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Bihamiltonian structures of PDEs and Frobenius manifolds

(Slides 1)

Boris Dubrovin
SISSA
Trieste, Italia

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

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Bihamiltonian structures of PDEs and Frobenius manifolds

Boris DUBROVIN

SISSA (Trieste)

Hamiltonian systems

$$\dot{x} = \{x, H\}$$

- a geometric framework for *conservative physical systems*
- simplify relationships between symmetries and conservation laws
- formulation of *integrability*
- quantization
- numerical algorithms

Integrable systems: commutative subalgebras in the Lie algebra of Hamiltonian vector fields + completeness

Modern theory of integrable systems: discovery of integrability in Hamiltonian systems with **infinite number of degrees of freedom**

Evolutionary PDEs as dynamical systems:

$$u_t = F(u, u_x, u_{xx}, \ldots) \tag{1}$$

Cauchy data

$$u|_{t=0} = u_0(x)$$

Solution $u(x,t) =$ "integral curve" of the vector field (1) on the space of functions in x beginning at the point $u_0(x)$

Lecture 1

2.1. Reminders:

- Poisson brackets
- Hamiltonian vector fields, first integrals
- Poisson cohomology
- Formalism of supermanifolds

2.2. Bihamiltonian structures, Magri chains and hierarchies

P be a N-dimensional smooth manifold. A *Poisson bracket* on P is a structure of a Lie algebra on the ring of functions $\mathcal{F} := \mathcal{C}^\infty(P)$

$$f, g \mapsto \{f, g\},$$

$$\{g, f\} = -\{f, g\}, \{af + bg, h\} = a\{f, h\} + b\{g, h\}, \tag{2}$$

$a,\ b \in \mathbb{R}, f,\ g,\ h \in \mathcal{F}$

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{3}$$

(Jacobi identity) satisfying the Leibnitz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

for arbitrary three functions $f,\ g,\ h \in \mathcal{F}$.

In a system of local coordinates $x^1, \ldots, x^N$ the Poisson bracket reads

$$\{f, g\} = \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \tag{4}$$

(summation over repeated indices will be assumed)

The bivector

$$\pi^{ij}(x) = -\pi^{ji}(x) = \{x^i, x^j\}$$

satisfies

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} \equiv \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s} \pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s} \pi^{si} = 0 \tag{5}$$

for any i, j, k (the Jacobi identity (3)). Such a bivector is called *a Poisson structure* on P.

Any bivector *constant* in some coordinate system is a Poisson structure. Vice versa, locally all solutions to (5) of the constant rank $2n = \mathrm{rk}(\pi^{ij})$ can be reduced to the *normal form*

$$\pi = \begin{pmatrix} \bar{\pi} & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

with a constant nondegenerate antisymmetric $2n \times 2n$ matrix $\bar{\pi} = \bar{\pi}^{ab}$. I.e., locally there exist coordinates $y^1, \ldots, y^{2n}$ (coordinates on the **symplectic leaves**) and $c^1, \ldots, c^k$ (**Casimir functions**), $2n + k = N$, s.t.

$$\bar{\pi}^{ab} = \{y^a, y^b\} = \mathrm{const}$$

and

$$\{f, c^j\} = 0, \quad j = 1, \ldots k \tag{7}$$

for an arbitrary function f.

For the case $2n = N$ the inverse matrix $\Big(\pi_{ij}(x)\Big) = \Big(\pi^{ij}(x)\Big)^{-1}$ defines on P a *symplectic structure*

$$\Omega = \sum_{i<j} \pi_{ij}(x) dx^i \wedge dx^j, \quad \Omega^n \neq 0.$$

For $2n < N$ one obtains on P a structure of *symplectic foliation* $P = \cup_{c_0} P_{c_0}$, $c_0 = (c_0^1, \dots c_0^k)$, of the codimension $k = N - 2n$

$$P_{c_0} := \{x \mid c^1(x) = c_0^1, \dots, c^k(x) = c_0^k\}. \qquad (8)$$

Example Let $\mathfrak{g}$ be n-dimensional Lie algebra. The *Lie - Poisson* bracket on the dual space $P = \mathfrak{g}^*$ reads

$$\{x^i, x^j\} = c_k^{ij} x^k. \tag{9}$$

The Casimirs are functions on $\mathfrak{g}^*$ invariant with respect to the co-adjoint action of the associated Lie group G. The symplectic leaves = the orbits of the coadjoint action with the Berezin - Kirillov - Kostant symplectic structure on them.

