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SCHOOL ON QUANTUM PHASE TRANSITIONS AND NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

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Strongly correlated bosons in 1D

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Strongly correlated bosons in 1D

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Outline

- One dimensional bosons with delta interactions.
  Bethe Ansatz solution.
- Quantum regimes at zero temperature.
  Tonks-Girardeau limit.
- Local correlation functions.
  Suppression of three body recombination.
- Tonks-Girardeau gas vs. free fermions
  in harmonic potential.
- Exact time evolution of the Tonks-Girardeau bosons.
- Conclusions and prospects

- **Enhanced interactions in 1D**
  $\longrightarrow$ strongly correlated regimes

- **Toy models for studying physics beyond mean field**

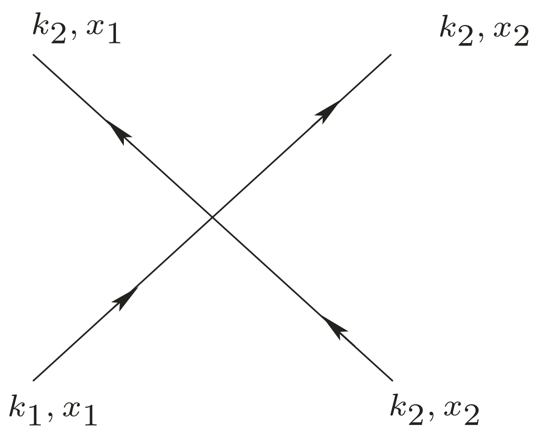
- **Interesting collective behaviour**

- **Exactly solvable**

- **Realizable in experiments with cold atoms (bosons, fermions). Tunable interactions**

One dimensional bosons with delta interactions: 2 particles

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + g\delta(x_1 - x_2)$$



$x_1 < x_2$ $k_1 > k_2$

$$\Phi(x_1, x_2) = A(12)e^{ik_1x_1+ik_2x_2} + A(21)e^{ik_2x_1+ik_1x_2}$$

$$A(21) = S(k_1 - k_2)A(12)$$

2-body scatterring matrix: $S(k) = -e^{-i\theta(k)} = \dfrac{k - ic}{k + ic}$

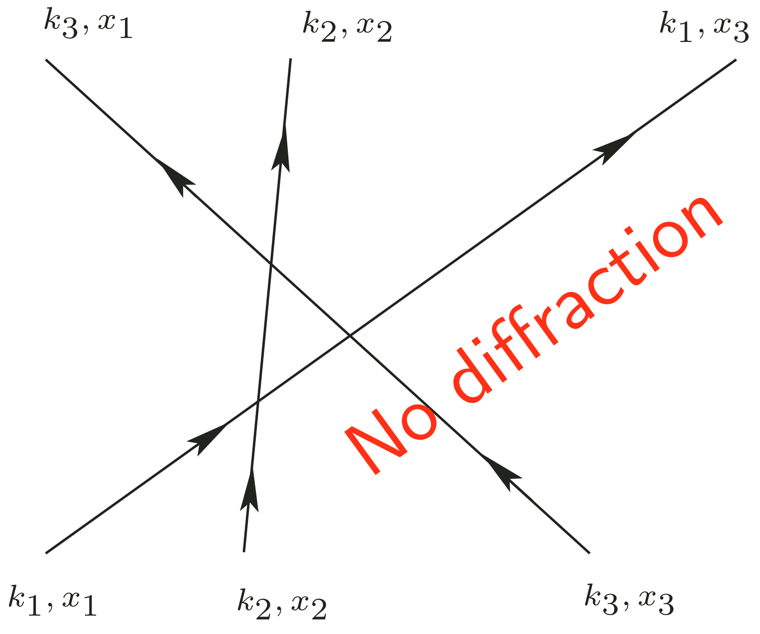
Phase shift: $\theta(k) = -2\arctan(k/c)$ $\quad 1/c = \hbar^2/mg$

