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**SCHOOL ON QUANTUM PHASE TRANSITIONS
AND
NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES**

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BCS - BEC Crossover: An introduction

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BCS-BEC CROSSOVER: AN INTRODUCTION

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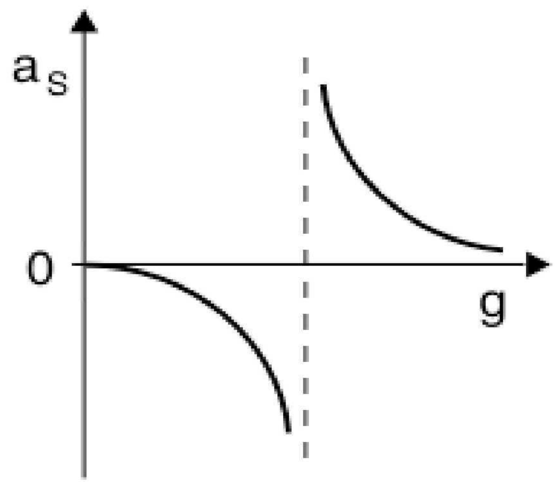
problem: Fermions $\uparrow\downarrow$ with density $n = k_F^3/3\pi^2$ and

attractive two particle interaction $V_{\uparrow\downarrow}(\vec{x}) = -\tilde{g} \cdot \delta(\vec{x})$

two-particle bound state : $g > g_c$

binding energy $\epsilon_b = \hbar^2/ma_s^2$



motivation: ultracold gases ^{6}Li, ^{40}K, $T_c \approx 10^{-7}$ K

idea goes back to H. Stoof et.al. PRL **76**, 10 (1996)

Outline: 1) BCS-theory, $T = 0$

2) Ginzburg-Landau theory, $T \approx T_c$

3) Unitarity limit and universality

4) Conclusion and open basic problems

Superfluidity of Fermions

1911 conventional SC's Hg, Al, ... $T_c \approx 1 - 23\,K$

1960 pairing in nuclei, $\Delta \approx MeV$

1972 superfluid ^3He, $T_c = 1\,mK$ (p-wave)

1975 neutron stars

1986 high-T_c SC's $La_2Cu\,O_4$, ... $T_c = 35 - 138$ K

1991 Alkali-doped C_{60} $T_c \approx 30$ K

1994 p-wave SC's in $Sr_2Ru\,O_4$ $T_c = 1.5$ K

1998 color SC's $\langle qq \rangle \neq 0$, $\Delta \approx GeV$

1) BCS-THEORY (1957) (only $T = 0$)

basic idea: Fermi-sea is unstable towards formation of

Cooper-pairs for arbitrary weak attraction

$$|\psi\rangle_C = \sum_{k>k_F} \frac{1}{2\epsilon_k - E} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |F\rangle$$

exp. small binding $\quad E = 2\epsilon_F - 2\hbar\omega_c \exp - \frac{\pi}{k_F|a_s|}$

BCS groundstate (variational Ansatz for arbitrary g)

$$\psi_{\mathsf{BCS},N} = \hat{A}\,\{\underbrace{\phi(1\,2)\,\phi(3\,4)\cdots\phi(N-1\,N)}\}$$

ideal Bose gas of identical pairs

pair-state $\quad \phi(1,2) = \psi(\vec{x})\cdot\chi(\vec{r})\cdot\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

internal wavefunction $\chi(\vec{r}) \sim \exp -r/\xi_b \rightarrow$ pair radius ξ_b

COM-coord. $\vec{x} \;\rightarrow$ inhomogeneous pair-amplitude $\psi(\vec{x})$

increase g: extended $\rightarrow$ local pairs $\;\rightarrow\;$ smooth crossover

(Keldysh 1965, Eagles 1969, Leggett 1980) model Hamiltonian: separable interaction between

Fermions of opposite spin (Gorkov)

$$\hat{H} - \mu \hat{N} = \sum_{k\,\sigma} (\epsilon_k - \mu)\, c^\dagger_{k\,\sigma}\, c_{k\,\sigma} - \frac{\tilde{g}}{V} \sum_q b^\dagger_q\, b_q$$

pair-operators $\quad b^\dagger_q = \sum_k c^\dagger_{k+q\,\downarrow}\, c^\dagger_{-k\,\uparrow}$

BCS Hamiltonian $\hat{H}_{\mathrm{BCS}} = -\frac{\tilde{g}}{V}\, b_0^\dagger b_0$ (only $q = 0$ pairs)

exact ground state for fixed μ is a coherent state

$$|\psi\rangle_{\mathrm{BCS}} = \prod_k (u_k|00\rangle_k + v_k|11\rangle_k) \sim \exp \sum_k \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |0\rangle$$

at fixed N (even): $|\psi_N\rangle_{\mathrm{BCS}} \sim \left(\sum_k \frac{v_k}{u_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right)^{N/2} |0\rangle$

