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Poisson-Lie groups and Poisson groupoids

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1 Poisson groups and Lie bialgebras

1.1 From Poisson groups to Lie bialgebras

Definition 1.1. A Poisson group is a Lie group endowed with a Poisson structure $\pi \in \mathfrak{X}^2(G)$ such that the multiplication $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Example 1.1. The reader may have in mind the following two trivial examples.

1. For any Lie algebra $\mathfrak{g}$, its dual $(\mathfrak{g}^*, +)$ is a Poisson group where *(i)* the Lie group structure is given by the addition *(ii)* the Poisson structure is the linear Poisson structure, i.e., Lie-Poisson structure.

2. Any Lie group G is a Poisson group with respect to the trivial Poisson bracket.

To impose that $m : G \times G \to G$ is a Poisson map is equivalent to impose any of the following two conditions

1. $\forall g, h \in G$, $m_*(\pi_g + \pi_h) = \pi_{gh}$ or,

2. $\forall g, h \in G$, $(R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh}$.

This leads to the following definition:

Definition 1.2. A bivector field π on G is said to be multiplicative if

$$(R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh}, \quad \forall g, h \in G. \tag{1}$$

In particular, $\pi \in \mathfrak{X}^2(G)$ is a Poisson group if and only if *i)* the identity $[\pi, \pi] = 0$ holds and *ii)* π is multiplicative.

Remark 1.1. Any multiplicative bivector π vanishes in $g = 1$, where 1 is the unit element of the group G. This can be seen from Eq. (1) by letting $g = h = 1$.

It is sometimes convenient to consider $\widetilde{\pi}(g) = (R_g)_*^{-1}\pi_g$, which is, by definition, a smooth map from G to $\wedge^2\mathfrak{g}$ (where, implicitely, we have identified the Lie algebra $\mathfrak{g}$ with the tangent space in $g=1$ of the Lie group G). When written with the help of $\widetilde{\pi}$, the condition $(R_h)_*\pi_g + (L_g)_*\pi_h = \pi_{gh}$ reads

$$\begin{array}{rcl} (R_{gh})_*^{-1}[(R_h)_*\pi_g + (L_g)_*\pi_h] &=& (R_{gh})_*^{-1}\pi_{gh} \\ \widetilde{\pi}(g) + Ad_g\widetilde{\pi}(h) &=& \widetilde{\pi}(gh) \end{array}$$

I.e., $\widetilde{\pi} : G \to \wedge^2\mathfrak{g}$ is a Lie group 1-cocycle, where G acts on $\wedge^2\mathfrak{g}$ by adjoint action.

Now, differentiating a Lie group 1-cocycle at the identity, one gets a Lie algebra 1-cocycle $\mathfrak{g} \to \wedge^2\mathfrak{g}$. For example, the 1-cocycle $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ associated to the Poisson structure $\widetilde{\pi}$ is given by $\forall X \in \mathfrak{g}$

$$\begin{array}{rcl} \delta(X) &=& \frac{d}{dt}_{|t=0}\widetilde{\pi}\big(exp(tX)\big) \\ &=& \frac{d}{dt}_{|t=0}(R_{exp(-tX)})_*\pi_{exp(tX)} \\ &=& (\phi_{-t})_*\pi_{\phi_t(1)} \\ &=& (L_{\overleftarrow{X}}\pi)_{|g=1} \end{array}$$

where $\overleftarrow{X}$ is the left invariant vector field on G corresponding to X and ϕ_t is its flow.

We have therefore determined $L_{\overleftarrow{X}}\pi$ at $g=1$, we now try to compute it at other points.

For all $g \in G$, since π is multiplicative, we have $\forall X \in \mathfrak{g}$,

$$\begin{array}{rcl} \pi_{g\,exp(tX)} &=& (R_{exp(-tX)})_*\pi_g + (L_g)_*\pi_{exp(tX)} \\ (R_{exp(-tX)})_*\pi_{g\,exp(tX)} &=& \pi_g + (R_{exp(-tX)})_*(L_g)_*\pi_{exp(tX)} \\ (\phi_{-t})_*\pi_{\phi_t(g)} &=& \pi_g + L_g(\phi_{-t})_*\pi_{\phi_t(1)} \end{array}$$

