The Abdus Salam
International Centre for Theoretical Physics

SMR 1666 - 9

SCHOOL ON QUANTUM PHASE TRANSITIONS AND NON-EQUILIBRIUM PHENOMENA IN COLD ATOMIC GASES

11 - 22 July 2005

Quantum information theory, quantum phase transitions and cold atoms

Presented by:

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Quantum Information Theory, Quantum Phase Transitions and Cold Atoms

Wanderin' quantum optics theory (Warsaw, Saclay, Hannover, Barcelona)

ICFO Institut de Ciències Fotòniques

Universität Hannover

Three Tales on Quantum Information Theory, Quantum Phase Transitions and Cold Atoms

- Lecture I – Introduction to QIT – theory of entanglement, entanglement criteria and measures, multiparty entaglement, entaglement detection, distillability (literature: bruss.pdf, lewen.pdf, lecture1.ppt)
- Lecture IIa – Entanglement and quantum phase transitions - entaglement in simple integrable models at the criticality, localizable entanglement, entanglement versus correlations (lecture2.pps).
- Lecture IIb – Generation of entanglement in many body systems, generation via quantum phase transitions, generation in complex and disordered systems.
- Lecture IIIa – Entaglement based codes, matrix product states, PEPS (Projected Entangled-Pair States) (lecture3.ppt, armand.pdf).
- Lecture IIIb – Examples – Spin ½ XY chain in a random X-oriented field

Entanglement in one-dimensional spin chains

Andreas Osterloh

MATIS – INFM & DMFCI Università di Catania

- L. Amico (MATIS-INFM & DMFCI)
- R. Fazio (NEST-INFM & SNS Pisa)
- F. Plastina (NEST-INFM & SNS Pisa)
- M. Palma (NEST-INFM & Milano)

Research Training Networks

HPRN-CT-2000-00144

OUTLINE

- Entanglement measures
- Anisotropic transverse XY models

Groundstate analysis at T=0

- Separable point
- CKW Conjecture

Based on: A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 608 (2002);
also T.J. Osborne and M.A. Nielsen, Phys. Rev. A 66, 032110 (2002).

Other Entanglement measures

Whole System in a pure state!

- von Neumann Entropy

Coffman, Kundu, Wootters PRA 61, 052306 (2000)

$-\mathrm{Tr}\, \rho_1 \log_2 \rho_1 \quad \leftrightarrow \quad 4 \det \rho_1$

- **CONCURRENCE**

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4,$ are square roots of the eigenvalues of

$$R = \rho_2 \cdot (\sigma_y \otimes \sigma_y) \cdot \rho_2^* \cdot (\sigma_y \otimes \sigma_y)$$

Anisotropic XY models

$$H = -J\sum_i (1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y - h\sigma_i^z$$

- Exactly solved
- Correlation functions accessible

$$\lambda = \frac{J}{2h}$$

Lieb, Schulz, Matthis Ann. Phys.NY 16, 407 (1961)

Barouch, McCoy, Dresden PRA 2, 1075 (1970)

Barouch, McCoy PRA 3, 786 (1971)

Pfeuty Ann. Phys.NY 57, 79 (1970)

Exact solution

$$H = h \cdot \left[\sum_k \Lambda_k \eta_k^+ \eta_k - 1/2 \sum_k \Lambda_k \right]$$

$$\Lambda_k = \sqrt{(1 - \lambda \cos k)^2 + \lambda^2 \gamma^2 \sin^2 k}$$

Ground state

Entanglement for the

Anisotropic XY models

Reduced Density Matrix from Symmetries

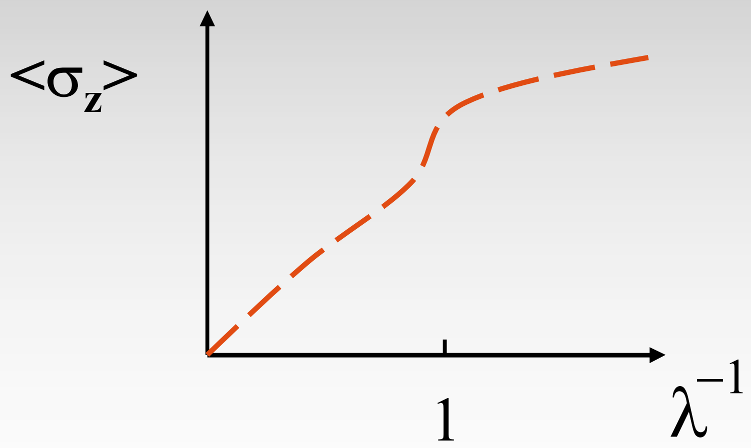
- Reality
- translational, and parity symmetry

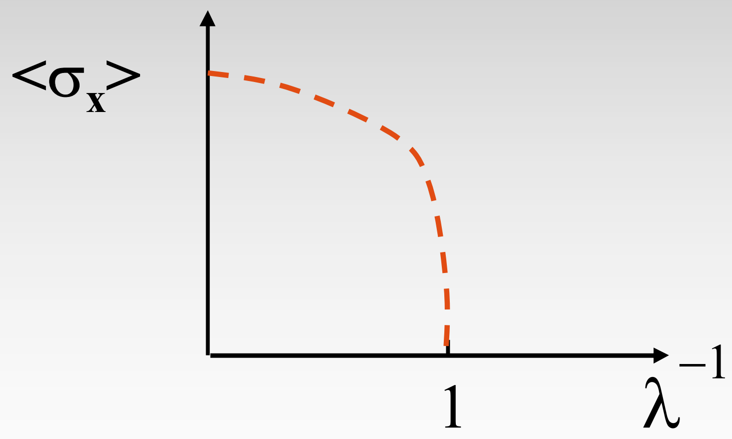
$$\rho = \begin{pmatrix} a & 0 & 0 & c \\ 0 & x & y & 0 \\ 0 & y & x & 0 \\ c & 0 & 0 & b \end{pmatrix}$$

