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SCHOOL ON QUANTUM PHASE TRANSITIONS AND NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

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Low-dimensional trapped gases

Presented by:

Georgy V. Shylapnikov

L.P.T.M.S., Orsay, France
University of Amsterdam

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Low-dimensional trapped gases

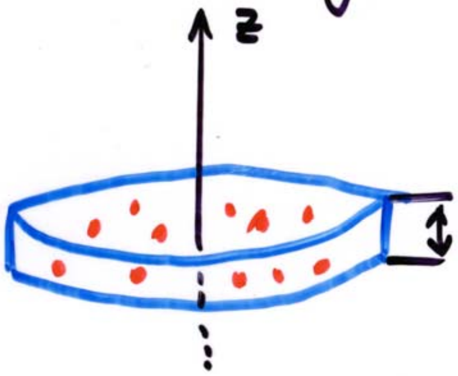
G. V. Shlyapnikov

LPTMS, Orsay, France
University of Amsterdam

Outline

1. What are low-D trapped gases
2. What is new? Examples.
3. Weakly interacting limit.
4. Long-range order.
5. Finite-T quasicondensates in 2D Bose gases.
6. Trapped 2D gas at finite T.

2/ What are low-D trapped gases?



$\sim \ell_0 \sim \left(\frac{\hbar}{m\omega_0}\right)^{1/2}$

$\hbar\omega_0 \gg \varepsilon$

$T \to 0$

$\varepsilon \sim \varepsilon_F$ fermions

$\varepsilon \sim ng$ bosons

Kinematically and statistically the gas is 2D

$\sim \ell_0 = \left(\frac{\hbar}{m\omega_0}\right)^{1/2}$

$\hbar\omega \gg \varepsilon$

Kinematically and statistically the gas is 1D

3/ What is new?

Finite size non-uniform systems. Interaction between particles

Density of states

Uniform ideal gas in 2D

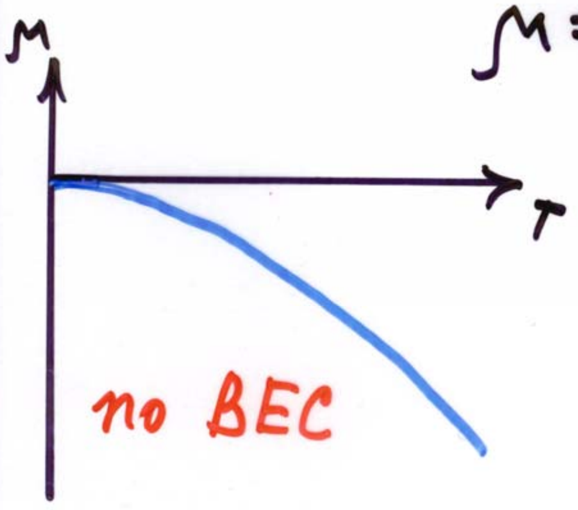
$$N = \sum_{\nu} N_{\nu}\left(\frac{E_{\nu}-\mu}{T}\right); \quad N_{\nu} = \left[\exp\left(\frac{E_{\nu}-\mu}{T}\right)-1\right]^{-1}$$

$$E_{\nu} = E_{k} = \frac{\hbar^2 k^2}{2m}$$

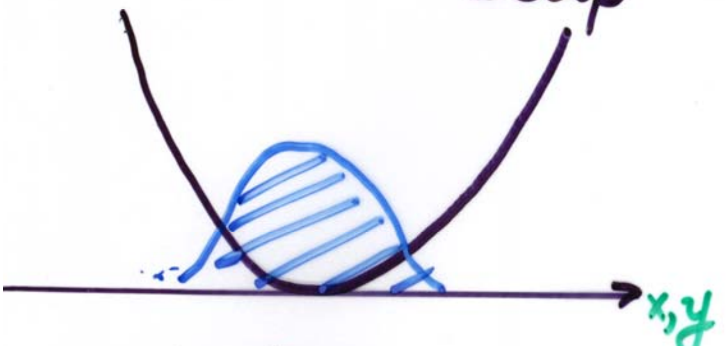
$$N = \int \frac{S\, d^2 k}{(2\pi)^2}\, N_k\left(\frac{E_k-\mu}{T}\right);$$

