The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR 1666 - 16

SCHOOL ON QUANTUM PHASE TRANSITIONS
AND
NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

11 - 22 July 2005

Entanglement based codes, matrix product states

Presented by:

Maciej Lewenstein

Institut de Ciencies Fotoniques, Barcelona

Quantum Information Theory, Quantum Phase Transitions and Cold Atoms

Wanderin' quantum optics theory (Warsaw, Saclay, Hannover, Barcelona)

ICFO
Institut de Ciències Fotòniques

Universität Hannover

Three Tales on Quantum Information Theory, Quantum Phase Transitions and Cold Atoms

- Lecture I – Introduction to QIT – theory of entanglement, entangle-ment criteria and measures, multiparty entaglement, entaglement detection, distillability (literature: bruss.pdf, lewen.pdf, lecture1.ppt)
- Lecture IIa – Entanglement and quantum phase transitions - entaglement in simple integrable models at the criticality, localizable entanglement, entanglement versus correlations (lecture2.ppt).
- Lecture IIb – Generation of entanglement in many body systems, generation via quantum phase transitions, generation in complex and disordered systems.
- Lecture IIIa – Entaglement based codes, matrix product states, PEPS (Projected Entangled-Pair States) (lecture3.ppt, armand.pdf).
- Lecture IIIb – Examples – Spin ½ XY chain in a random X-oriented field

Entanglement based codes, matrix product states

G. Vidal, PRL **93**, 040502 (2004); S.R. White and A.E. Feiguin, PRL **93**, 076401 (2004); F. Verstraete, D. Porras, and J.I. Cirac, PRL 93, 227205 (2004); A.J. Daley, C. Kollath, U. Schollwöck, and G. Vidal, cond-mat/0403313; J. Stat. Mech.: Theor. Exp. (2004) P04005; F. Verstraete and J.I. Cirac, cond-mat/0407066; M. Rigol and A. Muramatsu, PRL **94**, 240403 (2005); S.R. Manmana, A. Muramatsu, and R.M. Noack, cond-mat/0502396.

Many-body quantum systems

- Many-body quantum systems are difficult to describe.

$$|\Psi\rangle$$

$$|\Psi\rangle = \sum c_{i_1 \ldots i_N} \, |i_1, \ldots, i_N\rangle$$

We need 2^N coefficients to represent a state.

- To determine physical quantitites (expectation values) an exponential number of computations is required.

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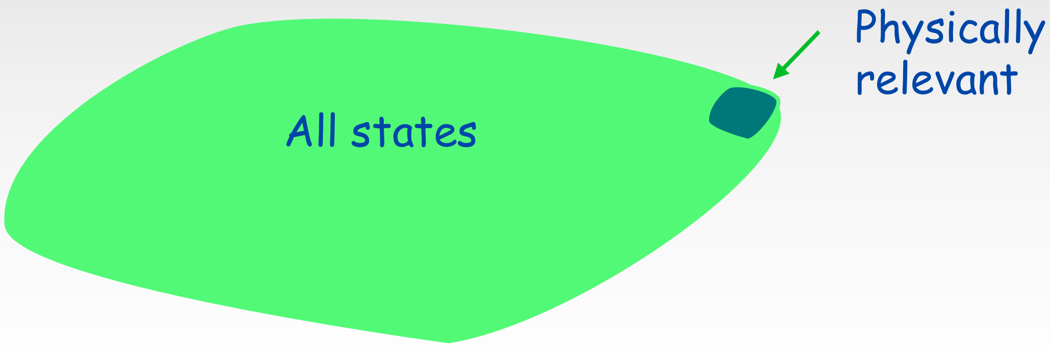
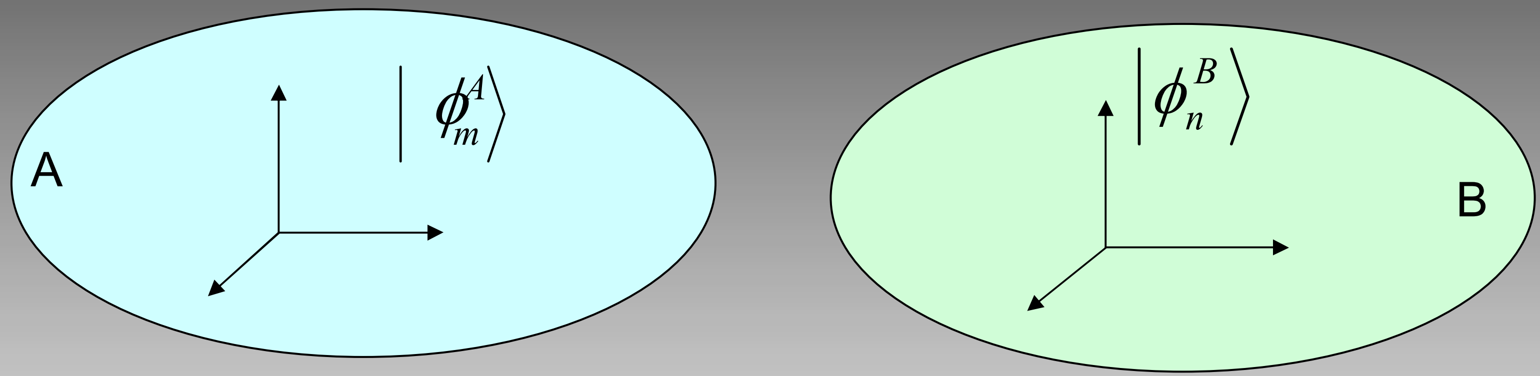


