The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR 1666 - 15

SCHOOL ON QUANTUM PHASE TRANSITIONS
AND
NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

11 - 22 July 2005

How supercurrents decay in optical lattices

Presented by:

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How supercurrents decay in optical lattices

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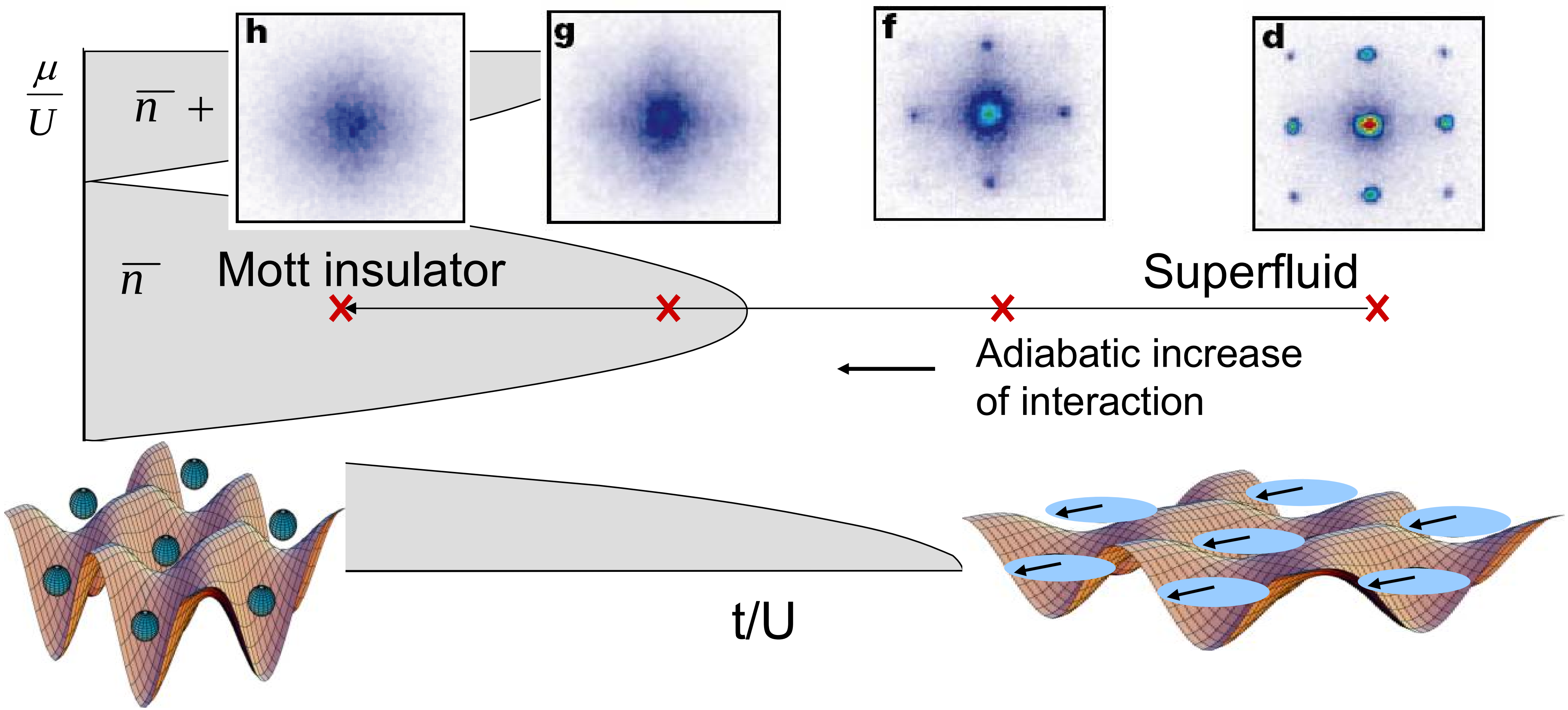
How do superfluids lose their super properties?

Equilibrium: Quantum Phase Transition

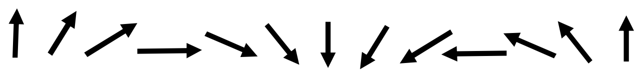
Greiner *et. al.* (I. Bloch group), Nature (02)

Ultra cold bosons on an optical lattice



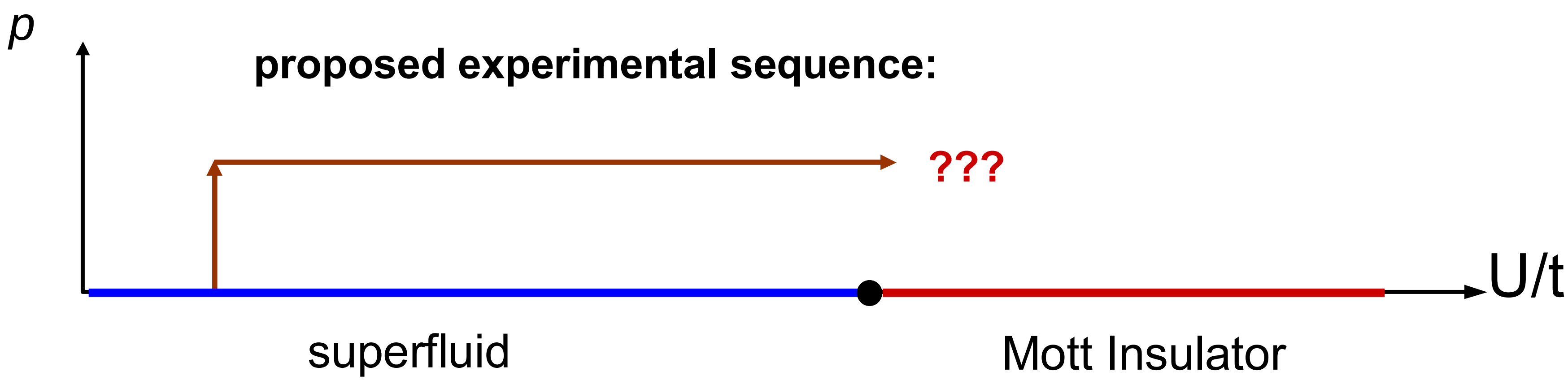
In this talk: Non equilibrium questions

1. Superfluid-Insulator transition of a moving condensate ?



Moving condensate:

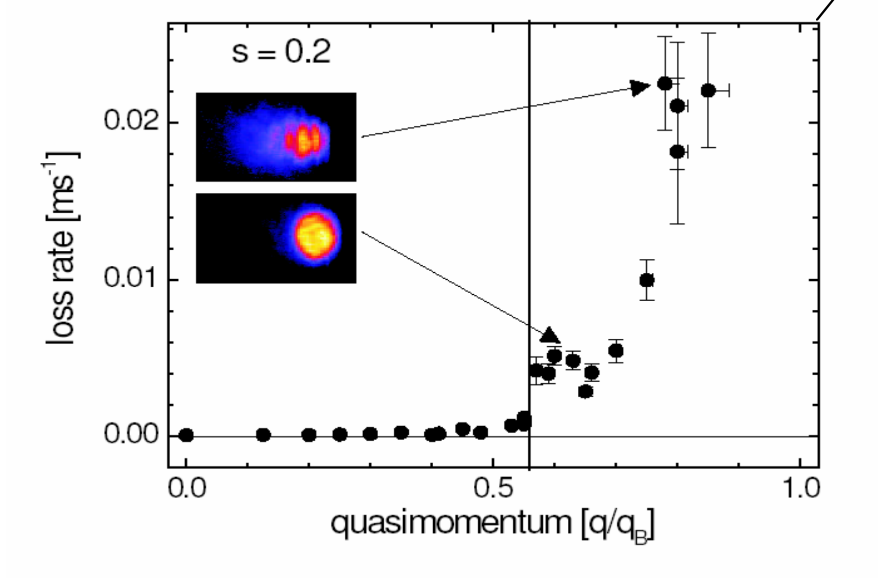
p = phase change/unit cell

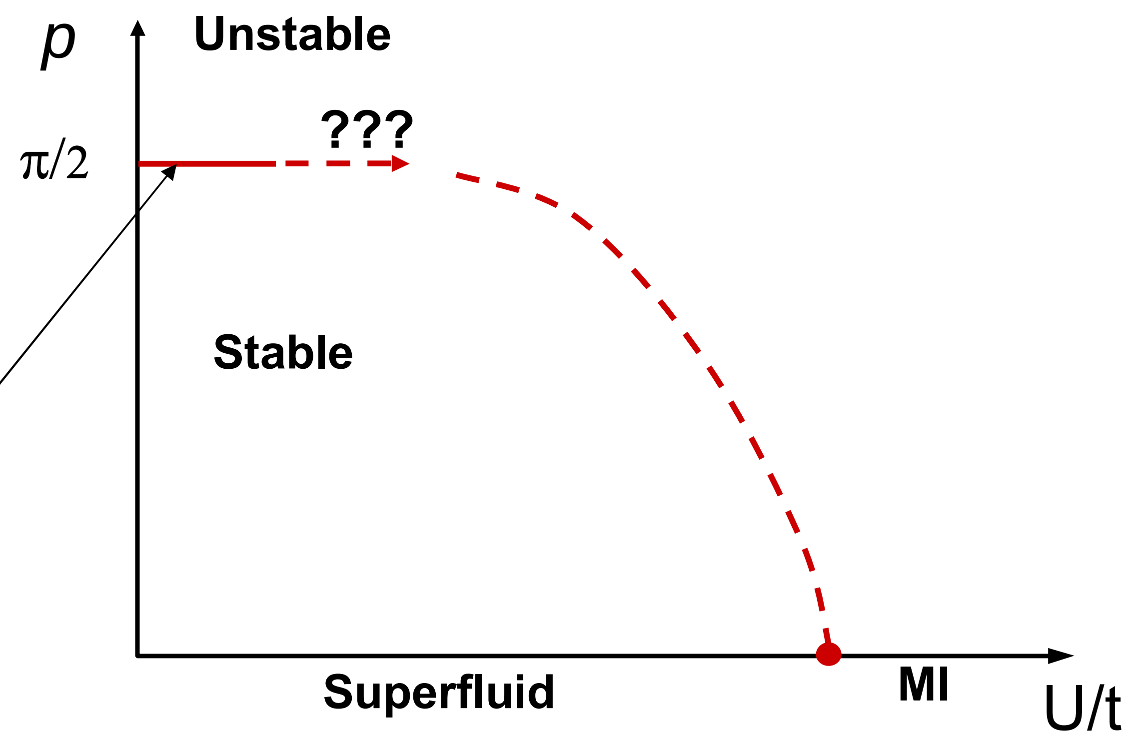


2. Dynamical stability of super-currents ?

Theory: Wu and Niu PRA (01); Smerzi et. Al. PRL (02).

Exp: Fallani et. al., (Florence) cond-mat/0404045



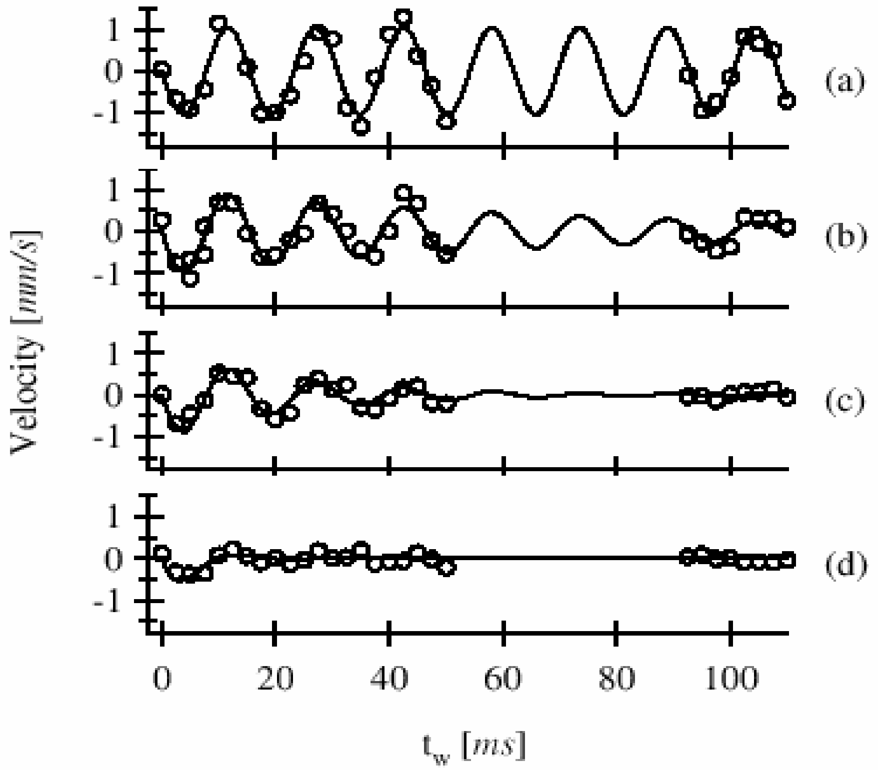


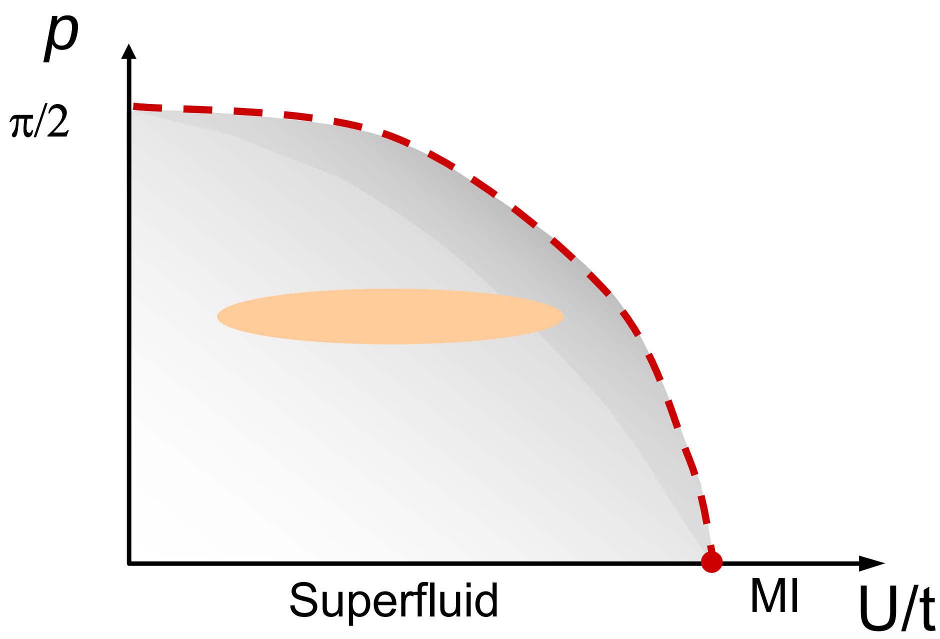
We find direct connection from classical instability to equilibrium transition

Mean field Stability "phase diagram" (critical current)

3. Decay of super-current below the critical current?

NIST exp: Decay of dipole oscillations 1d optical lattice (Fertig etal. 2004)



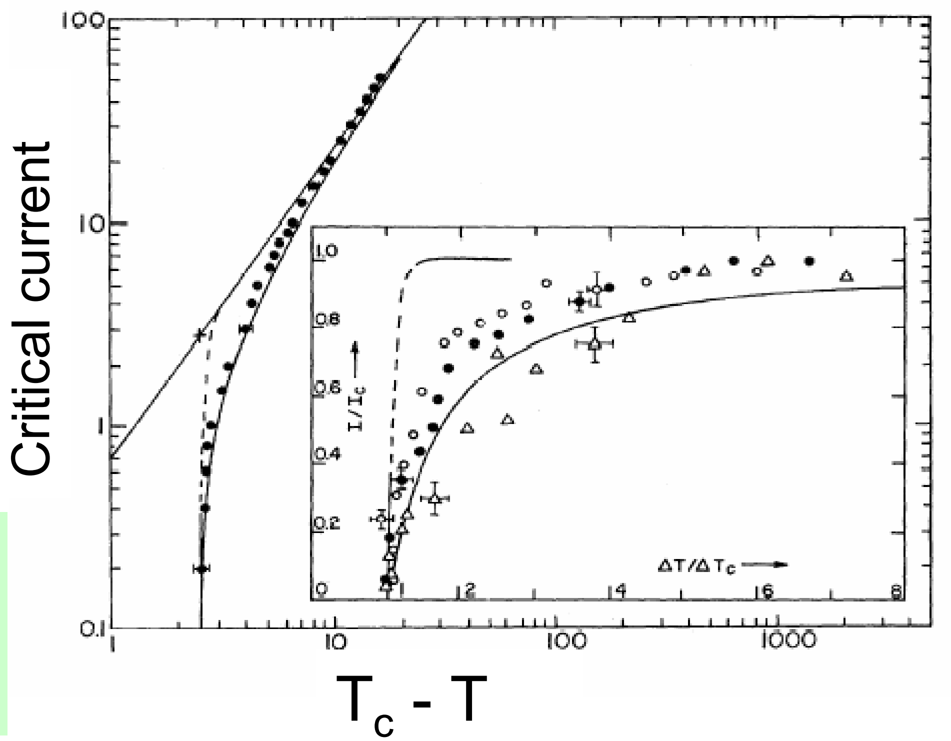


- Decay of current due to quantum fluctuations.
- Asymptotic decay rate near the classical instability from scaling approach

Related questions arise in superconductors

Reduction of T_C and the critical current in superconducting wires

Webb and Warburton, PRL (1968)



Theory (thermal phase slips):

Langer and Ambegaokar, Phys. Rev. (1967)
McCumber and Halperin, Phys Rev. (1970)