Taking the derivative of the previous identity at $t=0$, one obtains:

$$(L_{\overleftarrow{X}}\pi)_{|g} = (L_g)_*L_{\overleftarrow{X}}\pi_{|1} = (L_g)_*\delta(X)$$

which implies that $L_{\overleftarrow{X}}\pi$ is left invariant. For all $Y \in \wedge^k\mathfrak{g}$, by $\overleftarrow{Y}$ (resp. $\overrightarrow{Y}$) the left (resp. right) invariant k-vector field on G equal to Y at $g=1$. Then we obtain the following formula:

$$L_{\overleftarrow{X}}\pi = \overleftarrow{\delta(X)}.$$

and, for similar reasons

$$L_{\overrightarrow{X}}\pi = \overrightarrow{\delta(X)}.$$

We can now extend $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ to a derivation of degree $+1$ of the graded commutative associative algebra $(\wedge^*\mathfrak{g}$ that we denote by the same symbol $\delta : \wedge^\bullet\mathfrak{g} \to \wedge^{\bullet+1}\mathfrak{g}$.

Lemma 1.1. *1. The derivation δ has square zero.*

2. $\delta[X,Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \mathfrak{g}$

Proof. (1) For all $X \in \mathfrak{g}$,

$$\begin{array}{rcl} \big[\overleftarrow{X}, [\pi,\pi]\big] &=& 2\big[[\overleftarrow{X},\pi],\pi\big] \\ &=& 2\,[\overleftarrow{\delta(X)},\pi] \\ &=& 2\,\overleftarrow{\delta^2}(X) \end{array}$$

But $[\pi, \pi] = 0$, hence $\delta^2(X) = 0$.

(2) follows from the graded Jocobi identity:

$$[\overleftarrow{[X,Y]}, \pi] = [[\overleftarrow{X}, \overleftarrow{Y}], \pi] = [[\overleftarrow{X}, \pi], \overleftarrow{Y}] + [\overleftarrow{X}, [\pi, \overleftarrow{Y}]]$$

□

Definition 1.3. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ equiped with a degree 1-derivation δ of the graded commutative associative algebra $\wedge^\bullet \mathfrak{g}$ such that

1. $\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)]$ and
2. $\delta^2 = 0$.

Remark 1.2. Recall that a *Gerstenhaber algebra* $A = \oplus_{i\in\mathbb{N}} A^i$ is a graded commutative algebra s.t. $A = \oplus_{i\in\mathbb{N}} A^{(i)}$ where $A^{(i)} = A^{i+1}$ is a graded Lie algebra with the compatibility condition

$$[a, bc] = [a, b]c + (-1)^{(|a|+1)|b|} b[a, c]$$

for any $a \in A^{|a|}$, $b \in A^{|b|}$ and $c \in A^{|c|}$.

A *differential Gerstenhaber algebra* is a Gerstenhaber algebra equipped with a degree 1 derivation of square zero.

The Lie bracket on $\mathfrak{g}$ can be extented to a graded Lie bracket on $\wedge^\bullet \mathfrak{g}$ so that $(\wedge^\bullet \mathfrak{g}, \wedge, [\cdot,\cdot])$ is a Gerstenhaber algebra. Using this terminology, a Lie bialgebra is nothing else than a differential Gerstenhaber algebra $(\wedge^\bullet \mathfrak{g}, \wedge, [\cdot,\cdot], \delta)$.

Given a Lie bialgebra $(\mathfrak{g}, \delta)$, let us consider the dual $\delta^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ of the derivation δ.

Let $[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \wedge \eta)$ for all $\xi, \eta \in \mathfrak{g}^*$. The bilinear map $(\xi, \eta) \to [\xi, \eta]_{\mathfrak{g}^*}$ is skew-symmetric and

$$\delta^2 = 0 \quad \Leftrightarrow \quad [\cdot,\cdot]_{\mathfrak{g}^*} \text{ satisfies the Jacobi identity}$$

Therefore, the dual $\mathfrak{g}^*$ of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a Lie algebra again (which justifies the name). Conversely, a Lie bialgebras can be described again by:

Proposition 1.2. *A Lie bialgebra is equivalent to a pair of Lie algebras* $(\mathfrak{g}, \mathfrak{g}^*)$ *compatible in the sense that the following relation is satisfied: the coadjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g}^*$ *is a derivation of the bracket* $[\cdot,\cdot]_{\mathfrak{g}^*}$*, i.e.,*

$$ad^*_X[\alpha, \beta]_{\mathfrak{g}^*} = [ad^*_X\alpha, \beta]_{\mathfrak{g}^*} + [\alpha, ad^*_X\beta]_{\mathfrak{g}^*} \text{ for all } X \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*.$$

Remark 1.3. Note that Lie bialgebras are in duality: namely $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is a Lie bialgebra. This picture can be seen more naturally using Manin triples, which will be discussed in the next lecture.

