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Poisson-Lie groups and Poisson groupoids

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1 Poisson groups and Lie bialgebras

1.1 From Poisson groups to Lie bialgebras

Definition 1.1. A Poisson group is a Lie group endowed with a Poisson structure $\pi \in \mathfrak{X}^2(G)$ such that the multiplication $m : G \times G \rightarrow G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Example 1.1. The reader may have in mind the following two trivial examples.

1. For any Lie algebra $\mathfrak{g}$, its dual $(\mathfrak{g}^*, +)$ is a Poisson group where (i) the Lie group structure is given by the addition (ii) the Poisson structure is the linear Poisson structure, i.e., Lie-Poisson structure.

2. Any Lie group $G$ is a Poisson group with respect to the trivial Poisson bracket.

To impose that $m : G \times G \rightarrow G$ is a Poisson map is equivalent to impose any of the following two conditions

1. $\forall g, h \in G, m_* (\pi_g + \pi_h) = \pi_{gh}$ or,

2. $\forall g, h \in G, (R_h)_* \pi_g + (L_g)_* \pi_h = \pi_{gh}$.

This leads to the following definition:

Definition 1.2. A bivector field $\pi$ on $G$ is said to be multiplicative if

$$(R_h)_* \pi_g + (L_g)_* \pi_h = \pi_{gh}, \quad \forall g, h \in G. \quad (1)$$

In particular, $\pi \in \mathfrak{X}^2(G)$ is a Poisson group if and only if $i)$ the identity $[\pi, \pi] = 0$ holds and $ii)$ $\pi$ is multiplicative.

Remark 1.1. Any multiplicative bivector $\pi$ vanishes in $g = 1$, where 1 is the unit element of the group $G$. This can be seen from Eq. (1) by letting $g = h = 1$. 
It is sometimes convenient to consider \( \bar{\pi}(g) = (R_g)^{-1}\pi_g \), which is, by definition, a smooth map from \( G \) to \( \wedge^2 g \) (where, implicitly, we have identified the Lie algebra \( g \) with the tangent space in \( g = 1 \) of the Lie group \( G \)). When written with the help of \( \bar{\pi} \), the condition \( (R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh} \) reads

\[
\bar{\pi}(g) = (R_g)^{-1}[\bar{(R_h)_*\pi_g + (L_g)_*\pi_h}] = \bar{\pi}(gh)
\]

I.e., \( \bar{\pi} : G \to \wedge^2 g \) is a Lie group 1-cocycle, where \( G \) acts on \( \wedge^2 g \) by adjoint action.

Now, differentiating a Lie group 1-cocycle at the identity, one gets a Lie algebra 1-cocycle \( g \to \wedge^2 g \). For example, the 1-cocycle \( \delta : g \to \wedge^2 g \) associated to the Poisson structure \( \bar{\pi} \) is given by \( \forall X \in g \)

\[
\delta(X) = \frac{d}{dt}|_{t=0} \bar{\pi}(\exp(tX)) = \frac{d}{dt}|_{t=0} (R_{\exp(-tX)}\pi_{\exp(tX)}) = \bar{\pi}(\phi_t) \]

where \( \bar{X} \) is the left invariant vector field on \( G \) corresponding to \( X \) and \( \phi_t \) is its flow.

We have therefore determined \( L_{\bar{X}}\pi \) at \( g = 1 \), we now try to compute it at other points.

For all \( g \in G \), since \( \pi \) is multiplicative, we have \( \forall X \in g \),

\[
\pi_{g \exp(tX)} = (R_{\exp(-tX)}\pi_{\exp(tX)}) = \pi_g + (R_{\exp(-tX)})(L_g)\pi_{\exp(tX)}
\]

Taking the derivative of the previous identity at \( t = 0 \), one obtains:

\[
(L_{\bar{X}}\pi)_g = (L_g)_{\bar{X}}\pi|_1 = (L_g)_*\delta(X)
\]

which implies that \( L_{\bar{X}}\pi \) is left invariant. For all \( Y \in \wedge^k g \), by \( \bar{Y} \) (resp. \( \bar{Y}^r \)) the left (resp. right) invariant \( k \)-vector field on \( G \) equal to \( Y \) at \( g = 1 \). Then we obtain the following formula:

\[
L_{\bar{X}}\pi = \bar{\delta}(X)
\]

and, for similar reasons

\[
L_{\bar{X}}\pi = \bar{\delta}(X).
\]

We can now extend \( \delta : g \to \wedge^2 g \) to a derivation of degree +1 of the graded commutative associative algebra \( \wedge^\bullet g \) that we denote by the same symbol \( \delta : \wedge^\bullet g \to \wedge^{\bullet+1} g \).

