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Bihamiltonian structures of PDEs

and Frobenius manifolds

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Lecture 3

Problem of classification of local Poisson brackets

One more example of a local Poisson bracket on the space $\mathcal{L}(M)$, $M = \mathbb{R}$

$$\{u(x), u(y)\} = u(x)\delta'(x-y) + \frac{1}{2}u_x(x)\delta(x-y) - \delta'''(x-y).$$

Linear + constant (central extensions of Lie algebras)

Exercise. Check that the linear functionals on $\mathcal{L}(M)$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}u(x)\,dx$$

yield the Virasoro algebra

$$\frac{1}{i}\{c_n, c_m\} = \frac{1}{2}(n-m)c_{n+m} + n^3\delta_{n+m,0}$$

So $\mathcal{L}(M) = Vir^*$.

An alternative proof of Jacobi identity: reduce the bracket to a **constant** (i.e., u-independent) form by *Miura transform*

$$u = \frac{1}{4}v^2 + v', \quad \{v(x), v(y)\} = \delta'(x-y).$$

Indeed:

$$\{u(x), u(y)\} = \left\{\frac{1}{4}v^2(x) + v'(x), \frac{1}{4}v^2(y) + v'(y)\right\}$$

$$= \frac{1}{4}v(x)v(y)\{v(x), v(y)\} + \frac{1}{2}v(x)\partial_y\{v(x), v(y)\} + \frac{1}{2}v(y)\partial_x\{v(x), v(y)\} + \partial_x\partial_y\{v(x), v(y)\}$$

$$= \frac{1}{4}v(x)v(y)\delta'(x-y) - \frac{1}{2}v(x)\delta''(x-y) + \frac{1}{2}v(y)\delta''(x-y) - \delta'''(x-y)$$

Use

$$f(y)\delta(x-y) = f(x)\delta(x-y), \quad f(y)\delta'(x-y) = f(x)\delta'(x-y) + f'(x)\delta(x-y),$$

$$f(y)\delta''(x-y) = f(x)\delta''(x-y) + 2f'(x)\delta'(x-y) + f''(x)\delta(x-y)$$

obtain

$$= \left(\frac{1}{4}v^2 + v'\right)\delta'(x-y) + \frac{1}{2}\left(v'' + \frac{1}{2}v\,v'\right)\delta(x-y) - \delta'''(x-y)$$

$$= u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) - \delta'''(x-y).$$

Problems in general classification scheme:

- equivalences
- infinitesimal deformations
- higher obstructions
- rigid objects

Lecture 3

- grading of differential polynomials, evolutionary PDEs, and Poisson brackets. Extended formal loop space.

- Group of Miura-type transformations.

- $(n, 0)$-brackets. Darboux lemma.

- Riemannian geometry and classification of $(0, n)$-brackets

- On more general (p, q)-brackets

The aim: to classify local Poisson brackets w.r.t. general Miura type transformations of the form

$$u^i \to \tilde{u}^i = F^i(u; u_x, u_{xx}, \ldots). \tag{1}$$

The problem: this is **not a group**!

Extended formal loop space

Gradation on the ring $\mathcal{A}$ of differential polynomials:

$$\deg u^{i,k} = k, \quad k \geq 1, \quad \deg f(x;u) = 0. \tag{2}$$

Completion of the space of local functionals $\widehat{\Lambda}_0$:

$$\bar{f} = \int f(u; u_x, u_{xx}, \dots; \epsilon)\, dx,$$

$$f(u; u_x, u_{xx}, \dots) = \sum_{k=0}^{\infty} \epsilon^k f_k(u; u_x, \dots, u^{(k)}), \quad f_k \in \mathcal{A}, \quad \deg f_k = k. \tag{3}$$

Still called them differential polynomials

Taking the symmetric tensor algebra of $\widehat{\Lambda}_0$ we obtain the ring of functionals on the *extended* formal loop space denoted $\widehat{\mathcal{L}}(M)$.

Systems of evolutionary PDEs:

$$
\begin{aligned}
&u_t^i = \epsilon^{-1}\, a_0^i(u) + a_1^i(u; u_x) + \epsilon\, a_2^i(u; u_x, u_{xx}) + O(\epsilon^2) \\
&a_1^i(u; u_x) = v_j^i(u) u_x^j, \\
&a_2^i(u; u_x, u_{xx}) = b_j^i(u) u_{xx}^j + \frac{1}{2} c_{jk}^i(u) u_x^j u_x^k
\end{aligned} \tag{4}
$$

etc.

