

$$\mathcal{D}^*_{0,\infty} \ni \mathcal{D}^* = \frac{1}{2} \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \tag{12.10c}$$

$$\mathcal{A}^*_0 \ni \mathcal{A}^*_\nu = \begin{pmatrix} 0 & \sigma_\nu \\ 0 & 0 \end{pmatrix} \tag{12.10d}$$

One has the matrix coordinate map

$$\mathbb{M}_\infty \ni w \mapsto W \in Mat_{2\times 2}(\mathbb{C}) \tag{12.11}$$

defined by

$$w = \{\begin{pmatrix} W\zeta \\ \zeta \end{pmatrix} : \zeta \in \mathbb{C}^2\} \tag{12.12}$$

and $w = x \in \mathbb{M}^{00}_\infty$ iff $W = W^\dagger = X$. The element $(x, \mathcal{X}) \in pr_1^{-1}(\mathbb{M}^{00}_\infty)$ is parametrized by

$$(x, \mathcal{X}) \mapsto (X, \begin{bmatrix} XS & -XSX \\ S & -SX \end{bmatrix}), \tag{12.13}$$

where $X, S \in H(2)$ and $H(2)$ is the vector space of 2×2 Hermitian matrices.

The momentum maps (12.5) and (12.6) in the above defined coordinates are given by

$$J_0(X, S) = \begin{bmatrix} XS & -XSX \\ S & -SX \end{bmatrix} \tag{12.14}$$

$$J_\lambda(W) = i\lambda \begin{bmatrix} (W+W^\dagger)(W-W^\dagger)^{-1} & -2W(W-W^\dagger)^{-1}W^\dagger \\ 2(W-W^\dagger)^{-1} & -\sigma_0 - 2(W-W^\dagger)^{-1}W^\dagger \end{bmatrix}. \quad (12.15)$$

By decomposing $J_0(X,S)$ in the basis (12.10) $J_0(X,S) = p_\mu \mathcal{P}^*_\mu + m_{\mu\nu}\mathcal{L}^*_{\mu\nu} + a_\mu \mathcal{A}^*_\nu + d\mathcal{D}^*$ we obtain the expressions

$$p_\mu = s_\mu \quad (12.16)$$

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad (12.17)$$

$$d = x^\mu p_\mu \quad (12.18)$$

$$a_\mu = -2(x^\nu p_\nu)x_\mu + x^2 p_\mu \quad (12.19)$$

for the four-momentum p_μ, relativistic angular momentum $m_{\mu\nu}$, dilation d and four-acceleration a_ν respectively, where $S = s^\mu \sigma_\mu$, $X = x^\mu \sigma_\mu$.

In the coordinates x_μ, $p_\mu = s_\mu$ the symplectic form ω_0 assumes the canonical form

$$\omega_0 = dx^\mu \wedge dp_\mu. \quad (12.20)$$

Similarly, from $J_\lambda(W) = p_\mu \mathcal{P}^*_\mu + m_{\mu\nu}\mathcal{L}^*_{\mu\nu} + a_\mu \mathcal{A}^*_\nu + d\mathcal{D}^*$ we obtain

$$p^\nu = \lambda \frac{y^\nu}{y^2} \quad (12.21)$$

$$m_{\mu\nu} = x_\mu p_\nu - p_\nu x_\mu \quad (12.22)$$

$$d = x^\mu p_\mu \quad (12.23)$$

$$a_\mu = -2(x^\nu p_\nu)x_\mu + x^2 p_\mu - \frac{\lambda^2}{p^2} p_\mu, \quad (12.24)$$

where the real coordinates x_μ, y_μ on $\tilde{\mathbb{M}}$ are defined by $x^\nu + iy^\nu = w^\nu := \frac{1}{2}\operatorname{Tr}(W\sigma_\nu)$.

The coordinate description of ω_λ is the following

$$\omega_\lambda = i\lambda \frac{\partial^2}{\partial w^\mu \partial \bar{w}^\nu} \log(w-\bar{w})^2 dw^\mu \wedge d\bar{w}^\nu = dx^\nu \wedge dp_\nu. \quad (12.25)$$

Concluding, one has two models $(\mathbb{N}, \omega_0)$ and $(\mathbb{M}, \omega_\lambda)$ of the massive scalar conformal particle. Using the canonical coordinates (x^μ, p_ν) common for both models we obtain that

(i) the element $(x, \mathcal{X}) \in \mathbb{N}^{++}$ ($w \in \mathbb{M}^{++}$) iff $p^0 > 0$ and $(p^0)^2 - \overrightarrow{p}^2 > 0$, i.e. it describes the state of a conformal scalar massive particle;

(ii) the element $(x, \mathcal{X}) \in \mathbb{N}^{--}$ ($w \in \mathbb{M}^{--}$) iff $p^0 < 0$ and $(p^0)^2 - \overrightarrow{p}^2 > 0$, i.e. it describes the state of a conformal scalar massive anti-particle;

(iii) the element $(x, \mathcal{X}) \in \mathbb{N}^{+-}$ ($w \in \mathbb{M}^{+-}$) iff $(p^0)^2 - \overrightarrow{p}^2 < 0$, i.e. it describes the state of a conformal scalar tachyon.

The orbits $\mathbb{N}^{0+}$ ($\mathbb{M}^{0+}$) and $\mathbb{N}^{0-}$ ($\mathbb{M}^{0-}$) describe the states of massless particles and anti-particles but this case will not be discussed further.

Two above presented models do not differ if one considers them on the level of relativistic mechanics, since both of them behave towards Poincaré transformations in the same way. The difference appears if one considers the four-acceleration transformations parametrized by $C = c^\mu\sigma_\mu$, which in canonical coordinates $X = x^\mu\sigma_\mu$, $P = p^\mu\sigma_\mu$ are

$$\tilde{X} = X(CX + \sigma_0)^{-1}, \tag{12.26}$$

$$\tilde{P} = (CX + \sigma_0)P(XC + \sigma_0) \tag{12.27}$$

for the standard model $\tilde{\mathbb{N}}$ and

$$\tilde{X} = [XP + i\lambda\sigma_0 - i\lambda(XC - i\lambda P^{-1}C + \sigma_0)](CXP + i\lambda C + P)^{-1} \tag{12.28}$$

$$\tilde{P} = (CX + \sigma_0)P(XC + \sigma_0) + \lambda^2 CP^{-1}C \tag{12.29}$$

for the holomorphic model $\tilde{\mathbb{M}}$. We see from (12.29) that in the holomorphic model (opposite to the standard one) the four-momentum $P = p^\mu\sigma_\mu$ transforms in a non-linear way. This fact implies a lot of important physical consequences, e.g. the conformal scalar massive particle cannot be localized in the space-time in conformally invariant way. From (12.24), (12.28), (12.29) it follows that the holomorphic model corresponds to the nilpotent one when $\lambda \to 0$.

12.2 Conformally invariant quantum Kähler polarization

In this section we shall make the first step in the direction to construct quantum conformal phase space. Since the case of the antiparticle can be transformed by the charge conjugation map to the particle one, see [34], and the tachyon case is less interesting from physical point of view, we will work only with the phase space $\mathbb{M}^{++}$ of the conformal scalar massive particle.

The phase space $T^*\mathbb{M}^{00}$ has the real conformally invariant polarization defined by the leaves of its cotangent bundle structure. In canonical coordinates this polarization is spanned by the vector fields $\{\frac{\partial}{\partial p^\nu}\}_{\nu=0,\dots 3}$. For the holomorphic phase space $\tilde{\mathbb{M}}$ the conformally invariant polarization is Kähler and in the complex coordinate it is spanned by $(\frac{\partial}{\partial \bar{w}^\mu})_{\mu=0,\dots 3}$. The reason is that $SU(2,2)/\mathbb{Z}_4$ acts on $\tilde{\mathbb{M}}$ by biholomorphism. For $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$ and $w \in \mathbb{M}^{++}$ one has

$$\sigma_g W = (AW + B)(CW + D)^{-1}, \tag{12.30}$$

where $W \in Mat_{2\times 2}(\mathbb{C})$ is the matrix holomorphic coordinate of $w \in \mathbb{M}^{++}$. Using complex matrix coordinates (12.12) one identifies $\mathbb{M}^{++}$ with the future tube

$$\mathbb{T} := \{W \in Mat_{2\times 2} : \operatorname{im} W > 0\}. \tag{12.31}$$

Applying the Caley transform

$$Z = (W - iE)(W + iE)^{-1}, W = i(Z + E)(Z - E)^{-1} \tag{12.32}$$

we map $\mathbb{T}$ on the symmetric domain

$$\mathbb{D} := \{Z \in Mat_{2\times 2}(\mathbb{C}) : E - Z^* Z > 0\}. \tag{12.33}$$

Let us remark here that the coordinates $Z \in \mathbb{D}$ correspond to the diagonal representation of the twistor form $\eta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$. Below we use both systems of coordinates.

