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SCHOOL ON QUANTUM PHASE TRANSITIONS
AND
NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

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Introduction to the theory of atomic collisions

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Introduction to the theory of atomic collisions

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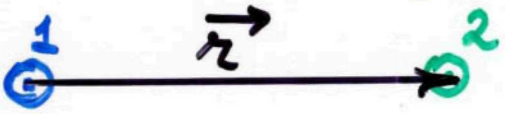
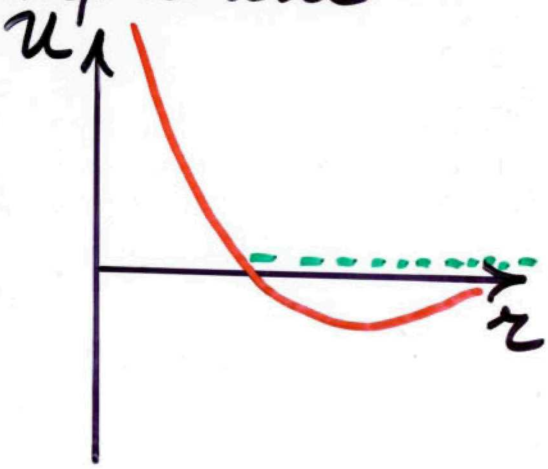
Outline

1. Scattering amplitude and phase
2. Ultracold limit
3. Resonance scattering
4. Wide and narrow Feshbach resonances
5. Interaction between particles
6. Problems

2) Scattering amplitude





$$\left[-\frac{\hbar^2}{2\mu}\Delta + U(r)\right]\psi = \frac{\hbar^2 k^2}{2\mu}\psi$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$r \to \infty$ $$\psi = e^{ikz} + \frac{f(\theta)}{r}e^{ikr}$$

$$P = \frac{|f(\theta)|^2}{r^2} v\, dS$$

$$dS = r^2 dO = 2\pi r^2 \sin\theta\, d\theta$$

$$\frac{P}{v} \equiv d\sigma = |f(\theta)|^2 dO$$

$$f(\theta) = \sum_{l=0}^{\infty} f_l\,(2l+1)\,P_l(\cos\theta)$$

3/ How to find f_ℓ?

$$\psi(r) = \sum_{\ell=0}^{\infty} i^\ell e^{i\delta_\ell} (2\ell+1) \chi_{k\ell}(r) P_\ell(\cos\theta)$$

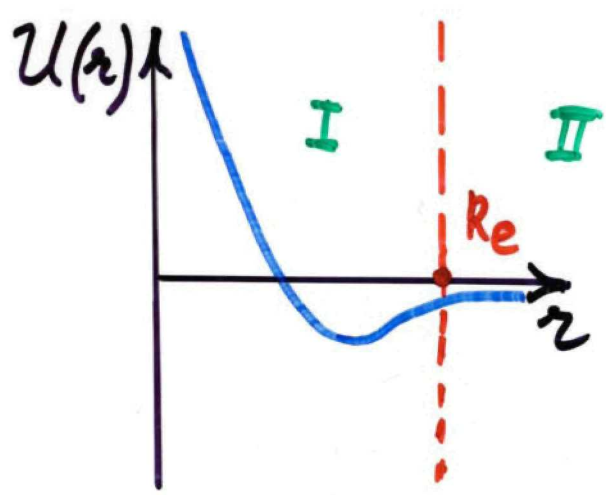
$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right)\chi_{k\ell}(r)$$

$$+ U(r)\chi_{k\ell}(r) = \frac{\hbar^2 k^2}{2\mu}\chi_{k\ell}(r)$$

$$r \to \infty \quad \Rightarrow \quad \chi_{k\ell} = \frac{\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell\right)}{kr}$$

$$f_\ell = \frac{\left(e^{2i\delta_\ell} - 1\right)}{2ik}$$

4)



$r \gg R_e \Rightarrow |U(r)| \ll \frac{\hbar^2}{2\mu r^2}$

$\boxed{k R_e \ll 1}$

I $\qquad \chi_{kl} = C(k)\,\chi_l(r)$

II $\qquad \chi_{kl} = A_1 \, j_l(kr) + A_2 \, \eta_l(kr)$

$r \ll \frac{1}{k}$ $\qquad j_l(kr) \to (kr)^l \qquad \eta_l(kr) \to (kr)^{-l-1}$

$$\frac{A_2}{A_1} \sim k^{2l+1}$$

II $\qquad r \gg \frac{1}{k}$ $\qquad \chi_{kl} = A_1 \frac{\sin(kr - \frac{\pi l}{2})}{kr} + \frac{A_2 \cos(\dots)}{kr}$