Linear *inhomogeneous* Poisson brackets

$$\{x^i, x^j\} = c_k^{ij} x^k + c_0^{ij} \tag{10}$$

$\Rightarrow$ central extension of $\mathfrak{g}$. c_0^{ij} is a 2-cocycle

$$c_0(b, a) = -c_0(a, b), \quad c_0([a, b], c) + c_0([c, a], b) + c_0([b, c], a) = 0$$

A Poisson bracket defines an (anti)homomorphism

$$\mathcal{F} \to Vect(P)$$

$$H \mapsto X_H := \{\cdot, H\}, \tag{11}$$

$$[X_{H_1}, X_{H_2}] = -X_{\{H_1,H_2\}}.$$

X_H is called **Hamiltonian vector field**. The corresponding dynamical system

$$\dot{x}^i = \{x^i, H\} = \pi^{ij}(x)\frac{\partial H}{\partial x^j} \tag{12}$$

is called **Hamiltonian system with the Hamiltonian** $H(x)$.

- X_H is a *symmetry of the Poisson bracket*

$$Lie_{X_H}\{\ ,\ \} = 0. \tag{13}$$

- Any function F commuting with the Hamiltonian

$$\{F, H\} = 0$$

is a *first integral* of the Hamiltonian system (12). The Hamiltonian vector fields X_H, X_F commute.

Poisson cohomology of (P, π) (introduced by Lichnerowicz). Recall the *Schouten - Nijenhuis bracket*. Denote

$$\Lambda^k = H^0(P, \Lambda^k TP)$$

the space of multivectors on P. The Schouten - Nijenhuis bracket is a bilinear pairing $a, b \mapsto [a, b]$,

$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}$$

uniquely determined by the properties of supersymmetry

$$[b, a] = (-1)^{kl}[a, b], \quad a \in \Lambda^k, \ b \in \Lambda^l \tag{14}$$

the graded Leibniz rule

$$[c, a \wedge b] = [c, a] \wedge b + (-1)^{lk+k} a \wedge [c, b], \quad a \in \Lambda^k, \ c \in \Lambda^l \tag{15}$$

and the conditions $[f, g] = 0$, $f, g \in \Lambda^0 = \mathcal{F}$,

$$[v, f] = v^i \frac{\partial f}{\partial x^i}, \quad v \in \Lambda^1 = Vect(P), \quad f \in \Lambda^0 = \mathcal{F},$$

$[v_1, v_2] =$ commutator of vector fields for $v_1, v_2 \in \Lambda^1$. In particular for a vector field v and a multivector a

$$[v, a] = Lie_v a.$$

Example. For two bivectors $\pi = (\pi^{ij})$ and $\rho = (\rho^{ij})$ their Schouten - Nijenhuis bracket is the following trivector

$$[\pi, \rho]^{ijk} = \frac{\partial \pi^{ij}}{\partial x^s}\rho^{sk} + \frac{\partial \rho^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s}\rho^{sj} + \frac{\partial \rho^{ki}}{\partial x^s}\pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s}\rho^{si} + \frac{\partial \rho^{jk}}{\partial x^s}\pi^{si}. \tag{16}$$

Observe that the l.h.s. of the Jacobi identity (5) reads

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} = \frac{1}{2}[\pi, \pi]^{ijk}.$$

The Schouten - Nijenhuis bracket satisfies the graded Jacobi identity

$$(-1)^{km}[[a,b],c] + (-1)^{lm}[[c,a],b] + (-1)^{kl}[[b,c],a] = 0, \qquad (17)$$

$a \in \Lambda^k,\ b \in \Lambda^l,\ c \in \Lambda^m.$ $\Rightarrow$ for a Poisson bivector π the map

$$\partial : \Lambda^k \to \Lambda^{k+1},\ \partial a = [\pi, a] \qquad (18)$$

is a **differential**, $\partial^2 = 0$. The cohomology of the complex (Λ^*, ∂) is called *Poisson cohomology* of (P, π)

$$H^*(P,\pi) = \oplus_{k \geq 0} H^k(P,\pi).$$

In particular,

- $H^0(P, \pi)$ = the ring of Casimirs of the Poisson bracket,
- $H^1(P, \pi)$ = the quotient of the Lie algebra of infinitesimal symmetries

$$v \in Vect(P),\ \ Lie_v \pi = 0$$

over the subalgebra of Hamiltonian vector fields,

- $H^2(P, \pi)$ = the quotient of the space of infinitesimal deformations of the Poisson bracket by those obtained by infinitesimal changes of coordinates (i.e., by those of the form $Lie_v \pi$ for a vector field v).