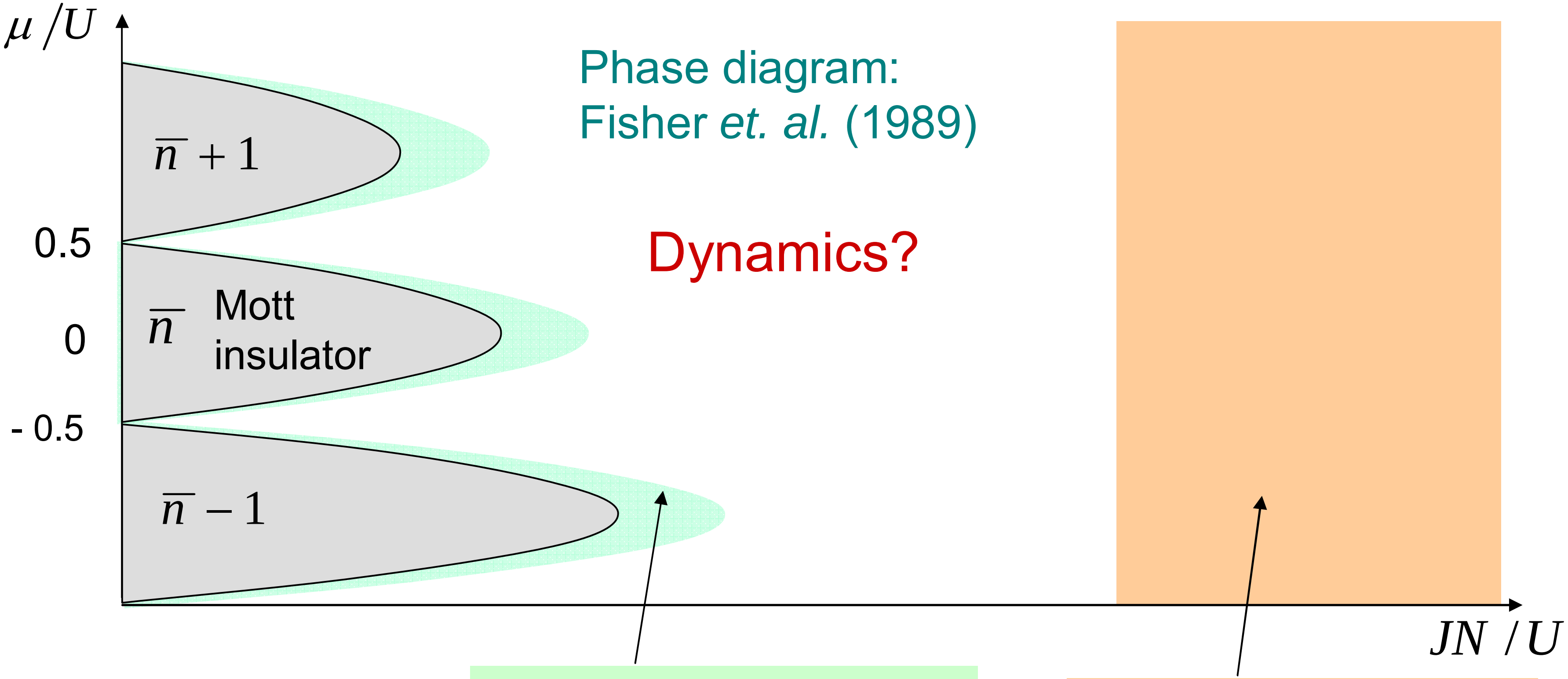
Role of quantum phase slips? Debated!
See Bezryadin, Lau and Tinkham, Nature (2000)

What is unique in ultra cold atomic systems?

- **Perfectly clean. Periodic potential.**
- **Isolated from the environment:**
  Underdamped dynamics.
  Quantum coherent.
- **Directly probe time evolution.**
- **Precise control over parameters.**
  Knob to control quantum fluctuations
  Access to quantum phase transition

Bose Hubbard Model

$$H = \frac{U}{2}\sum_i (n_i - \bar{n})^2 - J\sum_{\langle ij \rangle}(a_i^\dagger a_j + \mathrm{H.c.}) - \mu \sum_i (n_i - \bar{n})$$

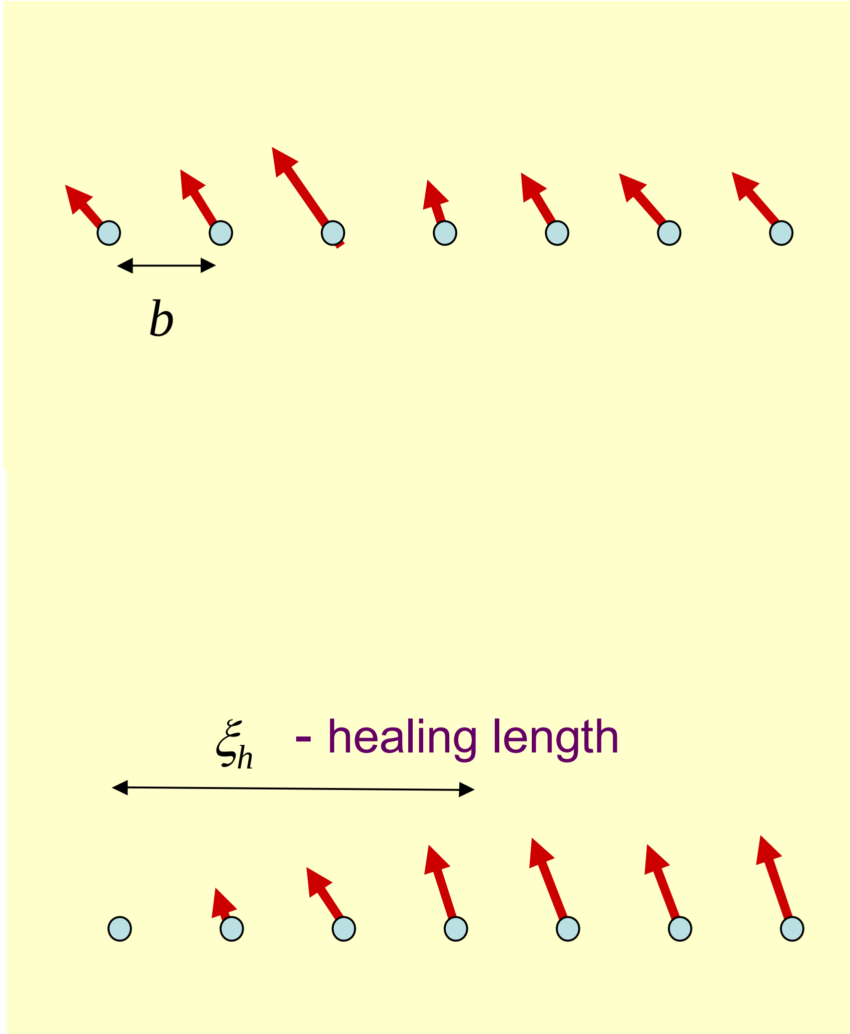


Classical dynamics:	Time dependent Ginzburg-Landau	Gross-Pitaevskii
Quantum corrections:	O(1/d)	$O\left(\sqrt{\frac{U}{JN}}\right)$

Weak coupling: Gross-Pitaevskii dynamics

$$i\frac{d\psi_j}{dt} = -J\sum_{\delta}\psi_{j+\delta} + U|\psi_j|^2\psi_j - \mu\psi_j$$

$$\psi_j = \sqrt{\rho_j}e^{i\theta_j}$$



Rescale:

$$\tilde{\psi}_j = \psi_j/\sqrt{N} \qquad \tilde{t} = \frac{UN}{\hbar}t$$

Continuum limit:

$$i\frac{d\tilde{\psi}}{dt} = -\underbrace{\left(b^2\frac{J}{UN}\right)}_{\xi_h^2}\nabla^2\tilde{\psi} + |\tilde{\psi}|^2\psi + \left(\frac{2J-\mu}{UN}\right)\tilde{\psi}$$

(valid while $\xi_h >> b$)

Suppression of density fluctuations

$$UN > J \qquad (\xi_h < b)$$

- Perturbations in density healed within a lattice constant.
- Density is uniform on the lattice.

$$i\frac{d\psi_j}{dt} = -J\sum_{\delta}\psi_{j+\delta} + U|\psi_j|^2\psi_j - \mu\psi_j$$

Phase only dynamics:

$$\frac{d^2\theta_j}{dt^2} = -2UJN\sum_{\delta}\sin(\theta_{j+\delta} - \theta_j)$$

Breakdown of GP and classical phase dynamics

Quantum phase model (equivalent to Hubbard for UN>>J , N>>1):

$$H = \sum_i \frac{U}{2} n_i^2 - JN \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \qquad [\theta_i, n_i] = i$$

For *U*<<*JN*, expand cos:

$$H = \sum_i \frac{U}{2} n_i^2 + \frac{JN}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \quad \Rightarrow \quad \langle \delta\theta^2 \rangle \sim \sqrt{\frac{U}{JN}}$$

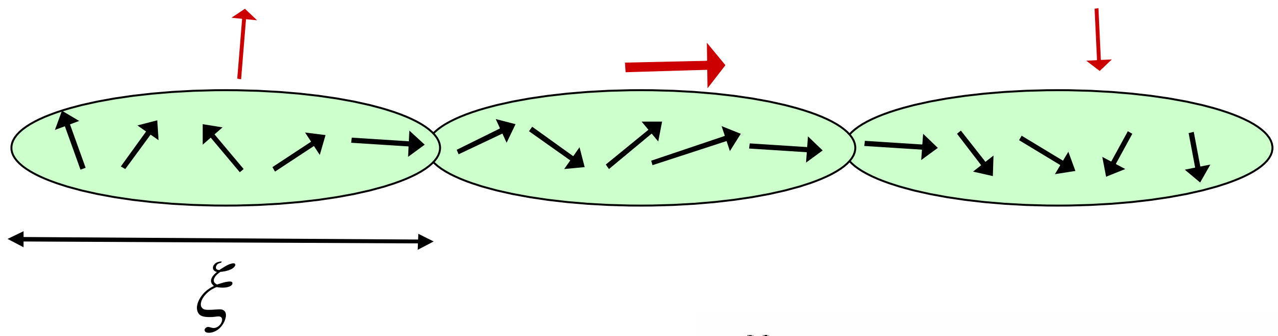
(Quantum fluctuation)

For *U*~*JN* : θ_j fluctuates strongly

Cannot be described by a classical variable !