$$\to \frac{\sin(kr - \frac{\pi l}{2} + \delta_l)}{kr}$$

$$\frac{A_2}{A_1} \sim \tan \delta_l \sim k^{2l+1}$$

5/ $\quad |\delta_\ell| \ll 1 \Rightarrow f_\ell \sim k^{2\ell}$

$kR_e \ll 1 \Rightarrow$ ultracold limit

$(\ell = 0)$ s-wave scattering dominates

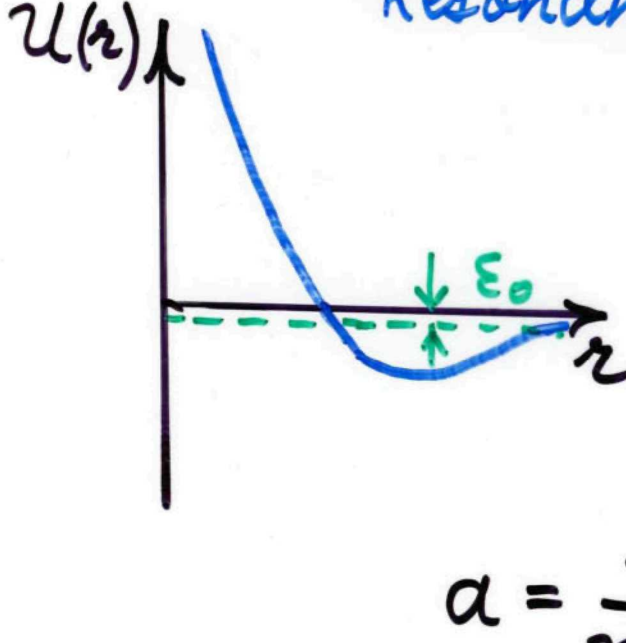
$$f_0 = \text{const} = -a$$

scattering length

$$\delta_0 = -ka$$

$$|\delta_0| \ll 1 \Rightarrow \boxed{k|a| \ll 1}$$

Resonance $\qquad k|a| \gtrsim 1$?