Maximal anisotropy γ=1
$\rightarrow$ the Ising model

$$H = -2J\sum_i \sigma_i^x \sigma_{i+1}^x - h\sum_i \sigma_i^z$$

- Critical point at $\lambda = J/2h = 1$



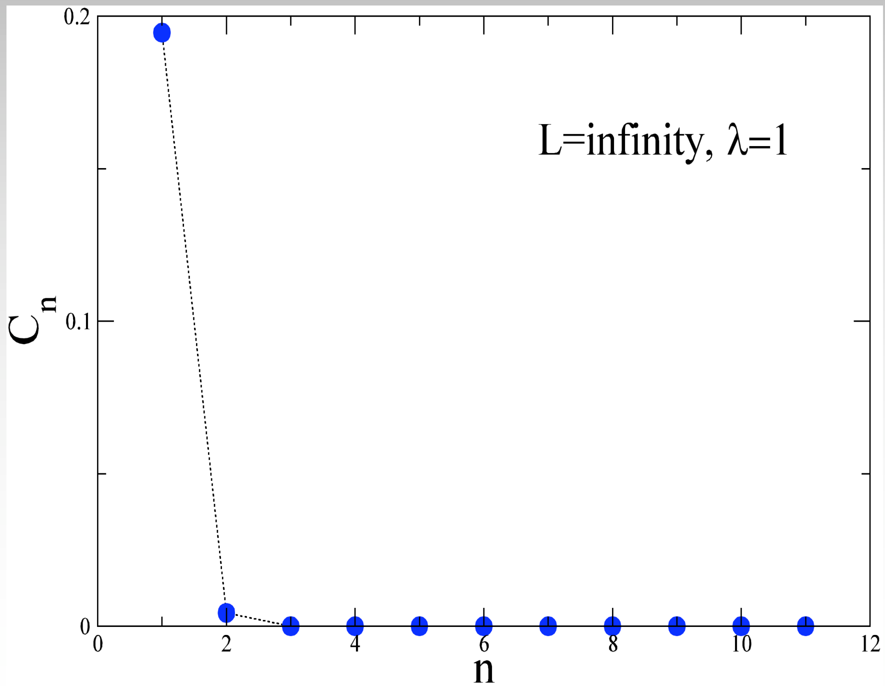


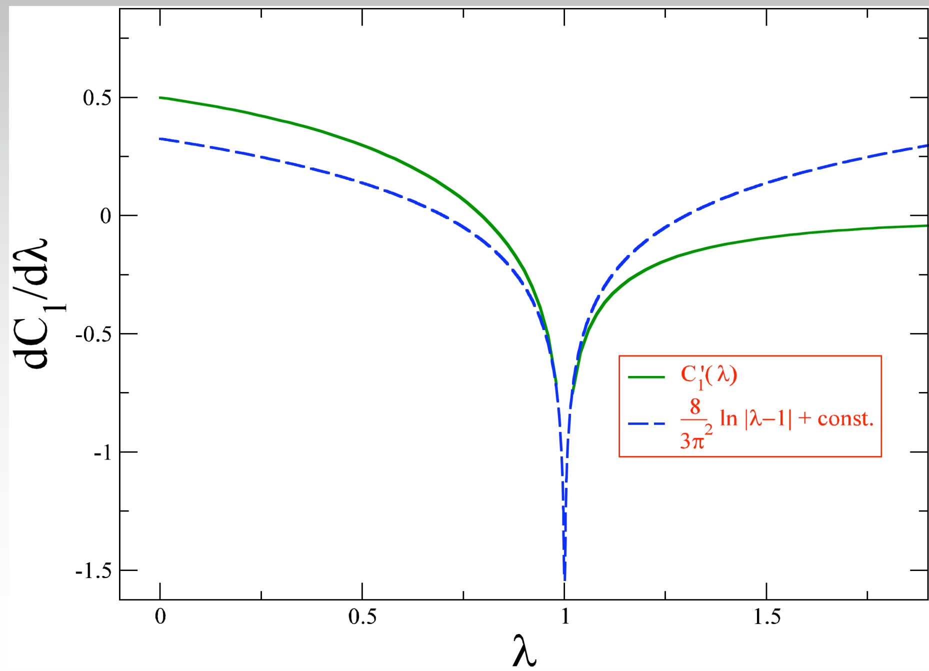
Pfeuty '70

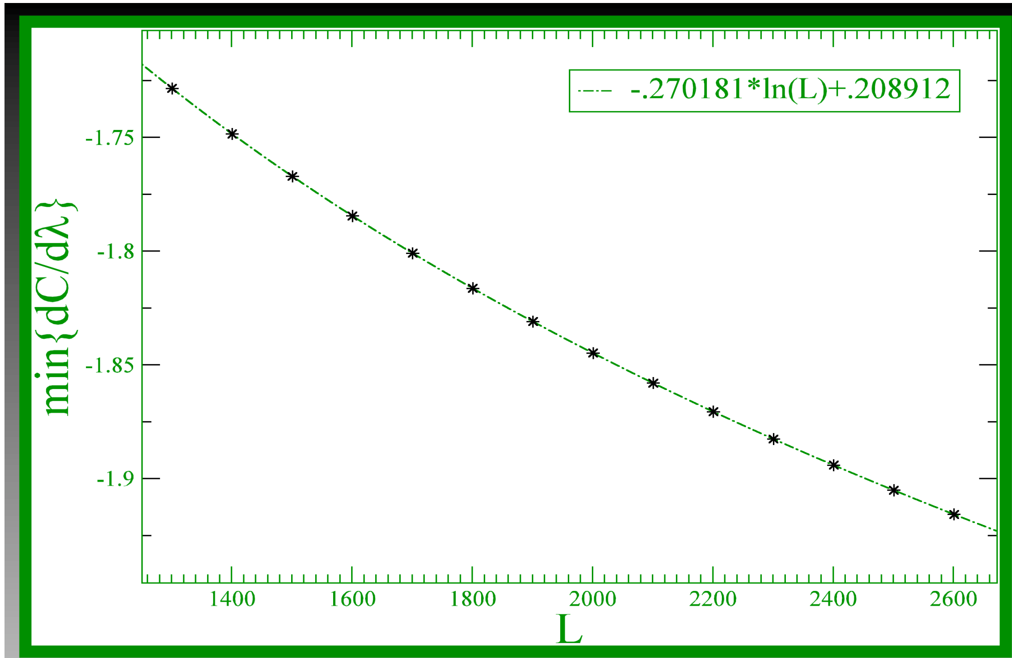
Quantum Phase Transitions

- T=0 phase transition driven by a coupling constant λ
- Correlation length diverges as $X \sim |g - g_c|^{-n}$; **ν=1**
- log-divergence of concurrence derivatives ↔ finite size scaling