. . . 3 particles

$$x_1 < x_2 < x_3 \qquad k_1 > k_2 > k_3$$

$$\begin{aligned}\Phi &= A(123)e^{ik_1x_1+ik_2x_2+ik_3x_3} + A(213)e^{ik_2x_1+ik_1x_2+ik_3x_3} \\ &+ A(231)e^{ik_2x_1+ik_3x_2+ik_1x_3} + \ldots\end{aligned}$$

$$A(231) = S(k_1 - k_2)S(k_1 - k_3)A(123)$$



Bethe Ansatz Solution

(Lieb, Liniger, '63):

$$H = -\sum_i \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{g}{2}\sum_{i\neq j}\delta(x_i - x_j)$$

$$x_1 < x_2 < x_3 < \ldots x_N \qquad k_1 > k_2 > k_3 > \ldots > k_N$$

$$\Phi(x_1, x_2, \ldots, x_N) = \sum_P A(P) e^{ik_{P_1}x_1 + ik_{P_2}x_2 + \ldots + ik_{P_N}x_N}$$

$$A(P) = \prod_{i<j} S(k_{P_i} - k_{P_j})$$

Periodic BC $\Rightarrow$ Bethe Ansatz equations $\hbar = 2m = 1$

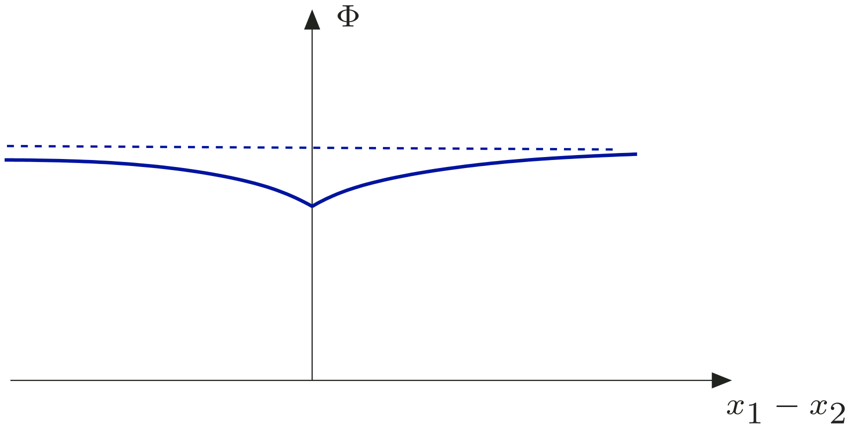
$$Lk_j = 2\pi I_j + \sum_l \theta(k_l - k_j); \qquad E = \sum_j k_j^2$$

Physical regimes at zero temperature

Dimensionless interaction parameter

$$\gamma = \frac{c}{n} = \frac{mg}{n\hbar^2}$$

Weak interactions $\gamma \to 0$



Mean-field (Gross-Pitaevskii) limit

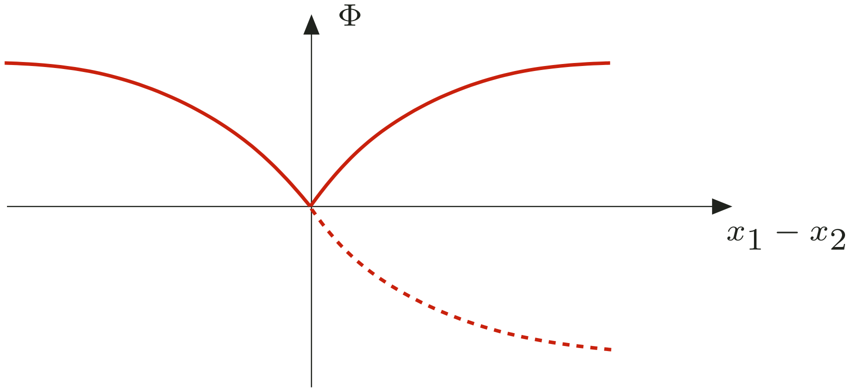
Quasi-condensate $$\frac{E_0}{N} = \frac{gn}{2}$$

Tonks-Girardeau regime

Strong interactions $\gamma \to \infty$

$1/\gamma = 0$ Free fermions Girardeau, '61

$$\Phi_0(x_1, \ldots, x_N) = |\Phi_0^{\text{fermi}}(x_1, \ldots, x_N)|$$



$$L k_j = 2\pi I_j + O(1/\gamma)$$

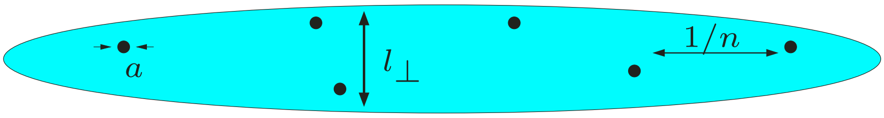
Fermi momentum

$$k_F = \pi N/L = \pi n$$

$$\frac{E_0}{N} = \frac{E_F}{3}$$

Experimental realization of Tonks-Girardeau regime

$$\frac{1}{c} = \frac{l_\perp^2}{2a}$$



- Tight confinement $n \ll 1/l_\perp$
- Strong interactions
- Number of particles $N \sim 100$
- Optical lattices ($\gamma \sim 200$)

Losses due to the three-body recombination ($\tau \simeq 1$ms) ?