Dynamics in the strongly correlated regime (U~JN)

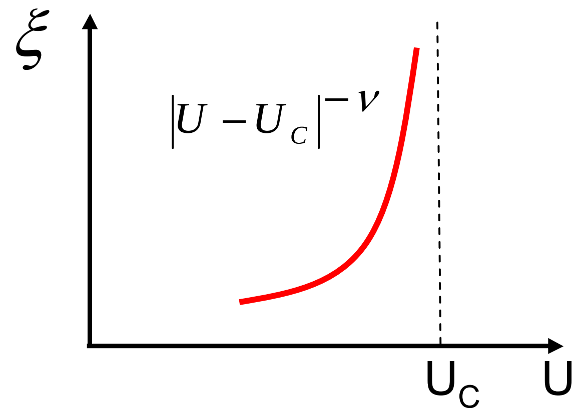
Semiclassical description possible after coarse graining:



Relativistic Ginzburg-Landau:

$$\ddot{\psi} = \nabla^2\psi + \psi(\xi^{-2} - |\psi|^2)$$

Sachdev, *Quantum phase transitions*
Altman & Auerbach, *PRL* (2002)



Remark: The coherence length ξ is a "healing length" of the order parameter. Not of density, which is uniform in this regime (recall $\xi_h \sim 1$)

Regimes of dynamics with increasing interaction

J/N — $JN \sim U_c$ — U

$$i\frac{d\psi_j}{dt} = -J\sum_{\delta}\psi_{j+\delta} + U|\psi_j|^2\psi_j - \mu\psi_j$$

Gross-Pitaevskii

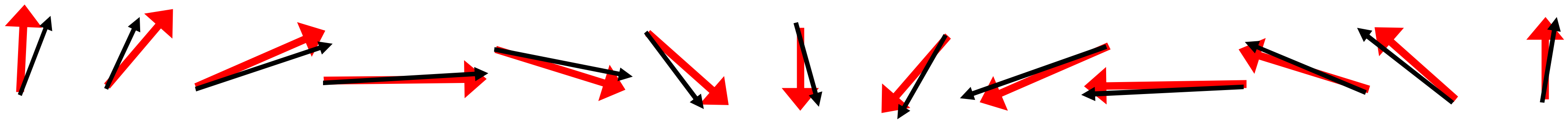
$$\frac{d^2\theta_j}{dt^2} = -2UJN\sum_{\delta}\sin\left(\theta_{j+\delta} - \theta_j\right)$$

Phase model

$$\ddot{\psi} = \left(\nabla^2 + \xi^{-2} - |\psi|^2\right)\psi$$

Relativistic GL

Dynamical instability in GP



- Expansion around uniform current $\psi_j = \sqrt{N} e^{ipx_j+\phi_j}$

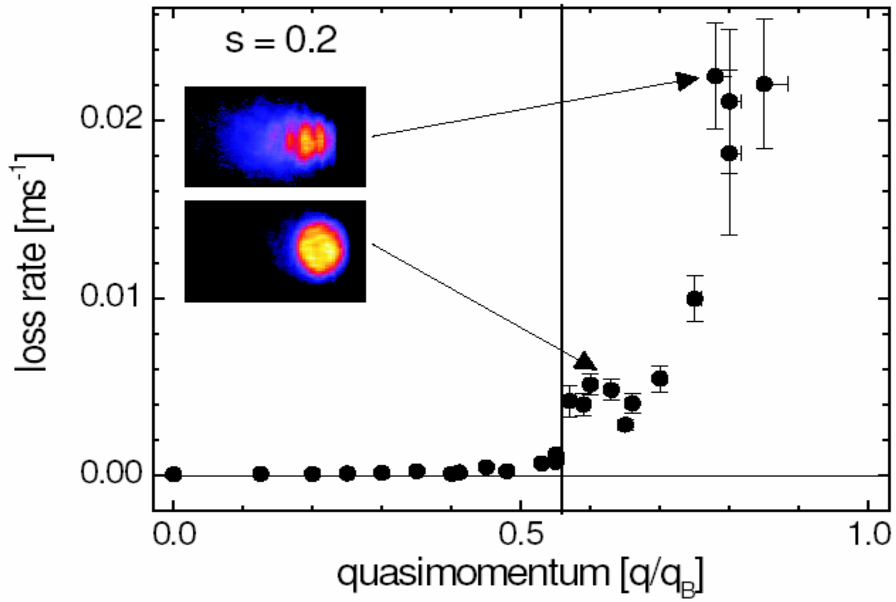
$$\frac{d^2\phi_j}{dt^2} = -2UJN\cos p(\phi_{j+1} - 2\phi_j + \phi_{j-1}) + O(\phi^2)$$

$p>\pi/2$ ⇨ negative spring constant

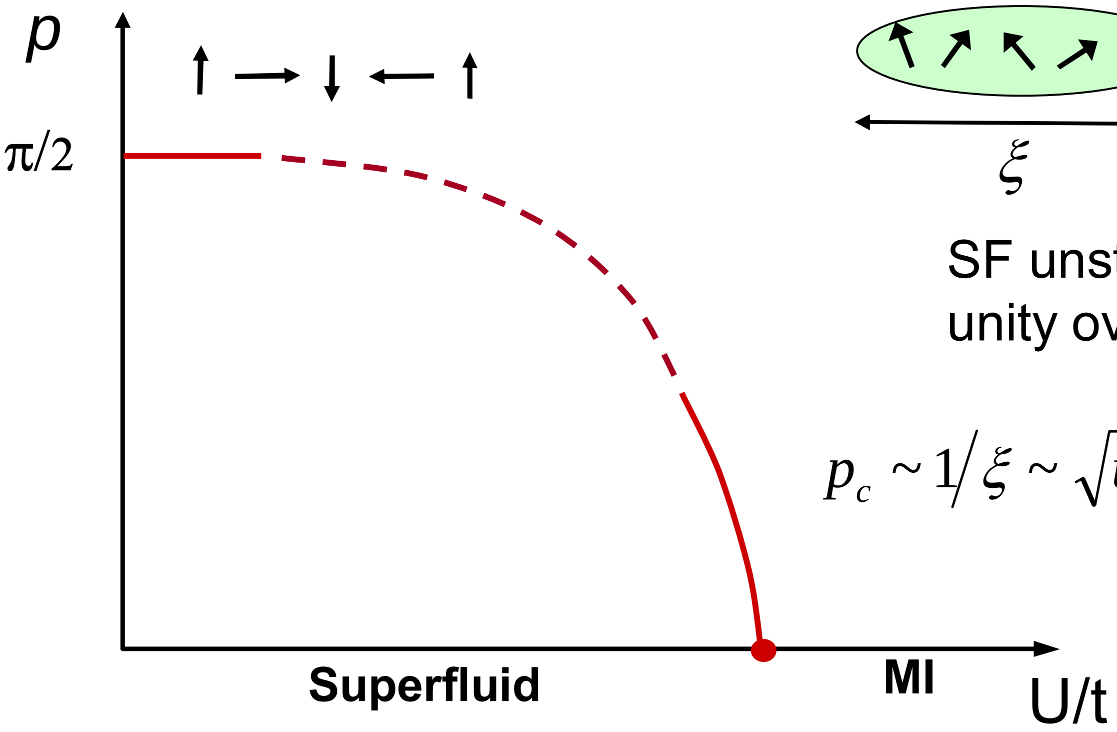
Imaginary normal mode frequencies

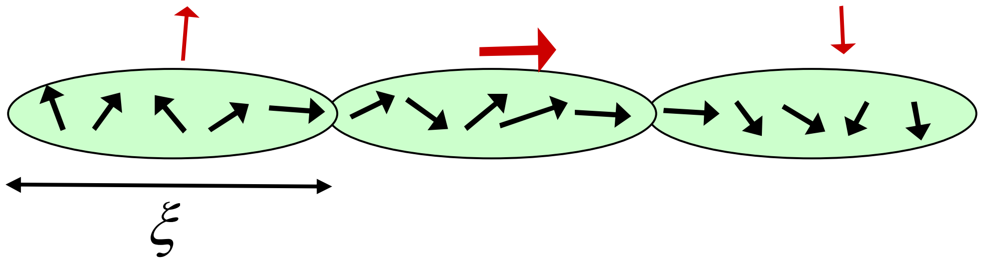
Theory: Wu and Niu PRA (01);
Exp. : Smerzi etal. PRL (02).