Rule of thumb: introduction of **slow variables**

$$x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t,$$

$$u^i \mapsto u^i, \quad u_x^i \mapsto \epsilon\, u_x^i, \quad u_{xx}^i \mapsto \epsilon^2 u_{xx}^i, \dots$$

So

$$u_t^i = a^i(u; u_x, u_{xx}, \dots) \mapsto u_t^i = \frac{1}{\epsilon} a^i(u; \epsilon u_x, \epsilon^2 u_{xx}, \dots)$$

Example 1 KdV

$$u_t + u\,u_x + \frac{\varepsilon^2}{12} u_{xxx} = 0$$

Example 2 Toda lattice

$$\ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}.$$

Continuous version:

$$u_n := q_{n+1} - q_n = u(n\varepsilon), \quad v_n := \dot{q}_n = v(n\varepsilon), \quad t \mapsto \varepsilon\, t$$

$$u_t = \frac{v(x+\varepsilon) - v(x)}{\varepsilon} = v_x + \frac{1}{2}\varepsilon\, v_{xx} + O(\varepsilon^2)$$

$$v_t = \frac{e^{u(x)} - e^{u(x-\varepsilon)}}{\varepsilon} = e^u u_x - \frac{1}{2}\varepsilon\,(e^u)_{xx} + O(\varepsilon^2)$$

Example 3 Camassa - Holm equation

$$u_t = \left(1 - \varepsilon^2 \partial_x^2\right)^{-1} \left\{ \frac{3}{2} u\, u_x - \varepsilon^2 \left[u_x u_{xx} + \frac{1}{2} u\, u_{xxx} \right] \right\}$$

$$= \frac{3}{2} u\, u_x + \varepsilon^2 \left(u\, u_{xxx} + \frac{7}{2} u_x u_{xx} \right) + O(\varepsilon^4)$$

Local bivectors in $\widehat{\Lambda}_{loc}$ represented by *infinite sums*

$$\{u^i(x), u^j(y)\} = \sum_{k=-1}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}^{[k]} \tag{5}$$

$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{ij}_{k,s}(u; u_x, \ldots, u^{(s)}) \delta^{(k-s+1)}(x-y),$$

$$A^{ij}_{k,s} \in \mathcal{A}, \quad \deg A^{ij}_{k,s} = s, \quad s = 0, 1, \ldots, k+1.$$

Another description: given

$$\bar{f} = \int f(u; u_x, \ldots)\, dx, \quad \bar{g} = \int g(u; u_x, \ldots)\, dx$$

then

$$\{\bar{f}, \bar{g}\}^{[k]} = \int h(u; u_x, \ldots)\, dx, \quad \deg h = \deg f + \deg g + k + 1$$

More explicitly, the first three terms in the expansion (5) read

$$\{u^i(x), u^j(y)\}^{[-1]} = \pi^{ij}(u(x))\delta(x-y) \tag{6}$$

$$\{u^i(x), u^j(y)\}^{[0]} = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u(x))u^k_x\delta(x-y) \tag{7}$$

$$\begin{aligned}\{u^i(x), u^j(y)\}^{[1]} &= a^{ij}(u(x))\delta''(x-y) + b^{ij}_k(u(x))u^k_x\delta'(x-y)\\ &+[c^{ij}_k(u(x))u^k_{xx} + \frac{1}{2}d^{ij}_{kl}(u(x))u^k_x u^l_x]\delta(x-y)\end{aligned} \tag{8}$$

where π^{ij}, $g^{ij}(u)$, $\Gamma^{ij}(u)$, $a^{ij}(u)$, $b^{ij}_k(u)$, $c^{ij}_k(u)$, $d^{ij}_{kl}(u)$ are some functions on the manifold M.

Exercise. The Schouten - Nijenhuis bracket gives a well-defined map

$$\epsilon\,[\ ,\ ]:\ \hat\Lambda^k_{\rm loc}\times\hat\Lambda^l_{\rm loc}\to\hat\Lambda^{k+l-1}_{\rm loc}.$$

Example 1. Bihamiltonian structure of KdV

$$u_t = u\,u_x + \frac{\epsilon^2}{12}u_{xxx} = \{u(x), H_1\}_1 = \frac{3}{2}\{u(x), H_0\}_2$$

$$\{u(x), u(y)\}_1 = \delta'(x-y)$$

$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}u_x\delta(x-y) + \frac{\epsilon^2}{8}\delta'''(x-y)$$

$$H_1 = \int \left(\frac{1}{6}u^3 - \frac{\epsilon^2}{24}u_x^2\right) dx, \quad H_0 = \int \frac{1}{2}u^2\, dx$$

Example 2. Toda lattice

$$\begin{aligned} \dot{u}_n &= v_{n+1} - v_n = \{u_n, H\} \\ \dot{v}_n &= e^{u_n} - e^{u_{n-1}} = \{v_n, H\} \end{aligned}$$

$$\{u_m, u_n\} = \{v_m, v_n\} = 0, \quad \{u_m, v_n\} = \delta_{m,n-1} - \delta_{mn}$$

$$H = \sum \left(\frac{1}{2} v_n^2 + e^{u_n} \right)$$

Interpolating obtain

$$\{u(x), v(y)\} = \frac{1}{\epsilon}[\delta(x-y+\epsilon) - \delta(x-y)] = \delta'(x-y) + \frac{\epsilon}{2}\delta''(x-y) + O(\epsilon^2) \tag{9}$$

$$H = \int \left[\frac{1}{2} v^2 + e^u \right] dx.$$

Miura group

The transformations

$$
\begin{aligned}
& u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u; u_x, \dots, u^{(k)}), \quad i = 1, \dots, n \\
& F_k^i \in \mathcal{A}, \quad \deg F_k^i = k, \\
& \det \left(\frac{\partial F_0^i(u)}{\partial u^j} \right) \neq 0.
\end{aligned} \tag{10}
$$