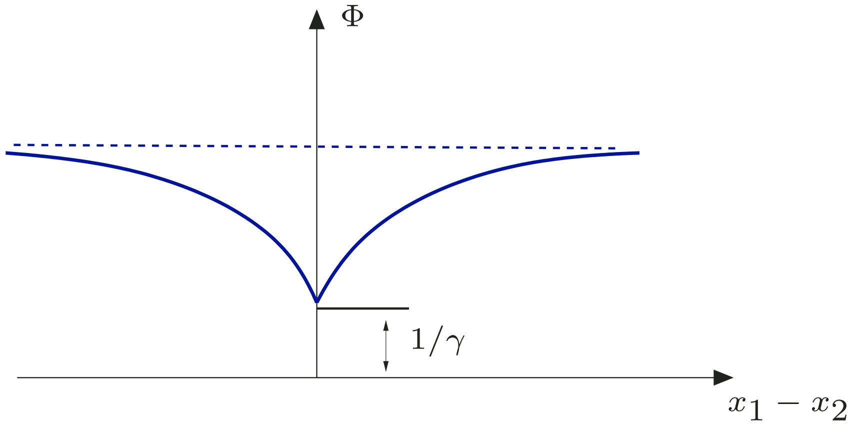
Need for three body local correlations

Local correlations

Local m - body correlation function

$$g_m = \langle \left(\psi^\dagger(0)\right)^m \left(\psi(0)\right)^m \rangle$$

Strong coupling $\gamma \gg 1$ 2-body Probability $\sim 1/\gamma^2$



Number of pairs: $\dfrac{m(m-1)}{2}$

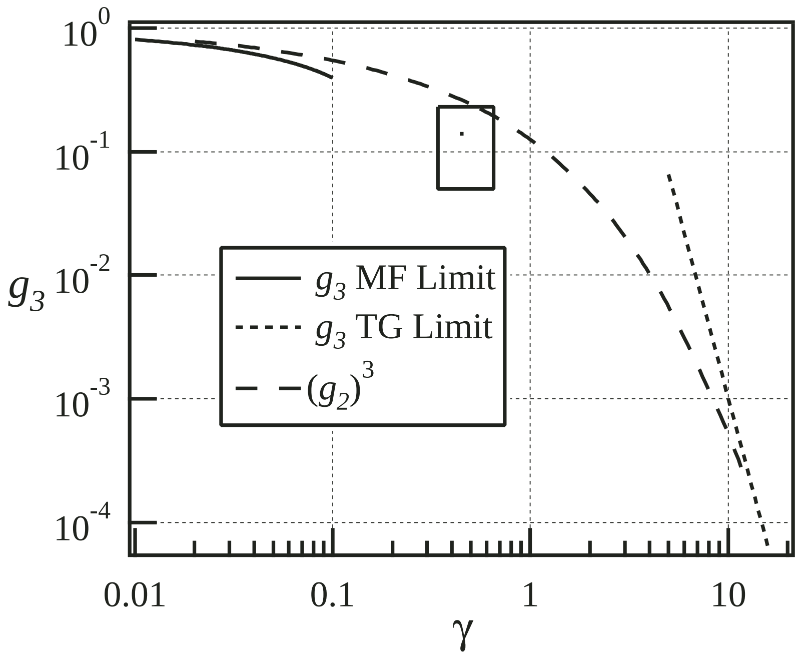
$$g_2 \sim \frac{1}{\gamma^2} \qquad\qquad g_3 \sim \frac{1}{\gamma^6}$$