$$\varepsilon_0 = \frac{\hbar^2 \varkappa^2}{2m}$$

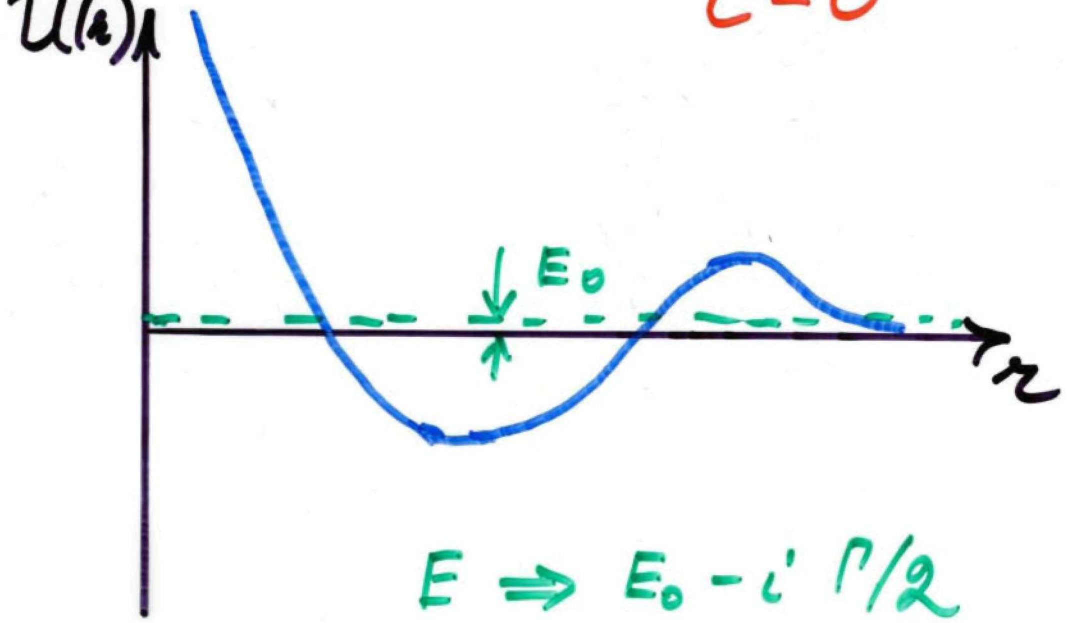
$$f = -\frac{1}{\varkappa + ik}$$

$$a = \frac{1}{\varkappa} \sim \frac{1}{\sqrt{\varepsilon_0}}$$

$$\sigma = \frac{4\pi}{\varkappa^2 + k^2}$$

6/ Resonance on a qusidiscrete level

$\ell = 0$



$$E \Rightarrow E_0 - i\,\Gamma/2$$

$$\chi_0 = \frac{1}{r}\left[B^*(E)e^{ikr} + B(E)e^{-ikr}\right]$$

$E=0 \rightarrow$ singular point

Absence of spherical incoming for $E = E_0 - i\Gamma/2$

$$B\left(E_0 - i\frac{\Gamma}{2}\right) = 0$$

Expansion of B near $E=0$ in $\sqrt{-E}$

$$B \sim \left(E - \mathcal{E}_0 + i\gamma\sqrt{E}\right)$$

$$E_0 = \varepsilon_0 \;;\quad \Gamma = 2\gamma\sqrt{\varepsilon_0}$$

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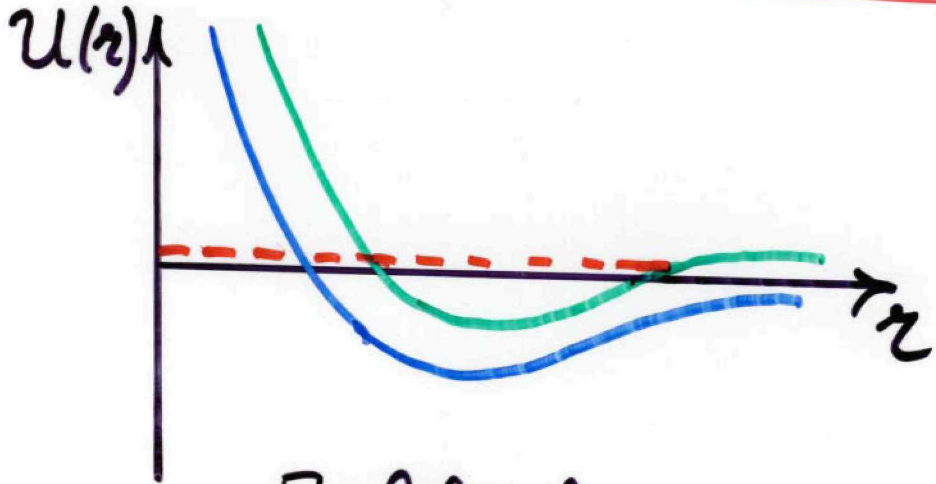
$$\mathrm{tg}\,\delta_0 = -\frac{\gamma\sqrt{E}}{(E-\varepsilon_0)}$$

$$f_0 = -\frac{\gamma\hbar}{\sqrt{2\mu}}\cdot\frac{1}{(E-\varepsilon_0+i\gamma\sqrt{E})}$$

$K \to 0 \qquad f_0 \to \mathrm{const}$

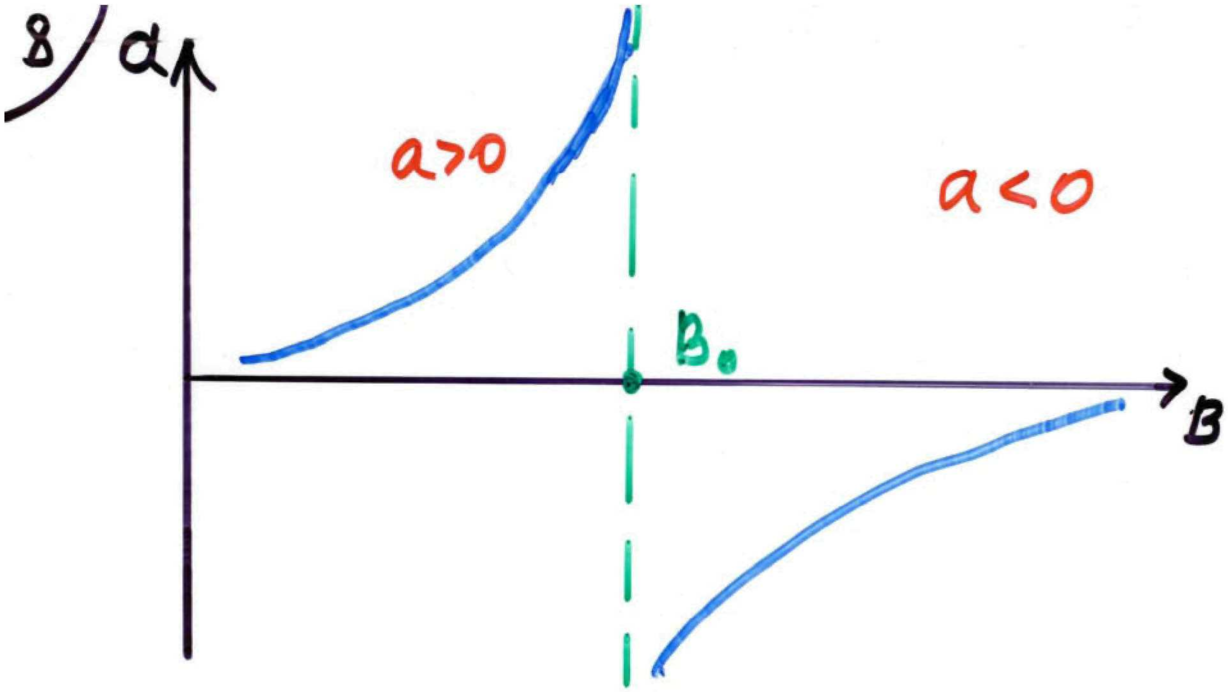
$$\boxed{f_0 = -\frac{1}{\frac{1}{a}+k^2R_* + ik}}$$