Lemma. *The transformations of the form (10) form a group.*

Example. The clasical Miura transformation

$$u = \frac{1}{4}v^2 + \epsilon v'$$

Inversion, $u \neq 0$

$$v = 2\sqrt{u - \epsilon v'} = 2\sqrt{u} - \epsilon\frac{v'}{\sqrt{u}} + O(\epsilon^2) = 2\sqrt{u} - \epsilon\frac{u'}{u} + O(\epsilon^2)$$

$\Rightarrow$ first two terms of the solution $v = F(u; u', \ldots; \epsilon)$.

Remark. Corresponds to the WKB solution to

$$\epsilon^2 y'' = \frac{1}{4}u\, y, \quad v = 4\epsilon\frac{y'}{y}.$$

$$y = u^{-1/4} e^{\frac{1}{2\epsilon}\int\sqrt{u}dx}\left(1 + O(\epsilon)\right).$$

Definition. The group $\mathcal{G}$ of all the transformations of the form (10) is called *Miura group* (“local diffeomorphisms” of the extended formal loop space $\widehat{\mathcal{L}}(M)$)

Lemma. *The class of local functionals, evolutionary PDEs, and local translation invariant multivectors on the extended formal loop space $\widehat{\mathcal{L}}(M)$ is invariant w.r.t. the action of the Miura group.*

Exercise 1. An arbitrary evolutionary PDE

$$u_t^i = \frac{1}{\epsilon} a_0^i(u) + A_j^i(u) u_x^j + O(\epsilon), \quad i = 1, \dots, n$$

with $a_0(u_0) \neq 0$ can be reduced, by a Miura-type transformation near $u = u_0$, to a constant form

$$a_0 = \text{const}, \quad a_i = 0 \text{ for } i > 0.$$

Exercise 2. Transformation law of Hamiltonian operators

$$A^{ij} = \sum A_k^{ij}(u; u_x, \dots)\partial_x^k$$

of local bivectors:

$$\tilde{A}^{ij} = L^{*i}_{k} A^{kl} L_l^j \tag{11}$$

where the matrix-valued operator L_k^i and the adjoint one L^{*i}_{k} are given by

$$L_k^i = \sum_s (-\partial_x)^s \circ \frac{\partial \tilde{u}^i}{\partial u^{k,s}}, \quad L^{*i}_{k} = \sum_s \frac{\partial \tilde{u}^i}{\partial u^{k,s}} \partial_x^s.$$

Main Problem. To describe the orbits of the action of the Miura group $\mathcal{G}$ on $\hat{\Lambda}^2$.

Darboux lemma on $\widehat{\mathcal{L}}(M)$, M = a ball

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon}\pi^{ij}(u(x))\delta(x-y) + \sum_{k\geq 0} \epsilon^k \{u^i(x), u^j(y)\}^{[k]}$$

$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{ij}_{k,s}(u; u_x, \dots, u^{(s)})\delta^{(k-s+1)}(x-y),$$

$$A^{ij}_{k,s} \in \mathcal{A}, \quad \deg A^{ij}_{k,s} = s, \quad s = 0, 1, \dots, k+1.$$

Assume:

$$\det\left(\pi^{ij}(u)\right) \neq 0$$

Theorem. The P.B. is equivalent to

$$\{\tilde{u}^i(x), \tilde{u}^j(y)\} = \frac{1}{\epsilon}\tilde{\pi}^{ij}\delta(x-y), \quad \tilde{\pi}^{ij} = \text{const}$$

Proof. Step 1: the leading term itself is itself a Poisson bracket on $\hat{\mathcal{L}}(M)$ $\Rightarrow$ finite-dimensional Jacobi identity for $\pi^{ij}(u)$

Step 2: try to kill all the terms with $k \geq 0$ of the expansion

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon}\pi^{ij}\delta(x-y) + \sum_{k\geq 0} \epsilon^k \{u^i(x), u^j(y)\}^{[k]}$$

by Miura-type transformations with $F_0^i(u) = u^i$, $i = 1, \ldots, n$. Triviality of Poisson cohomology $H^2(\hat{\mathcal{L}}(M), \varpi)$ of the bivector

$$\varpi = \int \pi^{ij}\theta_i\theta_j dx, \quad \pi^{ij} = -\pi^{ji} = \text{const}, \quad \det \pi^{ij} \neq 0$$

is needed.