Composite System

$$C = A \otimes B$$



$d_A = \dim(A)$ $\qquad$ $d_B = \dim(B)$

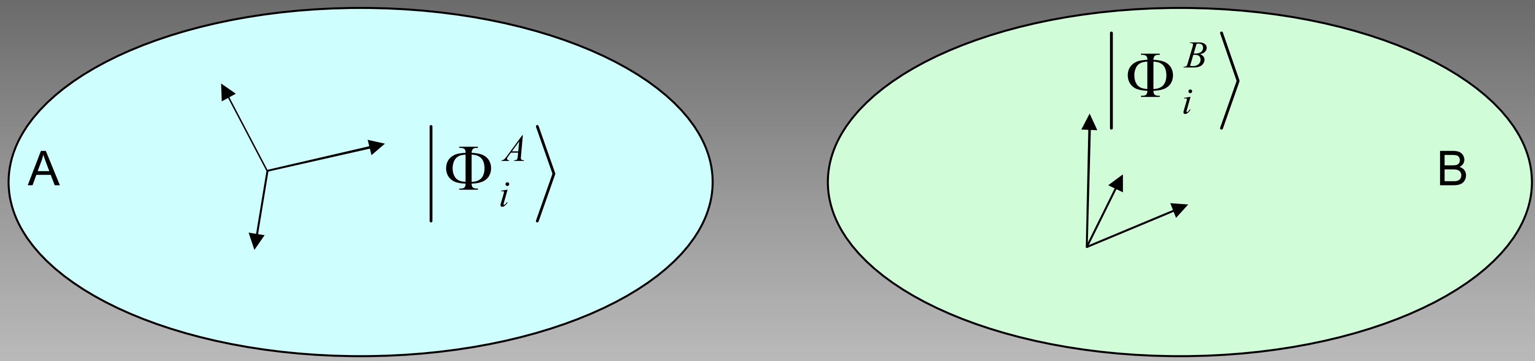
Some particles in A, the others in B. The total system is described by

$$\left| \psi \right\rangle = \sum_{m=1}^{d_A} \sum_{n=1}^{d_B} c_{m,n} \left| \phi_m^A \right\rangle \left| \phi_n^B \right\rangle$$

where $\displaystyle \sum_{m=1}^{d_A} \sum_{n=1}^{d_B} \left| c_{m,n} \right|^2 = 1$

$$\dim(C) = \dim(A) \cdot \dim(B)$$

Schmidt Decomposition (SD)



Theorem (Schmidt Decomposition)

Every state of a composite system can be decomposed as follows:

$$\left|\psi\right\rangle = \sum_{i=1}^{r} \lambda_i \left|\Phi_i^A\right\rangle \left|\Phi_i^B\right\rangle$$

$$r = \min(d_A, d_B)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$$

$$\sum_{i=1}^{r} \lambda_i^2 = 1$$

Follows from the Singular Value Decomposition (SVD) in linear algebra. Proof see notes.

SD versus Entanglement

- Consider a product state

$$\left|\psi\right\rangle = \left|\varphi^A\right\rangle\left|\varphi^B\right\rangle = \sum_i \lambda_i \left|\Phi_i^A\right\rangle\left|\Phi_i^B\right\rangle$$

product state: one single term!

- Consider a singlet state

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\left|1\right\rangle = \sum_i \lambda_i \left|\Phi_i^A\right\rangle\left|\Phi_i^B\right\rangle$$

singlet state: two terms!

The more terms in the SD, the more entangled the state

Approximation

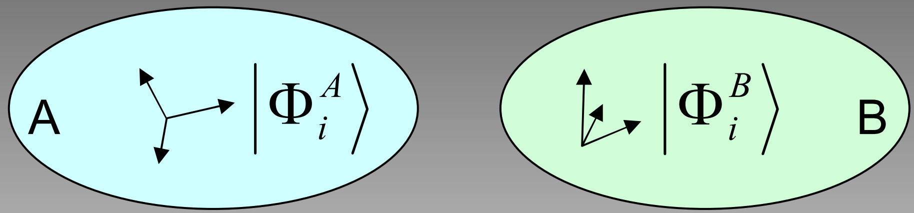
- Remark: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$
- Approximation: ***Neglect terms with very small coefficients!***
  Define $\chi_\varepsilon < r$, potentially $\chi_\varepsilon << r$

$$\left|\psi\right\rangle = \sum_{i=1}^{r} \lambda_i \left|\Phi_i^A\right\rangle \left|\Phi_i^B\right\rangle \approx \sum_{i=1}^{\chi_\varepsilon} \lambda_i \left|\Phi_i^A\right\rangle \left|\Phi_i^B\right\rangle$$

if (and only if) $\sum_{i=\chi_\varepsilon+1}^{r} \lambda_i^2 = \varepsilon << 1$

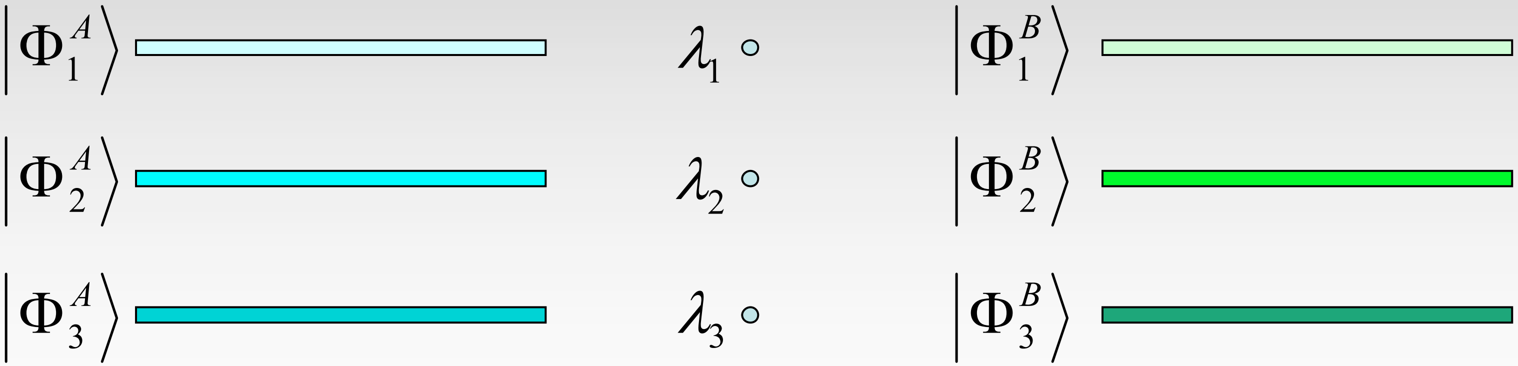
Approximation restricts entanglement in the system.