A. Osterloh, L. Amico, Falci, R. Fazio, Nature 416, 608 (2002)



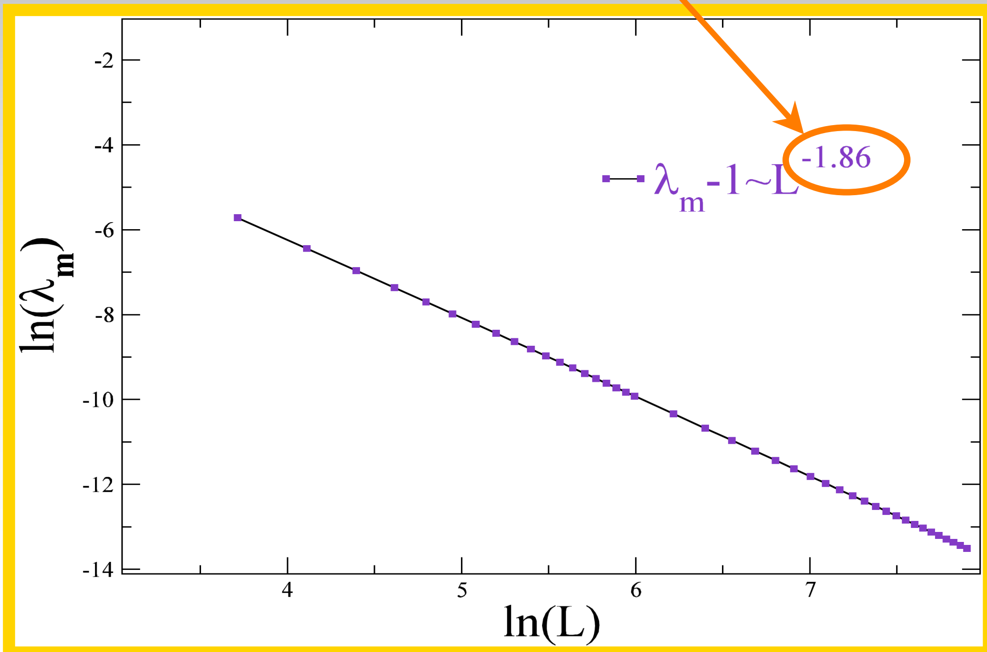




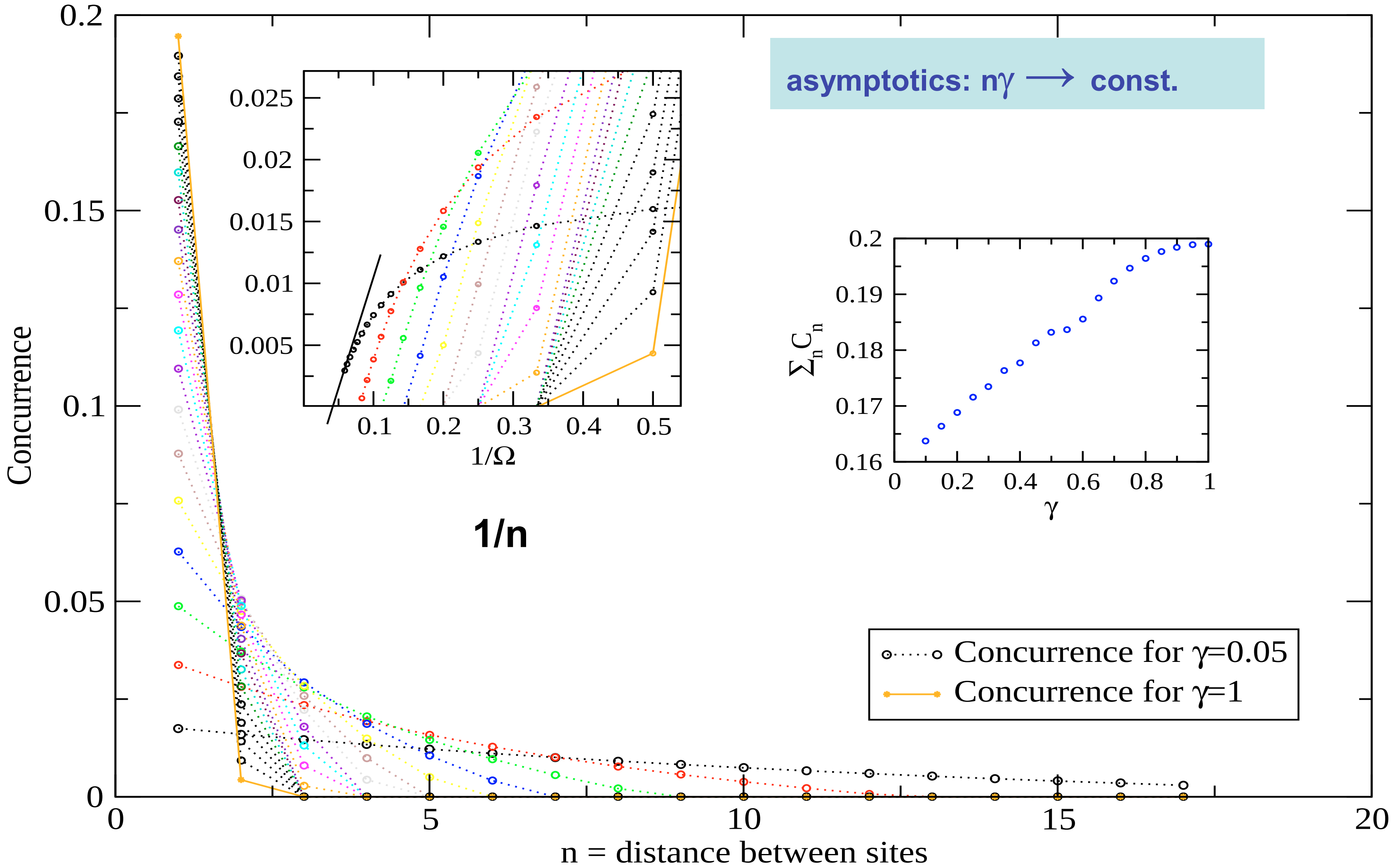
Value of the minimum

Must be < -1

Position of the minimum



Non-universal entanglement range



Different Entanglement Classes

$$|\Psi\rangle_{GHZ} = |\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle$$

$$|\Psi\rangle_{W} = |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle$$

$$\text{W} \qquad \Longrightarrow 4\det \rho_W^{(1)} = \frac{8}{9} \; ; \quad C_{12}^2 = C_{13}^2 = \frac{4}{9}$$