On a symplectic manifold (P, π) Poisson cohomology coincides with the de Rham one. The isomorphism is established by "lowering the indices": for a cocycle $a = (a^{i_1 \cdots i_k}) \in \Lambda^k$ the k-form

$$\sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \qquad \omega_{i_1 \ldots i_k} = \pi_{i_1 j_1} \ldots \pi_{i_k j_k} a^{j_1 \cdots j_k}$$

is closed. In particular, for $P =$ ball the Poisson cohomology is trivial.

In the general case rk$(\pi^{ij}) <$ dim P the Poisson cohomology does not vanish even locally. A simple criterion of triviality of 1- and 2-cocycles:

Lemma 1.1.*Let* $\pi = (\pi^{ij}(x))$ *be a Poisson structure of a constant rank* $2n < N$ *on a sufficiently small ball* U*.*
1). A one-cocycle $v = (v^i(x)) \in H^1(U, \pi)$ *is trivial* iff *the vector field* v *is tangent to the leaves of the symplectic foliation (8).*
2). A 2-cocycle $f = (f^{ij}(x)) \in H^2(U, \pi)$ *is trivial* iff

$$f(dc', dc'') = 0 \tag{19}$$

for arbitrary two Casimirs of π*.*

Language of supermanifolds: consider $\mathbf{N} = \Pi T^*P$. Coordinates

$$x^1, \dots, x^N,\ \theta_1, \dots, \theta_N, \quad x^j x^i = x^i x^j, \quad \theta_j \theta_i = -\theta_i \theta_j$$

Bivector

$$\pi = \frac{1}{2}\pi^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \mapsto \frac{1}{2}\pi^{ij}(x)\theta_i\theta_j =: \hat{\pi}$$

(a superfunction on $\mathbf{N} = \Pi T^*P$). (Super)Poisson bracket on $\mathbf{N}$

$$\{P, Q\} = \frac{\partial P}{\partial \theta_i}\frac{\partial Q}{\partial x^i} + (-1)^{|P|}\frac{\partial P}{\partial x^i}\frac{\partial Q}{\partial \theta_i} \tag{20}$$

$|P|$ = parity of the superfunction $|P|$.

Claim: Jacobi identity for $\pi \Leftrightarrow \{\hat{\pi}, \hat{\pi}\} = 0$

Proof.

$$\{\widehat{\pi}, \widehat{\pi}\} = \pi^{sk}\theta_k \frac{\partial \pi^{ij}}{\partial x^s}\theta_i\theta_j$$

$$= \frac{1}{3}\left(\frac{\partial \pi^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s}\pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s}\pi^{si}\right)\theta_i\theta_j\theta_k$$

More generally, for multivectors $a \in \Lambda^k$, $b \in \Lambda^l$,

$$\widehat{a} = \frac{1}{k!}a^{i_1 \ldots i_k}\theta_{i_1}\ldots\theta_{i_k}, \qquad \widehat{b} = \frac{1}{l!}b^{j_1 \ldots j_l}\theta_{j_1}\ldots\theta_{j_l}$$

the super-Poisson bracket

$$\{\widehat{a}, \widehat{b}\} = \widehat{[a, b]}$$

$[a, b]$= Schouten - Nijenhuis bracket.

2.2. Bihamiltonian structures

Definition. A *bihamiltonan structure* on the manifold P is a 2-dimensional linear subspace in the space of Poisson structures on P.