In order to quantize $\mathbb{M}^{++}$ we will use the method of coherent state map investigated in [36]. For the other construction of noncommutative manifolds by using coherent state method see also [16]. The essence of this method consists in replacing the classical state $m \in \mathbb{M}^{++}$ by the quantum pure state, which means, that one defines the map $\mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H})$ from the classical phase space $\mathbb{M}^{++}$ into the complex projective separable Hilbert space $\mathbb{CP}(\mathcal{H})$. We will call $\mathcal{K}_\lambda$ coherent state map and in our case we will postulate that it has the following properties:

(i) $\mathcal{K}_\lambda$ is consistent with the conformal symmetry, i.e. there exists an unitary irreducible representation $\mathbf{U}_\lambda : \mathrm{SU}(2,2) \mapsto \mathrm{Aut}\,\mathcal{H}$ with respect to which the coherent state map is equivariant:

$$\begin{array}{ccc} \mathbb{M}^{++} & \xrightarrow{\ \mathcal{K}_\lambda\ } & \mathbb{CP}(\mathcal{H}) \\ \downarrow{\scriptstyle \sigma_g} & & \downarrow{\scriptstyle [\mathbf{U}_\lambda(g)]\ \ \forall g \in \mathrm{SU}(2,2)} \\ \mathbb{M}^{++} & \xrightarrow[\ \mathcal{K}_\lambda\ ]{} & \mathbb{CP}(\mathcal{H}) \end{array} \tag{12.34}$$

(ii) $\mathcal{K}_\lambda$ is consistent with the holomorphic polarization $(\frac{\partial}{\partial w^\mu}{}_{\mu=0,\dots 3})$. This denotes that $\mathcal{K}_\lambda$ is a holomorphic map.

(iii) $\mathcal{K}_\lambda$ is symplectic, i.e.

$$\mathcal{K}_\lambda^* \omega_{FS} = \omega_\lambda, \tag{12.35}$$

where ω_{FS} is Fubini-Study form on $\mathbb{CP}(\mathcal{H})$. The projective Hilbert space is considered here as Kähler manifold (thus symplectic manifold). This condition one needs for the consistence of classical dynamics with quantum dynamics.

The coherent state map $\mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H})$ fulfilling the properties postulated above one obtains by the applying of the representation theory, see [44, 43]. We skip here the technical considerations and present only the final result. Let

$$\left\{ \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right\}, \tag{12.36}$$

where $m, 2j \in \mathbb{N} \cup \{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$, denote an orthonormal basis in $\mathcal{H}$, i.e.

$$\left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \middle| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle = \delta_{jj'} \delta_{mm'} \delta_{j_1 j_1'} \delta_{j_2 j_2'}. \tag{12.37}$$

Then the map $K_\lambda : \mathbb{M}^{++} \cong \mathbb{D} \mapsto \mathcal{H}$ given by

$$K_\lambda : Z \to |Z;\lambda\rangle := \sum_{j,m,j_1,j_2} \Delta^{jm}_{j_1 j_2}(Z) \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle, \tag{12.38}$$

where

$$\Delta^{jm}_{j_1 j_2}(Z) := (N^\lambda_{jm})^{-1} (\det Z)^m \sqrt{\frac{(j+j_1)!(j-j_1)!}{(j+j_2)!(j-j_2)!}} \times \tag{12.39}$$

$$\times \sum_{\substack{S \geqslant \max\{0, j_1+j_2\} \\ S \leqslant \min\{j+j_1, j+j_2\}}} \binom{j+j_2}{S} \binom{j-j_2}{S-j_1-j_2} z_{11}^S z_{12}^{j+j_1-S} z_{21}^{j+j_2-S} z_{22}^{S-j_1-j_2}$$

and

$$N^\lambda_{jm} := (\lambda-1)(\lambda-2)^2(\lambda-3) \frac{\Gamma(\lambda-2)\Gamma(\lambda-3) m!(m+2j+1)!}{(2j+1)!\Gamma(m+\lambda-1)\Gamma(m+2j+\lambda)}, \tag{12.40}$$

defines a coherent state map

$$[K_\lambda] =: \mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H}) \tag{12.41}$$

with the properties mentioned in assumptions: (i), (ii), (iii). The condition (i) restricts the variability of the parameter $\lambda > 3$ to integer numbers.

From now on, to simplify the notation, we will write $|Z\rangle$ instead of $|z;\lambda\rangle$. If the dependence on λ is relevant we will write $|z;\lambda\rangle$.

The projectors $\frac{|Z\rangle\langle Z|}{\langle Z|Z\rangle}$ representing the coherent states give the resolution of the identity

$$\mathbf{1} = \int_{\mathbb{D}} |Z\rangle\langle Z| \; d\mu_\lambda(Z, Z^\dagger) \tag{12.42}$$

with respect to the measure

$$d\mu_\lambda(Z, Z^\dagger) = c_\lambda [\det(E - Z^\dagger Z)]^{\lambda-4} |dZ|, \tag{12.43}$$

where $|dZ|$ is the Lebesgue measure on $\mathbb{D}$ and

$$c_\lambda = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3), \tag{12.44}$$

which is equivalent to $\int_{\mathbb{D}} d\mu_\lambda = 1$.

Hence, by the anti-linear monomorphism

$$I_\lambda : \mathcal{H} \ni |\psi\rangle \to \langle\psi| \cdot ; \lambda\rangle := \psi(\cdot) \in \mathcal{O}(\mathbb{D}) \tag{12.45}$$

one identifies $\mathcal{H}$ with the range of I_λ in $\mathcal{O}(\mathbb{D})$, which is equal to the Hilbert space of holomorphic functions $L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ square integrable with respect to the measure (12.43).

The representation $I_\lambda \circ U_\lambda \circ I_\lambda^{-1}$ acts on $L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ by

$$(I_\lambda \circ U_\lambda(g) \circ I_\lambda^{-1})\psi(Z) = [\det(CZ + D)]^{-\lambda}\psi(\sigma_g(Z)), \tag{12.46}$$

i.e. it is a discrete series representation of SU(2, 2) and acts on the coherent states by

$$U_\lambda(g)|Z\rangle = [\det(CZ + D)]^{-\lambda}|\sigma_g(Z)\rangle, \tag{12.47}$$

where $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(2, 2)$, see [15, 47].

The fifteen physical quantities p_ν, $m_{\mu\nu}$, d and a_ν, μ, $\nu = 0, 1, 2, 3$, which characterize the scalar massive conformal particle form the conformal Lie algebra $su(2, 2)$ with respect to the Poisson bracket

$$\{f, g\}_\lambda(\bar{w}, w) = \frac{i}{2\lambda}\left((w - \bar{w})^2\eta^{\mu\nu} - 2(w^\mu - \bar{w}^\mu)(w^\nu - \bar{w}^\nu)\right)\left(\frac{\partial f}{\partial w^\mu}\frac{\partial g}{\partial \bar{w}^\nu} - \frac{\partial g}{\partial w^\mu}\frac{\partial f}{\partial \bar{w}^\nu}\right) \tag{12.48}$$

defined by the symplectic form ω_λ. Each one of them defines a Hamiltonian flow $\sigma_{g(t)}$ on $\mathbb{M}^{++}$ realized by the corresponding one-parameter subgroup $g(t)$, $t \in \mathbb{R}$, of SU(2, 2). By the equivariance condition (12.112) this Hamiltonian flow $\sigma_{g(t)}$ is quantized to the Hamiltonian flow on $\mathbb{CP}(\mathcal{H})$ given by the one-parameter subgroup $\mathbf{U}_\lambda(g(t))$ of representation (12.47). The generators of these one-parameter subgroups are realized in $L^2\mathcal{O}(\mathbb{T}, d\mu_\lambda)$ as follows:

$$\hat{p_\mu} = -i\frac{\partial}{\partial w^\mu} \tag{12.49}$$

$$\hat{m_{\mu\nu}} = -i(w_\mu\frac{\partial}{\partial w^\nu} - w_\nu\frac{\partial}{\partial w^\mu}) \tag{12.50}$$

$$\hat{d} = -2iw^\mu\frac{\partial}{\partial w^\mu} - 2i\lambda \tag{12.51}$$

$$\hat{a_\nu} = -iw^2(\delta_\nu^\beta - 2w_\nu w^\beta)\frac{\partial}{\partial w^\beta} + 2i\lambda w_\nu, \tag{12.52}$$

see [35]. They are quantized versions of their classical counterparts given by (12.16)-(12.19). The measure $d\mu_\lambda$ in the future tube representation is given by

$$d\mu_\lambda(W, W^\dagger) = 2^{-4}[\det(W - W^\dagger)]^{\lambda-4}|dW|. \tag{12.53}$$

It was shown in [36] that the coherent state method of quantization is equivalent to the Kostant-Souriou geometric quantization.

Besides generators (12.16)-(12.19) of the conformal Lie algebra $su(2, 2)$ there is also reason to quantize other physically important observables. In particular case the ones belonging to the family $\mathcal{O}^{++}(\mathbb{D})$ consisting of complex valued smooth functions $f : \mathfrak{M}^{++} \to \mathbb{C}$ for whose there exists bounded operators $a(f) \in L^\infty(\mathcal{H})$ such that

$$a(f)|Z\rangle = f(Z)|Z\rangle \tag{12.54}$$

for any $Z \in \mathbb{D} \cong \mathfrak{M}^{++}$. Since the coherent states $|Z\rangle$ form a linearly dense subset of $\mathcal{H}$ one has correctly defined linear map $a : \mathcal{O}^{++}(\mathbb{D}) \to L^\infty(\mathcal{H})$ of $\mathcal{O}^{++}(\mathbb{D})$ in the Banach algebra of the bounded operators.