**Lemma 1.1.**

1. The derivation \( \delta \) has square zero.
2. \( \delta[X, Y] = [\delta X, Y] + [X, \delta Y], \ \forall X, Y \in g \)

**Proof.** (1) For all \( X \in g \),

\[
[[X, [\pi, \pi]] = 2[[[X, \pi], \pi] = 2[\delta(X), \pi] = 2\delta^2(X)
\]
But $[\pi, \pi] = 0$, hence $\delta^2(X) = 0$.

(2) follows from the graded Jacobi identity:

$$[[X, Y], \pi] = [[X, Y], \pi] = [[X, \pi], Y] + [X, [\pi, Y]]$$

\[\square\]

**Definition 1.3.** A Lie bialgebra is a Lie algebra $\mathfrak{g}$ equipped with a degree 1-derivation $\delta$ of the graded commutative associative algebra $\wedge^* \mathfrak{g}$ such that

1. $\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$ and
2. $\delta^2 = 0$.

**Remark 1.2.** Recall that a Gerstenhaber algebra $A = \oplus_{i \in \mathbb{N}} A^i$ is a graded commutative algebra s.t. $A = \oplus_{i \in \mathbb{N}} A^{(i)}$ where $A^{(i)} = A^{i+1}$ is a graded Lie algebra with the compatibility condition

$$[a, bc] = [a, b]c + (-1)^{|a|+1}b[a, c]$$

for any $a \in A^{|a|}$, $b \in A^{|b|}$ and $c \in A^{|c|}$.

A differential Gerstenhaber algebra is a Gerstenhaber algebra equipped with a degree 1 derivation of square zero.

The Lie bracket on $\mathfrak{g}$ can be extended to a graded Lie bracket on $\wedge^* \mathfrak{g}$ so that $(\wedge^* \mathfrak{g}, \wedge, [\cdot, \cdot])$ is a Gerstenhaber algebra. Using this terminology, a Lie bialgebra is nothing else than a differential Gerstenhaber algebra $(\wedge^* \mathfrak{g}, \wedge, [\cdot, \cdot], \delta)$.

Given a Lie bialgebra $(\mathfrak{g}, \delta)$, let us consider the dual $\delta^*: \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ of the derivation $\delta$.

Let $[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \wedge \eta)$ for all $\xi, \eta \in \mathfrak{g}^*$. The bilinear map $(\xi, \eta) \to [\xi, \eta]_{\mathfrak{g}^*}$ is skew-symmetric and

$$\delta^2 = 0 \iff [\cdot, \cdot]_{\mathfrak{g}^*} \text{ satisfies the Jacobi identity}$$

Therefore, the dual $\mathfrak{g}^*$ of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a Lie algebra again (which justifies the name). Conversely, a Lie bialgebras can be described again by:

**Proposition 1.2.** A Lie bialgebra is equivalent to a pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}^*)$ compatible in the sense that the following relation is satisfied: the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is a derivation of the bracket $[\cdot, \cdot]_{\mathfrak{g}^*}$, i.e.,

$$ad^*_{\alpha}[\beta]_{\mathfrak{g}^*} = [ad^*_{\alpha}\beta]_{\mathfrak{g}^*} + [\alpha, ad^*_{\beta}\beta]_{\mathfrak{g}^*} \text{ for all } X \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*.$$

**Remark 1.3.** Note that Lie bialgebras are in duality: namely $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is a Lie bialgebra. This picture can be seen more naturally using Manin triples, which will be discussed in the next lecture.
1.2 \( r \)-matrices

We now turn our attention to a particular class of Lie bialgebras, i.e., those coming from \( r \)-matrices.

We start from a Lie algebra \( g \). Assume that we are given an element \( r \in \wedge^2 g \). Then define \( \delta \) by, for all \( X \in \wedge^\ast g \), \( \delta(X) = [r, X] \). As can easily be checked, \( \delta \) is a derivation of \( \wedge^\ast g \). Note that, in terms of (Chevalley-Eilenberg) cohomology, \( \delta \) is the coboundary of \( r \).