Feshbach resonance

Breit-Wigner, Fano, Feshbach

Cold atomic collisions ⇒ Verhaar (1993)



$$R_* \sim \frac{1}{\gamma}$$

$kR_* \ll 1 \rightarrow$ Wide resonance

$$f_0 = -\frac{1}{\frac{1}{a} + ik}$$

$|a| \rightarrow \infty \Rightarrow$ universality

$a > 0$ and $R_* \ll a$

$k = i\varkappa$ Pole of the scattering amplitude

$$\varkappa = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_*}{a}}\right) \cdot \frac{1}{R_*}$$

$$\varkappa = \frac{1}{a}\,; \qquad \varepsilon_0 = -\frac{\hbar^2}{2ma^2}\,;$$

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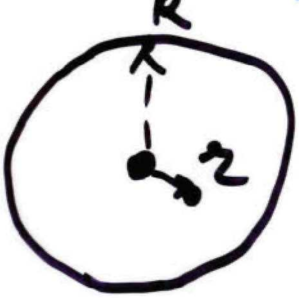
Interaction between particles in a gas



$$l = 0$$

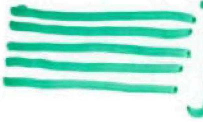
if $U(r) = 0$ then

$$\psi = \frac{\sin kr}{kr} \; ; \quad \boxed{kR = \pi j'}$$

if $U(r) \neq 0$ then

$$\psi = \frac{\sin(\tilde{k}r + \delta_0)}{\tilde{k}r} \; ; \quad \tilde{k}R + \delta_0 = \pi j'$$

$$\Delta E_{j'} = \frac{\hbar^2}{2m}\left(\tilde{k}_{j'}^2 - k_{j'}^2\right) = -\frac{\hbar^2}{m}\frac{k_{j'}^2}{R} \cdot \left(\frac{\delta_0}{k_{j'}}\right)$$



$\}dj'$

$$\sum_{j'} N_{j'} \cdot \Delta E_{j'} = \int dj' N_{j'} \cdot \Delta E(k_{j'})$$

$$= \int \frac{R}{\pi} dk \cdot \Delta E(k) N_k =$$

$$= -\int \frac{2\pi \hbar^2}{mV} \cdot \frac{\delta_0(k)}{k} \cdot N_k \frac{4\pi k^2 dk V}{(2\pi)^3}$$

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$$\Delta E_K = \frac{g_K}{V}; \quad g_K = -\frac{2\pi\hbar^2}{\mu} \cdot \frac{\delta_0}{K}$$

$$K|a| \ll 1 \Rightarrow g = \frac{2\pi\hbar^2}{\mu} \cdot a$$

N_1 ⊙ ⊙ N_2 binary approach

$$E_{int} = N_1 \cdot N_2 \cdot \left(\frac{g}{V}\right)$$

identical particles ($\mu = \frac{m}{2}$)

$$E_{int} = \frac{N^2}{2} \cdot \left(\frac{g}{V}\right)$$

$$\frac{\partial E}{\partial N} = ng = \frac{4\pi\hbar^2}{m} a n$$

Bogoliubov formula

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binary approach in a gas

$|a| \ll n^{-1/3}$ $\quad$ $n|a|^3 \ll 1$

or

$\Lambda \sim \frac{1}{K} \ll n^{-1/3}$ $\quad$ $n\Lambda^3 \ll 1$

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What to do in the strongly interacting regime? Which problems are solvable?