Choosing two nonproportional Poisson structures π_1 and π_2 in the subspace we obtain that the linear combination

$$a_1\pi_1 + a_2\pi_2 \tag{21}$$

with arbitrary constant coefficients a_1, a_2 is again a Poisson bracket. This reformulation is usually referred to as *the compatibility condition* of the two Poisson brackets. It is spelled out as vanishing of the Schouten - Nijenhuis bracket

$$[\pi_1, \pi_2] = 0. \tag{22}$$

An importance of bihamiltonian structures for recursive constructions of integrable systems was discovered by F.Magri (1978) in the analysis of the so-called Lenard scheme ($\leq$1974) of constructing the KdV integrals. The basic idea of these constructions is given by the following simple

Definition A sequence of functions H_0, H_1, ... satisfying the recursion relation

$$\{\ .\ , H_{p+1}\}_1 = \{\ .\ , H_p\}_2, \quad p = 0, 1, \dots \tag{23}$$

is called **Magri chain**

Lemma 1.2.

$$\{H_p, H_q\}_1 = \{H_p, H_q\}_2 = 0, \quad p, q = 0, 1, \dots$$

Proof. Let $p < q$. Using the recursion and antisymmetry of the brackets we obtain

$$\{H_p, H_q\}_1 = \{H_p, H_{q-1}\}_2 = -\{H_{q-1}, H_p\}_2 = -\{H_{q-1}, H_{p+1}\}_1 = \{H_{p+1}, H_{q-1}\}_1.$$

Assume $q - p = 2m$ for some $m > 0$. Iterating we arrive at

$$\{H_p, H_q\}_1 = \ldots = \{H_{p+m}, H_{q-m}\}_1 = 0$$

since $p + m = q - m$. Doing similarly in the case $q - p = 2m + 1$ we obtain

$$\{H_p, H_q\}_1 = \ldots = \{H_n, H_{n+1}\}_1 = \{H_n, H_n\}_2 = 0$$

where $n = p + m = q - m - 1$. The commutativity $\{H_p, H_q\}_2 = 0$ easily follows from the recursion. The Lemma is proved.

Two realizations of the recursive procedure (23).

The first case: the bihamiltonian structure is *symplectic*, i.e. $N = 2n$ and the Poisson structures of the affine line (21) do not degenerate for generic a_1, a_2. Without loss of generality one may assume nondegeneracy of π_1. The *recursion operator*

$$\mathcal{R} : TP \to TP$$

is defined by

$$\mathcal{R} := \pi_2 \cdot \pi_1^{-1}. \tag{24}$$

The main recursion relation (23) can be rewritten in the form

$$dH_{p+1} = \mathcal{R}^* dH_p, \quad p = 0, 1, \dots \tag{25}$$

where

$$\mathcal{R}^* : T^*P \to T^*P$$

is the adjoint operator, or, for the Hamiltonian vector fields X_{H_p}

$$X_{H_{p+1}} = \mathcal{R} X_{H_p} \tag{26}$$

Theorem *The Hamiltonians*

$$H_p := \frac{1}{p+1} \mathrm{tr}\, \mathcal{R}^{p+1}, \quad p \geq 0$$

satisfy the recursion (25).

Proof: exercise.

Clearly there are at most n independent of these commuting functions. We say that the bihamiltonian symplectic structure is *generic* if exactly n of these functions are independent. Let us denote $\lambda_i = \lambda_i(x)$ the eigenvalues of the recursion operator. Since the characteristic polynomial of $\mathcal{R}$ is a perfect square

$$\det(\mathcal{R} - \lambda) = \text{const} \cdot \det(\pi_2 - \lambda\,\pi_1) = \prod_{i=1}^{n} (\lambda - \lambda_i)^2.$$

only n of these eigenvalues can be distinct, say, $\lambda_1 = \lambda_1(x)$, ..., $\lambda_n = \lambda_n(x)$. For generic bihamiltonian symplectic structure these are independent functions on $P \ni x$.

Theorem *Let* $\{\ ,\ \}_{1,2}$ *be a generic symplectic bihamiltonian structure. Then*

1) All the commuting Hamiltonians

$$H_p = \frac{1}{p+1}\mathrm{tr}\,\mathcal{R}^{p+1} = \frac{1}{p+1}\sum_{i=1}^{n}\lambda_i^{p+1}(x), \quad p = 0, 1, \ldots, n-1$$

generate completely integrable systems on P.