The condition \( \delta^2(X) = 0 \) is equivalent to the relation \( [X, [r, r]] = 0 \), which itself holds if and only if \( [r, r] \) is \( \text{ad} \)-invariant. Conversely, any \( r \in \wedge^2 g \) such that \( [r, r] \) is \( \text{ad} \)-invariant defines a Lie bialgebra. Such an \( r \) is called an \( r \)-matrix. If moreover \( [r, r] = 0 \), then this Lie bialgebra is called triangular.

Here are two well-known examples of \( r \)-matrices.

Example 1.2. 1. Consider \( g \) a semi-simple Lie algebra of rank \( k \) over \( \mathbb{C} \) with Cartan sub-algebra \( h \). Let \( \{e_\alpha, f_\alpha, \alpha \in \Delta_+\} \cup \{h_i, i = 1, \ldots, k\} \) be a Chevalley basis. Then \( r = \sum_{\alpha \in \Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha \) with \( \lambda_\alpha = \frac{1}{(e_\alpha, f_\alpha)} \) is an \( r \)-matrix.

2. Consider now \( k \) a compact semi-simple Lie algebra over \( \mathbb{R} \). Let \( \{e_\alpha, f_\alpha, \alpha \in \Delta_+\} \cup \{t_i, i = 1, \ldots, k\} \) be a Chevalley basis (over \( \mathbb{C} \)) of the complexified Lie algebra \( g = k^\mathbb{C} \), that we assume to be constructed such that the family \( \{X_\alpha, Y_\alpha, \alpha \in \Delta_+\} \cup \{t_i, i = 1, \ldots, k\} \) is a basis of \( k \) (over \( \mathbb{R} \)) where

\[
\begin{align*}
X_\alpha &= e_\alpha - f_\alpha \quad \text{for all } \alpha \in \Delta_+ \\
Y_\alpha &= \sqrt{-1}(e_\alpha + f_\alpha) \quad \text{for all } \alpha \in \Delta_+ \\
t_i &= \sqrt{-1}h_i \quad \text{for all } i = 1, \ldots, k
\end{align*}
\]

Let \( \hat{r} = \sqrt{-1}r = \sqrt{-1} \sum_{\alpha \in \Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha \). Then \( \hat{r} \) is, according to the first example above, an \( r \)-matrix of \( g = k^\mathbb{C} \). However, by a direct computation, one checks that

\[
\hat{r} = \frac{1}{2} \sum_{\alpha \in \Delta_+} \lambda_\alpha X_\alpha \wedge Y_\alpha
\]

so that \( \hat{r} \) is indeed an element of \( \wedge^2 k \), and therefore is an \( r \)-matrix on \( k \). Hence, it defines a Lie bialgebra structure on the real Lie algebra \( k \).

1.3 Lie bialgebras and simply-connected Lie groups

We have already explained how to get a Lie bialgebra from a Poisson group. The inverse is true as well when the Lie group is connected and simply-connected.

Theorem 1.3 (Drinfeld). Assume that \( G \) is a connected and simply-connected Lie group. Then there exists a one-to-one correspondence

\[
\text{Poisson groups } (G, \pi) \leftrightarrow \text{Lie bialgebra } (g, \delta).
\]

Example 1.3. In particular, for a Lie bialgebra coming from an \( r \)-matrix \( r \), the corresponding Poisson structure on \( G \) is the bivector field \( \bar{r} - \bar{r} \).

Applying the theorem above to the previous two examples, we are lead to
**Proposition 1.4.** 1. Any complex semi-simple Lie group admits a natural (complex) Poisson group structure.

2. Soibelman, Lu-Weinstein Any compact semi-simple Lie group admits a natural Poisson group structure, called the Bruhat-Poisson structure.

**Remark 1.4.** Poisson groups come in pairs in the following sense. Given a Poisson group \((G, \pi)\), let \((g, g^\ast)\) be its Lie bialgebra, then we know \((g^\ast, g)\) is also a Lie bialgebra which gives rise to a Poisson group denoted \((G^\ast, \pi')\).