It follows immediately from (12.54) and the resolution of identity (12.42) that

i) $\mathcal{O}^{++}(\mathbb{D})$ is the commutative algebra and $f \in \mathcal{O}^{++}(\mathbb{D})$ is holomorphic;

ii) The map $a : \mathcal{O}^{++}(\mathbb{D}) \to L^\infty(\mathcal{H})$ is an isometric

$$\|a(f)\|_\infty = \|f\|_{\sup} = \sup_{Z\in\mathbb{D}} |f(Z)| \tag{12.55}$$

monomorphism of algebras;

iii) The image $a(\mathcal{O}^{++}(\mathbb{D}))$ is uniformly closed in $L^\infty(\mathcal{H})$ (i.e. with respect to operator norm $\|\cdot\|_\infty$).

Hence, $\mathcal{O}^{++}(\mathbb{D})$ is a Banach subalgebra of the Banach algebra $H^\infty(\mathbb{D})$ of functions which are holomorphic and bounded on $\mathbb{D}$. Let us remark here that completeness of $H^\infty(\mathbb{D})$ follows from the Weierstrass theorem, see e.g. [50].

Indeed one has:

Proposition 12.1. *The Banach algebra $\mathcal{O}^{++}(\mathbb{D})$ is equal to $H^\infty(\mathbb{D})$.*

Proof. Since $I_\lambda(\mathcal{H}) = L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ we have $f\langle\psi|\ \rangle\cdot \in I_\lambda(\mathcal{H})$ for any $f \in H^\infty(\mathbb{D})$. The multiplication operator $M_f : L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ is bounded. Thus there is a bounded operator $a(f)^* : \mathcal{H} \to \mathcal{H}$ such that

$$f(Z)\langle\psi|Z\rangle = \langle a(f)^*\psi|Z\rangle \tag{12.56}$$

for $Z \in \mathbb{D}$. The above shows that $a(f) = (a(f)^*)^*$ fulfills (12.54). □

According to [39] we shall call the commutative Banach algebra $\mathcal{P}^{++} := a(H^\infty(\mathbb{D}))$ the **quantum Kähler polarization** and its elements $a(f) \in \mathcal{P}^{++}$ the **annihilation operators**.

The coordinate functions $f_{kl}(Z) := z_{kl}$, where $k, l = 1, 2$ belong to $H^\infty(\mathbb{D})$. Therefore $a_{kl} := a(f_{kl}) \in \mathcal{P}^{++}$ and their action on the basis (12.36) is given by

$$\begin{aligned} a_{11}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle &= \sqrt{\tfrac{(j-j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle \\ &+\sqrt{\tfrac{(j+j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle \end{aligned} \tag{12.57}$$

$$\begin{aligned} a_{12}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle &= -\sqrt{\tfrac{(j-j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle \\ &+\sqrt{\tfrac{(j+j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle \end{aligned} \tag{12.58}$$

$$\begin{aligned} a_{21}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle &= -\sqrt{\tfrac{(j+j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle \\ &+\sqrt{\tfrac{(j-j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle \end{aligned} \tag{12.59}$$

$$a_{22}\begin{vmatrix} j & m \\ j_1 & j_2 \end{matrix}\Big\rangle = \sqrt{\tfrac{(j+j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\begin{vmatrix} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\Big\rangle$$
$$+\sqrt{\tfrac{(j-j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\begin{vmatrix} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\Big\rangle. \quad (12.60)$$

In the expressions above we put by definition $\Big|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\Big\rangle := 0$ if the indices do not satisfy the condition $m, 2j \in \mathbb{N}\cup\{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$.

The coordinate annihilation operators $a_{kl}, k, l = 1, 2$ generate Banach subalgebra $\mathcal{P}^{++}_{pol}$ of $\mathcal{P}^{++}$. Let us denote by $Pol(\overline{\mathbb{D}})$ the algebra of polynomials of variables $\{z_{kl}\}, k, l = 1, 2$ restricted to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ in $Mat_{2\times 2}(\mathbb{C})$.

For the following considerations let us fix the matrix notation

$$\mathbb{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{P}^{++}_{pol} \otimes Mat_{2\times 2}(\mathbb{C}), \quad (12.61)$$

$$\mathbb{A}^{+} := \begin{pmatrix} a^*_{11} & a^*_{21} \\ a^*_{12} & a^*_{22} \end{pmatrix} \in \overline{\mathcal{P}^{++}_{pol}} \otimes Mat_{2\times 2}(\mathbb{C}) \quad (12.62)$$

for the annihilation and creation operators. For example, in this notation the property (12.54) assumes the form

$$\mathbb{A}|Z\rangle = Z|Z\rangle. \quad (12.63)$$

Proposition 12.2.

i) $\mathcal{P}^{++}_{pol}$ *is isometrically isomorphic to the closure* $\overline{Pol(\overline{\mathbb{D}})}$ *of* $Pol(\overline{\mathbb{D}})$*, i.e.* $a(f) \in \mathcal{P}^{++}_{pol}$ *iff* f *is continuous on* $\overline{\mathbb{D}}$ *and holomorphic on* $\mathbb{D}$*. The space of maximal ideals of the* $\mathcal{P}^{++}_{pol}$ *(the spectrum) is homeomorphic to* $\overline{\mathbb{D}}$.

ii) $\mathcal{P}^{++}_{pol}$ *is a semisimple Banach algebra, i.e. if* $p \in \mathcal{P}^{++}_{pol}$ *is such that for each* $c \in \mathbb{C}$ *there exists* $(1+cp)^{-1}$ *then* $p = 0$.

iii) $\mathcal{P}^{++}_{pol} \subsetneq \mathcal{P}^{++}$*, i.e. it is proper Banach subalgebra of* $\mathcal{P}^{++}$.

iv) The vacuum state is cyclic with respect to the Banach algebra $\overline{\mathcal{P}^{++}_{pol}}$

Proof.

i) For $Z, W \in \overline{\mathbb{D}}$ and $\alpha \in [0, 1]$ one has

$$v^{\dagger}(E - [\alpha Z + (1-\alpha)W]^{\dagger}[\alpha Z + (1-\alpha)W])v = \|v\|^2 - \|[\alpha Z + (1-\alpha)W]v\|^2 \geqslant \quad (12.64)$$
$$\|v\|^2 - \{\alpha\|Zv\| + (1-\alpha)\|Wv\|\}^2 \geqslant \|v\|^2 - \{\alpha\|v\| + (1-\alpha)\|v\|\}^2 = 0,$$

for each $v \in \mathbb{C}^2$, what gives $\alpha Z + (1-\alpha)W \in \overline{\mathbb{D}}$. So, $\overline{\mathbb{D}}$ is convex bounded subset of $Mat_{2\times 2}(\mathbb{C})$. Thus $\overline{\mathbb{D}}$ is polynomially convex and compact. By definition $\mathcal{P}^{++}_{pol}$

has a finite number of generators. Hence statement *i)* is valid, see for example Chapter 7 of [3].

ii) We recall that the radical of algebra $\mathcal{P}^{++}$ is

$$\mathcal{R} = \{b \in \mathcal{P}^{++} : (b + \lambda\mathbb{I}) \text{ is invertible for any } \lambda \neq 0\}. \tag{12.65}$$

iii) To prove this it is enough to find a function $f \in H^\infty(\mathbb{D})$ such that $f \notin \overline{Pol(\overline{\mathbb{D}})}$. For example the function

$$f(Z) = \exp\frac{\mathrm{Tr}(Z+E)}{\mathrm{Tr}(Z-E)} \tag{12.66}$$

has this property.

iv) It is enough to check that

$$\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = \Delta^{jm}_{j_1 j_2}(\mathbb{A}^\dagger)\left|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\rangle \tag{12.67}$$

and notice that operator $\Delta^{jm}_{j_1 j_2}(\mathbb{A}^\dagger) \in \overline{\mathcal{P}^{++}_{pol}}$. □

We define the action of the $g \in \mathrm{SU}(2,2)$ on $\mathbb{A}$ by

$$\mathbf{U}_\lambda(g)\mathbb{A}\mathbf{U}_\lambda(g^{-1}) := \begin{pmatrix} U_\lambda(g)a_{11}U_\lambda(g^{-1}) & U_\lambda(g)a_{12}U_\lambda(g^{-1}) \\ U_\lambda(g)a_{21}U_\lambda(g^{-1}) & U_\lambda(g)a_{22}U_\lambda(g^{-1}) \end{pmatrix}, \tag{12.68}$$

where $\mathrm{SU}(2,2) \ni \mathrm{g} \to \mathrm{U}_\lambda(\mathrm{g}) \in \mathrm{Aut}(\mathcal{H})$ is discrete series representation defined by (12.46). Using the above notation we formulate the following statement.

Proposition 12.3. *One has*

i)

$$\sigma_g(\mathbb{A}) := (A\mathbb{A} + B)(C\mathbb{A} + D)^{-1} \in \mathcal{P}^{++}_{pol} \otimes Mat_{2\times 2}(\mathbb{C}), \tag{12.69}$$

ii)

$$\mathbf{U}_\lambda(g)\mathbb{A}\mathbf{U}_\lambda(g^{-1}) = \sigma_g(\mathbb{A}), \tag{12.70}$$

for $g \in \mathrm{SU}(2,2)$.