Graphical Representation:



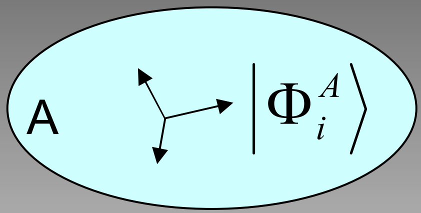
$$|\psi\rangle = \sum \lambda_i |\Phi_i^A\rangle |\Phi_i^B\rangle$$

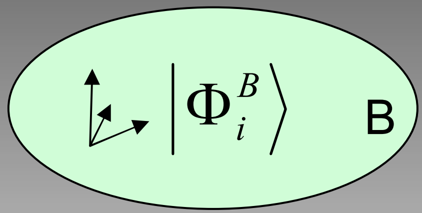
- Lines represent basis vectors
- Points represent coefficients



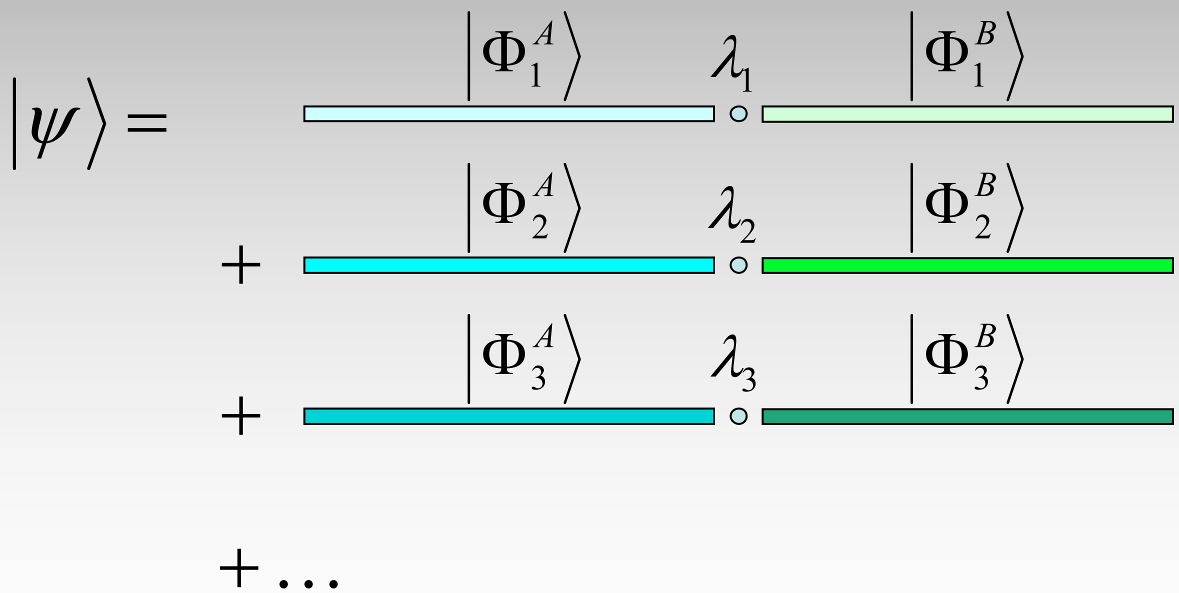
…and so forth

SD graphically

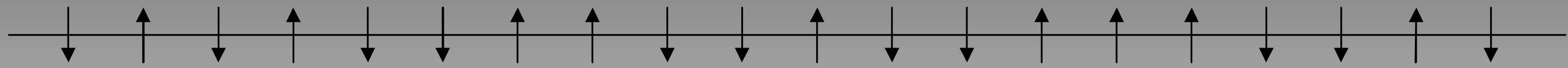




$$|\psi\rangle = \sum \lambda_i |\Phi_i^A\rangle |\Phi_i^B\rangle$$

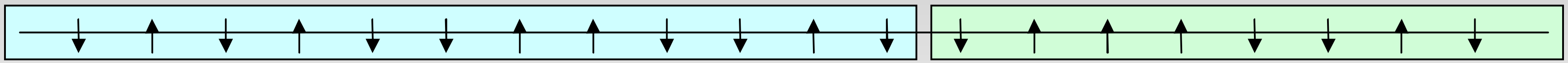


Spin chain

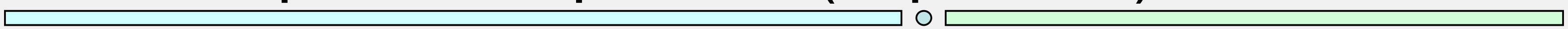


Decompose spin chain into two subsystems

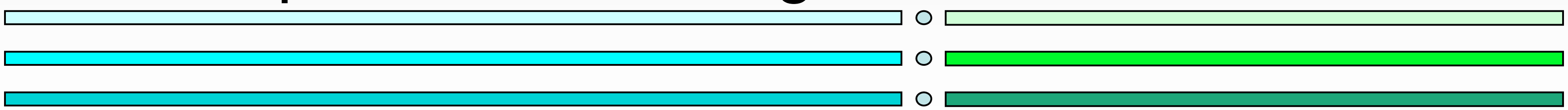
- Partition A:B = [1,2,…,j]:[(j+1),(j+2),…,N]



- Description of a product (separable) state:

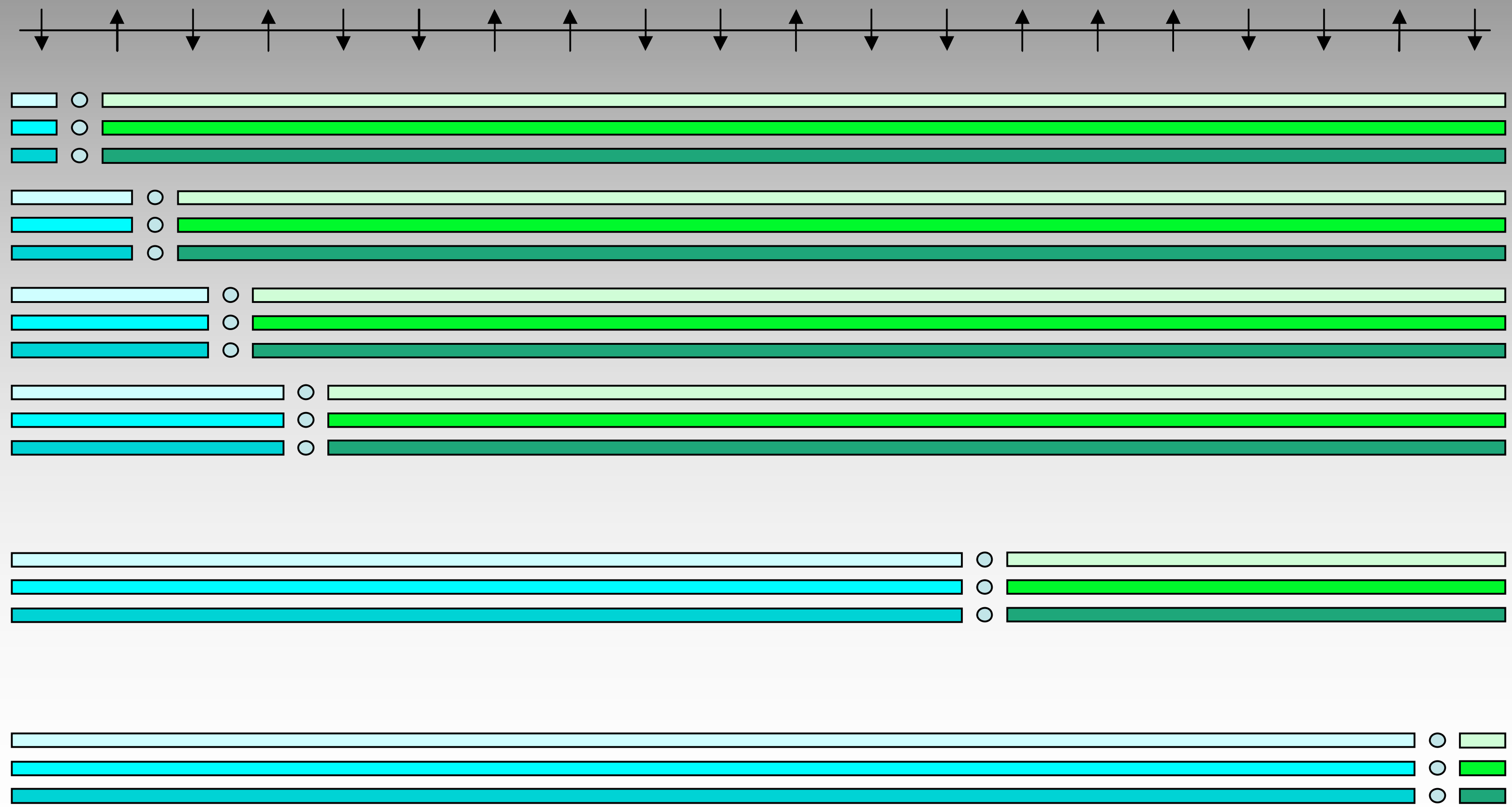


- Description of an entangled state:



Vidal's Decomposition (1/3)

1) Perform SD for all partitions [1,..,j]:[(j+1),..,N]



Vidal’s Decomposition (2/3)

2) Introduce computational basis $|0\rangle$ $|1\rangle$

3) Connect right vectors to subsequent bases

The decomposition for two subsequent decompositions is

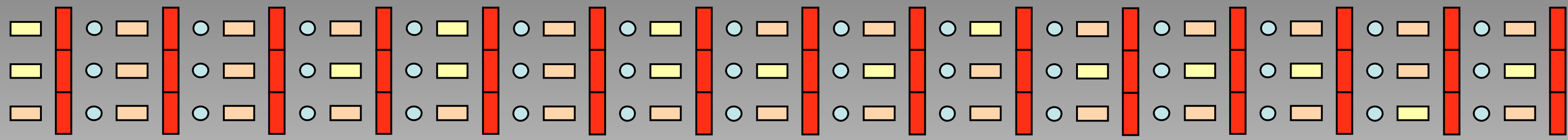
The first of these vectors (for example) is decomposed as

$\Gamma^{[\cdot]\,i}_{\alpha\beta}$

= Next-Neighbours Correlations

Vidal's Decomposition (3/3)

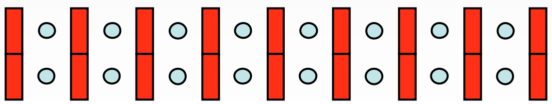
4) Repeat procedure for next site until done.



Finally: $$|\psi\rangle = \sum_{i_1}\sum_{i_2}\sum_{i_3}\cdots\sum_{i_N} c_{i_1 i_2 i_3 \ldots i_N} |i_1 i_2 i_3 \ldots i_N\rangle$$

$$c_{i_1 i_2 i_3 \ldots i_N} = \sum_{\{\alpha\}=1}^{\chi} \Gamma^{[1]i_1}_{\alpha_1} \lambda^{[1]}_{\alpha_1} \Gamma^{[2]i_2}_{\alpha_1\alpha_2} \lambda^{[2]}_{\alpha_2} \Gamma^{[3]i_3}_{\alpha_2\alpha_3} \cdots \lambda^{[N-1]}_{\alpha_{N-1}} \Gamma^{[N]i_N}_{\alpha_{N-1}}$$

$$c_{i_1 i_2 i_3 \ldots i_N} = \Gamma^{[1]i_1} \lambda^{[1]} \Gamma^{[2]i_2} \lambda^{[2]} \Gamma^{[3]i_3} \cdots \lambda^{[N-1]} \Gamma^{[N]i_N}$$



Comparison to full description

Description of chain of N spin ½ particles

- **Full description**

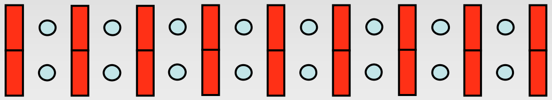
  Variables: $c_{i_1, i_2, i_3 \ldots, i_N}$

  Number of variables: 2^N

  Pro: Exact, universally applicable

  Contra: Huge number of variables

- **Vidal's decomposition**

  Variables: $\lambda_\alpha^{[\cdot]}, \Gamma_{\alpha\beta}^{[\cdot]i}$

  Number of variables: $\approx (\chi + 2\chi^2)N$

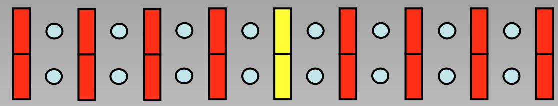
  Pro: Increase in number of variables linear in length

  Contra: only applicable for slightly entangled 1D systems with open boundary conditions

How to compute gates

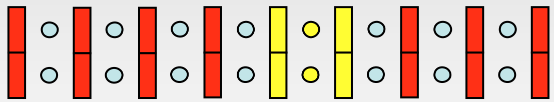
- **One Site Only**
  Only the corresponding Γ tensor has to be updated.

  $$\ldots \lambda^{[l-1]}\Gamma^{[l]i_l}\lambda^{[l]}\Gamma^{[l+1]i_{l+1}}\lambda^{[l+1]}\ldots$$

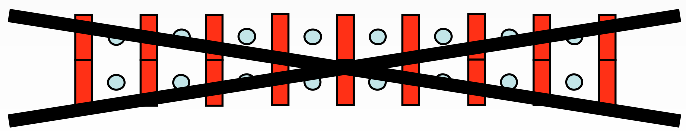
  

- **Two Neighbouring Sites**
  Only the Γ tensors corresponding to the two sites as well as the λ vector in between have to be updated.

  $$\ldots \lambda^{[l-1]}\Gamma^{[l]i_l}\lambda^{[l]}\Gamma^{[l+1]i_{l+1}}\lambda^{[l+1]}\ldots$$

- **Two or More Arbitrary Sites**
  Generally very complicated!