Fallani etal. cond-at/0404045



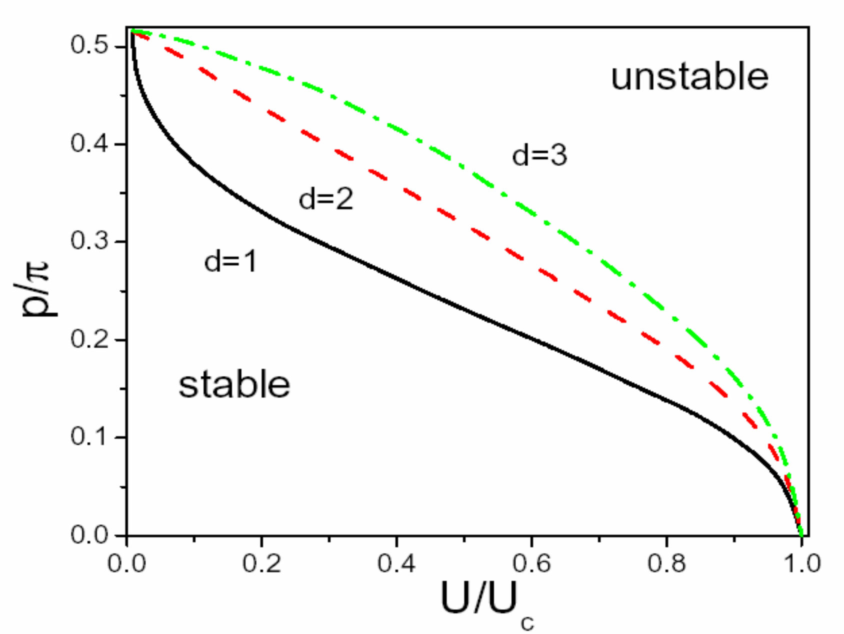
Dynamical Instability near the SF-MI transition





SF unstable if phase change is of order unity over a coherence length

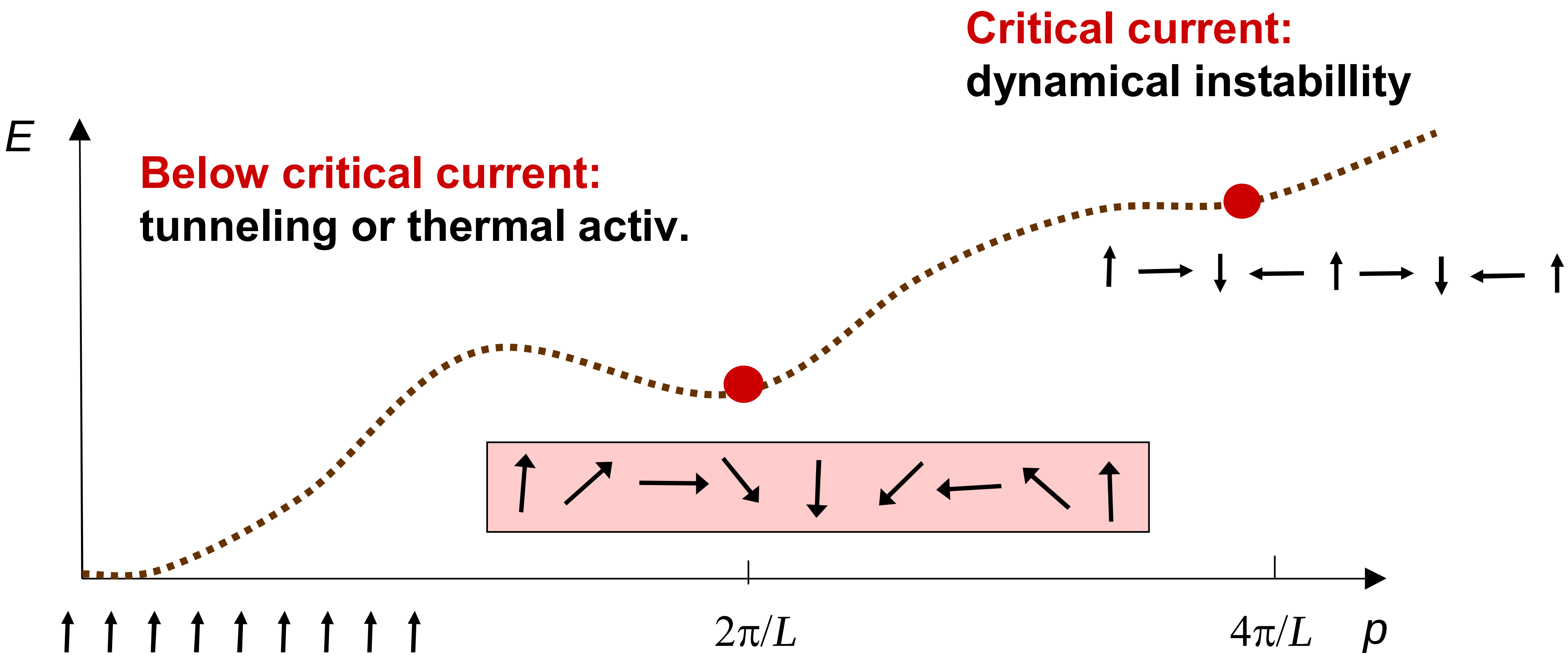
$$p_c \sim 1/\xi \sim \sqrt{u_c - u}$$



Interpolate between the two limits using time-dependent Gutzwiller approximation.

See: cond-mat/0411047

Decay of current below the critical current



Reminiscent of a 1st order transition. Irreversible.

Semiclassical instanton tunneling: $\Gamma_Q \propto e^{-S_{\text{inst}}}$

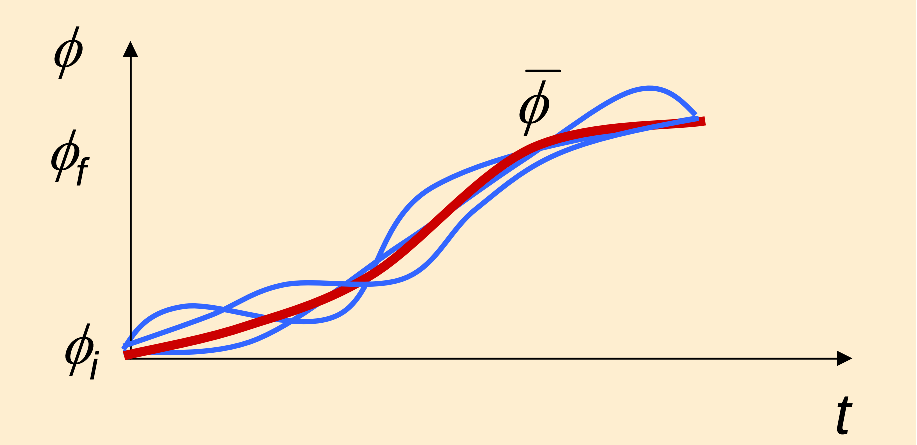
Single particle semiclassics

Path integral for the transition amplitude:

$$G_{i\to f} = \int_{\phi_I}^{\phi_F} \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\int_0^t dt' \left[\frac{1}{2}m\dot{\phi}^2 - V(\phi)\right]\right\}$$

Classical path:

$$\frac{\delta S}{\delta\phi}\Big|_{\bar{\phi}} = 0 \qquad m\ddot{\bar{\phi}} + V'(\bar{\phi}) = 0$$



Gaussian fluctuations:

$$\phi = \bar{\phi} + \eta \qquad \eta(0) = \eta(t) = 0$$

$$G_{i\to f} = e^{\frac{i}{\hbar}S(\bar{\phi})} \int \mathcal{D}\eta e^{-\frac{i}{2\hbar}\int dt'\eta(t')\left[m\partial_t^2 + V''(\bar{\phi})\right]\eta(t')} \equiv A_q e^{\frac{i}{\hbar}S(\bar{\phi})}$$

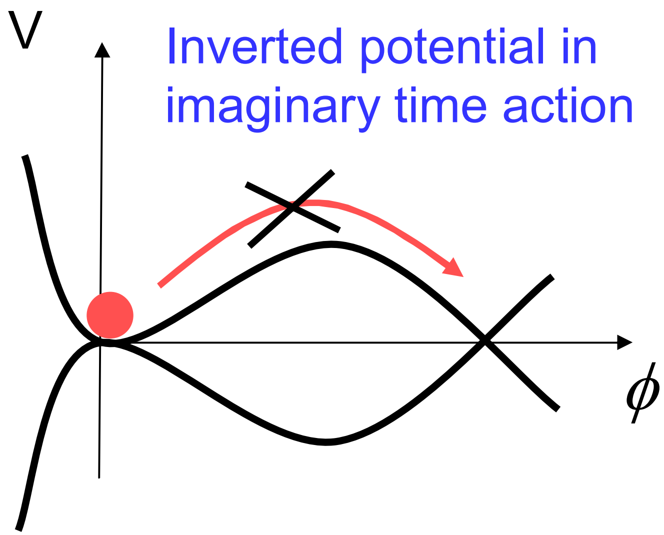
Single particle – semiclassical tunneling

No classical trajectory !