Proof.

i) For $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(2,2)$ one has

$$DD^\dagger = E + CC^\dagger. \tag{12.71}$$

So eigenvalues of $DD^\dagger$ satisfy $d_1, d_2 \geqslant 1$, which implies that

$$\left\|D^{-1}CZ\right\|^2 \leqslant \left\|D^{-1}C(D^{-1}C)^\dagger\right\| = \left\|D^{-1}(DD^\dagger - E){D^\dagger}^{-1}\right\| = \left\|E - D^{-1}{D^\dagger}^{-1}\right\| < 1 \tag{12.72}$$

for $Z \in \overline{\mathbb{D}}$. The above gives that $(D + CZ)^{-1} = (E + D^{-1}CZ)^{-1}D^{-1}$ exists for $Z \in \overline{\mathbb{D}}$. Since $\det(CZ + D)$ is continuous function function of Z and $\det(CZ +$

$D) \neq 0$ for $z \in \overline{\mathbb{D}}$ there exists $\Omega \supset \overline{\mathbb{D}}$ such that $\det(CZ + D) \neq 0$ for all $z \in \Omega$. This shows that the matrix function

$$\sigma_g(\mathbb{Z}) = (A\mathbb{Z} + B)(C\mathbb{Z} + D)^{-1} \tag{12.73}$$

is holomorphic on Ω. So, by the Oka-Weil theorem, see [3, 62], there is a sequence $\{p_n\}$ of polynomials in $z_{11}, z_{12}, z_{21}, z_{22}$ with $p_n \to \sigma_g$ uniformly on $\overline{\mathbb{D}}$. Since $\mathcal{P}^{++}_{pol} \cong \overline{Pol(\overline{\mathbb{D}})}$ one proves $\sigma_g(\mathbb{A}) \in \mathcal{P}^{++}_{pol} \otimes Mat_{2\times 2}(\mathbb{C})$.

ii) Let us note that for a linearly dense set of the coherent states $|Z\rangle$, $Z \in \mathbb{D}$,

$$\mathbf{U}_\lambda(g)\mathbb{A}\mathbf{U}_\lambda(g^{-1})|Z\rangle = \sigma_g(\mathbb{A})|Z\rangle \tag{12.74}$$

which gives (12.70). □

We conclude immediately from Proposition 12.3

Corrolary 12.4. *Banach subalgebra* $\mathcal{P}^{++}_{pol} \subset L^\infty(\mathcal{H})$ *is invariant* $U_\lambda(g)\mathcal{P}^{++}_{pol}U_\lambda(g^{-1}) \subset \mathcal{P}^{++}_{pol}$, $g \in \mathrm{SU}(2,2)$ *with respect to the discrete series representation.*

Let us make a closing remark that quantum polarization $\mathcal{P}^{++}$ gives holomorphic operator coordinatization for the classical phase space $\mathfrak{M}^{++}$ and subalgebra $\mathcal{P}^{++}_{pol} \subset \mathcal{P}^{++}$ gives the coordinatization of $\mathfrak{M}^{++}$ algebraic in the annihilation operators.

12.3 Conformal Kähler quantum phase space

The holomorphic quantum coordinatization of the classical phase space $\mathbb{M}^{++}$ by the operator Banach algebra $\mathcal{P}^{++}$ is not sufficient from the physical point of view. The reason is that the complete quantum description of the scalar conformal particle also requires self-adjoint operators, for example such as those given by (12.49)-(12.52). Therefore, we are obliged to include in our considerations the Banach algebra $\overline{\mathcal{P}^{++}}$ generated by the creation operators a^*_{kl}, $k, l = 1, 2$, which by definition are conjugated counterparts of the annihilation operators. The algebra $\overline{\mathcal{P}^{++}}$ gives anti-holomorphic quantum coordinatization of $\mathfrak{M}^{++}$. From Proposition 12.3 it follows that $\overline{\mathcal{P}^{++}}$ as well as $\mathcal{P}^{++}$ are conformally invariant **quantum Kähler polarizations** on $\mathfrak{M}^{++}$. Then, following [39], we shall call the operator C^*-algebra $\mathcal{M}^{++} \subset L^\infty(\mathcal{H})$ generated by $\mathcal{P}^{++}$ the **quantum Kähler phase space** of the scalar conformal particle. We shall denote by $\mathcal{M}^{++}_{pol}$ the proper C^*-subalgebra of $\mathcal{M}^{++}$ generated by $\mathcal{P}^{++}_{pol} \subsetneq \mathcal{P}^{++}$.

The relation between the quantum phase space $\mathcal{M}^{++}$ and its classical mechanical counterpart $\mathfrak{M}^{++}$ is best seen by the covariant and contravariant symbols description.

For any bounded operator $F \in L^\infty(\mathcal{H})$ one defines the **2-covariant symbol**

$$\langle F\rangle_2(Z^\dagger, V) := \frac{\langle Z|FV\rangle}{\langle Z|V\rangle}. \tag{12.75}$$

The **2-contravariant symbol** f is defined as an element of the space $\mathcal{B}_2(\mathbb{D}\times\mathbb{D})$ of complex valued functions on $\mathbb{D}\times\mathbb{D}$ for which the integral

$$F=\mathcal{F}_\lambda(f):=c_\lambda^2\int\limits_{\mathbb{D}\times\mathbb{D}} f(Z^\dagger,V)\frac{|Z\rangle\langle Z|V\rangle\langle V|}{\langle Z|Z\rangle\langle V|V\rangle}d\mu(Z^\dagger,Z)d\mu(V^\dagger,V) \tag{12.76}$$

exists weakly and $\mathcal{F}_\lambda(f)\in L^\infty(\mathcal{H})$, where the measure $d\mu$ is defined by

$$d\mu(Z^\dagger,Z)=\det(E-Z^\dagger Z)^{-4}\,|dz|\,. \tag{12.77}$$

We define:

i) the associative product

$$\begin{aligned}&(f\bullet_\lambda g)(Z^\dagger,W):= &(12.78)\\ &=c_\lambda^2\int\limits_{\mathbb{D}\times\mathbb{D}} f(Z^\dagger,V)g(S^\dagger,W)\frac{\langle Z|V\rangle\langle V|S\rangle\langle S|W\rangle}{\langle Z|W\rangle\langle V|V\rangle\langle S|S\rangle}d\mu(V^\dagger,V)d\mu(S^\dagger,S)=\\ &=\int\limits_{\mathbb{D}\times\mathbb{D}} f(Z^\dagger,V)g(S^\dagger,W)\frac{\langle Z|V\rangle\langle V|S\rangle\langle S|W\rangle}{\langle Z|W\rangle}d\mu_\lambda(V^\dagger,V)d\mu_\lambda(S^\dagger,S),\end{aligned}$$

of the 2-contravariant symbols $f,g\in\mathcal{B}_2(\mathbb{D}\times\mathbb{D})$;

ii) the seminorm

$$\|f\|:=\|\mathcal{F}_\lambda(f)\|_\infty \tag{12.79}$$

and the involution

$$f^*(Z^\dagger,V):=\overline{f(V,Z^\dagger)} \tag{12.80}$$

of the 2-contravariant symbol. The map $\mathcal{F}_\lambda:\mathcal{B}_2(\mathbb{D}\times\mathbb{D})\to L^\infty(\mathcal{H})$ is an epimorphism of algebras with involution and

$$\ker\mathcal{F}_\lambda=\{f\in\mathcal{B}_2(\mathbb{D}\times\mathbb{D}):\|f\|=0\}. \tag{12.81}$$

Thus the quotient algebra $\mathcal{B}_2(\mathbb{D}\times\mathbb{D})/\ker\mathcal{F}_\lambda$ and $L^\infty(\mathcal{H})$ are isomorphic as C^*-algebras. Since each equivalence class $[f]=f+\ker\mathcal{F}_\lambda$ is represented in a unique way by the 2-covariant symbol $\langle\mathcal{F}_\lambda(f)\rangle_2$, i.e. $[f]=\langle\mathcal{F}_\lambda(f)\rangle_2+\ker\mathcal{F}_\lambda$ and $\langle\mathcal{F}_\lambda(f)\rangle_2=\langle\mathcal{F}_\lambda(g)\rangle_2$ iff $f-g\in\ker\mathcal{F}_\lambda$, then the quotient vector space $\mathcal{B}_2(\mathbb{D}\times\mathbb{D})/\ker\mathcal{F}_\lambda$ is isomorphic with the vector space

$$\mathcal{B}^2(\mathbb{D}\times\mathbb{D}):=\{\langle F\rangle_2:F\in L^\infty(\mathcal{H})\} \tag{12.82}$$

of 2-covariant symbols of the bounded operators. Defining the product of the 2-covariant symbols $\langle F\rangle_2,\langle G\rangle_2\in\mathcal{B}^2(\mathbb{D}\times\mathbb{D})$ by

$$\langle F\rangle_2*_\lambda\langle G\rangle_2(Z^\dagger,V):=c_\lambda\int\langle F\rangle_2(Z^\dagger,W)\langle G\rangle_2(W^\dagger,V)\frac{\langle Z|W\rangle\langle W|V\rangle}{\langle W|W\rangle\langle Z|V\rangle}d\mu(W^\dagger,W) \tag{12.83}$$

one obtains the structure of C^*-algebra on $\mathcal{B}^2(\mathbb{D}\times\mathbb{D})$.