$$\mu = T \ln\left[1-e^{-n\Lambda^2}\right] < 0,$$

$$\Lambda = \left(\frac{2\pi\hbar^2}{mT}\right)^{1/2}$$



4/ Ideal 2D gas in harmonic trap



$$V(r) = \frac{m\omega^2 r^2}{2}$$

$$\rho(E) \sim E$$

$$E_\nu = \hbar\omega(n_x + n_y) \Rightarrow \rho(E) = \frac{E}{(\hbar\omega)^2}$$

$$N = N_0 + \int_0^\infty dE\, \rho(E)\, N_E\left(\frac{E-\mu}{T}\right)$$

$$N_0 = \left[\exp\left(-\frac{\mu}{T}\right) - 1\right]^{-1};$$

for $N_0 \gg 1 \Rightarrow -\frac{\mu}{T} \approx \frac{1}{N_0} \ll 1$

$$\int_0^\infty \rho(E)\, N_E\left(\frac{E-\mu}{T}\right) \approx \left(\frac{T}{\hbar\omega}\right)^2 \left(\frac{\pi^2}{6} - \frac{\ln N_0}{N_0}\right)$$

$$N\left[1 - \left(\frac{T}{T_c}\right)^2\right] = N_0 - \left(\frac{T}{\hbar\omega}\right)^2 \frac{\ln N_0}{N_0}$$

$$\boxed{T_c = \sqrt{\frac{6}{\pi^2}}\, N^{1/2} \cdot \hbar\omega}$$

Sharp crossover to the BEC regime

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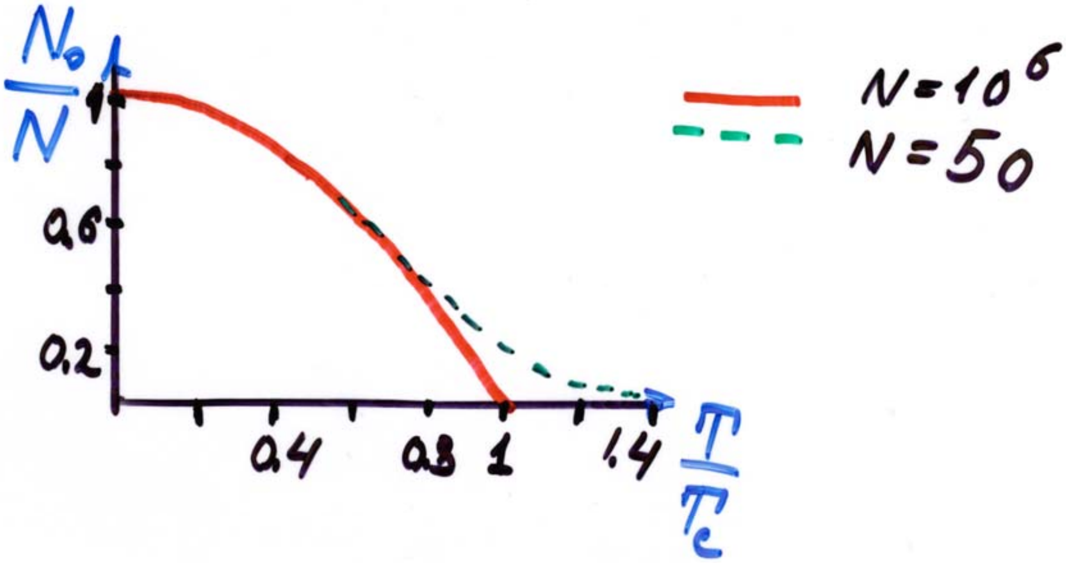
$$N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^2\right] \; ; \; T < T_c$$