- Rotate to imaginary time: $t = -i\tau$

$$G_{i\to f}(\tau) = \int_{\phi_I}^{\phi_F} \mathcal{D}\phi \exp\left\{-\frac{1}{\hbar}\int_0^{\tau} d\tau' \left[\frac{1}{2}m\dot{\phi}^2 + V(\phi)\right]\right\}$$

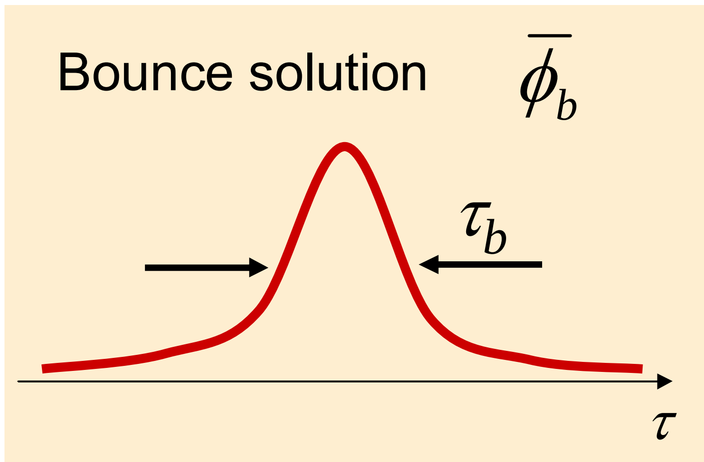
  - Analytically continue to real time at the end.



- Survival amplitude:

$$G_{i\to i}(t) \propto e^{-i\omega t/2} \exp\left[-tAe^{-S[\bar{\phi}_b]/\hbar}\right]$$

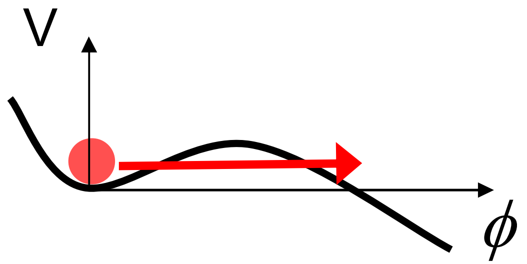
- Decay rate: $\Gamma = A^{-S_b/\hbar}$



Single particle - Limit of vanishing barrier

$$S = \int_0^\tau d\tau' \left[\frac{1}{2} m\dot{\phi}^2 + \epsilon\phi^2 - b\phi^3\right]$$

$$p - p_c \sim \epsilon \to 0$$



Rescale: $\phi = \frac{\epsilon}{b}\tilde{\phi} \qquad \tau = \tilde{\tau}\sqrt{\frac{m}{\epsilon}}$

$$S = \epsilon^{5/2} S_0$$

$$S_0 = \frac{\sqrt{m}}{b^2} \int_0^{\tilde{\tau}} d\tilde{\tau} \left[\frac{1}{2}\dot{\tilde{\phi}}^2 + \tilde{\phi}^2 - \tilde{\phi}^3\right]$$

$$\Gamma(\epsilon) = \exp\left(-\epsilon^{5/2} S_0\right)$$

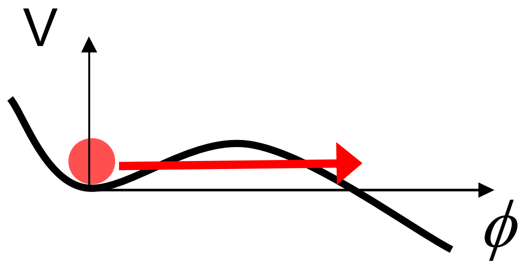
- Obtain parametric dependence on p_c-p near the instability
- Avoid calculation of precise tunneling action.

How does it work ?

Single particle - Limit of vanishing barrier

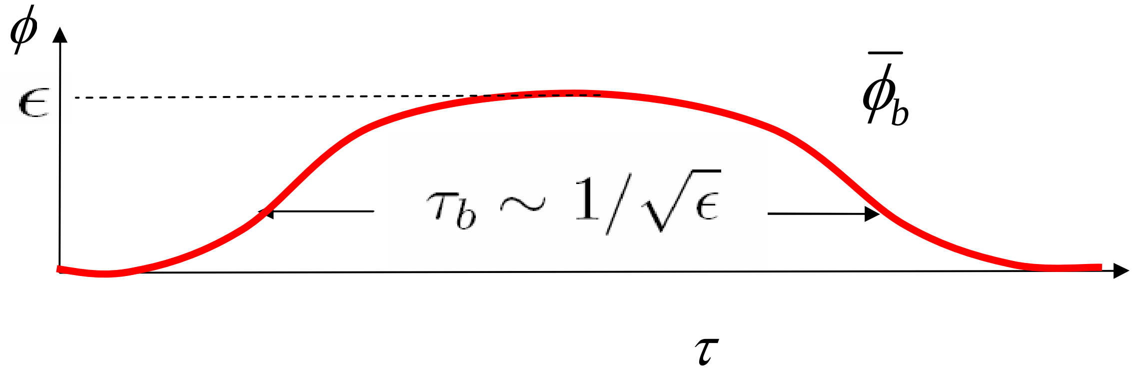
$$S = \int_0^{\tau} d\tau' \left[\frac{1}{2} m \dot{\phi}^2 + \epsilon \phi^2 - b \phi^3 \right]$$

$$p - p_c \sim \epsilon \to 0$$



Rescale: $\phi = \frac{\epsilon}{b} \tilde{\phi} \qquad \tau = \tilde{\tau} \sqrt{\frac{m}{\epsilon}}$

- Scaling facilitated by diverging time of critical bounce



- vanishing energy barrier: $E_{\text{barrier}} \sim \varepsilon \phi^2 \sim \varepsilon^3$

$\Rightarrow$ $$S_b \sim E_{\text{barrier}} \times \tau_b \sim \varepsilon^{5/2}$$

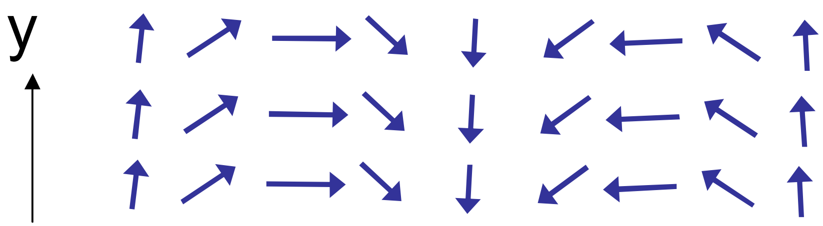
Quantum decay of current – weak coupling I

$$S = \sqrt{\frac{JN}{2U}} \int d\tau \left[\sum_j \left(\frac{d\theta_j}{d\tau} \right)^2 - \sum_{\langle j,j' \rangle} 2\cos(\theta_j - \theta_{j'}) \right]$$

"$1/\hbar$" Controls semiclassics

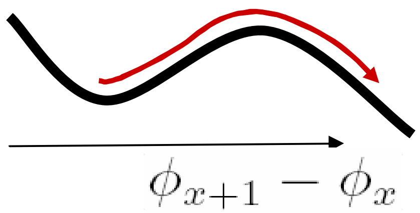
Expand about a uniform current:

$$\theta_j = p x_j + \phi_x(\mathbf{y})$$



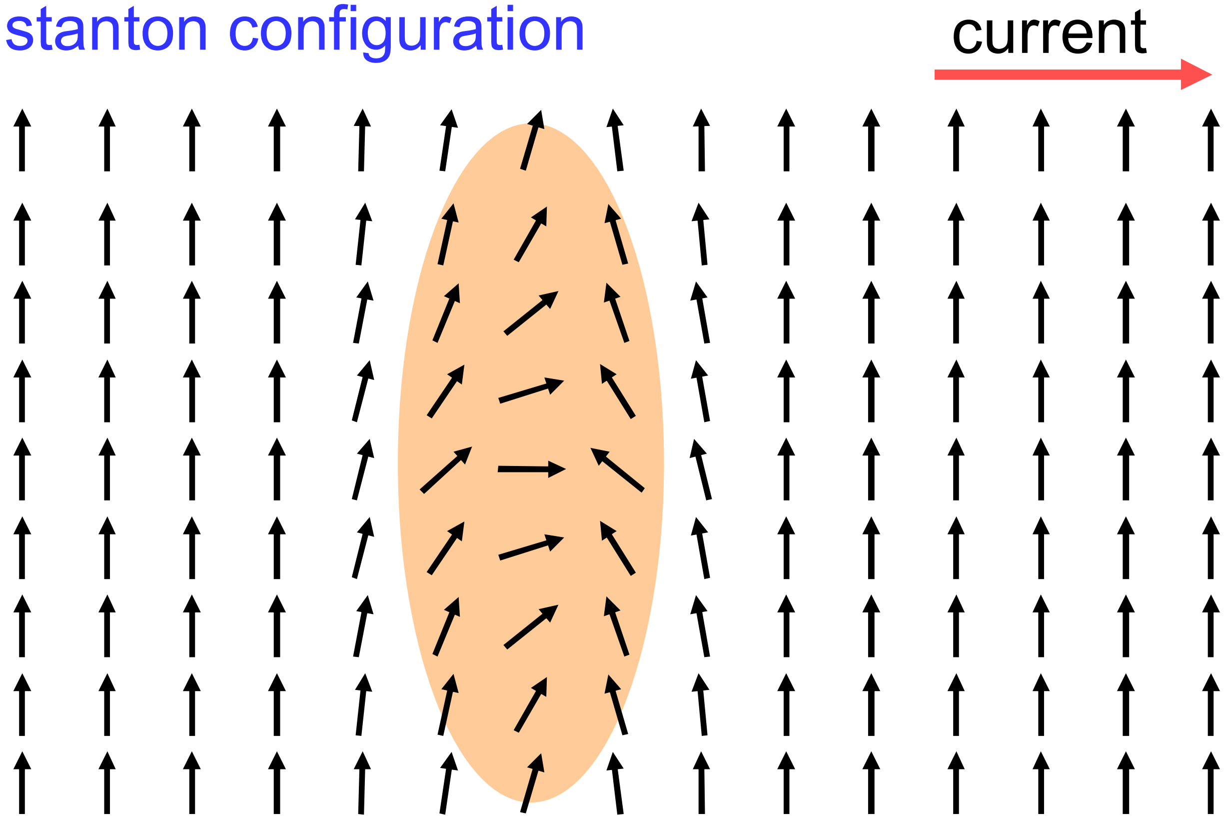
Continuum approximation in the transverse direction

$$S \approx \sqrt{\frac{JN}{2U}} \sum_x \int d\tau d^{d-1}y \left[\left(\frac{d\phi_j}{d\tau} \right)^2 + \left(\frac{d\phi_j}{d\mathbf{y}} \right)^2 + \cos(p)\,(\phi_{x+1} - \phi_x)^2 - \frac{1}{3}(\phi_{x+1} - \phi_x)^3 \right]$$



Quantum decay of current – weak coupling II

Instanton configuration



$$J_{\perp} = J$$

$$J_{\parallel} = J \cos p \to 0$$

phase slip occurs at the same time in many parallel chains.

Justifies continuum approximation in transverse direction.

Quantum decay of current – weak coupling III

Rescaling:

$$\phi = \tilde{\phi}\cos p,\ \tau = \frac{\tilde{\tau}}{\sqrt{\cos p}},\ y = \frac{\tilde{y}}{\sqrt{\cos p}}$$

Critical instanton has diverging time as well as size in the transverse direction.

$$\Gamma_Q \propto \exp\left[-s_d \sqrt{\frac{JN}{U}} (\pi/2 - p)^{3-d/2} \right]$$

$$s_d = \sum_x \int d^d y \left[\left(\frac{d\phi_j}{d\mathbf{y}} \right)^2 + (\phi_{x+1} - \phi_x)^2 - \frac{1}{3}(\phi_{x+1} - \phi_x)^3 \right]$$

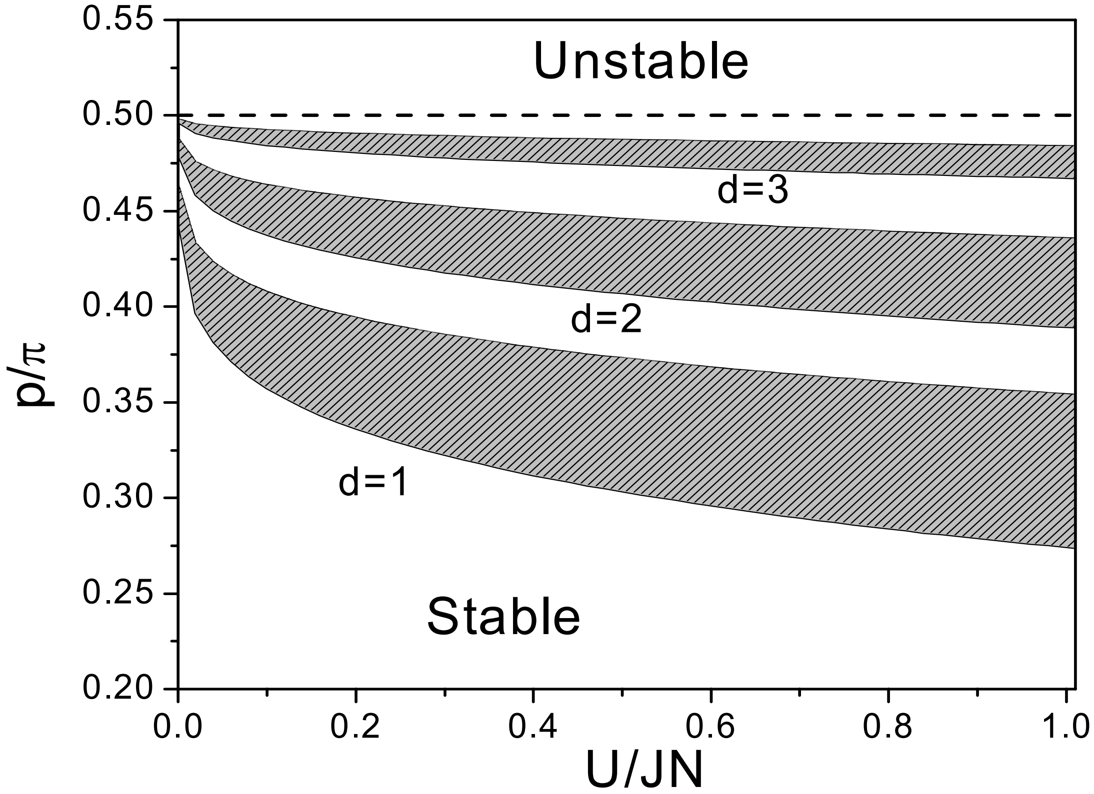
Variational calculation yields:

$s_1 \approx 7.1 \qquad s_2 \approx 25 \qquad s_3 \approx 93$

Quantum decay of current – weak coupling IV

$$\Gamma_Q \propto e^{-S_{\text{inst}}}$$

Instability → crossover ($1<S_{inst}<3$)



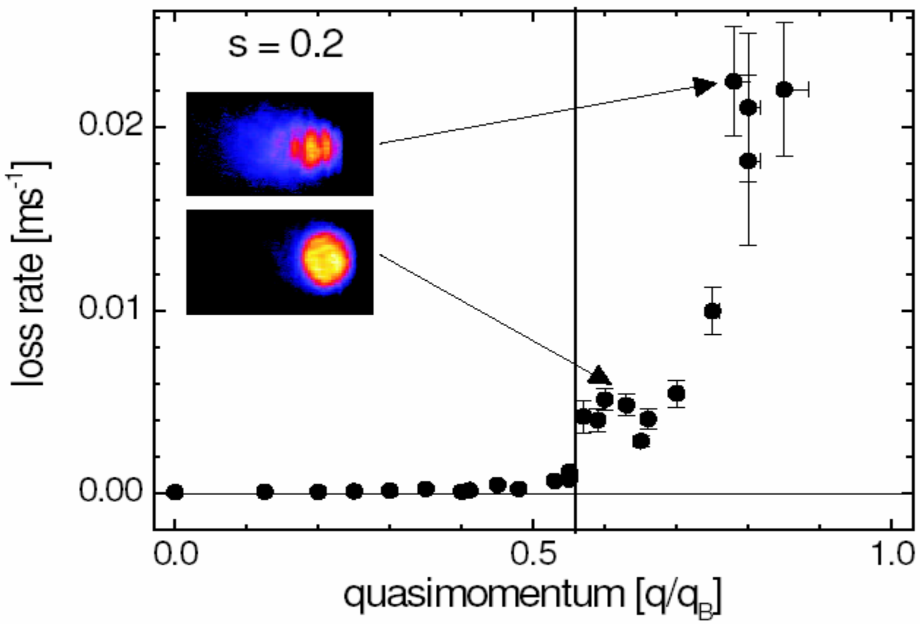
See: Cond-mat/0412497

Experiments in one dimensional traps

$$\Gamma \propto \exp\left[-7.1\sqrt{\frac{JN}{U}}\left(\frac{\pi}{2}-p\right)^{5/2}\right]$$