The quotient map $\mathcal{B}_2(\mathbb{D}\times\mathbb{D}) \to \mathcal{B}_2(\mathbb{D}\times\mathbb{D})/\ker\mathcal{F}_\lambda$ and the isomorphism $\mathcal{B}_2(\mathbb{D}\times\mathbb{D})/\ker\mathcal{F}_\lambda \cong \mathcal{B}^2(\mathbb{D}\times\mathbb{D})$ defines the epimorphism

$$\pi : \mathcal{B}_2(\mathbb{D}\times\mathbb{D}) \longrightarrow \mathcal{B}^2(\mathbb{D}\times\mathbb{D}) \tag{12.84}$$

of the algebra with involution $(\mathcal{B}_2(\mathbb{D}\times\mathbb{D}), \bullet_\lambda)$ on the C^*-algebra $(\mathcal{B}^2(\mathbb{D}\times\mathbb{D}), *_\lambda)$. Similarly the inclusion map

$$\iota : \mathcal{B}^2(\mathbb{D}\times\mathbb{D}) \hookrightarrow \mathcal{B}_2(\mathbb{D}\times\mathbb{D}) \tag{12.85}$$

is the monomorphism of C^*-algebra $(\mathcal{B}^2(\mathbb{D}\times\mathbb{D}), *_\lambda)$ to the algebra $(\mathcal{B}_2(\mathbb{D}\times\mathbb{D}), \bullet_\lambda)$.

In the case under consideration the coherent state map $\mathcal{K}_\lambda : \mathbb{D} \to \mathbb{CP}(\mathcal{H})$ is holomorphic and $\mathbb{D}$ is a simply connected domain. Hence one can recontruct the 2-covariant symbol $\langle F\rangle_2$ of the bounded operator $F \in L^\infty(\mathcal{H})$ from its **Berezin covariant symbol**

$$\langle F\rangle(Z^\dagger, Z) := \frac{\langle Z|FZ\rangle}{\langle Z|Z\rangle}. \tag{12.86}$$

The reconstruction is given by the analytic continuation of $\langle F\rangle$ from the diagonal $\delta : \mathbb{D} \cong \Delta \hookrightarrow \mathbb{D}\times\mathbb{D}$ to the product $\mathbb{D}\times\mathbb{D}$. As a result we obtain the linear isomorphism

$$c : \mathcal{B}(\mathbb{D}) \xrightarrow{\ \sim\ } \mathcal{B}^2(\mathbb{D}\times\mathbb{D}) \tag{12.87}$$

of the vector space $\mathcal{B}(\mathbb{D}) := \{\langle F\rangle : F \in L^\infty(\mathcal{H})\}$ of Berezin covariant symbols with $\mathcal{B}^2(\mathbb{D}\times\mathbb{D})$. The map (12.87) is inverse to the restriction map

$$\delta^* : \mathcal{B}^2(\mathbb{D}\times\mathbb{D}) \ni \langle F\rangle_2 \longrightarrow \langle F\rangle_2 \circ \delta \in \mathcal{B}(\mathbb{D}). \tag{12.88}$$

Hence one also defines the product

$$f *_\lambda g := \delta^*(c(f) *_\lambda c(g)) \tag{12.89}$$

of $f, g \in \mathcal{B}(\mathbb{D})$, which is given explicitly by

$$(f *_\lambda g)(Z^\dagger, Z) = c_\lambda \int_{\mathbb{D}} f(Z^\dagger, V) g(V^\dagger, Z) \left|a_\lambda(Z^\dagger, V)\right|^2 d\mu(V^\dagger, V), \tag{12.90}$$

where

$$a_\lambda(Z^\dagger, V) := \frac{\langle Z|V\rangle}{\sqrt{\langle Z|Z\rangle\langle V|V\rangle}} \tag{12.91}$$

is the transition amplitude between the coherent states $\mathcal{K}_\lambda(Z)$ and $\mathcal{K}_\lambda(V)$. For brevity, by f and g in (12.89) we denoted the Berezin covariant symbols of $F, G \in L^\infty(\mathcal{H})$.

Let us visualize the morphisms defined above in the following diagram

$$\begin{array}{ccc} (\mathcal{B}_2(\mathbb{D}\times\mathbb{D}),\bullet_\lambda) & \xrightarrow{\ \mathcal{F}_\lambda\ } & (L^\infty(\mathcal{H}),\circ) \\ \pi\searrow\ \nwarrow\iota & & \swarrow\langle\cdot\rangle_2 \\ & (\mathcal{B}^2(\mathbb{D}\times\mathbb{D}),*_\lambda) & \\ & \delta^*\downarrow\quad\uparrow c & \\ & (\mathcal{B}(\mathbb{D}),*_\lambda) & \end{array} \tag{12.92}$$

The notions of covariant and contravariant symbols were introduced by Berezin and their importance in various aspects of quantization was shown in [5, 6]. The 2-contravariant and 2-covariant symbols of Schatten class operators and bounded operators were studied in [37].

In the following proposition we will mention a few properties of the quantum scalar conformal phase space $\mathcal{M}^{++}$ and its C^*-subalgebra $\mathcal{M}^{++}_{pol}$.

Proposition 12.5.

(i) The autorepresentation of $\mathcal{M}^{++}_{pol}$ in $L^\infty(\mathcal{H})$ is irreducible and $\mathcal{P}^{++}_{pol}\cap\overline{\mathcal{P}^{++}_{pol}} = \mathbb{C}\mathbb{I}$.

(ii) $\mathcal{M}^{++}_{pol}$ is weakly (strongly) dense in $L^\infty(\mathcal{H})$.

(iii) $\mathcal{M}^{++}_{pol}$ contains the ideal $L^0(\mathcal{H})$ of compact operators. Thus any ideal of $\mathcal{M}^{++}_{pol}$, which autorepresentation in $\mathcal{H}$ is irreducible, also contains $L^0(\mathcal{H})$.

(iv) $\mathcal{M}^{++}_{pol}$ is conformally invariant, i.e. $U_\lambda(g)\mathcal{M}^{++}_{pol}U_\lambda(g)^\dagger \subset \mathcal{M}^{++}_{pol}$ for $g \in SU(2,2)$.

(v) $\mathcal{P}^{++}_{pol} \cap L^0(\mathcal{H}) = \{0\}$.

(vi) $L^0(\mathcal{H}) \subsetneq Comm\mathcal{M}^{++}_{pol}$, where $Comm\mathcal{M}^{++}_{pol}$ is commutator ideal of $\mathcal{M}^{++}_{pol}$.

(vii) The statements i), ii), iii), v), and vi) are valid also for $\mathcal{M}^{++}$ and $\mathcal{P}^{++}$

Proof.

(i) Let us denote by P the orthogonal projector defined by decomposition of $\mathcal{H}$ on the Hilbert subspaces irreducible with respect to $\mathcal{M}^{++}_{pol}$. Let us define $p \in L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ by

$$p(Z) := \overline{\left\langle Z \middle| P \middle| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\rangle}. \tag{12.93}$$

Since

$$a(f)^\dagger P = Pa(f)^\dagger \tag{12.94}$$

for each $f \in Pol(\overline{\mathbb{D}})$ then from (12.67) and (12.94)one has

$$(I \circ P \circ I^{-1}) I \left(\left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right) = pI \left(\left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right). \tag{12.95}$$

Since $p \in L^2\mathcal{O}(\mathbb{D}, d\mu\lambda)$ there exists a sequence of polynomials $\{p_n\}$ such that $p_n \xrightarrow[n\to\infty]{} p$ in $\|\cdot\|_2$-norm. The operator $I \circ P \circ I^{-1}$ is bounded, so we obtain from (12.95)

$$p = (I \circ P \circ I^{-1})p = (I \circ P \circ I^{-1}) \lim_{N\to\infty} p_N = \lim_{N\to\infty} (I \circ P \circ I^{-1}) p_N = \lim_{N\to\infty} p p_N. \tag{12.96}$$

For any compact subset $K \subset \mathbb{D}$ one has

$$\sup_{Z \in K} |\langle \psi | Z \rangle| \leqslant C_k \|\psi\|_2 , \tag{12.97}$$

where $C_k := \sup_{Z\in K} \sqrt{\langle Z|Z\rangle}$ and thus

$$\begin{aligned} 0 \leqslant \sup_{Z\in K} \left| p^2(Z) - p(Z)p_N(Z) \right| &\leqslant \sup_{Z\in K} |p(Z)| \sup_{Z\in K} |p(Z) - p_N(Z)| \leqslant \\ &\leqslant C_k^2 \|p\|_2 \|p - p_N\|_2 \xrightarrow[N\to\infty]{} 0. \end{aligned} \tag{12.98}$$

The above gives $p = \lim_{N\to\infty} p p_N = p^2 \in L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$. Thus $p \equiv 1$ and from (12.95) we obtain that $P = \mathbb{I}$, what proves irreducibility of $\mathcal{M}^{++}_{pol}$. If $x \in \mathcal{P}^{++}_{pol} \cap \overline{\mathcal{P}^{++}_{pol}}$ then it commutes with any element of $\mathcal{M}^{++}_{pol}$. So $x \in \mathbb{C}\mathbb{I}$.

(ii) It follows from i) and from the von Neumann bicommutant theorem.