Projected entangled-pair states

F. Verstraete, D. Porras, and J.I. Cirac, PRL 93, 227205 (2004); F. Verstraete and J.I. Cirac, cond-mat/0407066; F. Verstraete and J.I. Cirac, Phys. Rev. A 70, 060302 (Rap. Comm.) (2004).

For renormalization group aspects:

J.I. Latorre, C. A. Lütken, E. Rico, and G. Vidal, Phys. Rev. A 71, 034301 (2005); F. Verstraete, J.I. Cirac, J.I. Lattore, E. Rico, and M.M. Wolf, Phys. Rev. Lett. 94, 140601 (2005)

Projected Entangled-Pair States: properties and applications

F. Verstraete and J.I. Cirac

MAX-PLANCK INSTIUT FÜR QUANTENOPTIK

CALTECH, 22 February 2005

MPQ

Many-body quantum systems

- Many-body quantum systems are difficult to describe.

$$|\Psi\rangle$$

• • • • • •

$$|\Psi\rangle = \sum c_{i_1 \ldots i_N} |i_1, \ldots, i_N\rangle$$

We need 2^N coefficients to represent a state.

- To determine physical quantitites (expectation values) an exponential number of computations is required.

Numerical Methods:

- Monte-Carlo methods:

  - It works very well in 1,2, and 3D.
  - It has the „sign" problem:

    Problems with Fermions or frustration cannot be simulated.

  - It is difficult to simulate dynamics.

- Density Matrix Renormalization Group (DMRG):

  - It has no „sign" problem.
  - Based on certain kind of matrix product states.
  - Works for 1D systems:
    - S. White: Ground state.
    - G. Vidal: Dynamics.

      } Open boundary conditions.

    - Nishino: Finite temperature.

      } Infinite and homogeneous.

Many-body quantum systems

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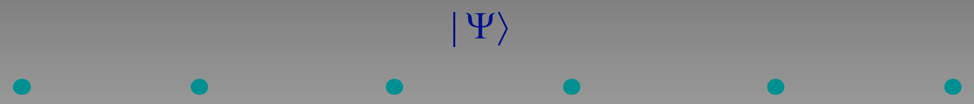

$$|\Psi\rangle = \sum c_{i_1 \dots i_N} |i_1, \dots, i_N\rangle$$

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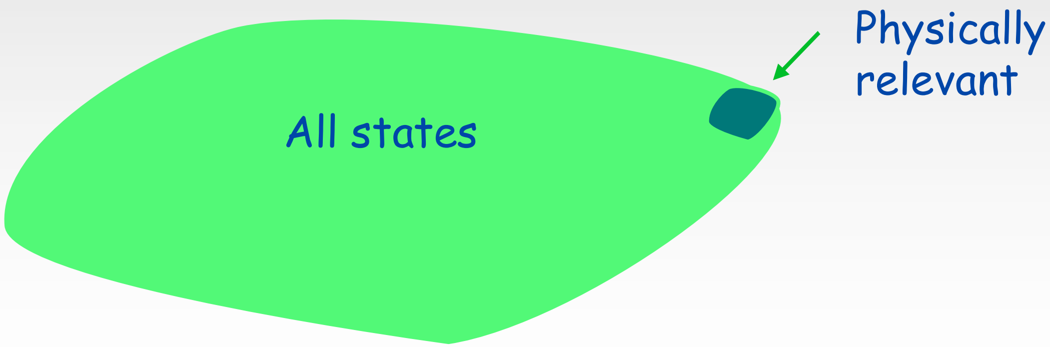
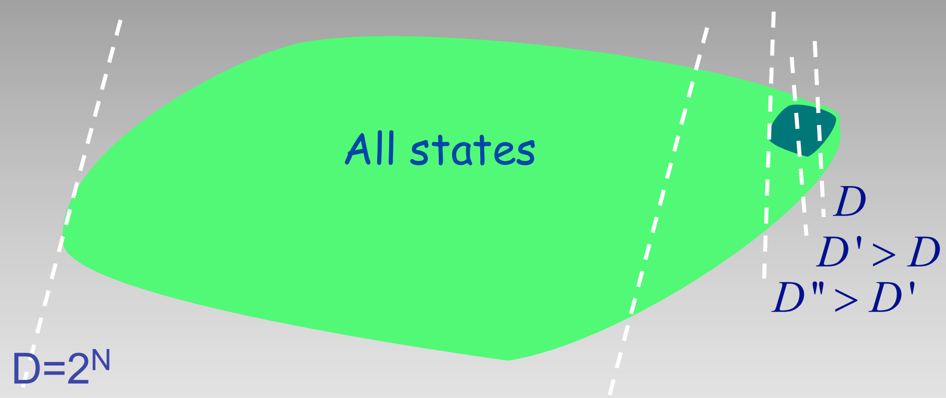


Projected entangled-pair states

- Different representation of states.
- Important quantity: D: Number of parameters characterizing the state.

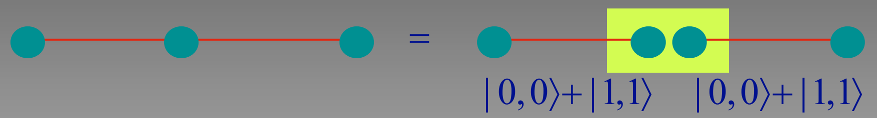


- With relatively small D, one can represent physically relevant states.
- One can determine physical properties in an efficient way.