$$\sqrt{\frac{JN}{U}} \leftrightarrow \frac{1}{"\hbar"}$$

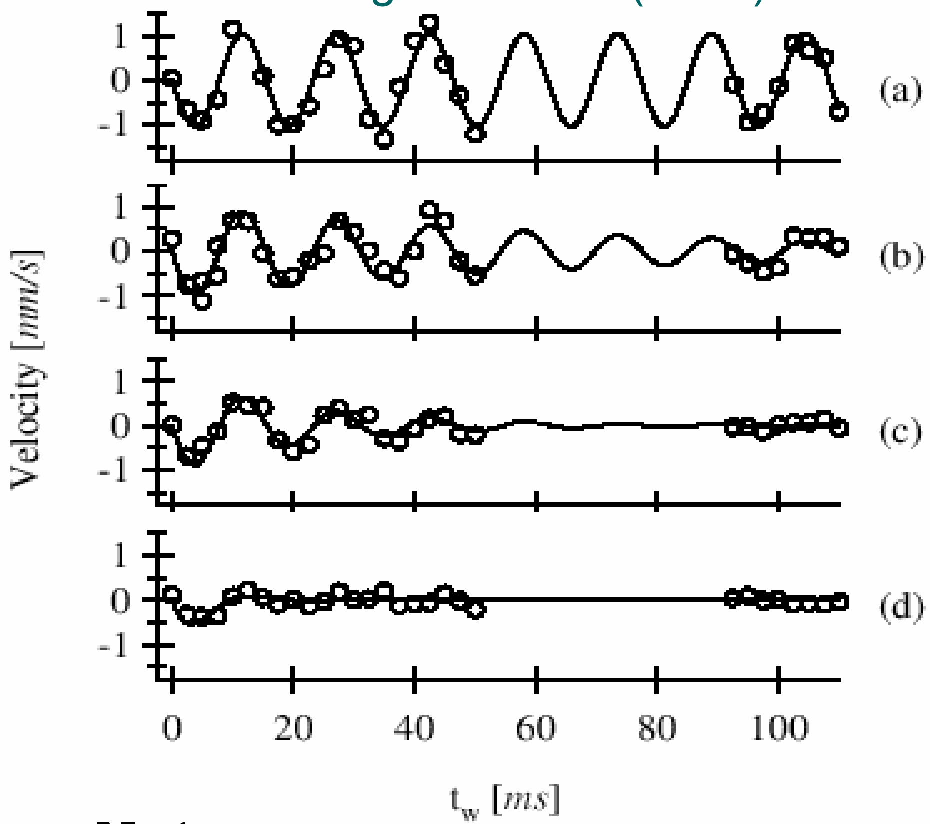
Fallani etal. 2004 (Florence)



N~10^2 - 10^3

Sharp instability at p=π/2

Fertig etal. 2004 (NIST)



N~1

Observe decay for p<π/2

Thermal decay of current – weak coupling

$$S = \sqrt{\frac{JN}{2U}} \int d\tau \left[\sum_j \left(\frac{d\theta_j}{d\tau} \right)^2 - \sum_{\langle j,j' \rangle} 2\cos(\theta_j - \theta_{j'}) \right]$$

$$\int d\tau \to \frac{\sqrt{2UJN}}{T}$$

substitute $d \to d$ -1
in quantum calculation:

$$\Gamma_T \propto \exp\left[- \frac{JN}{T} (\pi/2 - p)^{3.5-d/2} s_{d-1} \right]$$

Temperature scale for quantum→classical crossover :

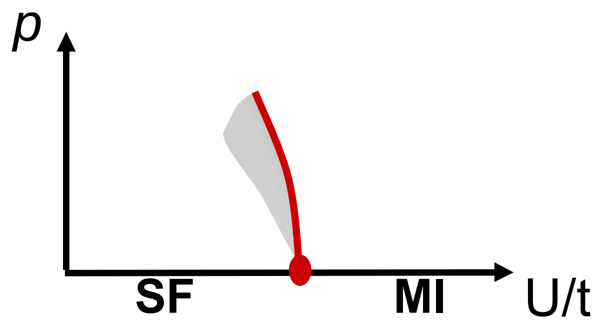
$$T^* \sim \sqrt{2UJN}$$

Quantum decay near the SF-MI transition I

$$S = A \int d^d x d\tau \left\{ \left| \frac{d\psi}{d\tau} \right|^2 + |\nabla \psi|^2 - \xi^{-2} |\psi|^2 + \frac{1}{2} |\psi|^4 \right\}$$

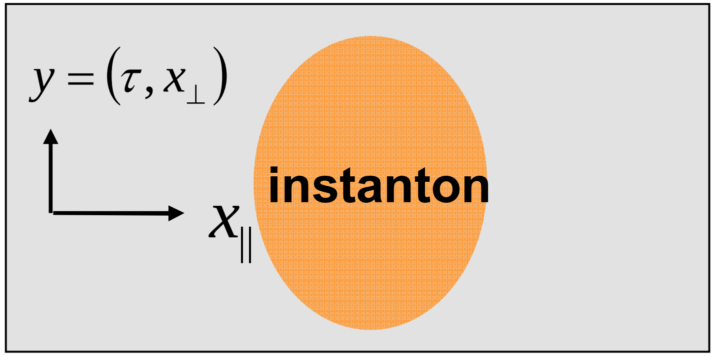
Current state + fluctuations (η, ϕ)

$$\psi(x) = \xi^{-1} \sqrt{1 - (p\xi)^2} (1 + \eta)\, \mathrm{e}^{\mathrm{i}(\mathrm{px} + \phi)}$$



Quantum decay near the SF-MI transition II

Critical instanton (bounce):



$$E_{barrier} \sim (p_c - p)^3$$

$$\tau_{bounce} \sim x^{\perp}_{bounce} \sim (p_c - p)^{-1}$$

$$x^{\parallel}_{bounce} \sim (p_c - p)^{-1/2}$$

Power counting: $S_b \sim x_{\perp b}^{d-1} \times x_{\parallel b} \times \tau_b \times E_{\text{barrier}} \sim (p_c - p)^{2.5-d}$

$$\Gamma \propto e^{-S_d} = \exp\left[-\frac{C_d}{\sqrt{\xi}}(p_c - p)^{2.5-d}\right]$$

⇒ Broad crossover in d=1,2

d=3 : Volume cost is overwhelming. It is cheaper to create noncritical (finite size) instantons, with finite action cost.

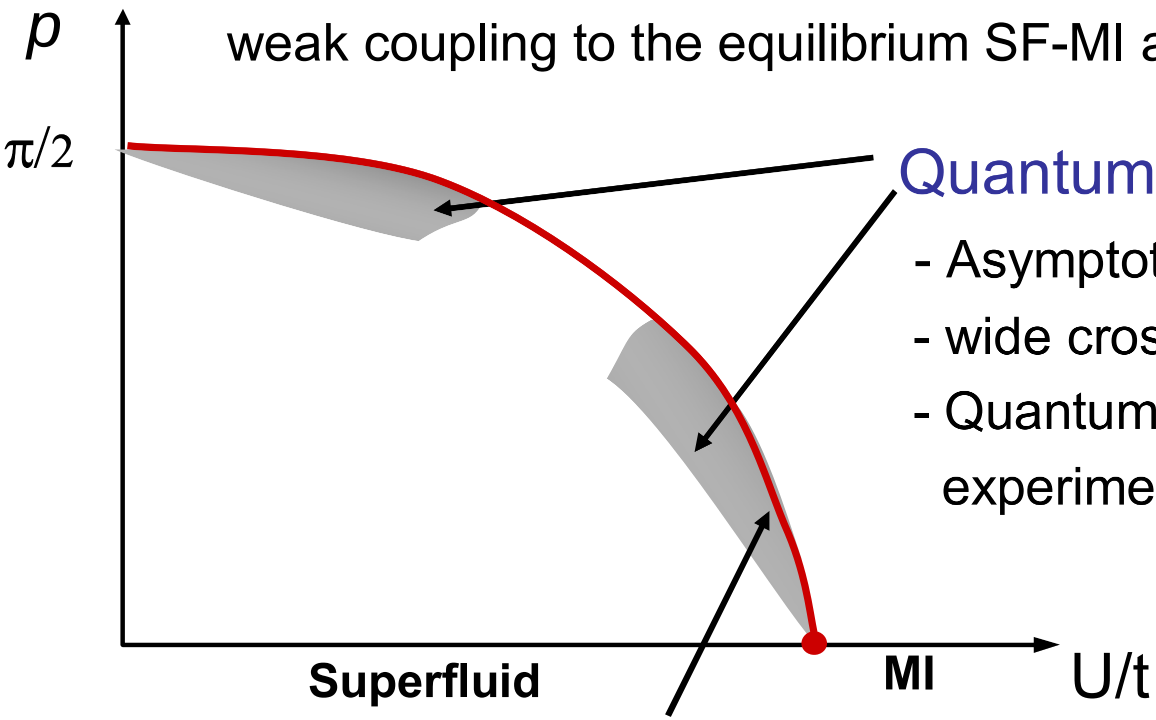
$$\Gamma \sim \begin{cases} e^{-4.3} & p < p_c \\ 1 & p > p_c \end{cases}$$

Discontinuity in the decay rate.
Mean-field instability is well defined!

Summary

Mean field stability phase diagram:

Continuous connection from the modulational instability at $p_c=p/2$ and weak coupling to the equilibrium SF-MI at strong coupling.



Quantum and thermal Fluctuations:

- Asymptotic decay rates in d=1,2,3
- wide crossover in d=1,2
- Quantum fluctuations dominate at experimentally relevant temperatures.

<u>d=3 (T=0):</u> discontinuity in the decay rate near SF-MI transition! Mean field instability is well defined.

Refs: Polkovnikov, Altman, Demler, Lukin and Halperin, Cond-mat/0412497

Altman, Polkovnikov, Demler, Lukin and Halperin, Cond-mat/0411047