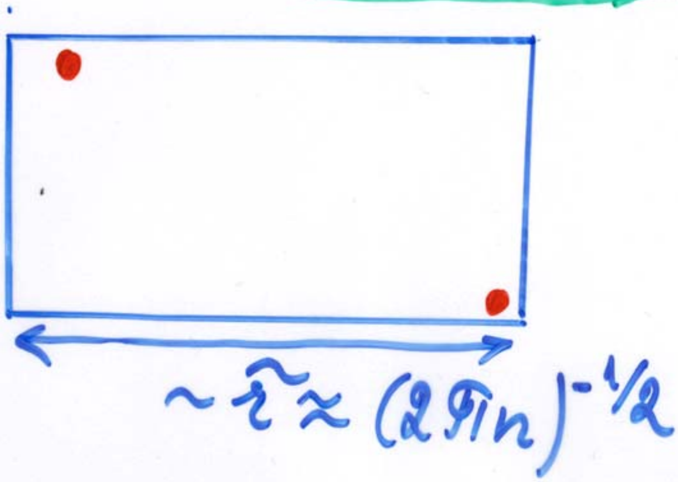


$$\frac{\Delta T}{T_c} \sim \sqrt{\frac{\ln N}{N}}$$

b) Weakly interacting limit

2D

$$n R_e^2 \ll 1$$



$$K \approx \frac{\hbar^2}{m \tilde{r}^2}$$

$$I = n g$$

$$K \gg I \Rightarrow \boxed{\frac{m g}{2\pi\hbar^2} \ll 1}$$

$$T \to 0 \qquad g \approx \frac{4\pi\hbar^2}{m} \cdot \frac{1}{\ln\left(\frac{1}{n d_*^2}\right)}$$

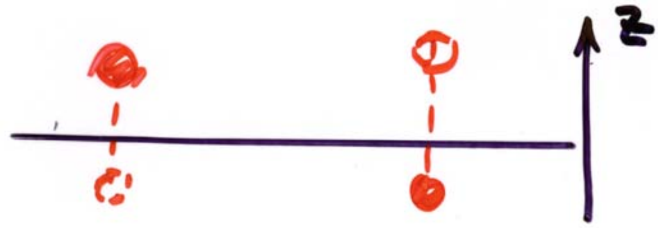
in the absence of resonances

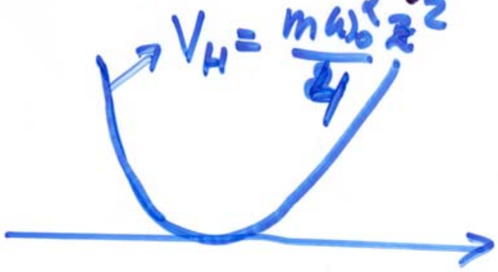
3D

$$K \approx \frac{\hbar^2}{m \tilde{r}^2} \approx \frac{2\pi\hbar^2}{m} n^{2/3} \; ; \quad I = n g$$

$$g = \frac{4\pi\hbar^2}{m} a \; ; \quad K \gg I \Rightarrow \boxed{n a^3 \ll 1}$$

7/ Interaction between particles in the quasi2D gas





$$f_0(z) = \left(\frac{1}{2\pi l_0^2}\right)^{1/4} \exp\left(-\frac{z^2}{4 l_0^2}\right)$$

$$l_0 = \left(\frac{\hbar}{m\omega_0}\right)^{1/2}$$

$$\hbar\omega_0 \gg E$$

$$\begin{cases} q R_e \ll 1 \\ l_0 \gg R_e \end{cases} \Rightarrow$$ ultracold limit s-wave scattering

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*) $\rho \to \infty$ $\quad \psi(\vec{r}) = f_0(z)\left(e^{i\vec{q}\vec{\rho}} - \frac{f(q,\varphi)\sqrt{i}}{\sqrt{8\pi q \rho}} e^{iq\rho}\right)$

$\varphi \to$ scattering angle

8) $$\left(-\frac{\hbar^2}{m}\Delta + U(r) + V_H(z) - \frac{\hbar\omega_0}{2}\right)\psi = E\psi$$

$r \gg R_e \Rightarrow U(r) \to 0$ and

$$\psi(\vec{r}) = f_0(z) J_0(q\rho) - \frac{f(q) G(\vec{r},0)}{f_0(0)}$$

for $\rho \to \infty$ $\quad G(\vec{r},0) = f_0(z) f_0(0) \frac{\sqrt{i}\, e^{iq\rho}}{\sqrt{8\pi q \rho}}$

and we get (*)

$l_0 \gg r \gg R_e \Rightarrow \psi \approx \eta f_0(0)\left(1 - \frac{a}{r}\right)$;

$a \to$ 3D scattering length

$r \to 0 \qquad G(\vec{r},0) = \frac{1}{4\pi r} + \frac{\ln\left(\frac{B\hbar\omega_0}{E}\right) + i\pi}{2(2\pi)^{3/2} l_0}$

$$f = 4\pi f_0^2(0)\eta = \frac{2\sqrt{2\pi}}{l_0/a + (1/\sqrt{2\pi})\left[\ln\left(\frac{B\hbar\omega_0}{E}\right) + i\pi\right]}$$