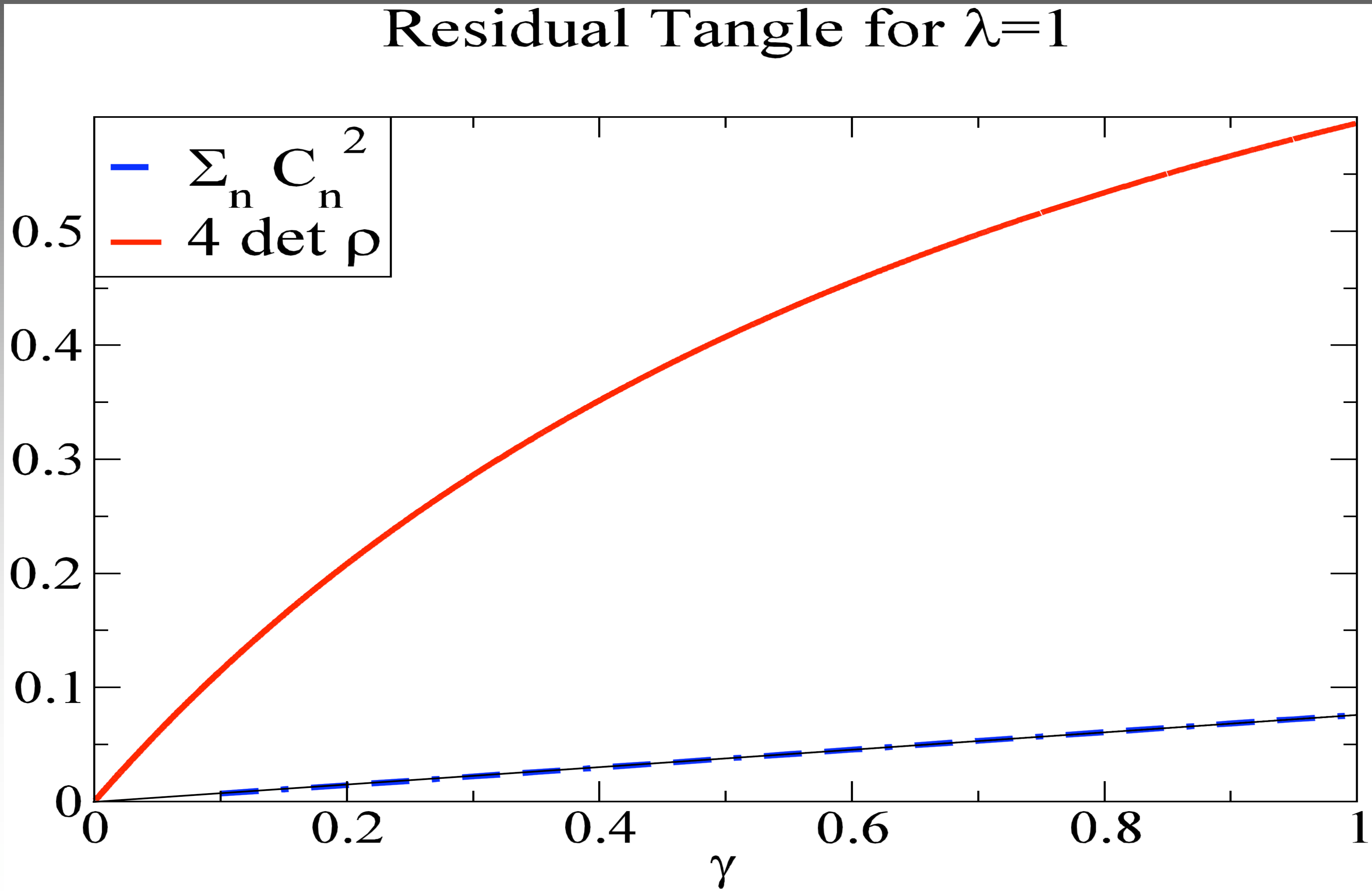
$$\text{GHZ} \qquad \Longrightarrow 4\det \rho_{GHZ}^{(1)} = 1 \; ; \quad C_{12}^2 = C_{13}^2 = 0$$

Residual entanglement measure (pure states)

$$\tau_1^{(3)} = 4\det \rho_1^{(1)} - C_{12}^2 - C_{13}^2$$

Coffman, Kundu, Wootters PRA 2000

Coffman-Kundu-Wootters



Results at Equilibrium

A.Osterloh, L.Amico, G.Falci, and R.Fazio 2002

- Critical change of concurrence at the quantum critical point
- Finite size scaling
- In general NO long range Concurrence
- Range NOT UNIVERSAL: $R \propto \gamma^{-1}$

$$\sum_n C_n^2 << 4 \det \rho^{(1)}$$

Coffman, Kundu, Wootters PRA 61, 052306 (2000)

- **Concurrence robust against mixing below λ_0**

Entanglement measures in N qubit systems

- **Entanglement of assistance**

  Let ρ $(|\Psi\rangle\langle\Psi|)$ be a many body state of the system and **M** - a measurement acting on qubits **k≠ij**. We define (**E=concurrence**)

$$\rho_{ij}(M) = M_{k\neq ij}\,(\rho)$$

$$E_{as}(ij) = \max_M E\,\rho_{ij}(M)$$

- **Localizable entanglement $E_{loc}(ij)$**

  The definition is the same as above, except that now **M** is a local measurement acting locally on qubits **k≠ij**. F. Verstraete, M. Popp, and J.I. Cirac, PRL **92,** 027901 (2004)

Entanglement versus correlation in spin systems

- Bounds on $E_{loc}(ij)$

Obviously,

$$E_{loc}(ij) \leq E_{as}(ij)$$

- Theorem: given a state of N qubits with classical correlation $Q_{\alpha\beta}(ij)$, there exist always directions in which one can measure other spins without decreasing $Q_{\alpha\beta}(ij)$. There exist always local measurements that keep, or increase classical correlations!
- Given a pure state, we measure N-2 spins without decreasing correlations; at the end we get pure state of 2 qubits with the same or bigger correlation. Since concurrence=maximal correlation

$$\max_{\alpha\beta} Q_{\alpha\beta}(ij) \leq E_{loc}(ij)$$

Entanglement versus correlation in spin systems

- We define entanglement length

$$\xi_E^{-1}=\lim_{n\to\infty}(-\ln E_{loc}(i,i+n)/n)$$

- Bounds on $E_{loc}(ij)$

$$\max_{\alpha\beta} Q_{\alpha\beta}(ij) \leq E_{loc}(ij) \leq E_{as}(ij)$$

imply that correlation length is not larger than entanglement lengh

$$\xi_C \leq \xi_E$$

- Whenever correlation length diverges, so does entanglement length!!!