Three body recombination suppression

DG and G.V. Shlyapnikov PRL 03, NJP 03

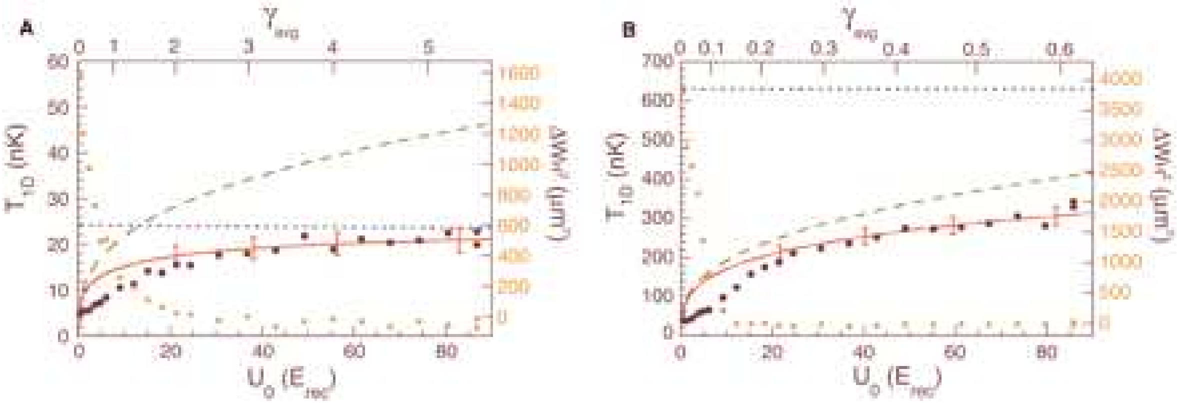
$$g_3/n^3 = \langle\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi\rangle = \begin{cases} 1-\frac{6}{\pi}\sqrt{\gamma}, & \gamma\to 0 \\ 16\pi^6/15\gamma^6, & \gamma\to\infty \end{cases}$$

B. Laburthe Tolra *et al.*, NIST group, PRL 04

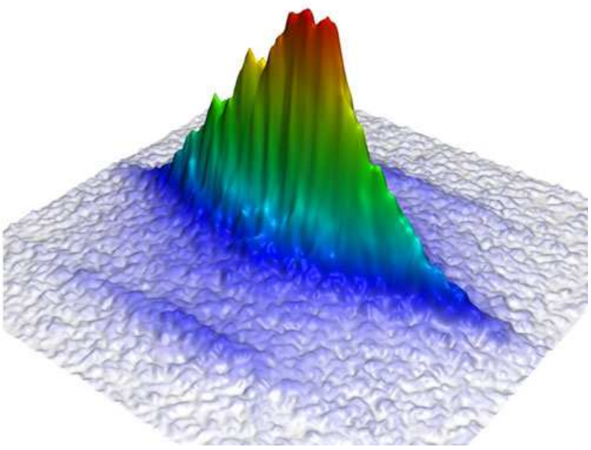


Experiments

T. Kinoshita, T. R. Wenger and D. S. Weiss, "Observation of a one-dimensional Tonks-Girardeau gas," Science **305**, 1125 (2004)



B. Paredes, A. Wildera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G.V. Shlyapnikov, T.W. Hänsch and I. Bloch, *"Tonks-Girardeau gas of ultracold atoms in an optical lattice"*, Nature **429**, 277 (2004)

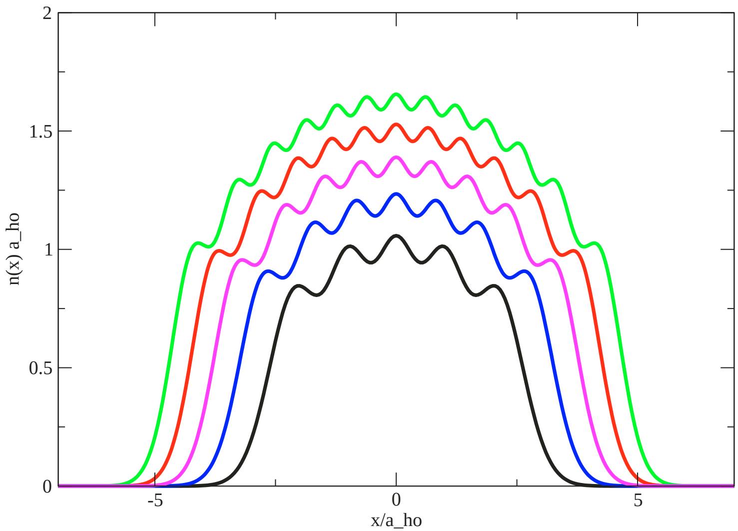


Tonks-Girardeau gas vs free Fermions in harmonic potential $V = \frac{1}{2}m\omega_0^2 x^2$