1) High-temperature regime
   $n\Lambda^3 \ll 1$ (Bet-Ulenbeck)

2) Heavy impurity in a metal
   (Fumi)

3) Few-body problems
   3 particles in a harmonic trap

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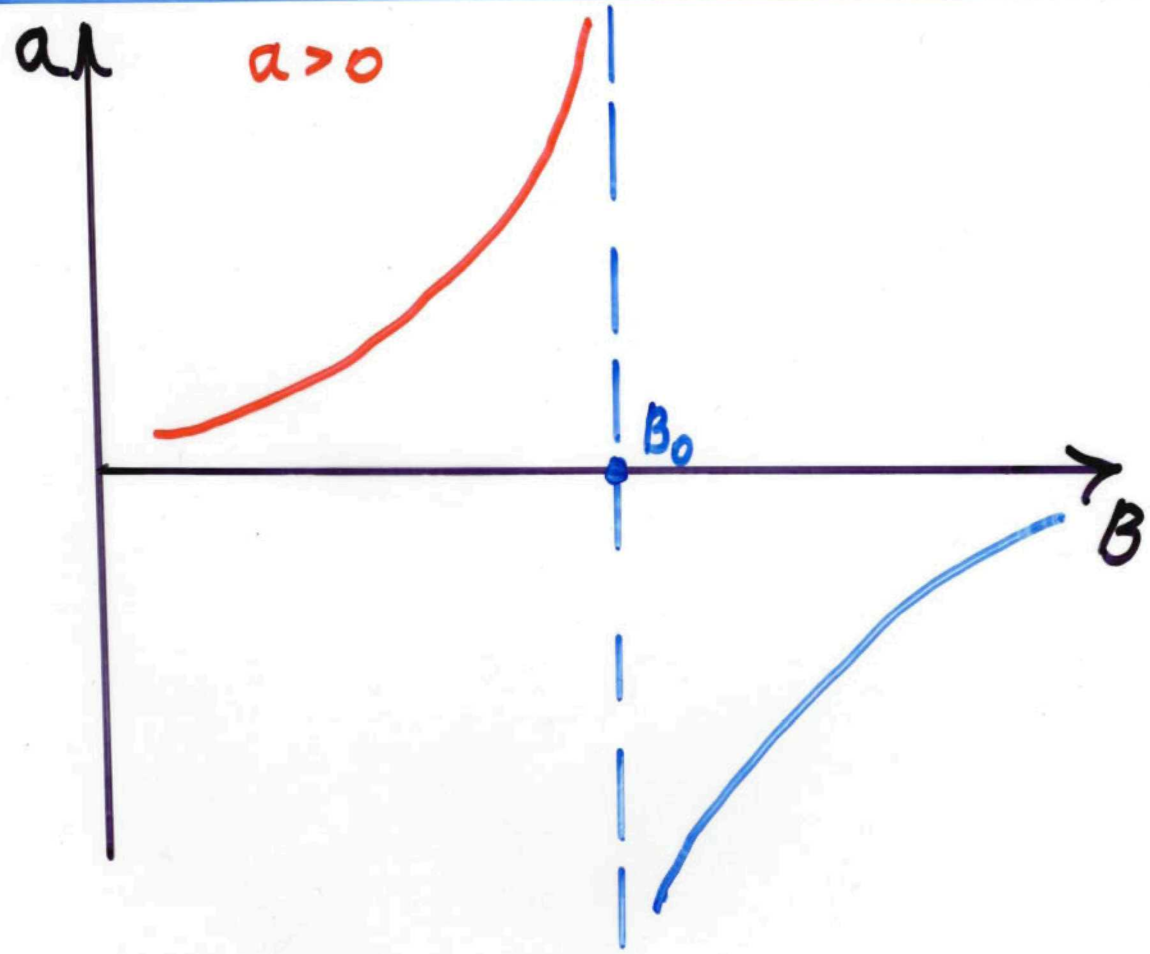
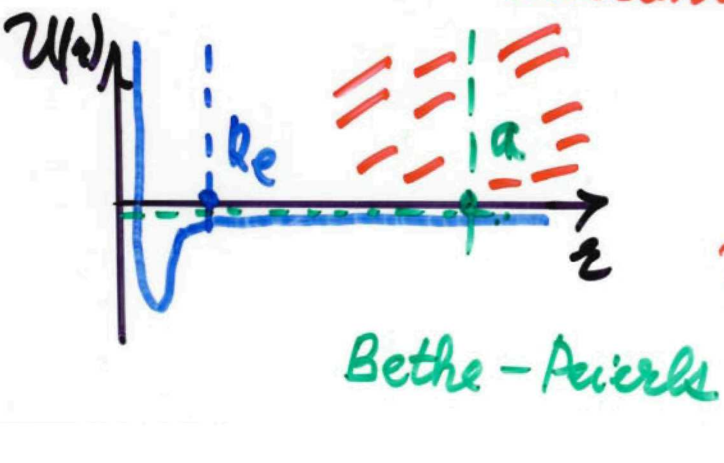
Other problems?