Natural decomposition

$$H^k = \oplus_{m\geq -1} H^{k,m}$$

with respect to ϵ^m. Denote

$$\tilde{H}^k := \oplus_{m\geq 0} H^{k,m}. \tag{12}$$

Lemma 1. *The first non-zero term in the expansion*

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon}\varpi + \sum_{k\geq 0} \epsilon^k \{u^i(x), u^j(y)\}^{[k]}$$

is a 2-cocycle in the Poisson cohomology $\tilde{H}^2$ of the ultralocal Poisson bracket ϖ.

To kill this first nonzero term we need to prove that, for any 2-cocycle $\in \hat{\Lambda}^2_{\text{loc}}$ of ϖ , there exists an evolutionary vector field a such that the Lie derivative $Lie_a \varpi$ gives the cocycle and $a|_{\epsilon=0} = 0$.

Main Lemma. *For* M = ball *the Poisson cohomologies* $\tilde{H}^k(\hat{\mathcal{L}}(M), \varpi)$ *vanish for* $k > 0$.

Let us prove first triviality of $\tilde{H}^1$. Let an evolutionary vector field a with the components a^1, ..., a^n be a cocycle. Denote

$$\omega_i = \pi_{ij} a^j$$

where the constant matrix π_{ij} is inverse to π^{ij}. The condition $\partial a = Lie_a \varpi = 0$ reads

$$\frac{\partial \omega_i}{\partial u^{j,s}} = \sum_{t \geq s} (-1)^t \begin{pmatrix} t \\ s \end{pmatrix} \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{i,t}}.$$

Indeed, the super-P.B. of

$$a = \int a^i \theta_i \, dx \quad \text{and} \quad \varpi = \frac{1}{2} \int \pi^{ij} \theta_i \theta_j \, dx$$

read

$$\{a, \varpi\} = -\int \frac{\delta a}{\delta u^s(x)} \frac{\delta \varpi}{\delta \theta_s(x)} \, dx = -\int \sum \frac{\partial \alpha^i}{\partial u^{s,k}} \pi^{sj} \theta_i \theta_j^{(k)} dx$$

(integration by parts have been used). Vanishing of the last integral can be written as

$$0 = \frac{\delta}{\delta \theta_i(x)} \int \sum \frac{\partial \alpha^i}{\partial u^{s,k}} \pi^{sj} \theta_i \theta_j^{(k)} dx$$

$$= \sum_k \frac{\partial \alpha^i}{\partial u^{s,k}} \pi^{sj} \theta_j^{(k)} - (-1)^k \partial_x^k \left(\frac{\partial \alpha^i}{\partial u^{s,k}} \pi^{sj} \theta_j \right).$$

This gives the above equations.

Using Helmholz criterion $\Rightarrow$ there exists a local functional $\bar{f} = \int f\, dx$ such that

$$\omega_i = \frac{\delta \bar{f}}{\delta u^i(x)}.$$

Therefore the vector field is a Hamiltonian one,

$$a^i = \pi^{ij} \frac{\delta \bar{f}}{\delta u^j(x)}.$$

Triviality of $\tilde{H}^2$: the bivector

$$\varpi + \varepsilon\alpha = \pi^{ij}\delta(x-y) + \varepsilon \sum_s A_s^{ij} \delta^{(s)}(x-y)$$

satisfies the Jacobi identity

$$[\varpi + \varepsilon\alpha, \varpi + \varepsilon\alpha] = 0 (\operatorname{mod} \varepsilon^2)$$

iff the inverse matrix is a closed differential form

$$\frac{1}{2}\pi_{ij} dx \wedge \delta u^i \wedge \delta u^j + \frac{1}{2}\,\varepsilon\, \omega_{i;js}\, dx \wedge \delta u^i \wedge \delta u^{j,s} \ (\mathrm{mod}\, \varepsilon^2)$$

where

$$\omega_{i;js} := \pi_{ip}\pi_{jq} A_s^{pq}. \tag{13}$$

Denote

$$\omega = \frac{1}{2}\omega_{i;js}\delta u^i \wedge \delta u^{j,s}.$$

From closedness $\delta(dx \wedge \omega) = 0 \in \Lambda_3$ we derive, due to Example 3 of Lecture 2 (see eq. (15)), existence of a one-form $dx \wedge \phi$, $\phi = \phi_i \delta u^i$ such that $\delta(dx \wedge \phi) = dx \wedge \omega$. The vector field a with

$$a^i = \pi^{ij}\phi_j$$

gives a solution to the coboundary equation

$$[\varpi, a] = \alpha.$$

Classification of $(0, n)$-brackets

$$\{u^i(x), u^j(y)\}^{[-1]} = 0$$

$$\{u^i(x), u^j(y)\}^{[0]} = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}(u)u^k_x\delta(x-y)$$

$$\text{satisfies } \det(g^{ij}(u)) \neq 0$$

As above, the first nonzero term $\{\ ,\ \}^{[0]}$ is itself a Poisson bracket.