Diverging entanglement length in gapped quantum spin system

- One considers Affleck-Kennedy-Lieb-Tasaki model

$$H^{AKTLT} = \sum_{k=0}^{N} X_{k,k+1}^{AKLT} = \sum_{k=0}^{N} \left(\vec{S}_k \vec{S}_{k+1} + \frac{1}{3} (\vec{S}_k \vec{S}_{k+1})^2 + \frac{2}{3} \right)$$

- One introduces 1 parameter family of models

$$H(\phi) = \sum_{k=0}^{N} \left(\left(\Sigma_k^{\phi}\right)^{-1} \otimes \left(\Sigma_{k+1}^{\phi}\right) \right) X_{k,k+1}^{AKLT} \left(\left(\Sigma_k^{\phi}\right)^{-1} \otimes \left(\Sigma_{k+1}^{\phi}\right) \right),$$

where

$$\Sigma_k^{\phi} = 1_k + \sinh(\phi) S_k^z + (\cosh(\phi) - 1)\left(S_k^z\right)^2$$

At T=0, entanglement length diverges at φ=0,

correlation length remains finite!!!

F. Verstraete, M.A. Martín-Delgado, and J.I. Cirac, PRL **92**, 087201 (2004)

Entanglement versus correlation in spin systems: morals!

- **In non-critical system bipartite entanglement of reduced density matrices, or blocks is of short range!**
- **Signatures of criticality are present in the properties of the reduced density operators!**
- **Von Neumann entropy of blocks of size L follows area law ($\sim L^{(d-1)}$)**

  G. Vidal, J.I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003); J.I. Latorre, E. Rico, and G. Vidal, Quantum Inf. Comput. 4, 48 (2004).

- **However, appropriately defined entanglement length diverges, which implies that entanglement must be present, and thus it has to be concentrated in a form of multipartite entanglement!!!**

Generation and propagation of entanglement in many body systems

Creating "cats" in 1D pipelines: generating entanglement via quatum phase transition

U. Dorner, P. Fedichev, D. Jaksch, M. Lewenstein and P. Zoller

PRL **91**, 073601 (2003)

see also U. Dorner, W.H. Żurek, and P. Zoller, cond-mat/050xxxx

Is Mott phase useful?

- Quantum information processing (following Innsbruck ideas, see recent experiments of I. Bloch)

- Possibility of creation multiparticle internal GHZ states locally

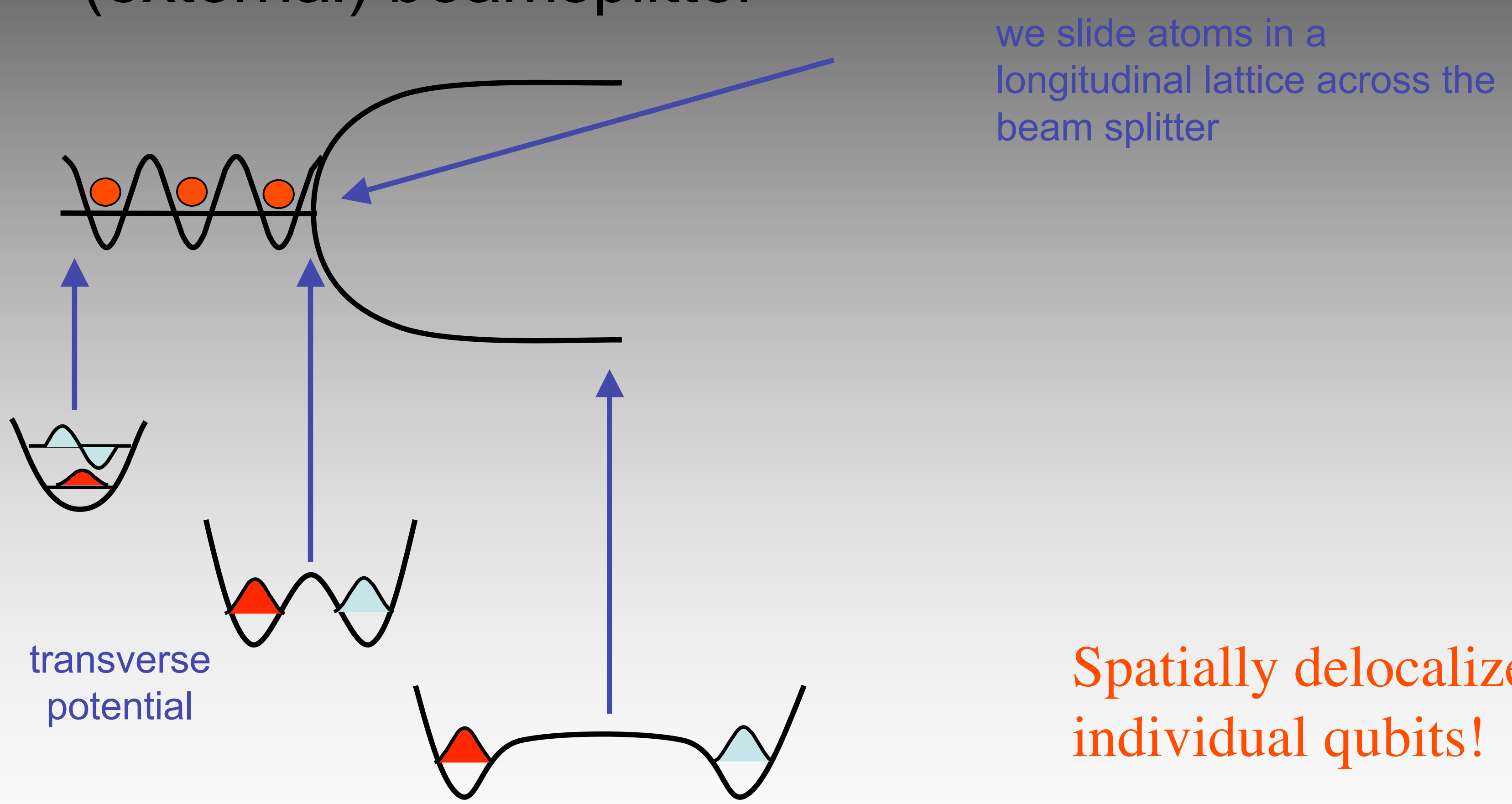
L. You, Phys. Rev. Lett. 90, 030402 (2003)