1.2 r-matrices

We now turn our attention to a particular class of Lie bialgebras, i.e., those coming from r-matrices.

We start from a Lie algebra $\mathfrak{g}$. Assume that we are given an element $r \in \wedge^2\mathfrak{g}$. Then define δ by, for all $X \in \wedge^\bullet\mathfrak{g}$, $\delta(X) = [r, X]$. As can easily be checked, δ is a derivation of $\wedge^\bullet\mathfrak{g}$. Note that, in terms of (Chevalley-Eilenberg) cohomology, δ is the coboundary of r.

The condition $\delta^2(X) = 0$ is equivalent to the relation $\big[X, [r, r]\big] = 0$, which itself holds if and only if $[r, r]$ is ad-invariant. Conversely, any $r \in \wedge^2\mathfrak{g}$ such that $[r, r]$ is ad-invariant defines a Lie bialgebra. Such an r is called an *r-matrix*. If moreover $[r, r] = 0$, then this Lie bialgebra is called *triangular*.

Here are two well-known examples of r-matrices.

Example 1.2.

1. Consider $\mathfrak{g}$ a semi-simple Lie algebra of rank k over $\mathbb{C}$ with Cartan sub-algebra $\mathfrak{h}$. Let $\{e_\alpha, f_\alpha, \alpha \in \Delta_+\} \cup \{h_i, i = 1, \ldots, k\}$ be a Chevalley basis. Then $r = \sum_{\alpha\in\Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha$ with $\lambda_\alpha = \frac{1}{(e_\alpha, f_\alpha)}$ is an r-matrix.

2. Consider now $\mathfrak{k}$ a compact semi-simple Lie algebra over $\mathbb{R}$. Let $\{e_\alpha, f_\alpha, \alpha \in \Delta_+\} \cup \{h_i, i = 1 \ldots, k\}$ be a Chevalley basis (over $\mathbb{C}$) of the complexified Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, that we assume to be constructed such that the family $\{X_\alpha, Y_\alpha, \alpha \in \Delta_+\} \cup \{t_i, i = 1, \ldots, k\}$ is a basis of $\mathfrak{k}$ (over $\mathbb{R}$) where

$$\begin{cases} X_\alpha = e_\alpha - f_\alpha & \text{for all } \alpha \in \Delta_+ \\ Y_\alpha = \sqrt{-1}(e_\alpha + f_\alpha) & \text{for all } \alpha \in \Delta_+ \\ t_i = \sqrt{-1}h_i & \text{for all } i \in \{1, \ldots, k\} \end{cases}$$

   Let $\hat{r} = \sqrt{-1}\, r = \sqrt{-1} \sum_{\alpha\in\Delta_+} \lambda_\alpha e_\alpha \wedge f_\alpha$. Then $\hat{r}$ is, according to the first example above, an r-matrix of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$. However, by a direct comptation, one checks that

$$\hat{r} = \frac{1}{2} \sum_{\alpha\in\Delta_+} \lambda_\alpha X_\alpha \wedge Y_\alpha$$

   so that $\hat{r}$ is indeed an element of $\wedge^2\mathfrak{k}$, and therefore is an r-matrix on $\mathfrak{k}$. Hence, it defines a Lie bialgebra structure on the real Lie algebra $\mathfrak{k}$.

1.3 Lie bialgebras and simply-connected Lie groups

We have already explained how to get a Lie bialgebra from a Poisson group. The inverse is true as well when the Lie group is connected and simply-connected.

Theorem 1.3 (Drinfeld)**.** *Assume that G is a connected and simply-connected Lie group. Then there exists a one-to-one correspondence*

$$\textit{Poisson groups } (G, \pi) \qquad \leftrightarrow \qquad \textit{Lie bialgebra } (\mathfrak{g}, \delta).$$

Example 1.3. In particular, for a Lie bialgebra coming from an r-matrix r, the corresponding Poisson structure on G is the bivector field $\overleftarrow{r} - \overrightarrow{r}$.