(iii) Let us take the operator $F \in L^\infty(\mathcal{H})$ which has finite number of nonzero matrix elements in the orthonormal basis (12.36). Then its 2-covariant symbol is given by

$$\langle F \rangle_2 (Z^\dagger, V) = \sum_{\substack{(j,m,j_1,j_2)\in\Phi \\ (j',m',j_1',j_2')\in\Phi}} (\det(E - Z^\dagger V)^\lambda \Delta^{jm}_{j_1 j_2}(Z^\dagger) \left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right| F \left| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle \Delta^{j'm'}_{j_1' j_2'}(V), \tag{12.99}$$

where Φ is a finite index set. The operator

$$\sum_{\substack{(j,m,j_1,j_2)\in\Phi \\ (j',m',j_1',j_2')\in\Phi}} \Delta^{jm}_{j_1 j_2}(\mathbb{A}^\dagger) \left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right| F \left| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle \Delta^{j'm'}_{j_1' j_2'}(\mathbb{A}) \tag{12.100}$$

belongs to $\mathcal{M}^{++}_{pol}$ and has the same 2-covariant symbol as operator F. Thus we gather that F is equal to (12.100) what implies that $F \in \mathcal{M}^{++}_{pol}$. So from the fact that $L^0(\mathcal{H}) \cap \mathcal{M}^{++}_{pol} \neq \{0\}$ and Theorem 2.4.9 in [30] we see that $L^0(\mathcal{H}) \subset \mathcal{M}^{++}_{pol}$.

(iv) Since $\mathcal{M}^{++}_{pol}$ is generated by $\mathcal{P}^{++}_{pol}$, the statement follows from the Proposition 12.3.

(v) Let $f \in C(\overline{\mathbb{D}})$ and $(\mathcal{F}_\lambda \circ \iota \circ c)(f)$ belongs to $L^0(\mathcal{H})$ and $\mathcal{P}^{++}_{pol}$ then its spectrum is discrete and equal to $f(\overline{\mathbb{D}})$ at the same time, which leads to a contradiction.

(vi) From iii) one has that $|\varphi\rangle\langle\psi| \in \mathcal{M}^{++}_{pol}$ for $\varphi, \psi \in \mathcal{H}$. Additionally one has

$$|\varphi\rangle\langle\psi| = (|u\rangle\langle v|)(|v\rangle\langle\varphi|) \tag{12.101}$$

$$|\varphi\rangle\langle v| = [|\varphi\rangle\langle\eta|, |\eta\rangle\langle v|] \tag{12.102}$$

if $v, \eta \in \mathcal{H}$ satisfy $\langle v|v\rangle = \langle\eta|\eta\rangle = 1$ and $\langle v|\eta\rangle = 0$. Hence $L^0(\mathcal{H}) \subset Comm\mathcal{M}^{++}_{pol}$.

In order to show that $L^0(\mathcal{H}) \subsetneq Comm\mathcal{M}^{++}_{pol}$ we observe that the operator $[a^\dagger_{11}, a_{11}] \in L^0(\mathcal{H}) \subset Comm\mathcal{M}^{++}_{pol}$ in the basis (12.36) assumes the form

$$[a^\dagger_{11}, a_{11}] \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle = \tag{12.103}$$

$$= \frac{(\lambda-2)((j_1+j_2)(m+2j+\lambda) - (m+2j+\lambda)(m+\lambda-2) - (j+j_1+1)(j+j_2+1))}{(m+2j+\lambda-1)(m+2j+\lambda)(m+\lambda-2)(m+\lambda-1)} \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle$$

Thus it is diagonal and $\frac{1}{4}\frac{2-\lambda}{(m+\lambda-2)(m+\lambda-1)}$ is the concentration point of its spectrum. So, it belongs to $Comm\mathcal{M}^{++}_{pol}$ and is not compact operator.

(vii) It follows from the fact that $\mathcal{P}^{++}_{pol} \subset \mathcal{P}^{++}$.

□

Now let us make few remarks about the Toeplitz (holomorphic) representation of $\mathcal{M}^{++}$, i.e. the representation in the Hilbert space $L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$. One obtains it using the anti-linear monomorphism $I : \mathcal{H} \to L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ given by (12.45):

$$\mathcal{T}_\lambda(X) := I \circ X \circ I^{-1} : L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda), \tag{12.104}$$

where $X \in \mathcal{M}^{++}$. In the particular case when $X \in \overline{\mathcal{P}^{++}}$ one has

$$\mathcal{T}_\lambda(X)\psi(Z) = \langle X\rangle(Z)\psi(Z). \tag{12.105}$$

So, $\mathcal{T}(\overline{\mathcal{P}^{++}_{pol}})$ is realized by multiplication operators M_f, $f \in H^\infty(\mathbb{D})$, having a continuous prolongation to $\overline{\mathbb{D}}$. Thus, the Toeplitz algebra $\mathcal{T}_\lambda(\mathcal{M}^{++}_{pol})$ is generated by the operators

$$\mathcal{T}_\lambda(f) = \Pi_\lambda \circ M_f \circ \Pi_\lambda, \tag{12.106}$$

where f is a real analytic polynomial. The operator $M_f : L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ is the operator of multiplication by $f \in C(\overline{\mathbb{D}})$ and

$$(\Pi_\lambda \psi)(Z) = \int\limits_{\mathbb{D}} \overline{\langle Z|V\rangle} \psi(V^\dagger, V) \frac{1}{\langle V|V\rangle} c_\lambda d\mu(V^\dagger, V) \tag{12.107}$$

is the orthogonal projector Π_λ of the Hilbert space $L^2(\mathbb{D}, d\mu_\lambda)$ on its Hilbert subspace $L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$.

Using the representation (12.104) one can investigate $\mathcal{M}^{++}_{pol}$ in the framework of theory of Toeplitz algebras related to bounded symmetric domains, which were intensively investigated in series of works [57, 58, 59].

The following basic statement can be viewed as a variant of the Coburn Theorem (see [11]).

Theorem 12.6. *One has the exact sequence*

$$0 \longrightarrow Comm\mathcal{M}^{++}_{pol} \xrightarrow{\ \iota\ } \mathcal{M}^{++}_{pol} \xrightarrow{\ \pi_\lambda\ } C(\mathbb{M}^{00}) \longrightarrow 0 \tag{12.108}$$

of C^*-algebra homomorphisms, where $C(\mathbb{M}^{00})$ is the C^*-algebra of continuous functions on the conformally compactified Minkowski space $\mathbb{M}^{00}$.

Proof. We begin observing that for $f \in C(\overline{\mathbb{D}})$ one has inequalities

$$\|\mathcal{T}_\lambda(f)\|_\infty \leqslant \|f\|_{sup} \leqslant \|Q_\lambda(f)\|_\infty \tag{12.109}$$

which follow from (12.106) and from (12.129) respectively. From the first inequality in (12.109) it follows that the map

$$C(\overline{\mathbb{D}}) \ni f \longrightarrow T_\lambda(f) := [\mathcal{T}_\lambda(f)] \in \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \tag{12.110}$$

is a continuous epimorphism of the C^*-algebra $C(\overline{\mathbb{D}})$ on the commutative quotient C^*-algebra $\mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol}$. Let us recall that the norm of $[x] \in \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol}$ is defined by

$$\|[x]\|_{inf} = \inf_{\xi \in Comm\mathcal{M}^{++}_{pol}} \|x + \xi\|\,. \tag{12.111}$$

Now let us consider the ideal $\ker T_\lambda \subset C(\overline{\mathbb{D}})$. It follows from $iv)$ of Proposition 12.5 that $U_\lambda(g)(Comm\mathcal{M}^{++}_{pol})U_\lambda(g)^\dagger \subset Comm\mathcal{M}^{++}_{pol}$, so the conformal group $SU(2,2)/\mathbb{Z}_4$ acts on the quotient C^*-algebra $\mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol}$ and the C^*-algebra epimorphism defined by (12.110) is a conformally equivariant map, i.e.

$$\begin{array}{ccc} C(\overline{\mathbb{D}}) & \xrightarrow{\ T_\lambda\ } & \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \\ \Big\downarrow{\scriptstyle \Sigma_g} & & \Big\downarrow{\scriptstyle [U_\lambda(g)]} \\ C(\overline{\mathbb{D}}) & \xrightarrow[\ T_\lambda\ ]{} & \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \end{array} \tag{12.112}$$

for any $g \in SU(2,2)/\mathbb{Z}_4$, where

$$(\Sigma_g f)(Z^\dagger, Z) := f((\sigma_g(Z))^\dagger, \sigma_g(Z)) \tag{12.113}$$

$$[U_\lambda(g)]([x]) := [U_\lambda(g) x U_\lambda(g)^\dagger]. \tag{12.114}$$

We conclude from the above that $\ker T_\lambda$ is an ideal in $C(\overline{\mathbb{D}})$ conformally invariant with respect to the action (12.113). Since any ideal in $C(\overline{\mathbb{D}})$ consists of functions vanishing on some compact subset $K \subset \overline{\mathbb{D}}$ the conformally invariant ideals correspond to the conformally invariant compact subsets: $\overline{\mathbb{D}}$, $\partial\mathbb{D} = \{Z \in Mat_{2\times 2}(\mathbb{C}) : \det(E - Z^\dagger Z) = 0 \text{ and } \mathrm{Tr}(E - Z^\dagger Z) \geqslant 0\}$ and $U(2) = \{Z \in Mat_{2\times 2}(\mathbb{C}) : Z^\dagger Z = E\}$, where the last one is the Šilov boundary of $\mathbb{D}$. In this way we show that $\ker T_\lambda$ is equal to one of the following three ideals