1. Definition

GHZ states:

$$|\mathrm{GHZ}\rangle = |0,0,0\rangle + |1,1,1\rangle$$

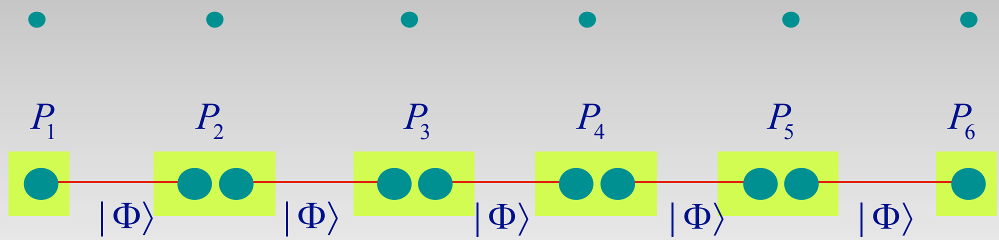


where $P = |0\rangle\langle 0,0| + |1\rangle\langle 1,1|$ maps $C^2 \otimes C^2 \to C^2$

1D states:

$|\Psi\rangle$

as:



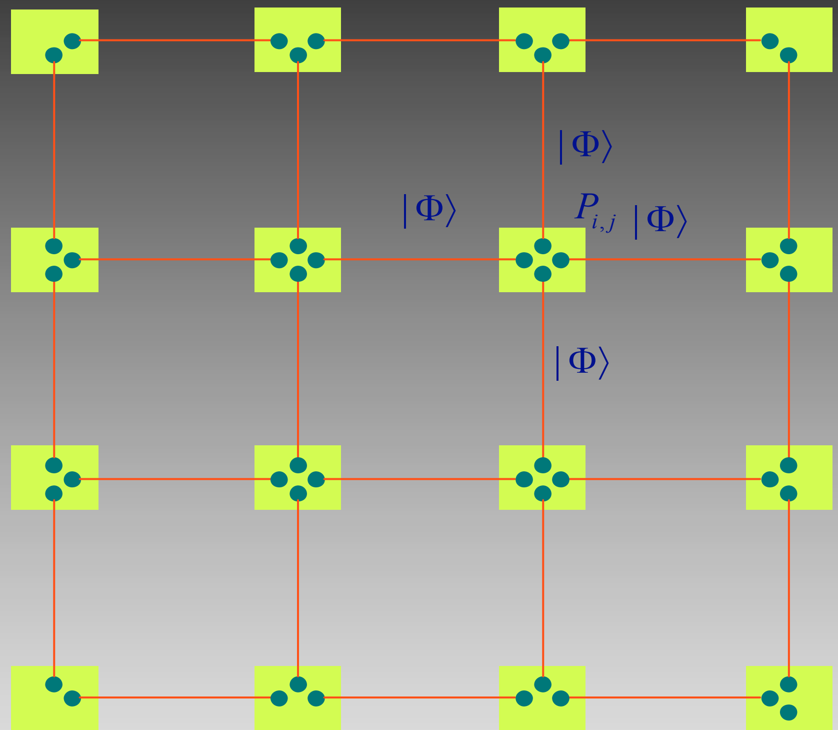
where

- D-dimensional

$$|\Phi\rangle = \sum_{m=1}^{D} |m,m\rangle$$ are maximally entangled states

$$P_k = \sum_{n=1}^{2} |n\rangle\langle \varphi_n^k|$$ maps $C^D \otimes C^D \to C^2$

2D states:

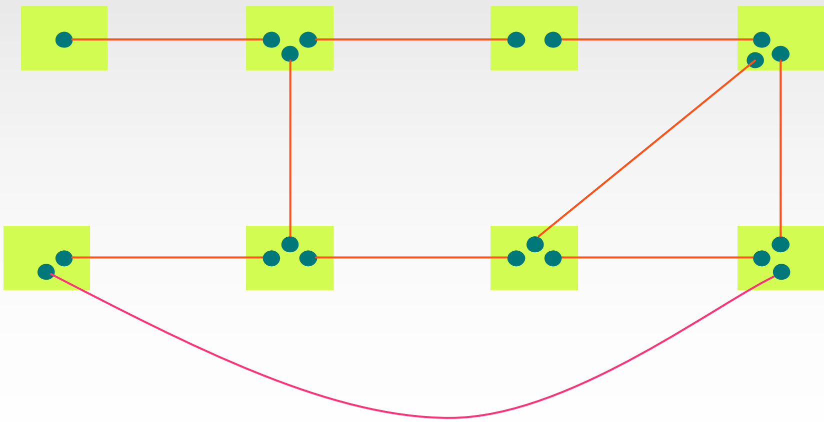


$$P_k = \sum_{n=1}^{2} |n\rangle\langle\varphi_n^k|$$

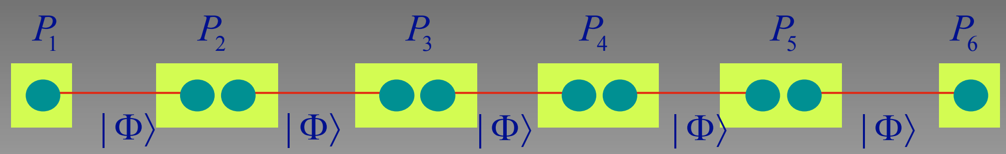
maps

$$C^2 \otimes C^2 \otimes C^2 \otimes C^2 \to C^2$$

General:



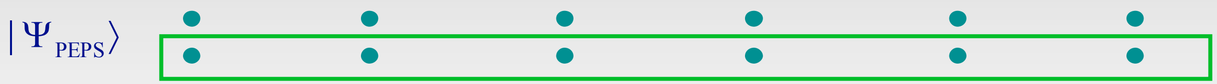
Mixed PEPS



where the P are now Completely Positive Maps

$$P_k : \quad B\left[C^2 \otimes C^2\right] \rightarrow B\left[C^2\right]$$

We can also use purifications

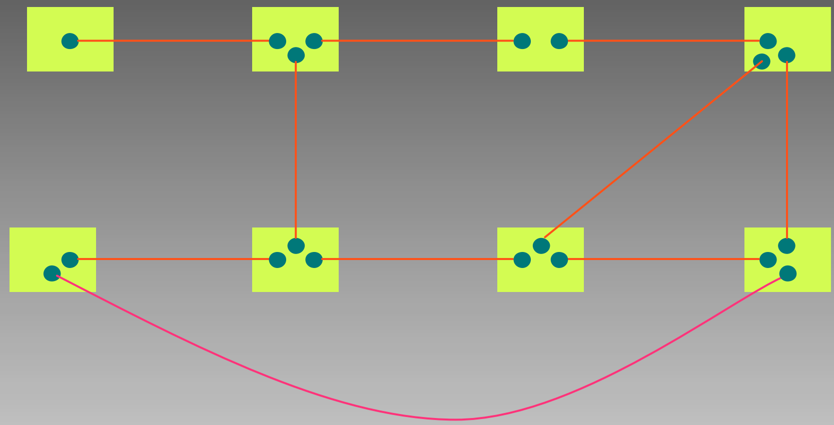


$$\rho = \mathrm{Tr}\left(|\Psi_{\mathrm{PEPS}}\rangle\langle\Psi_{\mathrm{PEPS}}|\right)$$

2. Properties:

- They are complete:

$|\Psi\rangle$



Proof: via teleportation

- They are ground states of local Hamiltonians: $H|\Psi\rangle = E_0|\Psi\rangle$

- They satisfy the area theorem: a requirement for describing physical states.