Applying the theorem above to the previous two examples, we are lead to

Proposition 1.4. 1. *Any complex semi-simple Lie group admits a natural (complex) Poisson group structure.*

2. ***Soibelmann, Lu-Weinstein*** *Any compact semi-simple Lie group admits a natural Poisson group structure, called the Bruhat-Poisson structure.*

Remark 1.4. Poisson groups come in pairs in the following sense. Given a Poisson group (G, π), let $(\mathfrak{g}, \mathfrak{g}^*)$ be its Lie bialgebra, then we know $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra which gives rise to a Poisson group denoted (G^*, π').

Example 1.4. On the Lie group $G = SU(2)$, define complex coordinates α, β by

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Note that these coordinates are not "free" since $|\alpha|^2 + |\beta|^2 = 1$. The Bruhat-Poisson structure is given by

$$\begin{array}{rcl} \{\alpha, \bar{\alpha}\} & = & 2\sqrt{-1}\,\beta\bar{\beta} \\ \{\alpha, \beta\} & = & -\sqrt{-1}\,\alpha\beta \\ \{\alpha, \bar{\beta}\} & = & -\sqrt{-1}\,\alpha\bar{\beta} \\ \{\beta, \bar{\beta}\} & = & 0 \end{array}$$

Example 1.5. Below are two examples of duals of Poisson groups.

1. For the Poisson group $G = SU(2)$, equipped with the Bruhat-Poisson structure, the dual group G^* is $SB(2)$, i.e. the subgroup of two-by-two matrices of the form

$$G^* = SB(2) \simeq \left\{ \begin{pmatrix} a & b + \sqrt{-1}c \\ 0 & \frac{1}{a} \end{pmatrix} \middle| \ b, c \in \mathbb{R}, a \in \mathbb{R}^+ \right\}$$

   Using these coordinates, the Poisson structure on G^* is given explicitly by

$$\begin{array}{rcl} \{b, c\} & = & a^2 - \frac{1}{a^2} \\ \{a, b\} & = & ab \\ \{a, c\} & = & ac \end{array}$$

2. For the Poisson group $G = SL_{\mathbb{C}}(n)$, equipped with the Poisson bracket constructed in Example 1.2, the dual group is

$$G^* \ = \ B_+ \star B_- \ \simeq \ \left\{ (A, B) \ \middle| \ \begin{array}{c} A \text{ upper triangular with determinant 1,} \\ B \text{ lower triangular with determinant 1,} \\ \text{s.t. } \operatorname{diag}(A) \cdot \operatorname{diag}(B) = 1 \end{array} \right\}$$

1.4 Poisson group actions

Definition 1.4. Let G be a Poisson group. Assume that G acts on a Poisson manifold X. The action is said to be a Poisson action if the action map

$$\begin{array}{rcl} G \times X & \rightarrow & X \\ (g, x) & \rightarrow & g \cdot x \end{array}$$

is a Poisson map, where $G \times X$ is equipped with the product Poisson structure.

Warning Note that in general, this definition *does not* imply that, for a fixed g in G, the action $x \to g \cdot x$ is a Poisson automorphism of X. The reader should not confuse Poisson actions with actions preserving the Poisson structure! Note, however, that when the Poisson structure on the Lie group G is the trivial one, then a Poisson action is an action of G on Xwhich preserves the Poisson structure.

Example 1.6. Any Lie group G acts on itself by left translation. If G is a Poisson group then this action is a Poisson action.

Proposition 1.5 (Lu-Weinstein)**.** *Let G be a Poisson group with Lie bialgebra $(\mathfrak{g}, \delta)$. Assume that G acts on a manifold X and let $\rho : \mathfrak{g} \to \mathfrak{X}^1(X)$ be the infinitesimal action. The action of G on X is a Poisson action if and only if the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}(X) \\ \downarrow \delta & & \downarrow [\pi, \cdot] \\ \wedge^2 \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}^2(X) \end{array}$$

In terms of Gerstenhaber algebras, the commutativity of the previous diagram has a clear meaning: it simply means that $\rho : \wedge^\bullet \mathfrak{g} \to \mathfrak{X}^\bullet(X)$ is a morphism of differential Gerstenhaber algebra.

Example 1.7. For the dual $SL_{\mathbb{C}}(3)^* = B_+ \star B_-$ of $G = SL_{\mathbb{C}}(3)$. Consider the Poisson manifold

$$X = \left\{ \left(\begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, y, z \in \mathbb{C} \right\}$$

equipped with the Poisson bracket

$$\begin{array}{rcl} \{x, y\} & = & xy - 2z \\ \{y, z\} & = & yz - 2x \\ \{z, x\} & = & zx - 2y \end{array}$$

The Lie group $G^* = B_+ \star B_-$ acts on X by

$$(A, B) \cdot U \to AUB^T$$

with $(A, B) \in B_+ \star B_- \simeq G^*$ and $U \in X$. This action turns to be a Poisson action.