Atom-molecule and molecule-molecule interactions



$a \gg R_e$ weakly bound diatomic molecules



$r \to 0$

$\Psi' \sim \left(\frac{1}{r} - \frac{1}{a}\right)$

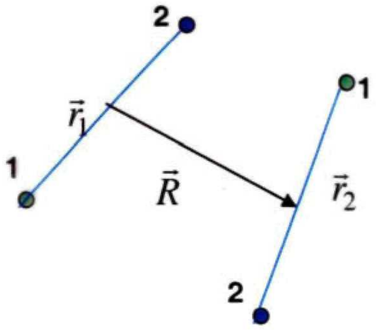
III Weakly interacting gas of bosonic dimers

Elastic interaction BEC stability

"Old answer" $\longrightarrow$ $2a$

4-body problem Exact solution for $a >> R_e$

Petrov et al, 2003



$\Psi \longrightarrow$ *9 variables*

Zero-range approximation

$\Psi \longrightarrow f(\vec{r}_2, \vec{R})(1/4\pi r_1 - 1/4\pi a)$

for $r_1 \to 0$

Integral equation for f

$k \longrightarrow 0$ *s*-wave scattering; *3* variables

$R \to \infty$
$$\Psi = \varphi_0(r_1)\varphi_0(r_2)(1 - a_{dd}/R)$$

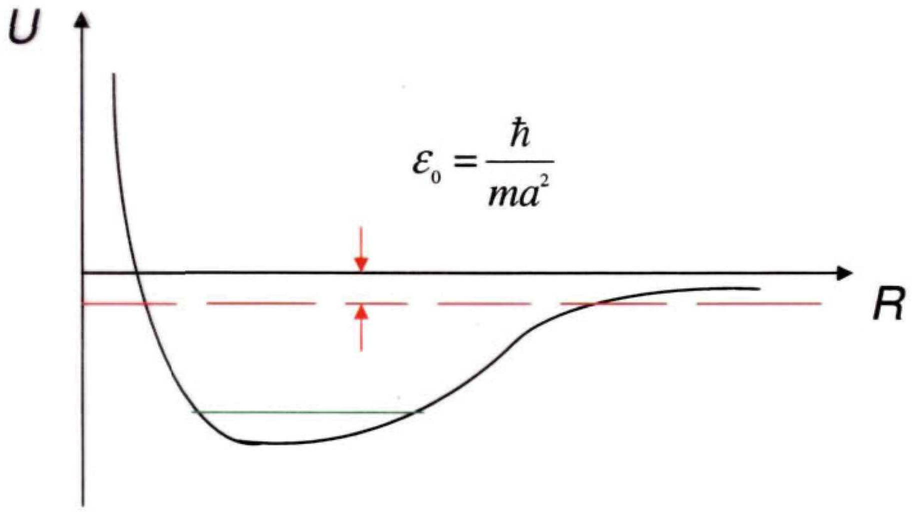
$$\varphi_0(r) = \frac{1}{r\sqrt{2\pi a}} \exp(-r/a)$$

$R \to \infty$
$$f(\vec{r}, \vec{R}) = \frac{2}{rR} \exp(-r/a)\,(1 - a_{dd}/R)$$

$$a_{dd} = 0.6\, a$$

Monte Carlo Giorgini/Astracharchik *2004*

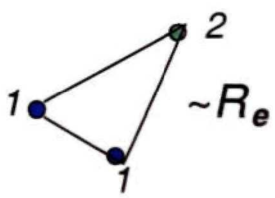
Weakly bound dimers $\longrightarrow$ The highest rovibrational state of the diatomic molecule



Collisional relaxation to deep bound states
(~1ms for Rb_2 at $n \sim 10^{13}$ cm^{-3})

Atom-dimer collisions ($a \gg R_e$)

Size $\longrightarrow$ Weakly bound dimer ~ a
Deep bound state ~ R_e(50 Å) $\ll a$