2) The eigenvalues $\lambda_i(x)$ *can be included in a coordinate system* $\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n$ *reducing the two Poisson structures to a block diagonal form where the* i*-th block in* π_1 *and in* π_2 *reads, respectively*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

The last formula gives the normal form of a generic symplectic bihamiltonian structure. Therefore all such structures are equivalent w.r.t. the group of local diffeomorphisms.

Idea of the proof: check that the recursion operator has **vanishing Nijenhuis torsion**

$$[\mathcal{R}, \mathcal{R}] = 0$$

where

$$[\mathcal{R}, \mathcal{R}](X, Y) = [\mathcal{R}\, X, \mathcal{R}\, Y] - \mathcal{R}[\mathcal{R}\, X, Y] - \mathcal{R}[X, \mathcal{R}\, Y] + \mathcal{R}^2[X, Y] \tag{27}$$

for arbitrary two vector fields X, Y, or, in local coordinates

$$[\mathcal{R}, \mathcal{R}]^i_{jk} = \mathcal{R}^p_j \mathcal{R}^i_{k,p} - \mathcal{R}^p_k \mathcal{R}^i_{j,p} - \mathcal{R}^i_p \left(\mathcal{R}^p_{k,j} - \mathcal{R}^p_{j,k} \right)$$

Conversely, given a Poisson tensor π_1 and a $(1,1)$-tensor $\mathcal{R}$ (not a scalar) with vanishing Nijenhuis torsion $\Rightarrow \pi_1$ and

$$\pi_2 := \mathcal{R}\,\pi_1$$

define on P a bihamiltonian structure.

Exercise Given a (1,1)-tensor $\mathcal{R}$ such that all eigenvalues $\lambda_1(x)$, ..., $\lambda_N(x)$ of $\mathcal{R}$ are pairwise distinct and $[\mathcal{R},\mathcal{R}] = 0$ prove existence of local coordinates y^1, ..., y^N such that $\mathcal{R}$ becomes diagonal and

$$\mathcal{R}\frac{\partial}{\partial y^i} = \lambda_i(y^i)\,\frac{\partial}{\partial y^i}, \quad i = 1, \dots, N$$

The degenerate situation.

Assume that the Poisson structure $a_1\pi_1+a_2\pi_2$ has *constant rank* for generic a_1 and a_2. Without loss of generality we may assume that

$$k = \text{corank}\,\pi_1 \equiv \text{corank}(\pi_1 + \epsilon\,\pi_2) \tag{28}$$

for an arbitrary sufficiently small ϵ.

Lemma 1.3 *Then the Casimirs of π_1 commute w.r.t. π_2.*

Proof. Let $2m$ = rank of π_1. Reduce the matrix of this bracket to the canonical constant block diagonal form. Denote (π^{ab}) the matrix of the second Poisson bracket in these coordinates. Let us choose two integers i, j such that $2m < i < j \leq N = 2m + k$ and form a $(2m+1) \times (2m+1)$ minor of the matrix $\pi_1 + \epsilon\pi_2$ by adding i-th column and j-th row to the principal $2m \times 2m$ minor standing in the first $2m$ columns and first $2m$ rows. The condition (28) is equivalent to vanishing of the determinants of all these minors. The determinant in question is equal to $-\epsilon\,\pi^{ij} + O(\epsilon^2)$. Therefore $\pi^{ij} = 0$ for all pairs (i, j) greater than $2m$.

Corollary *For a compatible pair of Poisson brackets of the constant* rank $(\pi_2 \ -\lambda\pi_1) = \text{rank}\,\pi_1$, $\lambda \to \infty$,

$$\pi_2 \in H^2(P, \pi_1) \text{ is a trivial cocycle.}$$

Proof. What π_2 is a cocycle w.r.t. the Poisson cohomology of (P, π_1) follows from $[\pi_1, \pi_2] = 0$. To prove triviality use commutativity of the Casimirs of the first Poisson bracket and also Lemma 1.1.

Poisson pencil:

$$\pi_\lambda := \pi_2 - \lambda\pi_1 \tag{29}$$

(marked π_1).