**Example 1.4.** On the Lie group \(G = SU(2)\), define complex coordinates \(\alpha, \beta\) by
\[
g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}
\]
Note that these coordinates are not "free" since \(|\alpha|^2 + |\beta|^2 = 1\). The Bruhat-Poisson structure is given by
\[
\{\alpha, \bar{\alpha}\} = 2\sqrt{-1} \beta \bar{\beta}
\]
\[
\{\alpha, \beta\} = -\sqrt{-1} \alpha \beta
\]
\[
\{\alpha, \bar{\beta}\} = -\sqrt{-1} \alpha \bar{\beta}
\]
\[
\{\beta, \bar{\beta}\} = 0
\]

**Example 1.5.** Below are two examples of duals of Poisson groups.

1. For the Poisson group \(G = SU(2)\), equipped with the Bruhat-Poisson structure, the dual group \(G^\ast\) is \(SB(2)\), i.e. the subgroup of two-by-two matrices of the form
\[
G^\ast = SB(2) \simeq \left\{ \begin{pmatrix} a & b + \sqrt{-1}c \\ 0 & \frac{1}{a} \end{pmatrix} \middle| b, c \in \mathbb{R}, a \in \mathbb{R}^+ \right\}
\]
Using these coordinates, the Poisson structure on \(G^\ast\) is given explicitly by
\[
\{b, c\} = a^2 - \frac{1}{a^2}
\]
\[
\{a, b\} = ab
\]
\[
\{a, c\} = ac
\]

2. For the Poisson group \(G = SL_C(n)\), equipped with the Poisson bracket constructed in Example 1.2, the dual group is
\[
G^\ast = B_+ \times B_- \simeq \left\{ (A, B) \middle| \begin{array}{c} A \text{ upper triangular with determinant 1,} \\ B \text{ lower triangular with determinant 1,} \\ \text{s.t. } \text{diag}(A) \cdot \text{diag}(B) = 1 \end{array} \right\}
\]

1.4 Poisson group actions

**Definition 1.4.** Let \(G\) be a Poisson group. Assume that \(G\) acts on a Poisson manifold \(X\). The action is said to be a Poisson action if the action map
\[
G \times X \rightarrow X \\
(g, x) \rightarrow g \cdot x
\]
is a Poisson map, where \(G \times X\) is equipped with the product Poisson structure.
Warning Note that in general, this definition does not imply that, for a fixed \( g \) in \( G \), the action \( x \to g \cdot x \) is a Poisson automorphism of \( X \). The reader should not confuse Poisson actions with actions preserving the Poisson structure! Note, however, that when the Poisson structure on the Lie group \( G \) is the trivial one, then a Poisson action is an action of \( G \) on \( X \) which preserves the Poisson structure.

Example 1.6. Any Lie group \( G \) acts on itself by left translation. If \( G \) is a Poisson group then this action is a Poisson action.

Proposition 1.5 (Lu-Weinstein). Let \( G \) be a Poisson group with Lie bialgebra \((\mathfrak{g}, \delta)\). Assume that \( G \) acts on a manifold \( X \) and let \( \rho : \mathfrak{g} \to \mathfrak{X}(X) \) be the infinitesimal action. The action of \( G \) on \( X \) is a Poisson action if and only if the following diagram commutes

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}(X) \\
\downarrow \delta & & \downarrow [\pi, \cdot] \\
\wedge^2 \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}^2(X)
\end{array}
\]

In terms of Gerstenhaber algebras, the commutativity of the previous diagram has a clear meaning: it simply means that \( \rho : \wedge^\bullet \mathfrak{g} \to \mathfrak{X}^\bullet(X) \) is a morphism of differential Gerstenhaber algebra.

Example 1.7. For the dual \( SL_\mathbb{C}(3)^* = B_+ \ast B_- \) of \( G = SL_\mathbb{C}(3) \). Consider the Poisson manifold

\[
X = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{C} \right\}
\]
equipped with the Poisson bracket

\[
\{x, y\} = xy - 2z \\
\{y, z\} = yz - 2x \\
\{z, x\} = zx - 2y
\]
The Lie group \( G^* = B_+ \ast B_- \) acts on \( X \) by

\[
(A, B) \cdot U \to AUB^T
\]
with \( (A, B) \in B_+ \ast B_- \cong G^* \) and \( U \in X \). This action turns to be a Poisson action.

2 Poisson groupoids and Lie bialgebroids

Idea Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. A Poisson groupoid structure on \( \Gamma \) should be a multiplicative Poisson structure on \( \Gamma \).