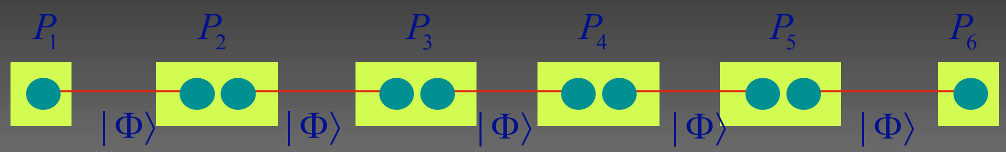
- In 1D they coincide with: Finitely correlated states (T. invariant, Fannes et al) Matrix product states (Römer and Ostlund)

- In 2D they extend FCS and MPS.

(Valence-bond states are a subclass of PEPS).

- Expectation values of observables have a simple form.

PEPS in 1D (OBC)



$$P_k = \sum_{n=1}^{2} |n\rangle\langle\varphi_n^k| = \sum_{i_k=1}^{2} \left[A_k^{i_k}\right]_{\alpha,\beta} |i_k\rangle\langle\alpha,\beta| \quad \Rightarrow \quad |\Psi\rangle = \sum_{i_1,\ldots,i_N=0}^{1} \vec{A}_1^{i_1,T} A_2^{i_2} \ldots \vec{A}_N^{i_N} |i_1,\ldots,i_N\rangle$$

Finitely correlated states (Fannes et al)
Matrix product states (Römer and Ostlund)

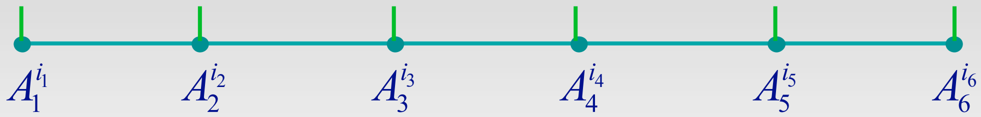
- Correlation functions:

$$\langle\sigma_x^i \sigma_y^j\rangle = \frac{\vec{E}_1^T E_2 \ldots X_i E_{i+1} \ldots Y_j E_{j+1} \ldots \vec{E}_N}{\vec{E}_1^T E_2 \ldots \vec{E}_N}$$

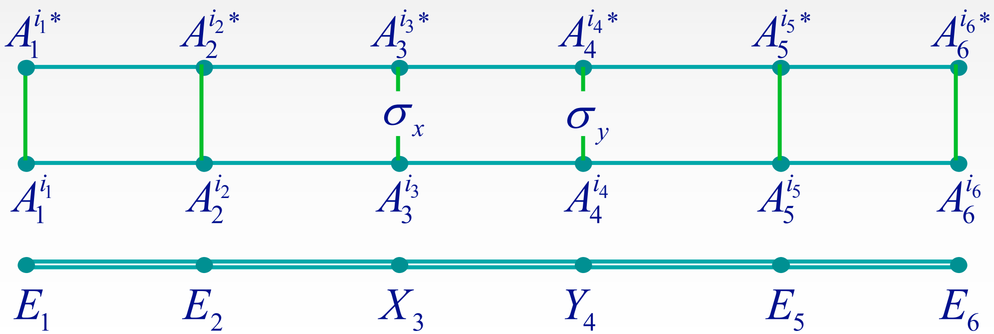
$\nwarrow$ $D^2 \times D^2$ matrices

Representation

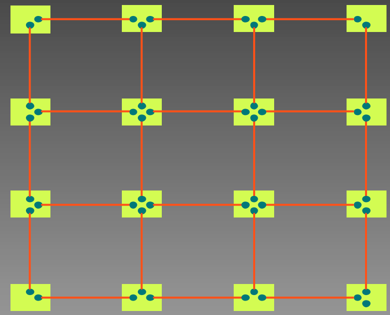
- States:



- Correlation functions:

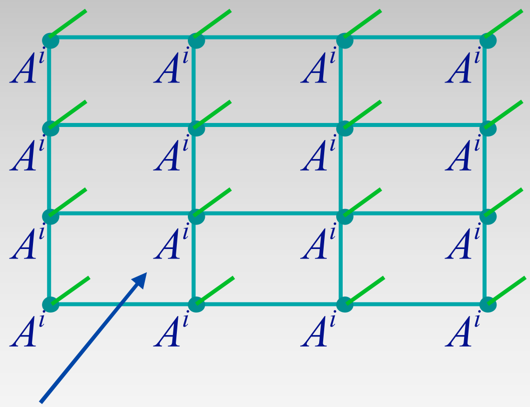


PEPS in 2D



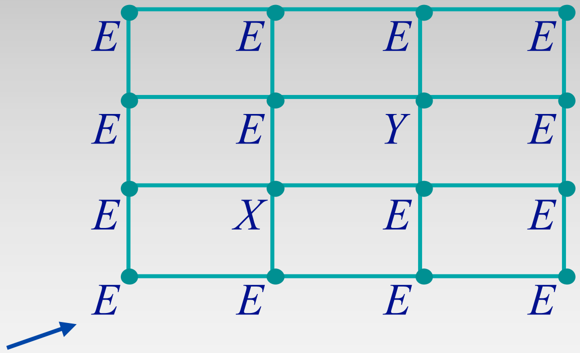
Representation

States:



$D \times D \times D \times D \times 2$ tensor

Correlation functions:



$D^2 \times D^2 \times D^2 \times D^2$ tensor

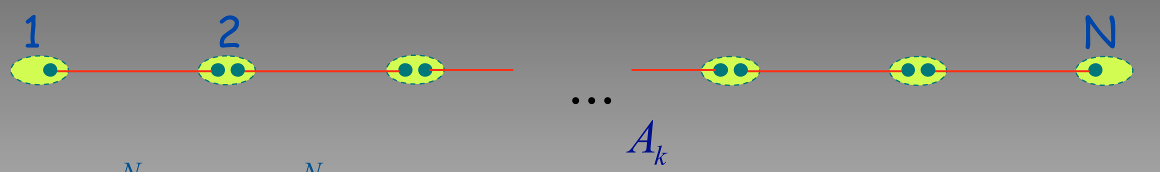
Problem: when contracting, the indices proliferate.