- Entangling chains of atoms in 1D pipelines

U. Dorner, P. Fedichev, D. Jaksch, M. Lewenstein, P. Zoller, Phys. Rev. Lett. 91, 073601 (2003), quant-ph/0212039

Atoms in 1D pipelines

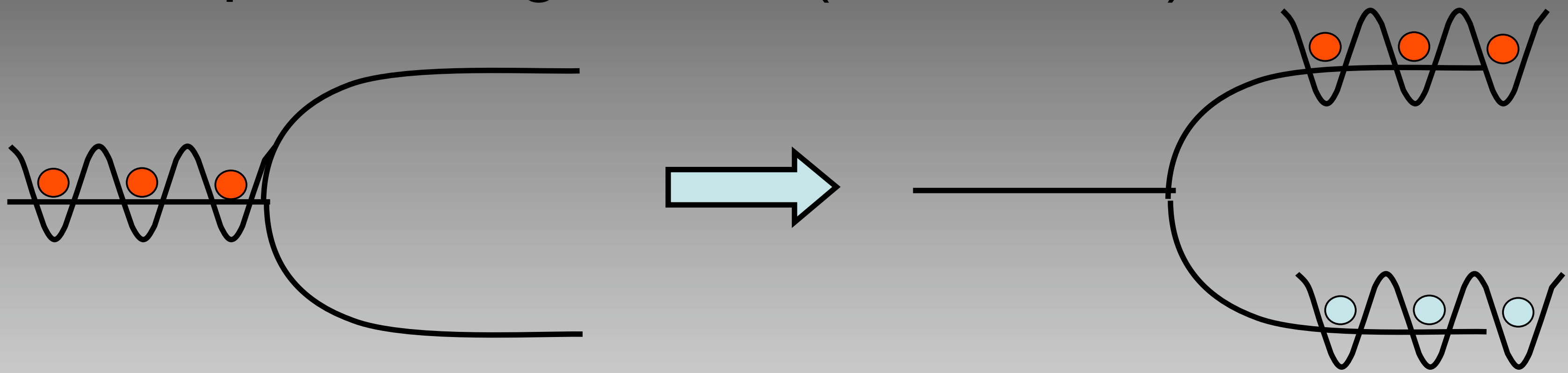
- (external) beamsplitter

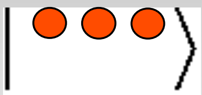


Spatially delocalized
individual qubits!

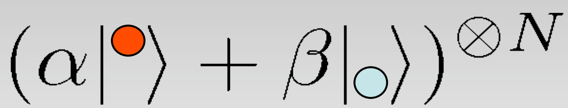
Atoms in 1D pipelines

- beamsplitter: single atom (= standard)