Antisymmetry conditions (see eq. (23) of Lecture 2) read

$$g^{ji}(u) = g^{ij}(u), \quad \Gamma^{ij}_k(u) + \Gamma^{ji}_k(u) = \partial_k g^{ij}(u). \tag{14}$$

Miura-type transformations reduce to local diffeomorfisms $u^i \mapsto \tilde{u}^i(u)$

$$\tilde{g}^{ij} = \frac{\partial \tilde{u}^i}{\partial u^k}\frac{\partial \tilde{u}^j}{\partial u^l} g^{kl}, \quad \tilde{\Gamma}^{ij}_k = \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k}\Gamma^{pq}_r + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^k} g^{pq}$$

So the leading coefficient $g^{ij}(u)$ determines a symmetric tensor field on M invariant w.r.t. the action of the Miura group. This gives a map

$$\hat{\Lambda}^2/\mathcal{G} \to \text{symmetric tensors on } M.$$

Theorem. *Let M be a ball. Then the only invariant of a $(0, n)$ Poisson bracket on $\hat{\Lambda}^2_{\rm loc}$ with respect to the action of the Miura group is the signature of the quadratic form $g^{ij}(u)$.*

Proof. The symmetric nondegenerate tensor $g^{ij}(u)$ defines a *pseudo-Riemannian metric*

$$ds^2 = g_{ij}(u)du^i du^j, \quad \left(g_{ij}\right) = \left(g^{ij}\right)^{-1}$$

on the manifold M. Moreover, the functions

$$\Gamma^k_{ij}(u) := -g_{js}(u)\Gamma^{sk}_i(u)$$

are Christoffel coefficients of an affine connection on M (cf. the transformation law for Γ^{ij}_k).

Main Lemma 1. 1) The affine connection Γ is compatible with the metric,

$$\nabla g_{ij} = 0.$$

2) Jacobi identity for the Poisson bracket

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}(u)u^k_x\delta(x-y)$$

is equivalent to

$$\Gamma^k_{ji} = \Gamma^k_{ij}, \quad R_{ijkl} = 0$$

i.e., Γ is the Levi-Civita connection for the metric, and the Riemann curvature of the metric vanishes.

Standard arguments of differential geometry $\Rightarrow$ local existence of coordinates $v^1(u)$, ..., $v^n(u)$ such that the metric becomes constant

$$\frac{\partial v^k}{\partial u^i}\frac{\partial v^l}{\partial u^j} g^{ij}(u) = \eta^{kl} = \text{const.}$$

and the Christoffel coefficients vanish. All constant symmetric matrices η^{kl} of a given signature are equivalent w.r.t. linear changes of coordinates.

The Poisson bracket $\varpi = \{\ ,\ \}^{[0]}$ takes the constant form

$$\varpi: \quad \{v^i(x), v^j(y)\} = \eta^{ij}\delta'(x-y). \tag{15}$$

We reduce the proof of the theorem to killing the ϵ-tail of the Poisson bracket

$$\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x-y) + \sum_{k\geq 1} \epsilon^k \{u^i(x), u^j(y)\}^{[k]} \qquad (16)$$

by the Miura-type transformations with $F_0^i =$id.

Main Lemma 2. *For M = ball all the cocycles in $H^k(\hat{\mathcal{L}}(M), \varpi)$ vanishing at $\epsilon = 0$ are trivial for $k > 0$.*

Before giving the proofs recall some basic differential geometry.

• Covariant derivatives of a vector field

$$\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k, \quad \partial_i = \frac{\partial}{\partial u^i}$$

and of (0,2) and (2,0) tensors

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^s_{ki} g_{sj} - \Gamma^s_{kj} g_{is}, \quad \nabla_k g^{ij} = \partial_k g^{ij} + \Gamma^i_{ks} g^{sj} + \Gamma^j_{ks} g^{is}.$$

• The *Levi-Civita* connection is uniquely determined by a symmetric non-degenerate tensor g_{ij} from the conditions

$$\Gamma^k_{ji} = \Gamma^k_{ij}, \quad \nabla_k g_{ij} = 0.$$

Let

$$\left(g^{ij}\right) = \left(g_{ij}\right)^{-1}, \quad \Gamma_k^{ij} = -g^{is}\Gamma_{ks}^{j}.$$

Then the definition of the Levi-Civita connection rewrites as

$$g^{is}\Gamma_s^{jk} = g^{js}\Gamma_s^{ik}, \quad \partial_k g^{ij} = \Gamma_k^{ij} + \Gamma_k^{ji}.$$

- The Riemann curvature of the Levi-Civita connection reads

$$R_l^{ijk} := g^{is}g^{jt}R_{slt}^{k} = g^{is}\left(\partial_l\Gamma_s^{jk} - \partial_s\Gamma_l^{jk}\right) + \Gamma_s^{ij}\Gamma_l^{sk} - \Gamma_s^{ik}\Gamma_l^{sj}.$$