This happens for tensor with more than 2 indices.

$$\sum_\alpha E_{\alpha\beta\gamma\delta} E_{\alpha\varepsilon\kappa\lambda} = M_{\beta\gamma\delta\varepsilon\kappa\lambda}$$

3. Ground state (1D)

IDEA: For a given D, find the optimal A which minimizes the energy.



Hamiltonian: $H = \sum_{x=1}^{N} H_x + \sum_{x,y=1}^{N} H_{x,y} + \ldots$

We want to find $\min_{\Psi \text{ PEPS}} \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \geq E_0$

Procedure:

- Fix all A's except for one: A_k
- Minimize with respect to A_k . The energy is quadratic on the coefficients of A_k

  One has to solve a (generalized) eigenvalue problem

- Iterate.

- The process converges.

Projected entangled-pair states – example: Spin models in random magnetic fields

A. Niederberger, L. Sanchez-Palencia, J. Wehr, and M. Lewenstein, in preparation.

Large effects by arbitraty small disorder

Classical spin model in random magnetic fields:

- Arbitrarily small random field (with the probability distribution respecting the Ising Z2 symmetry) destroys spontanous magnetization in the Ising spin model in 2D (i.e. at the lower critical dimension) at any temperature T.
- In XY spin model in 2D, according to Mermin-Wagner theorem there is no magnetisation at any finite T. Random, symmetrically distributed field of arbitrarily small strength in X direction breaks the continuous O(2) (U(1)) symmetry of the XY model, and prevents, obviously, magnetisation in the X direction. The model attains magnetisation in Y direction at T=0 (for sure) and at finite temperatures (for good?)
- How does quantum effects (quantum fluctuations, transverse fields) change these pictures?

Large effects by arbitrarily small disorder

Quantum spin chains in random magnetic fields:

- Armand Niederberger has applied Vidal's algorithm to the XY spin model in 1D in a random field in the X direction. At T=0 we expect appearance of (decaying algebraically, but very slowly) correlations in the Y direction:

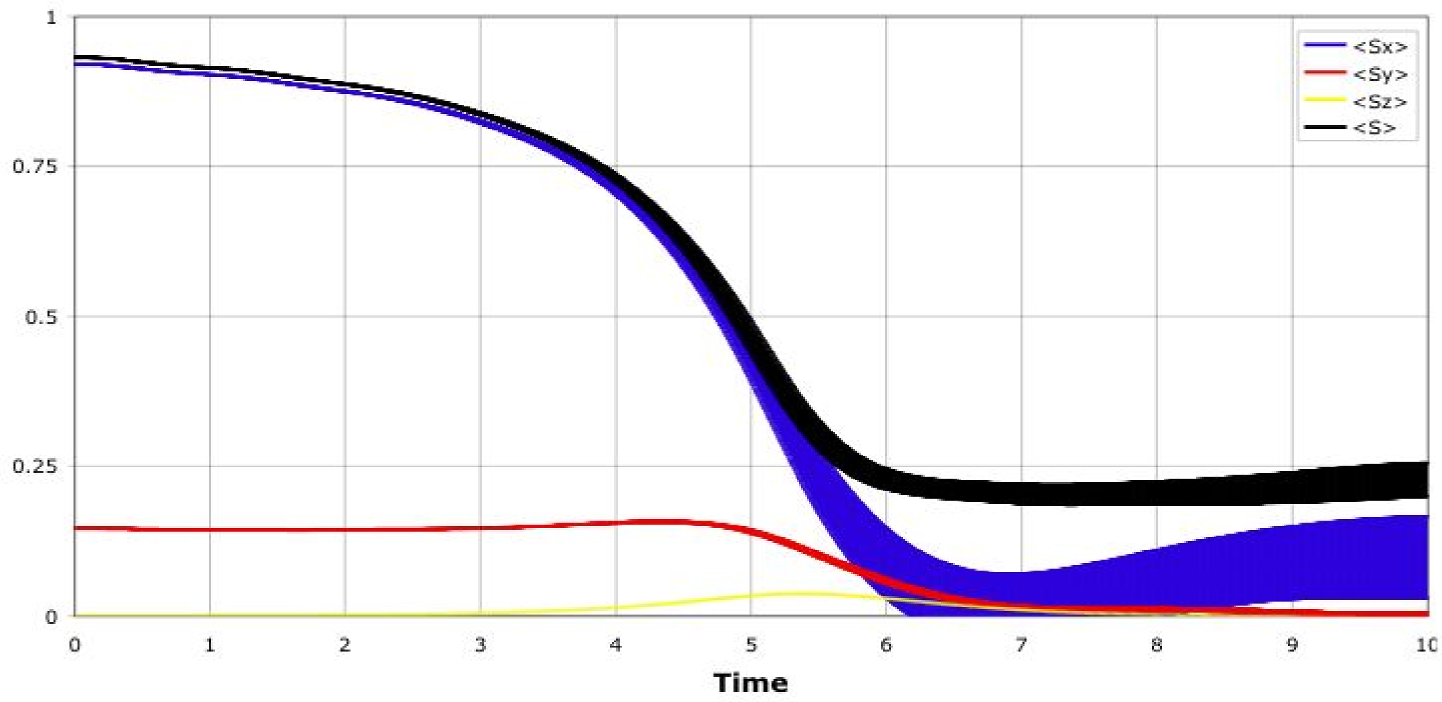
$$H = -J\sum_i \left(s_x^i s_x^{i+1} + s_y^i s_y^{i+1}\right) + \sum_i h_i s_x^i \ ,$$

- Changing X to Z, and Y to X, and performing the Jordan-Wigner transformation leads to a interacting spinless Fermi gas in a random potential (i..e. Thierry knows everything!):

$$H = -J\sum_i \left(2f_i^+ f_i - 1\right)\left(2f_{i+1}^+ f_{i+1} - 1\right) + \sum_i h_i \left(2f_i^+ f_i - 1\right)$$

$$-J\sum_i \left(f_i f_{i+1} + f_{i+1}^+ f_i^+ - f_i^+ f_{i+1} - f_{i+1}^+ f_i\right),$$

1D spin chain with random field



CONCLUSIONS (The Tragedy of Hamlet, by Shakespeare):

- *There are more thing in heaven and earth, Horatio, than are dreamt of in your philosophy.*

Wow!!!

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