product state

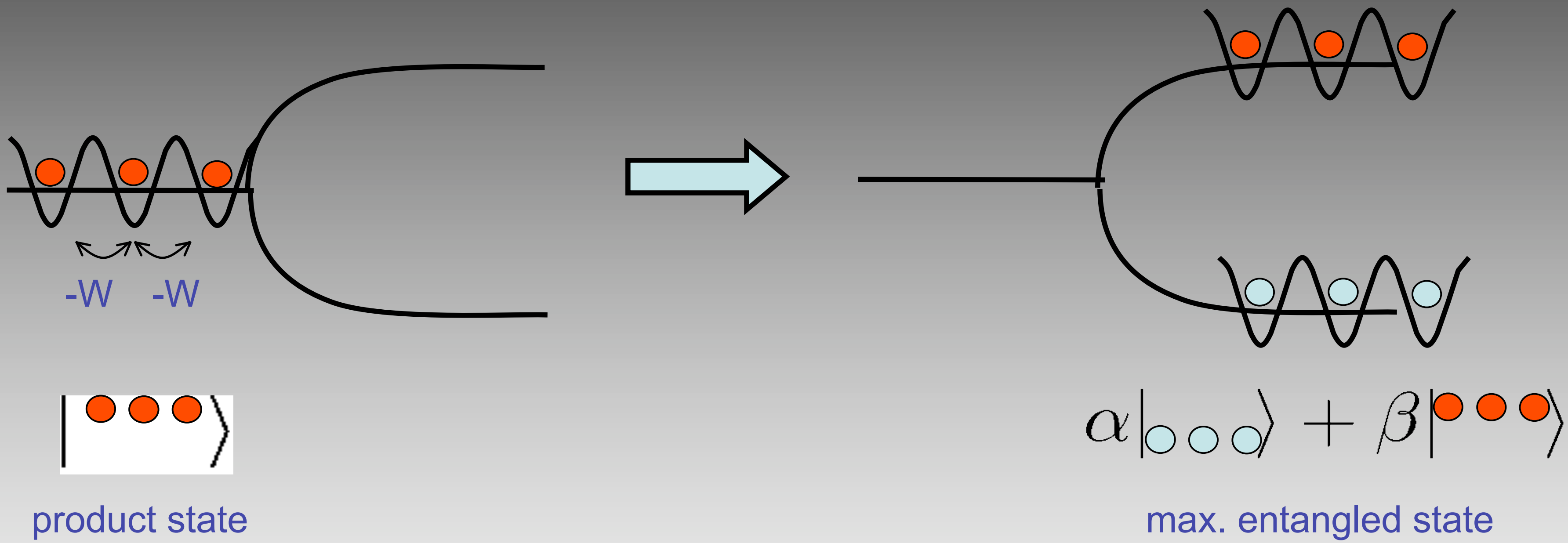


product state

$$
\begin{aligned}
|0\rangle &\longrightarrow |0\rangle + |1\rangle \\
|1\rangle &\longrightarrow |0\rangle - |1\rangle
\end{aligned}
$$

Atoms in 1D pipelines

- beamsplitter: generate a cat state



- cat splitter

$$|000\rangle \longrightarrow |000\rangle + |111\rangle$$
$$|111\rangle \longrightarrow |000\rangle - |111\rangle$$

Note: we perform adiabatically quantum phase transition in an Ising model in a transverse field!

Generating entanglement in XY chain in transverse time dependent field

U. Sen, A. Sen (De), and M. Lewenstein

PRA (Rap. Comm.) **70**, 060304 (2003)

A. Sen (De), U. Sen, and M. Lewenstein

quant-ph/0505006

Nonergodicity of entanglement in XY spin chain