$$f(q) = \frac{2\pi}{\ln\left(\frac{1}{q d_*}\right) + i\pi/2}$$

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$$d_* = \sqrt{\frac{\pi}{B}}\, l_0 \exp\left\{-\sqrt{\frac{\pi}{2}}\,\frac{l_0}{a}\right\}$$

small $a<0 \Rightarrow$ large d_*

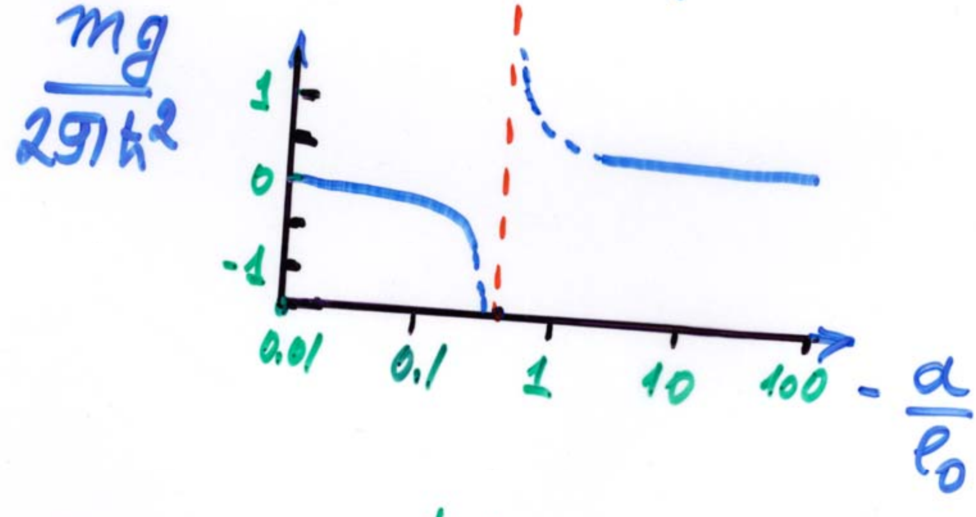
weakly bound quasi2D state

$$\varepsilon_0 = \frac{\hbar^2}{m d_*^2}$$

$$g = \int d^3r\, \varphi_0(z)\, U(r)\, \psi(\vec{r}) = \frac{\hbar^2 f(E)}{m}$$

Assuming BEC $\rightarrow E = 2|\mu|$

Thermal ga $\rightarrow E \approx T$



$$\frac{\hbar\omega_0}{E} = 10^3$$

Confinement-induced resonance
Petrov et al (2000)

10/ Interacting 2D Bose gas

Uniform gas

Kosterlitz-Thouless phase transition (superfluid)

$$T_{KT} = \frac{\pi \hbar^2}{2m} n_s$$

$T \leq T_{KT} \Rightarrow$ formation of bound vortex-antivortex pairs

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Trapped gas, problem

KTT or ordinary BEC?

$ng \ggg \hbar\omega \longrightarrow$ KTT

$ng \ll \hbar\omega \longrightarrow$ BEC

What is in between?

11/ BEC problem in finite-T 2D Bose gases

Uniform 2D gas. Assume BEC

$$n' = \langle \hat{\psi}^{+\prime} \hat{\psi}' \rangle = \int \frac{d^2 k}{(2\pi)^2} \langle \hat{\psi}_k^{+\prime} \hat{\psi}_k' \rangle N_k$$

$$N_k = \left[\exp\left(\frac{\varepsilon_k}{T}\right) - 1 \right]^{-1};$$

$$\varepsilon_k = \sqrt{E_k^2 + 2\mu E_k} \; ; \; E_k = \frac{\hbar^2 k^2}{2m} \; ; \; \mu = ng$$

$$k \to 0 \Rightarrow \varepsilon_k = \hbar c k$$

$$k \to 0 \quad \begin{cases} \langle \hat{\psi}_k^{+\prime} \hat{\psi}_k' \rangle \to \dfrac{mc}{\hbar k} \\ N_k \to \dfrac{T}{\hbar c k} \end{cases}$$

$$n' \xRightarrow{k \to 0} \int \frac{d^2 k}{(2\pi)^2} \cdot \frac{mT}{\hbar^2 k^2} \sim \int \frac{d^2 k}{k^2}$$

infrared divergency

Absence of true BEC

Bogoliubov, Mermin/Wagner, Hohenberg

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Density-phase representation

$$\hat\psi = \exp(i\hat\phi)\sqrt{\hat n}\,;\quad \hat\psi^+ = \sqrt{\hat n}\exp(-i\hat\phi)$$

$$[\hat n(\vec r)\hat\phi(\vec r\,')] = i\delta(\vec r - \vec r\,')$$

$$i\hbar\frac{\partial\hat\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\hat\psi + V(r)\hat\psi + g\hat\psi^+\hat\psi\hat\psi$$

small density fluctuations!