$\chi_k = v_k/u_k$ = Fouriertransf. of internal wavefunction gap parameter Δ_k follows from gap-equation

$$\frac{1}{\tilde{g}} = \frac{1}{V}\sum_k \frac{1}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}}$$

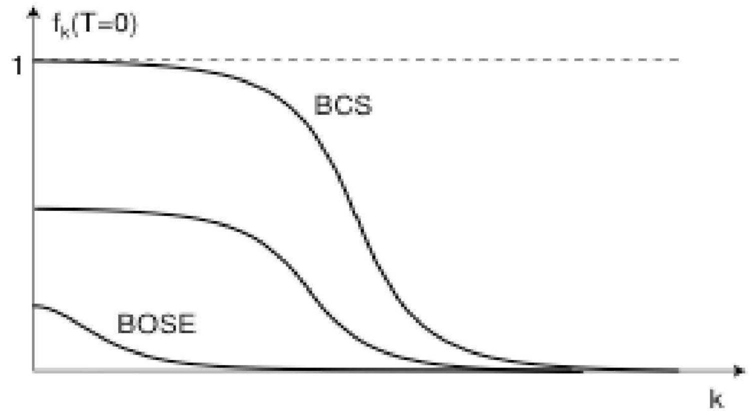
chemical potential μ from number equation ($\mu_{\mathrm{BCS}} = \epsilon_F$)

$$\langle N \rangle = 2\sum_k v_k^2 = \sum_k \left(1 - \frac{\epsilon_k - \mu}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}}\right)$$

momentum distrib.

$$f_k = v_k^2 \rightarrow \frac{4\pi n \xi_b^3}{1 + (k\xi_b)^2}$$

since $\mu_{\mathrm{BEC}} \rightarrow -\epsilon_b/2$



exact solution of $\hat{H}_{\text{BCS}}$ at $T = 0$ and fixed N

Richardson 1963, Ortiz/Dukelsky cond-mat/0503664

are Cooper pairs Bosons ?

antisymmetrization $\hat{A}$ reduces the condensate fraction N_0/N to $\approx \Delta_0/\epsilon_F \ll 1$ in weak coupling, nevertheless

BCS-superfluidity is BEC of pairs !

BCS-LIMIT: weak coupling if $k_F|a_s| < 0.5$ (in practice)

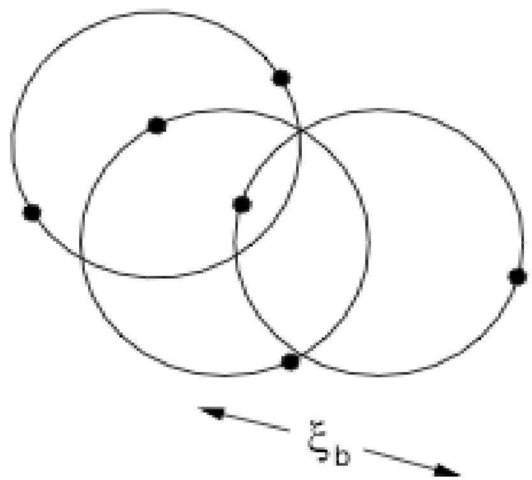
pairs form and condense at

$T_c \approx 0.3\, T_F \exp - \frac{\pi}{2k_F|a_s|} \ll T_F$ $\quad 2\Delta_0 = 3.52\, k_B T_c$

superfluidity is destroyed by pair- breaking

pair size $\quad \xi_b = \xi_0 = \frac{\hbar v_F}{\pi \Delta_0}$ is large $\quad \boxed{k_F \xi_b \gg 1}$

$$k_F \xi_0 \approx \begin{cases} 10^3 & \text{supercond.} \\ 10^2 & {}^3\text{He} \\ 10^1 & \text{high-}T_c \end{cases}$$



BCS-transition in a harmonic trap

local density approx. $\xi_b \ll R_{TF} \quad (\Delta_0 \gg \hbar\omega)$ valid

if $N \gg N^* \approx \exp 3\pi/2k_F|a_s| = 10^5$ at $k_F|a_s| = 0.4$

$N \ll N^*$: pairs formed within a single shell $\Delta_0 \ll \hbar\omega$

as in atomic nuclei (Bruun, Heiselberg, Mottelson 02)