2 Poisson groupoids and Lie bialgebroids

Idea Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A Poisson groupoid structure on Γ should be a multiplicative Poisson structure on Γ.

Recall In the Poisson group case,

$$\begin{array}{ll} & \pi \text{ is multiplicative} \\ \Leftrightarrow & m : G \times G \to G \text{ is a Poisson map} \\ \Leftrightarrow & \{(x, y, xy) | x, y \in G\} \subset G \times G \times \overline{G} \text{ is coisotropic} \end{array}$$

where $\overline{G}$ denotes (G, pi). This motivates the following definition.

Definition 2.1. A groupoid Γ with a Poisson structure π is said to be a Poisson groupoid if the graph of the groupoid multiplication

$$\Lambda = \{(x, y, xy)|(x, y) \in \Gamma_2 \text{ composable pair}\} \subset \Gamma \times \Gamma \times \overline{\Gamma}$$

is coisotropic. Here $\overline{\Gamma}$ means that Γ is equipped with the opposite Poisson structure $-\pi$.

Example 2.1.

1. If P is a Poisson manifold, then $P \times \overline{P} \rightrightarrows P$ is a Poisson groupoid.
2. Let A be the Lie algebroid of a Lie groupoid Γ and $\Lambda \in \Gamma(\wedge^2 A)$ be an element satisfying $\mathcal{L}_X[\Lambda, \Lambda] = 0$, $\forall X \in \Gamma(A)$. Then $\pi = \overleftarrow{\Lambda} - \overrightarrow{\Lambda}$ defines a Poisson groupoid structure on Γ.

Definition 2.2. A symplectic groupoid is a Poisson groupoid $(P \rightrightarrows M, \pi)$ such that π is non-degenerate. In other words, $\Lambda \subset \Gamma \times \Gamma \times \overline{\Gamma}$ is a Lagrangian submanifold.

Example 2.2.

1. $T^*M \rightrightarrows M$ with the canonical cotangent symplectic structure is a symplectic groupoid
2. If G is a Lie group, then $T^*G \rightrightarrows \mathfrak{g}^*$ is a symplectic groupoid. Here the symplectic structure on T^*G is the canonical cotangent symplectic structure. The groupoid structure is as follows. Right translations give an isomorphism between T^*G and the transformation groupoid $G \times \mathfrak{g}^*$ where G acts on $\mathfrak{g}^*$ by coadjoint action.
3. In general, if $\Gamma \rightrightarrows M$ is a Lie groupoid with Lie algebroid A, then $T^*\Gamma \rightrightarrows A^*$ is a symplectic groupoid. Let $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ denote the graph of the multiplication and $N^*\Lambda \subset T^*\Gamma \times T^*\Gamma \times T^*\Gamma$ its conormal space.

   Exercise 2.1. Show that $\overline{N^*\Lambda} = \{(\xi, \eta, \delta)|(\xi, \eta, -\delta) \in N^*\Lambda\}$ is the graph of a groupoid multiplication on $T^*\Gamma$ with corresponding unit space isomorphic to $A^* \simeq N^*M$. This defines a groupoid structure on $T^*\Gamma \rightrightarrows A^*$.

Question Why symplectic groupoids ?
Symplectic groupoids are used in

1. quantization
2. symplectic realization

Given a Poisson manifold M, can one embed the Poisson algebra $C^\infty(M)$ into a Poisson subalgebra of $C^\infty(S)$ where S is some symplectic manifold ?

$$C^\infty(M) \hookrightarrow C^\infty(S)$$

Note that locally, there exists local coordinates $(p_1, \ldots, p_k, q_1, \ldots, q_k)$ in which the Poisson bracket on $C^\infty(S)$ has the following form: $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = 0 = \{q_i, q_j\}$.

This question was first investigated in 1890 by Sophus Lie under the name of "Function groups". It leads to the following definition.

Definition 2.3. A symplectic realization of a Poisson manifold (M, π) consists of a pair (X, Φ), where X is a symplectic manifold and $\Phi : X \to M$ is a Poisson map which is a surjective submersion.