$$\mathcal{I}_{\overline{\mathbb{D}}} = \{0\} \subset \mathcal{I}_{\partial\overline{\mathbb{D}}} \subset \mathcal{I}_{U(2)}, \tag{12.115}$$

where by $\mathcal{I}_K$ we denote the ideal of functions equal to zero on K. The polynomial

$$\varphi(Z^\dagger, Z) := \mathrm{Tr}(E - Z^\dagger Z) \tag{12.116}$$

generates the ideal $\mathcal{I}_{U(2)}$ and maps $\overline{\mathbb{D}}$ on the interval $[0, 2]$. Let us consider the positive operator

$$: \mathrm{Tr}(E - A^\dagger A) := 2 - a_{11}^\dagger a_{11} - a_{12}^\dagger a_{12} - a_{21}^\dagger a_{21} - a_{22}^\dagger a_{22}, \tag{12.117}$$

which is diagonal, with

$$: \mathrm{Tr}(E - A^\dagger A) : \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle = \frac{2(\lambda - 2)(m + j + \lambda - 1)}{(m + \lambda - 1)(m + 2j + \lambda)} \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle, \tag{12.118}$$

in the basis (12.36). We see from (12.118) that the spectrum σ of $: \mathrm{Tr}(E - A^\dagger A) :$ is contained in the interval $[0, 2]$ and the set

$$\sigma_a := \left\{ \frac{\lambda - 2}{m + \lambda - 1} : m \in \mathbb{N} \cup \{0\} \cup \{\infty\} \right\} \tag{12.119}$$

is its approximative spectrum. The continuous function $F : [0, 2] \to \mathbb{R}$ defined by

$$F(x) := x \sin \frac{(\lambda - 2)\pi}{x} \tag{12.120}$$

vanishes on σ_a and $F \circ \varphi \in \mathcal{I}_{U(2)}$. Since $F_{|\sigma_a} \equiv 0$ and F assumes the same value at most on a finite subset of $\sigma \setminus \sigma_a$, we conclude that $F(: \mathrm{Tr}(E - A^\dagger A) :)$ is a compact operator. Thus, by *iii)* of Proposition 12.5 $F(: \mathrm{Tr}(E - A^\dagger A) :)$ belongs to $Comm\mathcal{M}_{pol}^{++}$. Let us take the sequence $\{P_n(x)\}_{n \in \mathbb{N}}$ of polynomials which uniformly approximate $P_n \to F$ the function $F \in C([0, 2])$. Thus one has

$$\|P_n \circ \varphi - F \circ \varphi\|_{sup} \xrightarrow[n \to \infty]{} 0 \tag{12.121}$$

From (12.121) and the first inequality of (12.109) we obtain

$$\|\mathcal{T}_\lambda(P_n \circ \varphi) - \mathcal{T}_\lambda(F \circ \varphi)\|_\infty \xrightarrow[n\to\infty]{} 0. \tag{12.122}$$

On the other hand, from the Gelfand-Naimark theorem and (12.121) we have

$$\left\| P_n(:\mathrm{Tr}(E - A^\dagger A):) - F(:\mathrm{Tr}(E - A^\dagger A):) \right\|_\infty \xrightarrow[n\to\infty]{} 0. \tag{12.123}$$

The operators $\mathcal{T}_\lambda(P_n \circ \varphi)$ are polynomials of the creation and annihilation operators taken in the anti-normal ordering, so they differ from the polynomials $P_n(:\mathrm{Tr}(E - A^\dagger A):)$ modulo elements of $Comm\mathcal{M}^{++}_{pol}$. Thus, using also (12.122) and (12.123), we obtain that

$$0 = \left\| [\mathcal{T}_\lambda(F \circ \varphi)] - [F(:\mathrm{Tr}(E - A^\dagger A):)] \right\|_{inf} = \|[\mathcal{T}_\lambda(F \circ \varphi)]\|_{inf} = \|T_\lambda(F \circ \varphi)\|_{inf}. \tag{12.124}$$

Summing up we conclude that $F \circ \varphi \in \ker T_\lambda \cap \mathcal{I}_{U(2)}$. Since it is easy to check that $F \circ \varphi \notin \mathcal{I}_{\partial\mathbb{D}}$ and that $\ker T_\lambda$, is conformally invariant it follows that $\ker T_\lambda = \mathcal{I}_{U(2)} = \mathcal{I}_{\mathbb{M}^{00}}$.

Taking into account that (12.110) is an epimorphism of C^*-algebras, we state the following isomorphisms $\mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \cong C(\overline{\mathbb{D}})/\mathcal{I}_{\mathbb{M}^{00}} \cong C(\mathbb{M}^{00})$. These isomorphisms give the epimorphism $\pi_\lambda : \mathcal{M}^{++}_{pol} \to C(\mathbb{M}^{00})$. □

Ending this section, let us remark that "neglecting" the non-commutativity of quantum complex Minkowski space $\mathcal{M}^{++}_{pol}$ we come back to the commutative C^*-algebra $C(\mathbb{M}^{00})$ whose spectrum is given by the conformally compactified Minkowski space $\mathbb{M}^{00}$.

12.4 Quantization and physical interpretation

Analogously to the classical coordinate observables $(Z, Z^\dagger)$ on $\mathbb{M}^{++}$ we shall use quantum coordinate observables $(\mathbb{A}, \mathbb{A}^\dagger)$ for the quantum phase space $\mathcal{M}^{++}$. Superposing morphisms from diagram (12.92) we obtain the extension of this correspondence. In such a way we get the isomorphism

$$Q_\lambda := \mathcal{F}_\lambda \circ \iota \circ c : \mathcal{B}(\mathbb{D}) \longrightarrow L^\infty(\mathcal{H}), \tag{12.125}$$

which extends the quantization map,

$$a : H^\infty(\mathbb{D}) \ni f \longrightarrow a(f) \in L^\infty(\mathcal{H}), \tag{12.126}$$

discussed in the previous section. Taking into account the properties

$$Q_\lambda(f *_\lambda g) = Q_\lambda(f) Q_\lambda(g), \tag{12.127}$$

$$Q_\lambda(\bar{f}) = Q_\lambda(f)^*, \tag{12.128}$$

$$\langle Q_\lambda(f) \rangle_\lambda = f, \tag{12.129}$$

for $f, g \in \mathcal{B}(\mathbb{D})$, we see that the isomorphism Q_λ gives a **quantization procedure** inverse to the mean value map.

According to relation (12.129), Berezin covariant symbols are the classical observables corresponding to the quantum observables realized by the bounded operators. As a particular case the quantum phase space $\mathcal{M}^{++} \subset L^\infty(\mathcal{H})$ is obtained from $\langle \mathcal{M}^{++} \rangle \subset \mathcal{B}(\mathbb{D})$ by the quantization (12.125). However for physical reasons we are interested in the extension of $Q_\lambda : \mathcal{B}(\mathbb{D}) \to L^\infty(\mathcal{H})$ to a larger algebra of observables. For example it is reasonable to include in this scheme the elements of the enveloping algebra of the conformal Lie algebra $su(2,2)$. The latter ones are represented by unbounded operators in $\mathcal{H}$ which, according to the equivariance property (12.112), possess the common domain given by the linear span $\mathcal{L}(\mathcal{K}_\lambda(\mathfrak{M}^{++}))$ of the set $\mathcal{K}_\lambda(\mathfrak{M}^{++})$ of the coherent states. Let us then define the vector space $\mathcal{A}^{++}$ of operators in $\mathcal{H}$ closed with respect to the operation of conjugation and all elements of which possess $\mathcal{L}(\mathcal{K}_\lambda(\mathfrak{M}^{++}))$ as a common domain. Therefore for any operator $F \in \mathcal{A}^{++}$ the 2-covariant and Berezin covariant symbols have sense.

In the following we will use the coherent state weak topology, i.e. $A_n \xrightarrow{coh} A$ if $\langle Z|A_n|V\rangle \to \langle Z|A|V\rangle$ for all $Z, V \in \mathbb{D}$. It is a weaker topology than the weak one, as can be seen from the following example. Let $\mathbb{D} \ni Z_n = (1 - \frac{1}{n})E$, $n \in \mathbb{N}$. We define the sequence of operators

$$A_n := n \frac{|Z_n\rangle\langle Z_n|}{\langle Z_n|Z_n\rangle}. \tag{12.130}$$

It is easily observed that

$$\forall Z, V \in \mathbb{D} \quad \lim_{n\to\infty} \langle Z|A_n|V\rangle = 0, \tag{12.131}$$

thus $A_n \xrightarrow{coh} 0$. On the other hand $\sup_{n\in\mathbb{N}} \|A_n\| = \infty$, thus A_n is not weakly convergent.