Recall In the Poisson group case,

\[
\pi \text{ is multiplicative} \iff m : G \times G \to G \text{ is a Poisson map} \iff \{(x, y, xy)|x, y \in G\} \subset G \times G \times \overline{G} \text{ is coisotropic}
\]
where \( \overline{G} \) denotes \((G, \pi)\). This motivates the following definition.
Definition 2.1. A groupoid $\Gamma$ with a Poisson structure $\pi$ is said to be a Poisson groupoid if the graph of the groupoid multiplication

$$
\Lambda = \{(x, y, xy) | (x, y) \in \Gamma \times \Gamma \text{ composable pair} \} \subset \Gamma \times \Gamma \times \Gamma
$$

is coisotropic. Here $\bar{\Gamma}$ means that $\Gamma$ is equipped with the opposite Poisson structure $-\pi$.

Example 2.1. 1. If $P$ is a Poisson manifold, then $P \times \bar{P} \Rightarrow P$ is a Poisson groupoid.

2. Let $A$ be the Lie algebroid of a Lie groupoid $\Gamma$ and $\Lambda \in \Gamma(\wedge^2 A)$ be an element satisfying $L_X[\Lambda, \Lambda] = 0, \forall X \in \Gamma(A)$. Then $\pi = \Lambda - \bar{\Lambda}$ defines a Poisson groupoid structure on $\Gamma$.

Definition 2.2. A symplectic groupoid is a Poisson groupoid $(P \Rightarrow M, \pi)$ such that $\pi$ is non-degenerate. In other words, $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ is a Lagrangian submanifold.

Example 2.2. 1. $T^*M \Rightarrow M$ with the canonical cotangent symplectic structure is a symplectic groupoid.

2. If $G$ is a Lie group, then $T^*G \Rightarrow g^*$ is a symplectic groupoid. Here the symplectic structure on $T^*G$ is the canonical cotangent symplectic structure. The groupoid structure is as follows. Right translations give an isomorphism between $T^*G$ and the transformation groupoid $G \times g^*$ where $G$ acts on $g^*$ by coadjoint action.

3. In general, if $\Gamma \Rightarrow M$ is a Lie groupoid with Lie algebroid $A$, then $T^*\Gamma \Rightarrow A^*$ is a symplectic groupoid. Let $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ denote the graph of the multiplication and $N^*\Lambda \subset T^*\Gamma \times T^*\Gamma \times T^*\Gamma$ its conormal space.

Exercise 2.1. Show that $N^*\Lambda = \{(\xi, \eta, \delta) | (\xi, \eta, -\delta) \in N^*\Lambda \}$ is the graph of a groupoid multiplication on $T^*\Gamma$ with corresponding unit space isomorphic to $A^* \cong N^*M$. This defines a groupoid structure on $T^*\Gamma \Rightarrow A^*$.

Question Why symplectic groupoids?

Symplectic groupoids are used in

1. quantization

2. symplectic realization

Given a Poisson manifold $M$, can one embed the Poisson algebra $C^\infty(M)$ into a Poisson subalgebra of $C^\infty(S)$ where $S$ is some symplectic manifold?

$$
C^\infty(M) \hookrightarrow C^\infty(S)
$$

Note that locally, there exists local coordinates $(p_1, \ldots, p_k, q_1, \ldots, q_k)$ in which the Poisson bracket on $C^\infty(S)$ has the following form: $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = 0 = \{q_i, q_j\}$.

This question was first investigated in 1890 by Sophus Lie under the name of "Function groups". It leads to the following definition.

Definition 2.3. A symplectic realization of a Poisson manifold $(M, \pi)$ consists of a pair $(X, \Phi)$, where $X$ is a symplectic manifold and $\Phi : X \rightarrow M$ is a Poisson map which is a surjective submersion.
Given a Poisson manifold, does there exist a symplectic realization. And if so, is it unique?

1. Local existence:
   (a) Lie (regular Poisson)
   (b) Weinstein 1983 (using splitting theorem)

2. Global existence: Karasev and Weinstein 1987
   (a) Symplectic realizations exist globally for any Poisson manifold.
   (b) There exists a distinguished symplectic realization, which admits a compatible local groupoid structure, i.e. a symplectic local groupoid.

Idea Find local symplectic realizations. Patch them together.

Puzzle Why do symplectic and groupoid structures arise in the context of Poisson manifolds in such a striking manner?
Recall that in the case of a Poisson group \((G, \pi)\), the associated infinitesimal object is a Lie bialgebra \((g, \delta)\).
Half-way between Poisson groups and symplectic groupoids, there should be a notion of Poisson groupoids. Such a notion could help to better understand symplectic groupoids by imitating Poisson group theory.