$$\hat n = n + \delta n(\vec r, t)$$

shift the phase by $-\mu t/\hbar$

Expand around $\hat n = n$; $\nabla\phi = 0$

$$-\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}} + V(r) + gn = \mu$$

GP equation

$$\hbar\frac{\partial}{\partial t}\left(\frac{\delta n}{\sqrt{n}}\right) = \left(-\frac{\hbar^2}{2m}\Delta + V(r) + gn - \mu\right)\left(2\sqrt{n}\,\hat\phi\right)$$

$$-\hbar\frac{\partial}{\partial t}\left(2\sqrt{n}\,\hat\phi\right) = \left(-\frac{\hbar^2}{2m}\Delta + V(r) + 3gn - \mu\right)\left(\frac{\delta\hat n}{\sqrt{n}}\right)$$

$$\delta\hat n(\vec r, t) = n^{1/2}\sum_\nu f_\nu^-(\vec r)\,e^{-i\varepsilon_\nu t/\hbar}\,\hat a_\nu + h.c.$$

$$\hat\phi(\vec r, t) = [4n]^{-1/2}\sum_\nu -i f_\nu^+(\vec r)\,e^{-i\varepsilon_\nu t/\hbar}\,\hat a_\nu + h.c.$$

13)

$\varepsilon_\nu \to$ Bogoliubov spectrum

$f_\nu^\pm = u_\nu \pm v_\nu \to$ Bogoliubov functions

Uniform gas

$$\delta\hat{n} = \delta\hat{n}_p + \delta\hat{n}_f \qquad p \quad \varepsilon < \mu$$
$$\hat{\phi} = \hat{\phi}_p + \phi_f \qquad f \quad \varepsilon > \mu$$

$$\langle \hat{\psi}_f^+ \hat{\psi}_f \rangle \approx \int_{\varepsilon_k > \mu} \frac{d^2k}{(2\pi)^2} N_k < n \frac{T}{T_d} \ln\frac{T}{\mu}$$

$\downarrow$ small fluctuations for T well below $T_d = \frac{2\pi\hbar^2}{m} n$

$$\frac{\langle \delta\hat{n}_p(\vec{r})\,\delta\hat{n}(0)\rangle}{n^2} = \frac{1}{n}\int_{\varepsilon_k < \mu} \frac{d^2k}{(2\pi)^2} \cdot \frac{E(k)}{\varepsilon(k)} (1 + 2N_k) \times$$
$$\times \cos(\vec{k}\vec{r}) < \max\left\{\frac{T}{T_d};\ \frac{mg}{2\pi\hbar^2}\right\}$$

zero order in perturbation theory

$$\hat{\psi} = \sqrt{n}\, \exp\{i\hat{\phi}_p\}$$

$$g_1(r) = \langle \hat{\psi}^{+}(\vec{r})\, \hat{\psi}(0)\rangle = n e^{-\frac{1}{2}\langle(\phi(r)-\phi(0))^2\rangle}$$