Question Given a Poisson manifold, does there exist a symplectic realization. And if so, is it unique ?

1. Local existence :
   (a) Lie (regular Poisson)
   (b) Weinstein 1983 (using splitting theorem)
2. Global existence : Karasev and Weinstein 1987
   (a) Symplectic realizations exist globally for any Poisson manifold.
   (b) There exists a distinguished symplectic realization, which admits a compatible local groupoid structure, i.e. a symplectic local groupoid.

Idea Find local symplectic realizations. Patch them together.

Puzzle Why do symplectic and groupoid structures arise in the context of Poisson manifolds in such a striking manner ?
Recall that in the case of a Poisson group (G,π), the associated infinitesimal object is a Lie bialgebra $(\mathfrak{g},\delta)$.
Half-way between Poisson groups and symplectic groupoids, there should be a notion of Poisson groupoids. Such a notion could help to better understand symplectic groupoids by imitating Poisson group theory.

$$\text{Poisson groupoids} \begin{cases} \text{Poisson groups} \\ \text{symplectic groupoids} \end{cases}$$

Theorem 2.1. *Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$ be a Lie groupoid. Let $\pi \in \mathfrak{X}^2(\Gamma)$ be a Poisson tensor. Then (Γ,π) is a Poisson groupoid if and only if all the following hold.*

1. *For all $(x,y) \in \Gamma_2$,*
$$\pi(xy) = R_Y\pi(x) + L_X\pi(y) - R_Y L_X \pi(w),$$
where $w = \beta(x) = \alpha(y)$ and X, Y are (local) bisections through x and y respectively.
2. *M is a coisotropic submanifold of Γ*
3. *For all $x \in \Gamma$, $\alpha_*\pi(x)$ and $\beta_*\pi(x)$ only depend on the base points $\alpha(x)$ and $\beta(x)$ respectively.*
4. *For all $\alpha, \beta \in C^\infty(M)$, one has $\{\alpha^* f, \beta^* g\} = 0$, $\forall f, g \in C^\infty(M)$.*
5. *The vector field $X_{\beta^* f}$ is left invariant for all $f \in C^\infty(M)$.*

Remark 2.1. If M is a point, then

1. $\Leftrightarrow$ multiplicativity condition,
2. $\Leftrightarrow \pi(1) = 0$,

3. is automatic,
4. is automatic,
5. is automatic.

And one gets the characterization of a Poisson group: a Lie group equipped with a multiplicative Poisson tensor.

Question What is the infinitesimal object associated to a Poisson groupoid ?

Corollary 1. *Given a Poisson groupoid $(\Gamma \rightrightarrows M, \pi)$, we have*

1. *for all $X \in \Gamma(A)$, $\left[\overleftarrow{X}, \pi\right]$ is still left invariant*
2. *$\pi_M := \alpha_* \pi$ (or $-\beta_* \pi$) is a Poisson tensor on M*

Proof. For all $X \in \Gamma(A)$, take $\xi_t = \exp tX \in U(\Gamma)$ (the space of bisections of Γ), $u_t = (\exp tX)(u)$ and $x \in \Gamma$ with $\beta(x) = u$. In other words, u_t is the flow of $\overleftarrow{X}$ initiated at u. Let K be any bisection through x. One gets

$$\begin{aligned} \pi(xu_t) &= R_{\xi_t}\pi(x) + L_K \pi(u_t) - L_K R_{\xi_t}\pi(u) \\ \Rightarrow R_{\xi_t^{-1}}\pi(xu_t) &= \pi(x) + L_K R_{\xi_t^{-1}}\pi(u_t) - L_K\pi(u) \in \wedge^2 T_x\Gamma \end{aligned}$$

and, differentiating with respect to t at 0,

$$(\mathcal{L}_{\overleftarrow{X}}\pi)(x) = L_K\big((\mathcal{L}_{\overleftarrow{X}}\pi)(u)\big).$$

This implies that $\mathcal{L}_{\overleftarrow{X}}\pi$ is left invariant. □

Now, we can define $\delta : \Gamma(\wedge^i A) \to \Gamma(\wedge^{i+1} A)$. For $i = 0$,

$$C^\infty(M) \to \Gamma(A) : f \mapsto X_{\beta_* f} = [\beta^* f, \pi].$$

For $i = 1$,

$$\Gamma A \to \Gamma(\wedge^2 A) : X \mapsto \overleftarrow{\delta X} = \left[\overleftarrow{X}, \pi, .\right]$$

The following lemma can be easily verified.