Poisson groupoids \{ Poisson groups, symplectic groupoids \}

**Theorem 2.1.** Let \(\Gamma \xrightarrow{a} M\) be a Lie groupoid. Let \(\pi \in \mathfrak{X}^2(\Gamma)\) be a Poisson tensor. Then \((\Gamma, \pi)\) is a Poisson groupoid if and only if all the following hold.

1. For all \((x, y) \in \Gamma_2\),
   \[\pi(xy) = R_Y \pi(x) + L_X \pi(y) - R_Y L_X \pi(w),\]
   where \(w = \beta(x) = \alpha(y)\) and \(X, Y\) are (local) bisections through \(x\) and \(y\) respectively.

2. \(M\) is a coisotropic submanifold of \(\Gamma\)

3. For all \(x \in \Gamma\), \(\alpha_* \pi(x)\) and \(\beta_* \pi(x)\) only depend on the base points \(\alpha(x)\) and \(\beta(x)\) respectively.

4. For all \(\alpha, \beta \in C^\infty(M)\), one has \(\{\alpha^* f, \beta^* g\} = 0\), \(\forall f, g \in C^\infty(M)\).

5. The vector field \(X_{\beta^* f}\) is left invariant for all \(f \in C^\infty(M)\).

**Remark 2.1.** If \(M\) is a point, then

1. \(\Leftrightarrow\) multiplicativity condition,

2. \(\Leftrightarrow\pi(1) = 0\),
3. is automatic,
4. is automatic,
5. is automatic.

And one gets the characterization of a Poisson group: a Lie group equipped with a multiplicative Poisson tensor.

**Question** What is the infinitesimal object associated to a Poisson groupoid?

**Corollary 1.** Given a Poisson groupoid \((\Gamma \rightrightarrows M, \pi)\), we have

1. for all \(X \in \Gamma(A)\), \([\overline{X}, \pi]\) is still left invariant
2. \(\pi_M := \alpha_* \pi \) (or \(-\beta_* \pi\)) is a Poisson tensor on \(M\)

**Proof.** For all \(X \in \Gamma(A)\), take \(\xi_t = \text{exp}_tX \in U(\Gamma)\) (the space of bisections of \(\Gamma\)), \(u_t = (\text{exp}_tX)(u)\) and \(x \in \Gamma\) with \(\beta(x) = u\). In other words, \(u_t\) is the flow of \(\overline{X}\) initiated at \(u\). Let \(K\) be any bisection through \(x\). One gets

\[
\pi(xu_t) = R_{\xi_t}\pi(x) + L_K\pi(u_t) - L_KR_{\xi_t}\pi(u)
\]

\[
\Rightarrow R_{\xi^{-1}_t}\pi(xu_t) = \pi(x) + L_KR_{\xi^{-1}_t}\pi(u_t) - L_K\pi(u) \in \wedge^2 T_x\Gamma
\]

and, differentiating with respect to \(t\) at 0,

\[
(L_{\overline{X}}\pi)(x) = L_K((L_{\overline{X}}\pi)(u)).
\]

This implies that \(L_{\overline{X}}\pi\) is left invariant.

Now, we can define \(\delta : \Gamma(\wedge^i A) \to \Gamma(\wedge^{i+1} A)\). For \(i = 0\),

\[
C^\infty(M) \to \Gamma(A) : f \mapsto X_{\beta_* f} = [\beta^* f, \pi].
\]

For \(i = 1\),

\[
\Gamma A \to \Gamma(\wedge^2 A) : X \mapsto \overline{\delta X} = [\overline{X}, \pi].
\]

The following lemma can be easily verified.

**Lemma 2.2.**  1. \(\delta(fg) = g\delta f + f\delta g\), \(\forall f, g \in C^\infty(M)\)
   2. \(\delta(fX) = f \wedge X + f\delta X\), \(\forall f \in C^\infty(M)\) and \(X \in \Gamma(A)\)
   3. \(\delta [X, Y] = [\delta X, Y] + [X, \delta Y]\), \(\forall X, Y \in \Gamma(A)\)
   4. \(\delta^2 = 0\)

**Definition 2.4.** A Lie bialgebroid is a Lie algebroid \(A\) equipped with a degree 1 derivation \(\delta\) of the associative algebra \((\Gamma(\wedge^i A), \wedge)\) satisfying conditions 3 and 4 of the previous lemma.