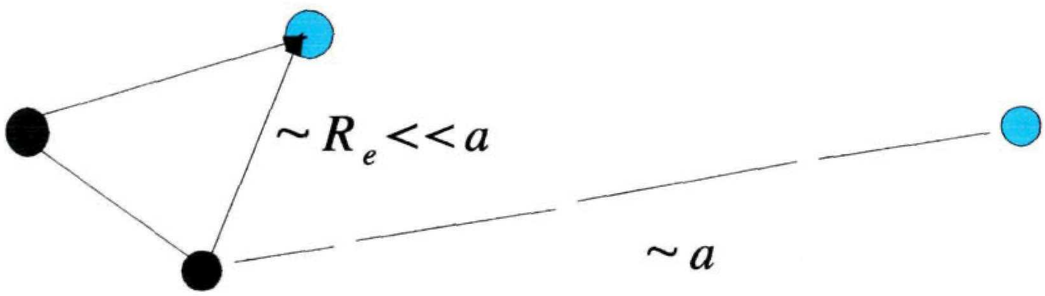


2 particles are identical fermions

Pauli principle

$$\alpha_{rel} \sim (k_{eff} R_e)^{2?} \sim (R_e / a)^{2?}$$

Molecule-molecule collisions

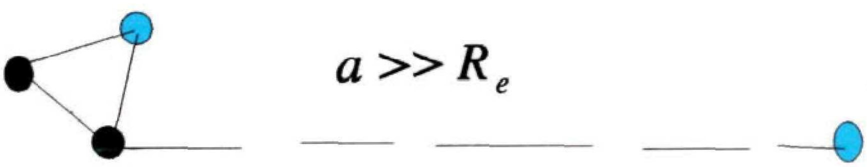


$$m = M \qquad \alpha_{rel} = C\frac{\hbar R_e}{m}\left(\frac{R_e}{a}\right)^s; \qquad s = 2.55$$

Petrov et al 2003

$$\tau \sim \frac{1}{\alpha_{rel} n} \sim \qquad \text{seconds}$$

Molecules of bosonic atoms



Resonance enhancement $\alpha_{rel} \sim \frac{\hbar}{m} a \qquad \tau < 1\text{ms}$

Suppressed collisional relaxation

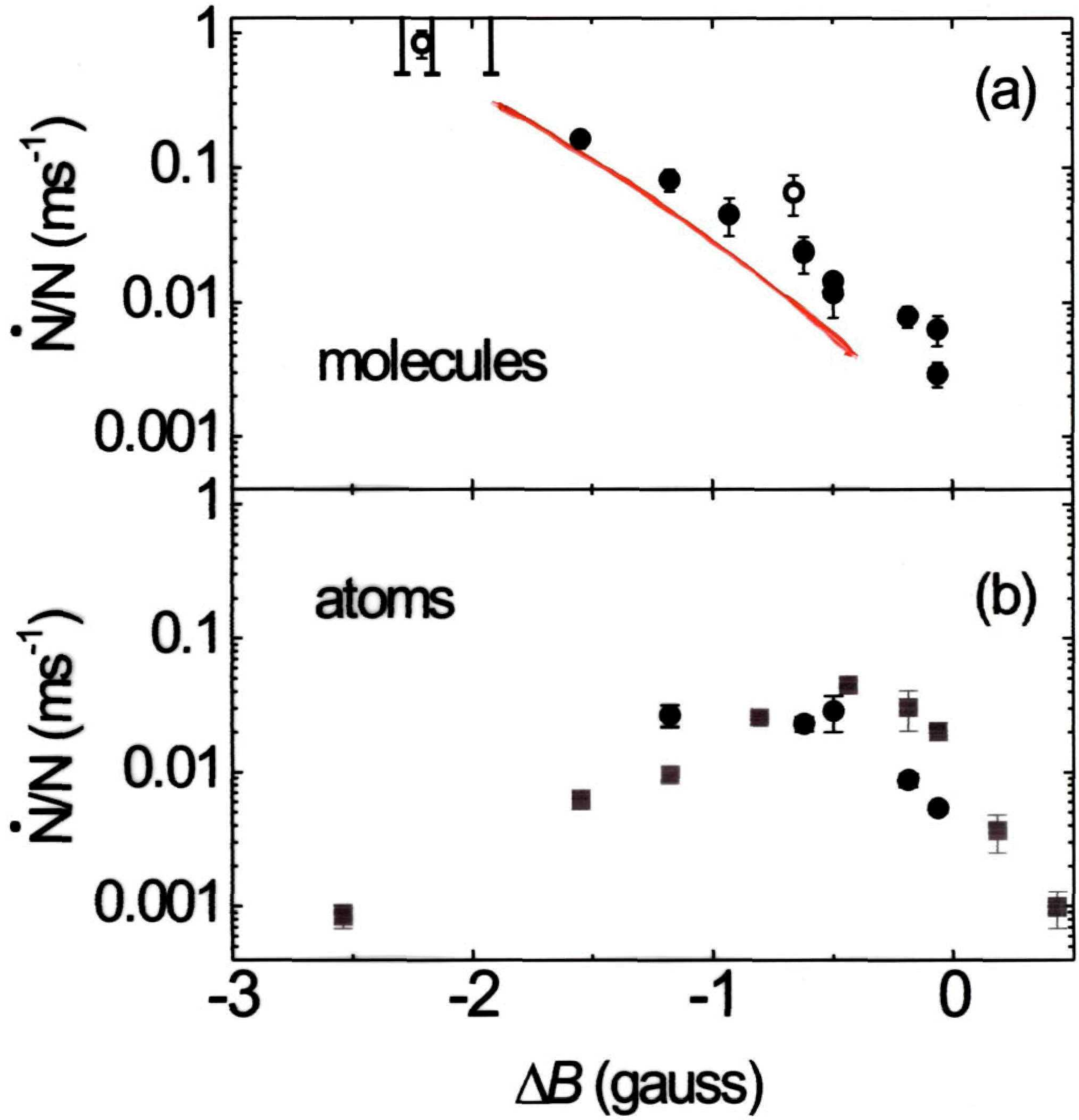
Fast elastic collisions $\longrightarrow$ $a_{dd} = 0.6a$