Proof of Main Lemma 1. For

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u)u^k_x\delta(x-y)$$

the conditions of skew symmetry (14) imply compatibility of the connection Γ^{ij}_k with the metric g^{ij}. Let us prove vanishing of torsion and curvature. The super-functional ϖ reads

$$\varpi = \int \theta_i \left(g^{ij}\theta'_j + u^k_x \Gamma^{ij}_k \theta_j \right) dx.$$

So

$$\frac{\delta\varpi}{\delta\theta_s(x)} = 2\left[g^{si}\theta'_i + u^k_x\Gamma^{si}_k\theta_i \right].$$

Compute now

$$\frac{\delta\varpi}{\delta u^s(x)} = \theta_i \left[\left(\Gamma^{ij}_s + \Gamma^{ji}_s \right) \theta'_j + u^k_x \Gamma^{ij}_{k,s}\theta_j \right] - \partial_x \left(\Gamma^{ij}_s \theta_i\theta_j \right)$$

$$= 2\Gamma^{ji}_s\theta_i\theta'_j + \left(\Gamma^{ij}_{k,s} - \Gamma^{ij}_{s,k} \right) u^k_x\theta_i\theta_j$$

So

$$\{\varpi, \varpi\} = \int \left[A^{ijk}\theta_i'\theta_j'\theta_k + B^{ijk}\theta_i\,\theta_j\,\theta_k' + C^{ijk}\theta_i\theta_j\theta_k\right]\,dx$$

$$A^{ijk} = -g^{is}\Gamma_s^{jk} + g^{js}\Gamma_s^{jk},$$

$$2B^{ijk} = \left[2\Gamma_s^{kj}\Gamma_l^{si} - g^{sk}\left(\Gamma_{l,s}^{ji} - \Gamma_{s,l}^{ji}\right)\right]u_x^l - (i \leftrightarrow j)$$

$$C^{ijk} = \text{antisymmetrization of } \Gamma_l^{si}\left(\Gamma_{m,s}^{jk} - \Gamma_{s,m}^{jk}\right)u_x^l u_x^m.$$

Vanishing of the leading coefficients yields the symmetry of the connection

$$g^{si}\Gamma_s^{jk} = g^{sj}\Gamma_s^{ik}.$$

The remaining conditions of vanishing of the integral read

$$C^{ijk} = \frac{1}{3}\partial_x B^{ijk}$$

Since B^{ijk} depends linearly on u^l_x, it must vanish. Using the identity

$$g^{ks}\left(\Gamma^{ji}_{l,s} - \Gamma^{ji}_{s,l}\right) = g^{ks}\left(\Gamma^{ij}_{s,l} - \Gamma^{ij}_{l,s}\right)$$

we arrive at

$$B^{ijk} = R^{kji}_l u^l_x.$$

Main Lemma 1 is proved.

Proof of Main Lemma 2. Denote, like above

$$\tilde{H}^k = \oplus_{m>0} H^{k,m}. \tag{17}$$

We want to prove that $\tilde{H}^1 = \tilde{H}^2 = 0$.

Let us begin with proving triviality of $\tilde{H}^1$. From

$$\{a, \varpi\} = \int \sum_k \frac{\partial a^i}{\partial u^{s,m}} \eta^{sj} \theta_i \theta_j^{(m+1)} dx$$

we obtain the condition $[a, \varpi] = 0$ in the form

$$\sum_m \frac{\partial a^k}{\partial u^{s,m}} \eta^{sj} \theta_j^{(m+1)} - (-1)^{m+1} \partial_x^{m+1} \left(\frac{\partial a^j}{\partial u^{s,m}} \eta^{sk} \theta_j \right) = 0 \tag{18}$$

Collecting the coefficient of θ_j yields

$$\partial_x \left[\eta^{ks} \frac{\delta}{\delta u^s(x)} \int a^j \, dx \right] = 0.$$

Therefore

$$a^j = \sum c_t^j u^t + \text{total derivative.}$$

But $a^j|_{\epsilon=0} = 0$. So there exist differential polynomials b^j s.t.

$$a^j = \partial_x b^j, \quad j = 1, \ldots, n.$$

This is the **crucial point** in the proof: we have shown that the vector field a is **tangent** to the level surface of the Casimirs

$$\bar{u}^k = \int u^k \, dx, \quad k = 1, \ldots, n$$

$$i_a \delta \bar{u}^k = \int a^j \frac{\delta \bar{u}^k}{\delta u^j(x)} \, dx = \int \partial_x b^k \, dx = 0.$$