Poisson pencils of constant rank: the corank of π_λ equals k for $\lambda \to \infty$. The recursive construction of the commuting flows in this case is given by

Theorem 1.4 *Under the assumption (28) the coefficients of the Taylor expansion*

$$c^\alpha(x,\lambda) = c^\alpha_{-1}(x) + \frac{c^\alpha_0(x)}{\lambda} + \frac{c^\alpha_1(x)}{\lambda^2} + \dots, \quad \lambda \to \infty \tag{30}$$

of the Casimirs $c^\alpha(x,\lambda)$, $\alpha = 1, \dots, k$ *of the Poisson bracket* $\{\ ,\ \}_\lambda$ *commute with respect to both the Poisson brackets*

$$\{c^\alpha_p, c^\beta_q\}_{1,2} = 0, \quad \alpha, \beta = 1, \dots, k, \quad p, q \geq -1.$$

Proof. Spelling out the definition of the Casimirs

$$\{\ \cdot\ , c^\alpha\}_\lambda = 0$$

for the coefficients of the expansion (30) we must have first that

$$\{\ \cdot\ , c^\alpha_{-1}\}_1 = 0. \tag{31}$$

That is, the leading coefficients of the Taylor expansions are Casimirs of $\{\ ,\ \}_1$. For the subsequent coefficients we get the recursive relations

$$\{\ \cdot\ , c^\alpha_{p+1}\}_1 = \{\ \cdot\ , c^\alpha_p\}_2, \quad p = -1, 0, 1, \ldots \tag{32}$$

From (32) and Lemma 1.2 it follows that

$$\{c^\alpha_p, c^\alpha_q\}_{1,2} = 0, \quad p, q \geq -1.$$

The commutativity $\{c_p^\alpha, c_q^\beta\}_{1,2} = 0$ for $\alpha \neq \beta$ easily follows from the same recursion trick and from commutativity of the Casimirs

$$\{c_{-1}^\alpha, c_{-1}^\beta\}_2 = 0 \tag{33}$$

proved in Lemma 1.3. The theorem is proved.

Example According to triviality of the cohomology class $\pi_2 \in H^2(\pi_1)$ there exists a vector field Z such that

$$Lie_Z \pi_1 = \pi_2.$$

We say that the bihamiltonian structure is *exact* if the vector field Z can be chosen in such a way that

$$(Lie_Z)^2 \pi_1 = 0. \tag{34}$$

For an exact bihamiltonian structure the generating functions (30) of the commuting Hamiltonians $c_p^\alpha(x)$ have the form

$$c^\alpha(x;\lambda) = \exp\left(-Z/\lambda\right) c^\alpha_{-1}(x) = c^\alpha_{-1}(x) - \frac{1}{\lambda}\partial_Z c^\alpha_{-1}(x) + \frac{1}{\lambda^2}\partial_Z^2 c^\alpha_{-1}(x)\dots \tag{35}$$

for every $\alpha = 1, \dots, k$ (**exercise**!).

Conversely, if, in a given coordinate system, $\{\ ,\ \}_1$ depends linearly on one of the coordinates and $\{\ ,\ \}_2$ does not depend on this coordinate then the bihamiltonian structure is exact.

In particular for the standard linear Lie - Poisson structures on the dual spaces to Lie algebras

$$\{x^i, x^j\} = c^{ij}_k x^k \mapsto \{x^i, x^j\}_\lambda = c^{ij}_k x^k - \lambda \pi^{ij}, \quad \pi^{ij} = c^{ij}_k a^k$$

$$x^i \mapsto x^i - \lambda\, a^i$$

(the method of argument translation).

A Poisson pencil of constant corank k $\Rightarrow$ k chains of pairwise commuting bihamiltonian flows

$$\frac{dx}{dt^{\alpha,p}} = \{x, c_p^\alpha\}_1 = \{x, c_{p-1}^\alpha\}_2, \quad \alpha = 1, \ldots, k, \quad p = 0, 1, 2, \ldots \quad (36)$$

Labels of the chains α $\rightarrow$ the Casimirs c_{-1}^α of the first Poisson bracket.

The level p $\rightarrow$ the number of iterations of the recursive procedure

All the family of commuting flows organized by the above recursion procedure is called *the hierarchy* determined by the bihamiltonian structure.

Different choice of the second Poisson bracket in the pencil produces a triangular linear transformation of the commuting Hamiltonians, i.e., to the Hamiltonians of the level p it will be added a linear combination of the Hamiltonians of the lower levels.