**Exercise 2.2.** Show that a Lie bialgebroid structure is equivalently characterized as a degree 1 derivation \(\delta\) of the Gerstenhaber algebra \((\Gamma(\wedge^i A), \wedge, [\cdot, \cdot])\) such that \(\delta^2 = 0\). This is also called a differential Gerstenhaber algebra.
Remark 2.2. Given a Lie bialgebroid \((A, \delta)\), there is a natural Lie algebroid structure on \(A^*\) defined as follows.

1. The anchor map \(\rho_\star : A^* \to TM\) is
   \[ \langle \rho_\star \xi, f \rangle = \langle \xi, \delta f \rangle, \quad \forall f \in C^\infty(M). \]

2. The bracket \([,]\) is given by
   \[ \langle [\xi, \eta], X \rangle = (\delta X)(\xi, \eta) + (\rho_\star \xi) \langle X, \eta \rangle - (\rho_\star \eta) \langle X, \xi \rangle, \quad \forall \xi, \eta \in \Gamma(A^*), \forall X \in \Gamma(A). \]

Indeed, equivalently, a Lie bialgebroid is a pair of Lie algebroids \((A, A^*)\) such that
\[
\delta [X, Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \Gamma(A),
\]
where \(\delta : \Gamma(A) \to \Gamma(\wedge^2 A)\) is defined by the above equation \((2)\).

Remark 2.3. If \((A, A^*)\) is a Lie bialgebroid, then \((A^*, A)\) is also a Lie bialgebroid and it is called its dual.

Example 2.3. 1. If \(\pi\) is a Poisson tensor on \(M\), then \(A = TM\) with \(\delta = [\pi, \cdot] : \mathfrak{X}^*(M) \to \mathfrak{X}^{*+1}(M)\) is a Lie bialgebroid. In this case, \(A^* = T^*M\) is the canonical cotangent Lie algebroid.

2. The dual to the previous one: \(A = T^*M\), the cotangent Lie algebroid of a Poisson manifold \((M, \pi)\), together with \(\delta^\star = d_{DR} : \Omega^*(M) \to \Omega^{*+1}(M)\).

3. Coboundary Lie bialgebroid. Take \(A\) a Lie algebroid admitting a \(\Lambda \in \Gamma(\wedge^2 A)\) satisfying
   \[ \mathcal{L}_X [\Lambda, \Lambda] = 0, \forall X \in \Gamma(A). \]
   Let \(\delta = [\Lambda, \cdot] : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)\). Then \((A, \delta)\) defines a Lie bialgebroid.

4. Dynamical \(r\)-matrix. Consider the Lie algebroid \(A = T\mathfrak{h}^* \oplus \mathfrak{g} \to \eta\) where \(\mathfrak{h}\) is an abelian subalgebra of \(\mathfrak{g}\) and the Lie algebroid structure on \(A\) is the product Lie algebroid. Choose a map \(r : \mathfrak{h}^* \to \wedge^2 \mathfrak{g}\) and consider it as a element \(\Lambda\) of \(\Gamma(\wedge^2 A)\). Then \(L_X [\Lambda, \Lambda] = 0\) if and only if
   \[
   \sum h_i \wedge \frac{dr}{\partial \lambda_i} + \frac{1}{2} [r, r] \in (\wedge^3 \mathfrak{g})^0
   \]
   is a constant function over \(\mathfrak{h}^*\). Here \(\{h_1, \ldots, h_k\}\) is a basis of \(\mathfrak{h}\) and \((\lambda_1, \ldots, \lambda_k)\) are the dual coordinates on \(\mathfrak{h}^*\).
   In particular, if \(\mathfrak{g}\) is a simple Lie algebra and \(\mathfrak{h} \subset \mathfrak{g}\) is a Cartan subalgebra, one can take
   \[ r(\lambda) = \sum_{\alpha \in \Delta^+} \frac{\lambda_\alpha}{(\alpha, \lambda)} e_\alpha \wedge f_\alpha \]
or
   \[ r(\lambda) = \sum_{\alpha \in \Delta^+} \lambda_\alpha \coth (\alpha, \lambda) e_\alpha \wedge f_\alpha, \]
   where \((e_\alpha, f_\alpha, h_i)\) is a Chevalley basis.