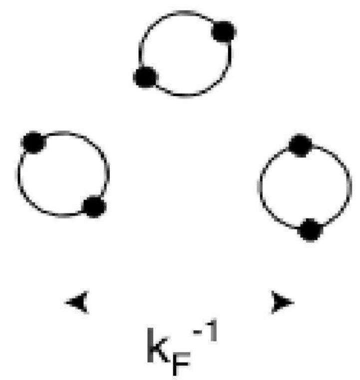
BEC limit: strong coupling $|\epsilon_b| \gg \varepsilon_F \rightarrow k_F a_s \ll 1$

point bosons $\xi_b \approx a_s \rightarrow 0$ form far above T_c

gap parameter $\Delta_0 \rightarrow \sqrt{2|\epsilon_b|\epsilon_F} \gg \epsilon_F$ is irrelevant

pair size $\boxed{k_F \xi_b \ll 1}$

$\xi_b \neq$ coherence length ξ_0



condensation temperature $T_c = 0.218\, T_F \sim n^{2/3}$

from ideal Bose gas with $n_B = n/2$ and $m_B = 2m$

BEC is destroyed by excitations to finite momenta $\rightarrow$

BCS-theory does not describe crossover at finite T !

still, T_c increases monotonically (?) with coupling:

Nozieres/Schmitt-Rink JLTP **59**, 195 (1985)

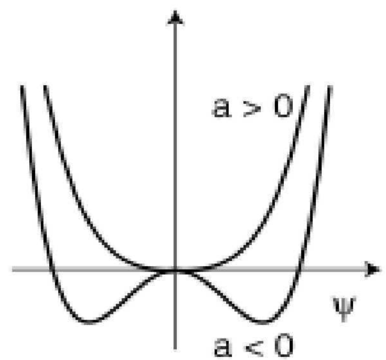
2) Ginzburg–Landau theory

Drechsler/Zw. Ann. Physik **1**, 15, 1992, $d = 2$

Randeria et.al. PRL **71**, 3202, 1993, $d = 3$

idea: condensation is associated with broken gauge symmetry $\rightarrow$ complex order parameter ψ

$$F[\psi] = \int d^d x \left[a\,|\psi|^2 + \frac{b}{2}|\psi|^4 + c\,|\vec{\nabla}\psi|^2 \right]$$



Thermodynamics $Z = \text{Tr}\ \exp -\beta(\hat{H} - \mu\hat{N})$

linearize $\sum_q b_q^\dagger b_q$ via Hubbard-Stratonovich $\rightarrow$

slowly varying order parameter $\psi(\vec{x}, \tau)$

$$Z = Z_0 \int D\psi(\vec{x}, \tau)\ \exp\left(-F[\psi]\right)$$

expansion up to ψ^4 near T_c

quantum GL-functional $\qquad F[\psi] =$

$$\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int d^d x \left[-\mu^\star |\psi|^2 + \frac{\hbar^2}{2m^\star} |\vec{\nabla}\psi|^2 + \frac{g^\star}{2} |\psi|^4 + \hbar\, \psi^\star \partial_\tau \psi \right]$$

GL-theory is coherent state representation of

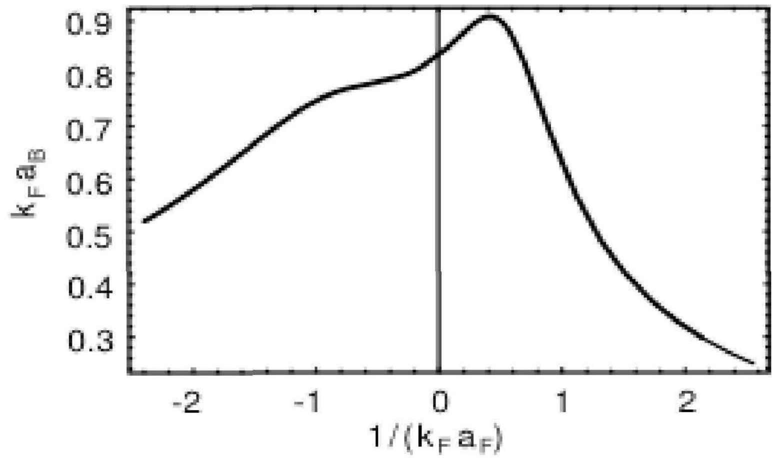
interacting Bosons, smooth crossover from Gorkov's