$$\Phi(x_1, x_2, \ldots, x_N) = \left| \det_{k,l} \phi_k(x_l) \right|$$

$\phi_l(x)$ – Hermite polynomials

- Density $\rho(x)$



- Collective excitation spectrum

One-body density matrix and momentum distribution

- One-body density matrix

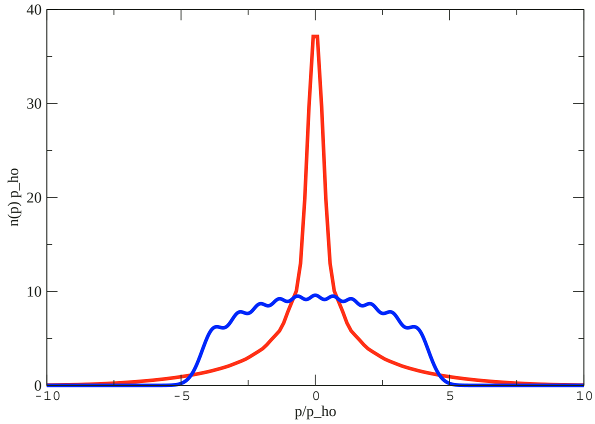
(P.J. Forrester *et al.*, PRA 02; DG, JPA 04)

$$g_1(x, y; t) = N \int dx_2..dx_N \, \Phi^*(x, x_2.., x_N; t)\Phi(y, x_2, .., x_N; t)$$

quasi long-range order:

$$g_1(x, y; 0) \sim \frac{[n(x)n(y)]^{1/4}}{|x - y|^{\frac{1}{2}}}$$

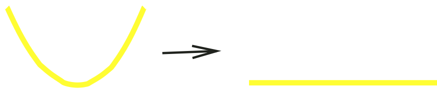
- Momentum distribution $n(p, t) = \int dxdy \, e^{ip(x-y)} g_1(x, y; t)$



Important: fermions in harmonic potential $n(p) \sim \rho(x = pl_0^2)$

Dynamical evolution under arbitrary time-dependent frequency $\omega(t)$

- 1D expansion



- Large-amplitude oscillations

Methods

- We use the time-dependent Bose-Fermi mapping
- We use a scaling transformation $X = x/b(t)$, $\tau = \tau(t)$ on the single-particle orbitals

$$\phi_j(x,t) = \frac{1}{\sqrt{b}}\phi_j(X,0)\exp\left[i\frac{mx^2}{2\hbar}\frac{\dot{b}}{b} - iE_j\tau(t)\right]$$

with

$$\ddot{b} + \omega^2(t)b = \omega_0^2/b^3 \text{ and } \tau(t) = \int_0^t dt'/b^2(t')$$

General results

(A. Minguzzi and DG, PRL 05)

- The density profile obeys a simple scaling law

$$\rho(x,t) = \frac{1}{b}\rho\left(\frac{x}{b};0\right)$$

- One-body density matrix

$$g_1(x,y;t) = \frac{1}{b}g_1\left(\frac{x}{b},\frac{y}{b};0\right)\exp\left(-\frac{i}{b}\frac{\dot{b}}{\omega_0}\frac{x^2-y^2}{2l_0^2}\right)$$

- Time-dependent momentum distribution

$$n(p,t) = b\int dxdy\, g_1(x,y;0)\; e^{-ib\left[\frac{\dot{b}}{\omega_0}\frac{x^2-y^2}{2l_0^2}-p(x-y)\right]}$$

For large expansion $b \gg 1$: dynamical phase is stationary for $x = y = x^* = (\omega_0/\dot{b})pl_0^2$ and