The remaining part of the proof is rather straightforward. Using

$$\frac{\partial}{\partial u^{i,s}} \partial_x = \partial_x \frac{\partial}{\partial u^{i,s}} + \frac{\partial}{\partial u^{i,s-1}}$$

and also the Pascal triangle identity

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

we rewrite the coefficient of $\theta^{(r+1)}$ in the form

$$\partial_x \left[\frac{\partial \omega_k}{\partial u^{l,r}} - \sum_{t \geq r} (-1)^t \binom{t+1}{r} \left(\frac{\partial \omega_l}{\partial u^{k,t}} \right)^{(t-r)} \right]$$

$$+ \frac{\partial \omega_k}{\partial u^{l,r-1}} + (-1)^r \frac{\partial \omega_l}{\partial u^{k,r-1}} = 0.$$

Here

$$\omega_k = \eta_{li} b^i, \quad (\eta_{ij}) = (\eta^{ij})^{-1},$$

the last two terms are not present for $r = 0$. As above, for $r = 0$ we derive that

$$\frac{\partial \omega_k}{\partial u^l} = \sum_{t \geq 0} (-1)^t \left(\frac{\partial \omega_l}{\partial u^{k,t}} \right)^{(t)}.$$

Proceeding by induction in r we prove that the 1-form $\int dx \wedge \omega_i \, \delta u^i$ is closed. Using the Helmholz criterion we derive existence of a differential polynomial f s.t. $\omega = \delta \int f \, dx$. Hence

$$a^i = \eta^{ij} \partial_x \frac{\delta \bar{f}}{\delta u^j(x)}.$$

We proved triviality of $\tilde{H}^1$.

Let us proceed to prove the triviality of $\tilde{H}^2$. The condition $\partial \alpha = 0$ for α of the form

$$\alpha = \int A_k^{ij}(u; u_x, \dots; \epsilon) \theta_i \theta_j^{(k)} dx$$

can be computed similarly to the ultralocal case:

$$\{\alpha, \varpi\} = \int \sum_q \frac{\partial A_p^{ij}}{\partial u^{s,q}} \eta^{sk} \theta_i \theta_j^{(p)} \theta_k^{(q+1)} \, dx.$$

We obtain a system of equations

$$\frac{\partial A_t^{ij}}{\partial u^{l,s-1}} \eta^{lk} + \sum (-1)^{q+r+s} \begin{pmatrix} q+r+s \\ q \quad r \end{pmatrix} \left(\frac{\partial A_{q+r+s}^{ki}}{\partial u^{l,t-q-1}} \right)^{(r)} \eta^{lj}$$

$$+ \sum (-1)^{q+r+t} \begin{pmatrix} q+r+t \\ q \quad r \end{pmatrix} \left(\frac{\partial A_{s-q}^{jk}}{\partial u^{l,q+r+t-1}} \right)^{(r)} \eta^{li} = 0$$

for any i, j, k, s, t (19)

(it is understood that the terms with $s-1$, $t-q-1$ or $t+q+r-1$ negative do not appear in the sum). Recall that the crucial point in the proof of triviality of the 2-cocycle is to establish vanishing of α on differentials of Casimirs

$$\alpha(\delta\bar{u}^i, \delta\bar{u}^j) = \int A_0^{ij} dx = 0 \quad \text{for any } i, j. \tag{20}$$

We first use (19) for $s = t = 0$ to prove that

$$\partial_x \sum_r (-1)^r \left(\frac{\partial A_0^{jk}}{\partial u^{l,r}} \right)^{(r)} \eta^{li} = 0.$$

Hence

$$A_0^{jk} = \partial_x B^{jk}$$

for some differential polynomial B^{jk} (here we use that $A_k^{ij}|_{\epsilon=0} = 0$). This implies (20).

The rest of the proof is identical to the proof of vanishing of cohomology in the ultralocal case. We first construct the vector field z (see the proof of Lemma 1.1 of Lecture 1) such that for the cohomologous cocycle $\tilde\alpha = \alpha + [z, \varpi]$ the functionals $\bar u^1, \ldots,$ $\bar u^n$ are Casimirs too. To this end we use the equation (19) for $s = 0$, $t > 0$:

$$\sum_{q,r}(-1)^{q+r}\binom{q+r}{r}\left(\frac{\partial A^{ki}_{q+r}}{\partial u^{l,t-q-1}}\right)^{(r)}\eta^{lj}+\sum_{r}(-1)^{t+r}\binom{t+r}{r}\left(\frac{\partial A^{jk}_{0}}{\partial u^{l,t+r-1}}\right)^{(r)}\eta^{li}=0.$$

Differentiating the antisymmetry condition

$$A_0^{ik} = \sum (-1)^{r+1}\left(A_r^{ki}\right)^{(r)}$$

w.r.t. $u^{l,t-1}$ we identify the first term of the previous equation with

$$-\frac{\partial A_0^{ik}}{\partial u^{l,t-1}}\eta^{lj}.$$

The resulting equation coincides with the condition $\partial a^k = 0$ of closedness of the 1-cocycle

$$(a^k)^i = A_0^{ik}$$

for every $k = 1, \ldots, n$ (see (18) for the explicit form of this condition). Using the first part of Lemma we arrive at existence of n differential polynomials q^1, . . . , q^n s.t.