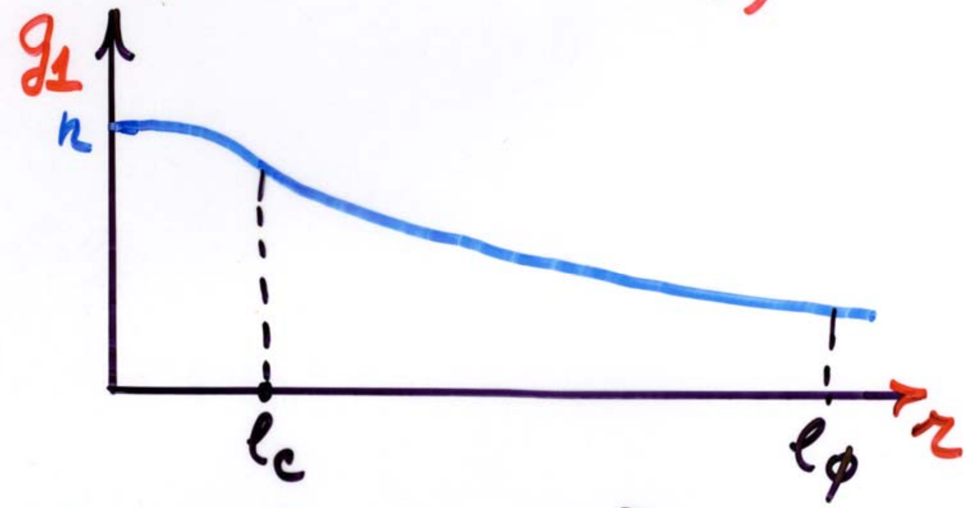
$$T \ll T_d$$

$$\langle (\hat{\phi}(\vec{r}) - \hat{\phi}(0))^2 \rangle_T \overset{r \gg \lambda_T}{\approx} \frac{2T}{T_d} \ln\left(\frac{r}{\lambda_T}\right)$$

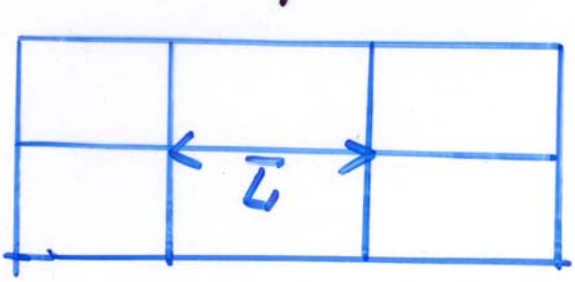
$$\lambda_T = l_c = \frac{\hbar}{\sqrt{mng}} \quad \text{for } T \gg \mu$$

$\lambda_T \Rightarrow$ de Broglie wavelength for $T \ll \mu$

$r \gg \lambda_T$ $$g_1(r) = n\left(\frac{\lambda_T}{r}\right)^{T/T_d}$$



$$l_\phi = \lambda_T \exp\left(\frac{T_d}{2T}\right) \gg l_c$$



$$l_c \ll \bar{L} \ll l_\phi$$

Quasicondensate

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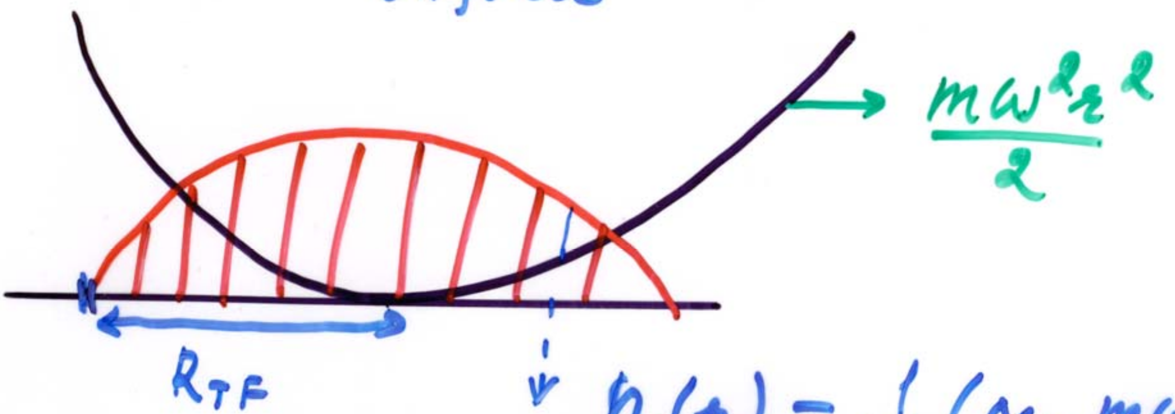
Trapped 2D Bose gas at finite T

MIT, LENS, Innsbruck
Oxford



$$n(r) = \frac{1}{g}\left(\mu - \frac{m\omega^2 r^2}{2}\right)$$

Thomas–Fermi regime

$$R_{TF} = \left(\frac{2\mu}{m\omega^2}\right)^{1/2}$$

$$\left[\delta\phi(r \sim R_{TF})\right]^2 \approx \left(\frac{mg}{2\pi\hbar^2}\right)\cdot\frac{T}{\mu}\ln N$$

MIT (Na) $\Rightarrow \frac{mg}{2\pi\hbar^2} \approx 10^{-2}$; $N \sim 10^5$

true BEC

H↑ on liquid He (Turku)

$T \approx 100\,mK$; $n \approx 10^{13}\,cm^{-2}$

$N \sim 10^9 \rightarrow$ expected quasiBEC