$$^6\mathrm{Li}_2 \rightarrow \frac{\alpha_{rel}}{\alpha_{el}} \leq 10^{-4}$$

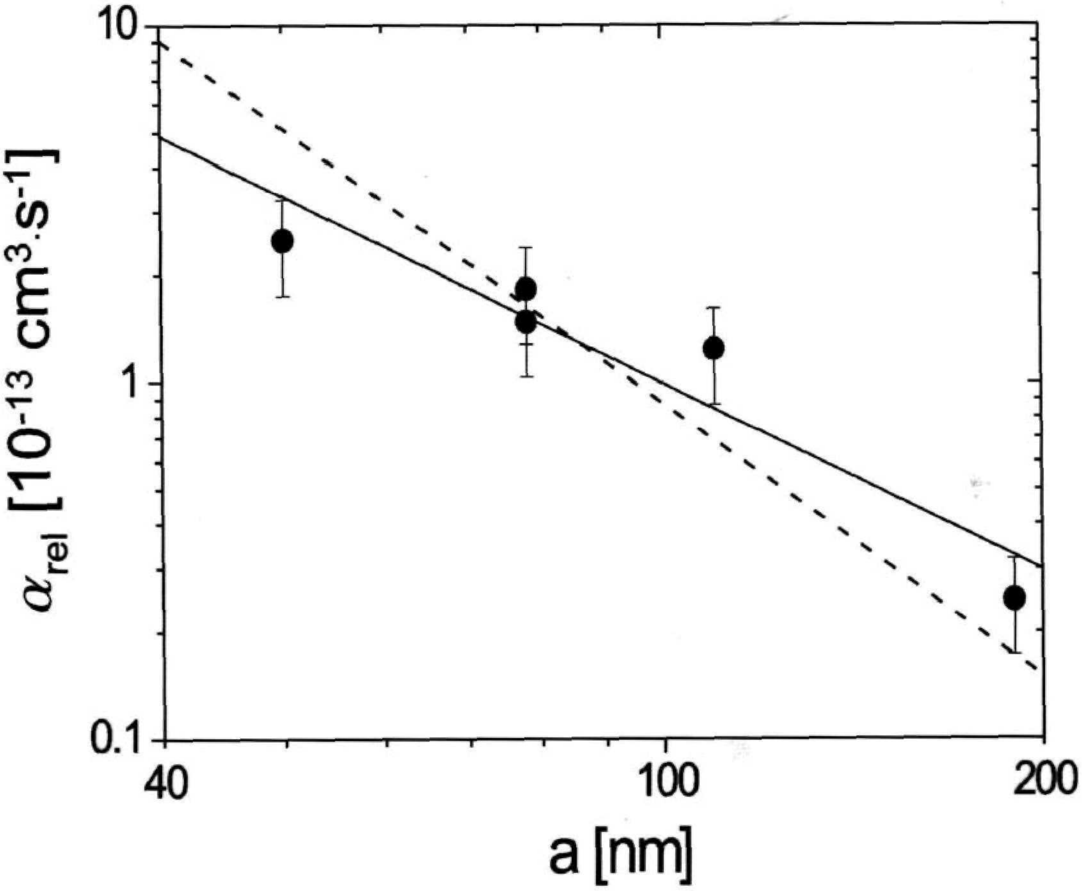
Efficient evaporative cooling $\longrightarrow$ BEC

JILA, Innsbruck, MIT

ENS, Rice



1
0.1
0.01
0.001
(a)
molecules
1
0.1
0.01
0.001
atoms
(b)
Ṅ/N (ms⁻¹)
Ṅ/N (ms⁻¹)
-3
-2
-1
0
ΔB (gauss)



10
1
0.1
40
100
200
a [nm]
α_{rel} [10^{-13} cm^3·s^{-1}]