$$A_0^{ik} = \eta^{is} \partial_x \frac{\delta \bar{q}^k}{\delta u^s(x)}. \tag{21}$$

Now we are able to change the cocycle α to a cohomological one to obtain a 2-cocycle

$$\alpha \mapsto \alpha + \partial z =: \alpha'$$

for

$$z = q^i \frac{\partial}{\partial u^i}.$$

The new 2-cocycle α' will have the same form as above with $A_0^{ij} = 0$. Denote

$$g_{i;js} := \eta_{ip}\eta_{jq}A_s^{ij}, \quad s \geq 1.$$

We will now show existence of differential polynomials $\omega_{i;j0}$, $\omega_{i;j1}, \ldots$ s.t.

$$\begin{aligned} g_{i;j1} &= \partial_x \omega_{i;j0}, \\ g_{i;js} &= \partial_x \omega_{i;j,s-1} + \omega_{i;j,s-2} \quad \text{for } s \geq 2. \end{aligned} \tag{22}$$

From (19) for $s = 1$, $t = 0$ we obtain

$$\partial_x \sum_r (-1)^r \left(\frac{\partial A_1^{jk}}{\partial u^{l,r}} \right)^{(r)} = 0.$$

As we already did many times, from the last equation it follows that

$$\sum_r (-1)^r \left(\frac{\partial A_1^{jk}}{\partial u^{l,r}} \right)^{(r)} = 0.$$

This shows existence of $\omega_{i;j0}$. Using (19) for $s = 1$ and $t > 0$ we inductively prove existence of the differential polynomials $\omega_{i;j,t-1}$. Actually, we can obtain

$$\omega_{i;jl} = \sum_{s \geq l+2} \partial_x^{s-l-2} g_{i;js}. \tag{23}$$

From this it readily follows that the coefficients $\omega_{i;js}$ satisfy the antisymmetry conditions

$$\omega_{i;js} = \sum_{t \geq s} (-1)^{t+1} \begin{pmatrix} t \\ s \end{pmatrix} \partial_x^{t-s} \omega_{j;it}$$

Thus they determine a 2-form ω.

Let us prove that the 2-form ω is closed. Denote

$$J_{ijk;st} := \left(\sum_{m=s}^{t+s} \sum_{r=0}^{m-s} + \sum_{m \geq t+s+1} \sum_{r=0}^{t} \right) (-1)^m \left(\begin{array}{c} m \\ r\ s \end{array} \right) \partial_x^{m-r-s} \frac{\partial \omega_{j;k,t-r}}{\partial u^{i,m}}$$

$$+ \frac{\partial \omega_{i;j,s}}{\partial u^{k,t}} - \frac{\partial \omega_{i;k,t}}{\partial u^{j,s}}$$

the l.h.s. of the equation (15) of Lecture 2 (the conditions of closedness of a 2-form). Let us show that the l.h.s. in (19) is equal to

$$\partial_x J_{ijk;t-1,s-1} + J_{ijk;t-1,s-2} + J_{ijk;t-2,l-1}. \tag{24}$$

To this end we replace the second sum in (19) by

$$-\frac{\partial A_s^{ik}}{\partial u^{l,t-1}} \, \eta^{lj}.$$

Lowering the indices by means of η_{ij} and using (22) we obtain (24). From vanishing of (24) we inductively deduce that $J_{ijk;st} = 0$ for all $i, j, k = 1, \ldots, n$ and all $s, t \geq 0$ (observe that the coefficients $J_{ijk;t0} = J_{ijk;0s} = 0$ due to our assumption $A_0^{ij} = 0$. This proves that the 2-form ω is closed. So $\omega = \delta \int dx \wedge \phi$ for some 1-form $\phi = \phi_i \delta u^i$. Introducing the vector field

$$a^i = \eta^{ik} \phi_k$$

we finally obtain, for the original cocycle α,

$$\alpha = \partial(a - z).$$

Theorem is proved.

Example. The Poisson brackets (9) of the "interpolated" Toda lattice can be reduced to the canonical form

$$\{u(x), w(y)\} = \delta'(x - y), \quad \{u(x), u(y)\} = \{w(x), w(y)\} = 0$$

by the Miura-type transformation $(u, v) \mapsto (u, w)$,

$$v'(x) = \frac{1}{\epsilon}\left[w(x+\epsilon) - w(x)\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\epsilon\,\partial_x)^n w(x). \tag{25}$$

The inverse transformation reads

$$w(x) = \epsilon\,\partial_x \left[e^{\epsilon\,\partial_x} - 1\right]^{-1} v(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!}(\epsilon\,\partial_x)^n v(x). \tag{26}$$

Here B_n are Bernoulli numbers.