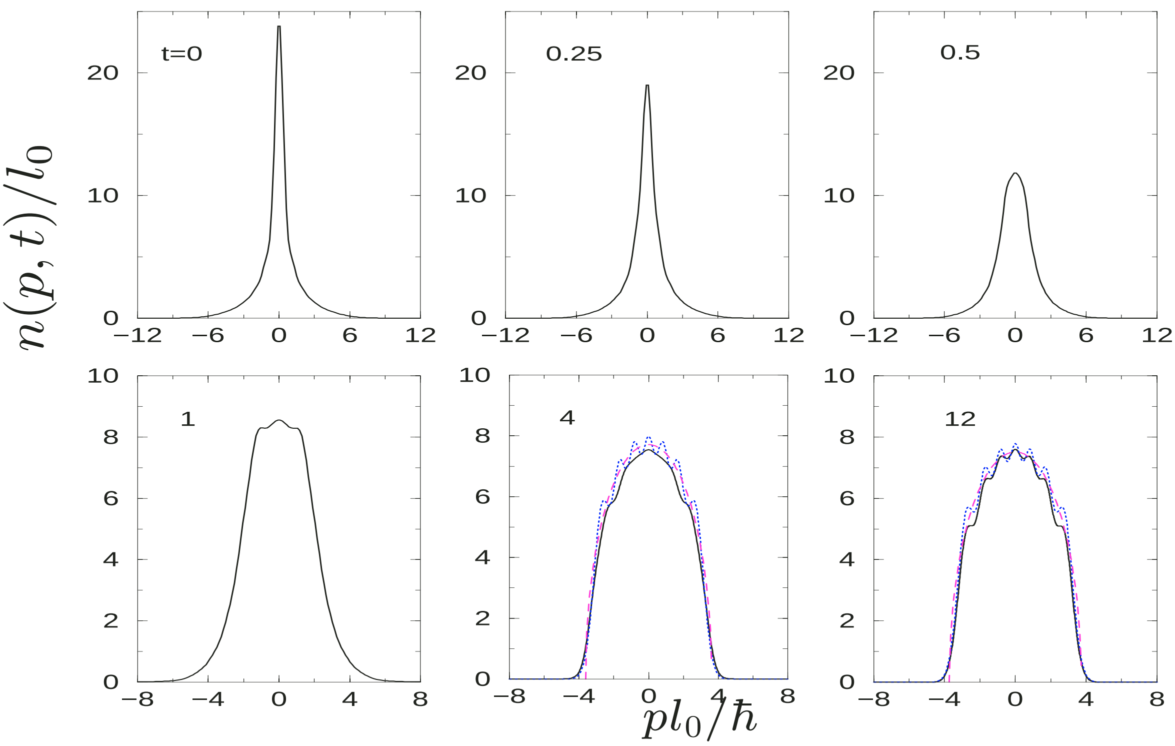
$$\boxed{n(p,t) \sim \rho(x = x^*(p);0)}$$

Expansion

Scaling parameter

$$b(t) = \sqrt{1 + \omega_0^2 t^2}$$

Momentum distribution for $N = 7$ particles



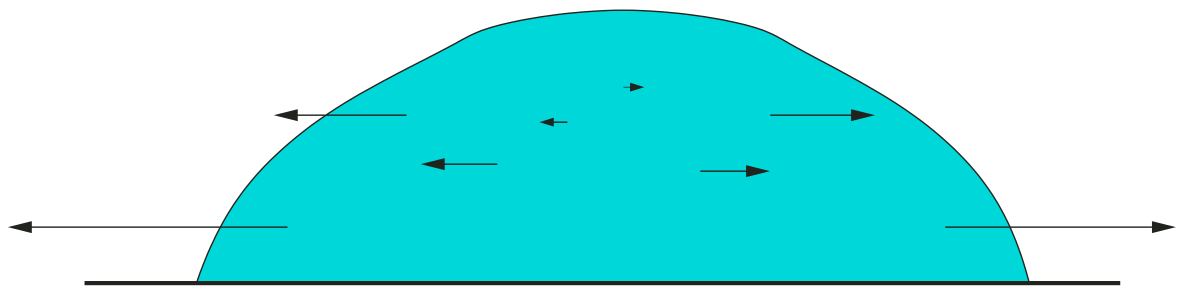
"fermionization time" $t_F = 1/N\omega_0 = \hbar/E_F$

Velocity (momentum) distribution

- Velocity field $v = v_0 + v_{\text{hydr}}$
- Hydrodynamical velocity for a harmonically confined gas

$$v_{\text{hydr}} \sim x$$

- For sufficiently long times $v_{\text{hydr}} \gg v_0$ and velocity (momentum) distribution becomes distribution of positions, *i.e.* density profile