The space $\mathcal{A}^{++}$ is closed with respect to coherent state weak topology. The quantum phase space $\mathcal{M}^{++}$ is contained in $\mathcal{A}^{++}$ as a dense subset with respect to the coherent state weak topology. For any $F \in \mathcal{A}^{++}$ its Berezin symbol $f = \langle F \rangle \in \mathcal{RO}^{++}(\mathbb{D})$ is the real analytic function

$$f(Z^\dagger, Z) = \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}} \bar{Z}_{11}^{i_{11}} \bar{Z}_{12}^{i_{12}} \bar{Z}_{21}^{i_{21}} \bar{Z}_{22}^{i_{22}} Z_{11}^{j_{11}} Z_{12}^{j_{12}} Z_{21}^{j_{21}} Z_{22}^{j_{22}} \tag{12.132}$$

of the variables $(Z^\dagger, Z)$. One extends the quantization (12.125) naturally to the space $\mathcal{RO}^{++}(\mathbb{D})$ of real analytic functions on $\mathbb{D}$ by setting

$$\begin{aligned} Q_\lambda(f) = & \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}} a_{11}^{\dagger\ i_{11}} a_{12}^{\dagger\ i_{12}} a_{21}^{\dagger\ i_{21}} a_{22}^{\dagger\ i_{22}} a_{11}^{i_{11}} a_{12}^{i_{12}} a_{21}^{i_{21}} a_{22}^{i_{22}} = \\ = & \ : f(\mathbb{A}^\dagger, \mathbb{A}) :, \end{aligned} \tag{12.133}$$

where as usual, the colons $: \cdot :$ denote normal ordering. The infinite sum in (12.133) is taken in the sense of coherent state weak topology. The extension of the product $*_\lambda$, see (12.90), to the real analytic Berezin symbols

$f, g \in \mathcal{RO}^{++}(\mathbb{D})$ is defined by

$$(f *_\lambda g)(Z^\dagger, Z) := \frac{\langle Z^\dagger| : f(\mathbb{A}^\dagger, \mathbb{A}) :: g(\mathbb{A}^\dagger, \mathbb{A}) : |Z\rangle}{\langle Z^\dagger|Z\rangle}. \tag{12.134}$$

As an illustration let us consider the Berezin symbols

$$\langle U_\lambda(g)\rangle(Z^\dagger, Z) = (\det(CZ + D))^{-\lambda} \left(\frac{\det(E - Z^\dagger\sigma_g(Z))}{\det(E - Z^\dagger Z)}\right)^{-\lambda} \tag{12.135}$$

and their quantum $(\mathbb{A}^\dagger, \mathbb{A})$-coordinate representation

$$U_\lambda(g) = Q_\lambda(\langle U_\lambda(g)\rangle) =: \left(\frac{\det(E - \mathbb{A}^\dagger\sigma_g(\mathbb{A}))}{\det(E - \mathbb{A}^\dagger\mathbb{A})}\right)^{-\lambda} : (\det(C\mathbb{A} + D))^{-\lambda} \tag{12.136}$$

for the conformal group elements $g \in \mathrm{SU}(2, 2)$. In order to express the quantum 4-momentum, relativistic angular momentum, dilation and 4-acceleration in terms of quantum coordinates $(\mathbb{A}^\dagger, \mathbb{A})$ we differentiate $U_\lambda(g(t))$ given by (12.136) with respect to the parameter $t \in \mathbb{R}$ for an appropriate choice of one-parameter subgroup $\mathbb{R} \ni t \to g(t) \in \mathrm{SU}(2, 2)$. As a result one obtains

$$Q_\lambda(p_\mu) = i\lambda : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) : \tag{12.137}$$

$$\begin{aligned} Q_\lambda(m_{\mu\nu}) = i\lambda \Big(&\frac{1}{2} \operatorname{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma_\nu(\mathbb{W} - \mathbb{W}^\dagger)) : - \\ &- \frac{1}{2} \operatorname{Tr}(\sigma_\nu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) : \Big) \end{aligned} \tag{12.138}$$

$$Q_\lambda(d) = i\lambda \operatorname{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma^\mu(\mathbb{W} - \mathbb{W}^\dagger)) : -2i\lambda\mathbb{I} \tag{12.139}$$

$$\begin{aligned} Q_\lambda(a_\nu) = {} & i\lambda \det(\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma_\nu(\mathbb{W} - \mathbb{W}^\dagger)) : - \\ & - i\lambda\frac{1}{2} \operatorname{Tr}(\sigma_\nu\mathbb{W}^\dagger) \operatorname{Tr}(\sigma^\beta\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1} \operatorname{Tr}(\sigma_\beta(\mathbb{W} - \mathbb{W}^\dagger)) : + \\ & + i\lambda \operatorname{Tr}(\sigma_\nu\mathbb{W}^\dagger), \end{aligned} \tag{12.140}$$

where $(\mathbb{W}^\dagger, \mathbb{W})$ are matrix operator coordinates in $\mathcal{A}^{++}$ obtained from $(\mathbb{A}^\dagger, \mathbb{A})$ by the Caley transform

$$\mathbb{W} = i(\mathbb{A} + E)(\mathbb{A} - E)^{-1}, \tag{12.141}$$

which has sense in the coherent state weak topology. After passing to the representation in the Hilbert space $L^2\mathcal{O}(\mathbb{T}, d\mu_\mu)$ of holomorphic functions on the future tube $\mathbb{T}$, square integrable with respect to the measure (12.53), we rediscover from (12.137)-(12.140) the operators (12.49)-(12.52) obtained by the Kostant-Souriau geometric quantization.

It follows from (12.49) that

$$[Q_\lambda(p_\mu), Q_\lambda(p_\nu)] = 0. \tag{12.142}$$

Using (12.21), we see from (12.142) that

$$[Q_\lambda(y_\mu), Q_\lambda(y_\nu)] = 0. \tag{12.143}$$

The creation operators

$$Q_\lambda(\bar{w}^\mu) = \frac{1}{2}\operatorname{Tr}(\sigma_\mu \mathbb{W}^\dagger) \tag{12.144}$$

in $L^2\mathcal{O}(\mathbb{T}, d\mu_\lambda)$ are given as multiplication by the complex coordinate functions w^μ, so they commute. Thus, in addition to (12.142), we have

$$[Q_\lambda(x^\mu), Q_\lambda(x^\nu)] = 0 \tag{12.145}$$

$$[Q_\lambda(x^\mu), Q_\lambda(p_\nu)] = -i\delta^\mu_\nu 1 \tag{12.146}$$

for the quantum canonical coordinates $(Q_\lambda(x^\mu), Q_\lambda(p_\nu))$.

Therefore we see that Heisenberg algebra generated by unbounded operators of 4-momenta $Q_\lambda(p_\nu)$ and 4-positions $Q_\lambda(x^\mu) = \frac{1}{2}\operatorname{Tr}(\sigma_\mu(\mathbb{W} + \mathbb{W}^\dagger))$ is included in $\mathcal{A}^{++}$. The creation operators (12.144) and the annihilation ones

$$Q_\lambda(w_\nu) = \frac{1}{2}\operatorname{Tr}(\sigma_\nu \mathbb{W}) \tag{12.147}$$

generate the Caley transforms of quantum polarizations $\overline{\mathcal{P}^{++}_{pol}}$ and $\mathcal{P}^{++}_{pol}$ respectively. However their commutators $[Q_\lambda(\bar{w}^\mu), Q_\lambda(w_\nu)] \neq 0$ do not have so simple form as it has place in the case of quantum real polarization given by the canonical commutation relation (12.146).

Let us now discuss the physical sense of the parameter $\lambda \in \mathbb{R}$. So far, for technical reasons, we assumed that it was dimensionless. However, as one sees from (12.21), λ has dimensions of action. We therefore assume the Planck constant h as the natural unit for λ. After this we obtain

$$w^\mu = x^\mu + i\lambda \frac{h}{mc}\frac{p^\mu}{mc}, \tag{12.148}$$

where $mc = \sqrt{p_0^2 - \vec{p}^2}$. The quantity $\frac{h}{mc}$ is the Compton wavelength of the conformal particle. For example for the proton $\frac{h}{mc} \cong 10^{-13}cm$.

The quantities $\frac{p^\mu}{mc}$ denote the components of relativistic 4-velocity measured with the speed of light as the unit. Dimensional analysis shows that in the limit $\lambda \to \infty$ the theory describes physical phenomena characterized by a space-time scale much bigger than the Compton scale characteristic for the quantum phenomena. This physical argument is consistent with the following asymptotic behavior of of $*_\lambda$-product

$$f *_\lambda g \sim fg \tag{12.149}$$

$$f *_\lambda g - g *_\lambda f \sim i\lambda\{f, g\} \tag{12.150}$$

for $\lambda \to \infty$, where the right hand side of (12.149) is usual multiplication of functions and the right side of (12.150) is the Poisson bracket (12.48). In order to show these asymptotic formulae we apply the method used for the case of a general symmetric domain in [5]. The expressions (12.149), (12.150) show the correspondences of the quantum description of the massive scalar conformal particle to its classical mechanical description in the large space-time scale limit.

The quantum effects are described by the transition amplitude (12.91), which in the coordinates $(\bar{w}^\mu, w^\nu)$ is given by

$$a_\lambda(v^\dagger, w) = \left(\frac{((w-\bar{w})^2(v-\bar{v})^2)^{\frac{1}{2}}}{(w-\bar{v})^2} \right)^\lambda, \tag{12.151}$$

where $(w-\bar{v})^2 = \eta_{\mu\nu}(w^\mu - \bar{v}^\mu)(w^\nu - \bar{v}^\nu)$ and $\lambda > 3$. One sees from (12.151) that the transition probability $\left|a_\lambda(v^\dagger, w)\right|^2$ from w to v as a function of v forms a narrow peak around the coherent state $w \in \mathbb{T}$ if $\lambda \frac{h}{mc} \approx 0$. A more detailed physical discussion can be found in [35].

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