BCS GL-functional to Gross-Pitaevski for BEC's

$g^\star = 4\pi\hbar^2 a_B/m^\star$ defines dimer scatt. length a_B

Born approximation

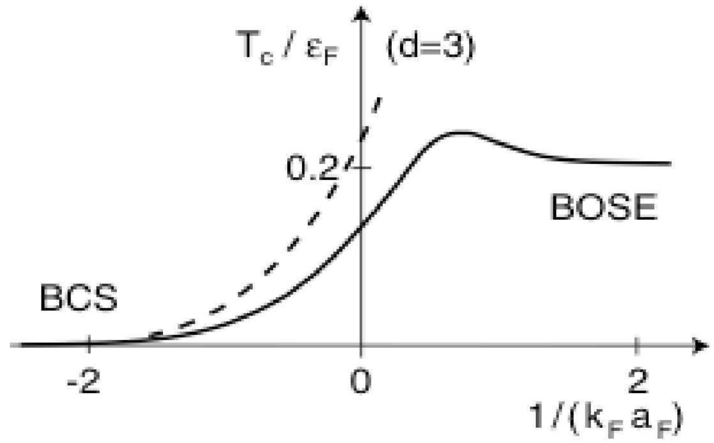
$$a_B = \begin{cases} 6|a_F| & \text{BCS} \\ 2a_F & \text{Bose} \end{cases}$$



Pauli principle gives statistical, repulsive inter-action between composite Bosons critical temperature

in Gaussian approx.

(Thouless criterion)



BCS-limit: density int. $T_c = T_c^{(0)}/2.2$ (Gorkov 1961)

BEC-limit: dilute Bose-gas $\Delta T_c/T_0 \approx 1.3\, n_B^{1/3} a_B > 0$!

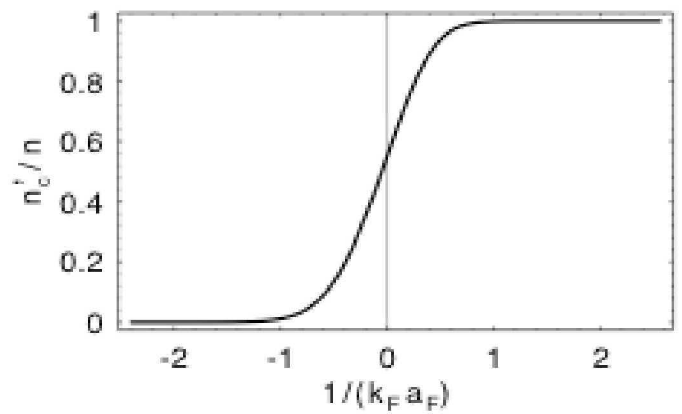
exact result $a_B = 0.6\, a_F$ (Petrov/Shlyap./Salomon 2003) pair formation is just a matter of shifting the

equilibrium from $\bullet + \bullet$ to $\bullet\bullet$ it coincides

with the superfluid transition only in weak coupling

preformed pairs

at condensation



nature of the transition changes in a narrow

range of coupling !

optimally doped high-T_c superconductors

determine $\xi(0)$ from $\left.\dfrac{dH_{c2}}{d\ln T}\right|_{T_c} = \dfrac{\phi_0}{2\pi\xi^2(0)}$ gives

$$k_F\,\xi(0) \approx \begin{cases} 5 & \text{YBCO} \\ 8 & \text{BSCCO} \end{cases}$$

interaction strength g determines

$$\rightarrow \quad \kappa = \frac{\lambda}{\xi} \approx \begin{cases} 99 & \text{YBCO} \\ 87 & \text{BSCCO} \end{cases} \qquad \text{as observed}$$

conclusion optimally doped high-T_c's are far from

weak coupling but still on the BCS-side

(Stintzing/ Zw. PR **B56**, 9004 (1997))

pseudogap in underdoped high-T_c's is not related

to the BCS-BEC crossover (precursor fluctuations)

3) Unitarity limit and universality

interacting Fermions with scattering amplitude

$$f(k) = \frac{-1}{a^{-1} + r^* k^2/2 + ik} \to i/k$$

if $k_F r^* \ll 1$ broad Feshbach resonance

$\to$ $k_F \sim n^{1/3}$ and ϵ_F are the only scales left $\to$

universal thermodynamics $F(T)/N = \epsilon_F \cdot a(T/T_F)$

(e.g. specific heat, expansion energy,... Thomas 2004)

and dynamics, e.g. viscosity $\eta \approx 0.1\hbar n$ (Shuryak 2004)

universal numbers at resonance $(k_F a)^{-1} = 0$

$\epsilon_{int} = -0.34\epsilon_F,\ \ \Delta_0 \approx 0.5\epsilon_F,\ \ T_c \approx 0.1\epsilon_F,\ \ v_s = \ldots$

equation of state at $T = 0$ from QMC

(Carlson et.al. PRL **91**, Astrakharchik et.al. PRL **93**)

4) Conclusion and open basic problems

1) BCS–BEC is a smooth crossover for s-waves,
   quantitative results exist only asymptotically

2) Exact solution for arb. coupling is possible only for
   $\hat{H}_{\mathrm{BCS}}$ or in 1D (Fuchs/Recati/Zw. PRL **93**)

3) quantitative theory at all $0 < T < T_c$

4) interacting fermions at the unitarity limit

5) p-wave crossover $\longrightarrow$ quantum phase transition