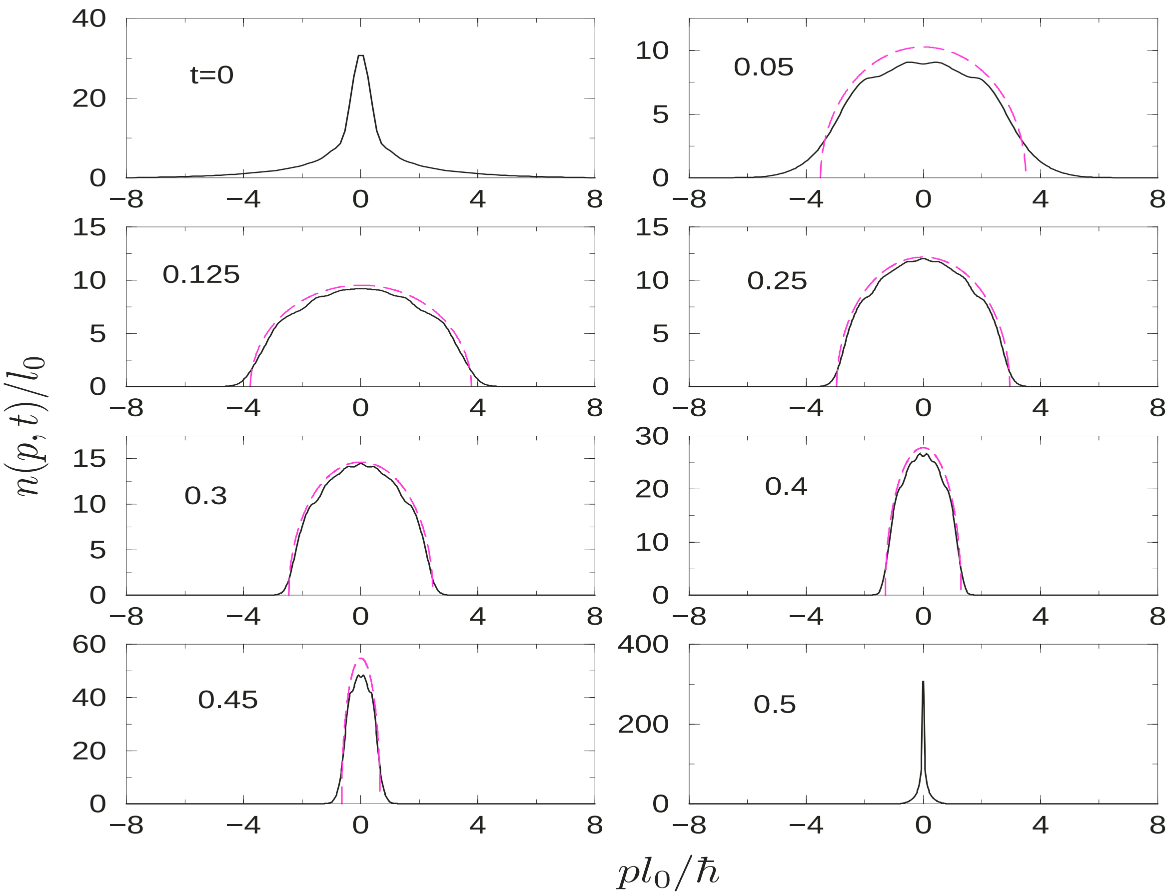


Large-amplitude oscillations

Sudden change of the trap frequency from ω_0 to ω_1

scaling parameter $b(t) = \sqrt{1 + \frac{\omega_0^2 - \omega_1^2}{\omega_1^2} \sin^2 \omega_1 t}$

Reversible dynamical fermionization $N = 9$, $\omega_0/\omega_1 = 10$



Conclusions and prospects

- Exact solution for equilibrium
  and out of equilibrium properties.
- Explicit dependence on microscopic parameters
- Theoretical results in the strongly correlated (Tonks) limit

- Effects of finite γ.
  Perturbation theory around free fermionic limit.
- Effects of different external potentials.
  Semiclassical approximation.
- Approach to equilibrium in exactly solvable models

"...However, my personal reason for working on one-dimensional problems is merely that they are fun. A man grows stale if he works all the time on the insoluble, and a trip to the beautiful world of one dimension will refresh his imagination better than a dose of LSD. If Hans Bethe in his youth had not wasted his time solving the one-dimensional Heisenberg model of an antiferromagnet, I doubt whether he would have created the theory of energy production in stars any sooner..."

F.J. Dyson, Physics Today **9**, 83 (1967)