Lemma 2.2.
1. *$\delta(fg) = g\delta f + f\delta g, \quad \forall f, g \in C^\infty(M)$*
2. *$\delta(fX) = \delta f \wedge X + f\delta X, \quad \forall f \in C^\infty(M)$ and $X \in \Gamma(A)$*
3. *$\delta[X, Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \Gamma(A)$*
4. *$\delta^2 = 0$*

Definition 2.4. A Lie bialgebroid is a Lie algebroid A equiped with a degree 1 derivation δ of the associative algebra $(\Gamma(\wedge^\bullet A), \wedge)$ satisfying conditions 3 and 4 of the previous lemma.

Exercise 2.2. Show that a Lie bialgebroid structure is equivalently characterized as a degree 1 derivation δ of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [,])$ such that $\delta^2 = 0$. This is also called a *differential Gerstenhaber algebra*.

Remark 2.2. Given a Lie bialgebroid (A, δ), there is a natural Lie algebroid structure on A^* defined as follows.

1. The anchor map $\rho_* : A^* \to TM$ is

$$\langle \rho_* \xi, f \rangle = \langle \xi, \delta f \rangle, \quad \forall f \in C^\infty(M).$$

2. The bracket $[,]$ is given by

$$\langle [\xi, \eta], X \rangle = (\delta X)(\xi, \eta) + (\rho_* \xi) \langle X, \eta \rangle - (\rho_* \eta) \langle X, \xi \rangle, \quad \forall \xi, \eta \in \Gamma(A^*), \ \forall X \in \Gamma(A). \tag{2}$$

Indeed, equivalently, a Lie bialgebroid is a pair of Lie algebroids (A, A^*) such that

$$\delta [X, Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \Gamma(A),$$

where $\delta : \Gamma(A) \to \Gamma(\wedge^2 A)$ is defined by the above equation (2)

Remark 2.3. If (A, A^*) is a Lie bialgebroid, then (A^*, A) is also a Lie bialgebroid and it is called its dual.

Example 2.3.

1. If π is a **Poisson tensor** on M, then $A = TM$ with $\delta = [\pi, \cdot] : \mathfrak{X}^*(M) \to \mathfrak{X}^{*+1}(M)$ is a Lie bialgebroid. In this case, $A^* = T^*M$ is the canonical cotangent Lie algebroid.

2. The **dual** to the previous one: $A = T^*M$, the cotangent Lie algebroid of a Poisson manifold (M, π), together with $\delta^* = d_{\mathrm{DR}} : \Omega^*(M) \to \Omega^{*+1}(M)$.

3. **Coboundary Lie bialgebroid.** Take A a Lie algebroid admitting a $\Lambda \in \Gamma(\wedge^2 A)$ satisfying

$$\mathcal{L}_X [\Lambda, \Lambda] = 0, \forall X \in \Gamma(A).$$

Let $\delta = [\Lambda, \cdot] : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)$. Then (A, δ) defines a Lie bialgebroid.

4. **Dynamical r-matrix.** Consider the Lie algebroid $A = T\mathfrak{h}^* \oplus \mathfrak{g} \to \eta$ where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ and the Lie algebroid structure on A is the product Lie algebroid. Choose a map $r : \mathfrak{h}^* \to \wedge^2 \mathfrak{g}$ and consider it as a element Λ of $\Gamma(\wedge^2 A)$. Then $\mathcal{L}_X [\Lambda, \Lambda] = 0$ if and only if

$$\sum h_i \wedge \tfrac{dr}{d\lambda_i} + \tfrac{1}{2} [r, r] \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$$

is a constant function over $\mathfrak{h}^*$. Here $\{h_1, \ldots, h_k\}$ is a basis of $\mathfrak{h}$ and $(\lambda_1, \ldots, \lambda_k)$ are the dual coordinates on $\mathfrak{h}^*$.
In particular, if $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, one can take

$$r(\lambda) = \sum_{\alpha \in \Delta^+} \frac{\lambda_\alpha}{(\alpha, \lambda)} e_\alpha \wedge f_\alpha$$

or

$$r(\lambda) = \sum_{\alpha \in \Delta^+} \lambda_\alpha \coth{(\alpha, \lambda)} e_\alpha \wedge f_\alpha,$$

where $(e_\alpha, f_\alpha, h_i)$ is a Chevalley basis.