(PRA, (Rapid Comm.) 70, 060304 (2004))

Hamiltonian:

$$H=\sum_i [(1+\gamma)\, \sigma^x_i \sigma^x_{i+1} + (1-\gamma)\, \sigma^y_i \sigma^y_{i+1} - h(t)\sigma^z_i],\ \gamma \neq 0$$

Initial state: $\rho^{eq}_{\beta}(0)$, where $\rho^{eq}_{\beta}(t) = \exp(-\beta H(t))/Z$;
$Z = tr(\exp(-\beta H(t))$; $\beta = 1/KT$,
K= Boltzmann constant; T= absolute temperature

Evolved state: $\rho_{\beta}(t)$

To check ergodicity of entanglement:
Compare entanglement of the evolved state with the equilibrium state for large time for which

$$tr(H\, \rho_{\beta}(t)) = tr\,(H\, \rho^{eq}_{\alpha}(t))$$

Choice of the field: **h(t) = a, t≤0**

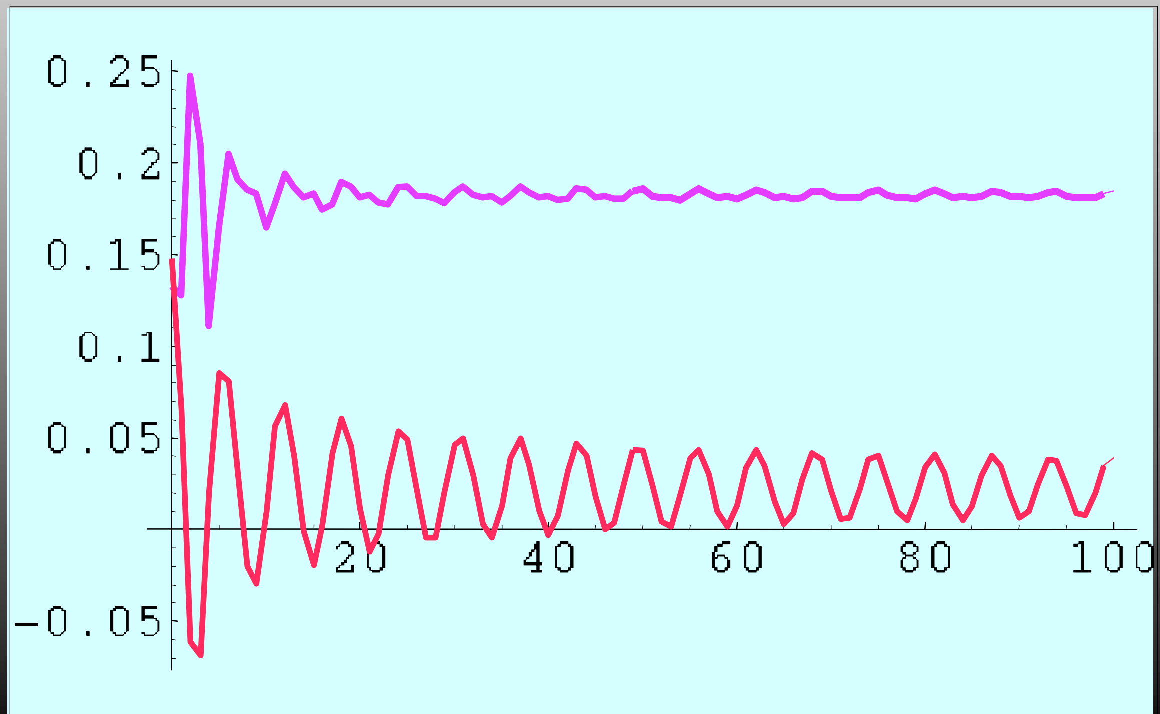
= b, t>0

For γ= 0.5, a= 0.5, b=0, β= 200

Initial equilibrium entanglement: **0.133**

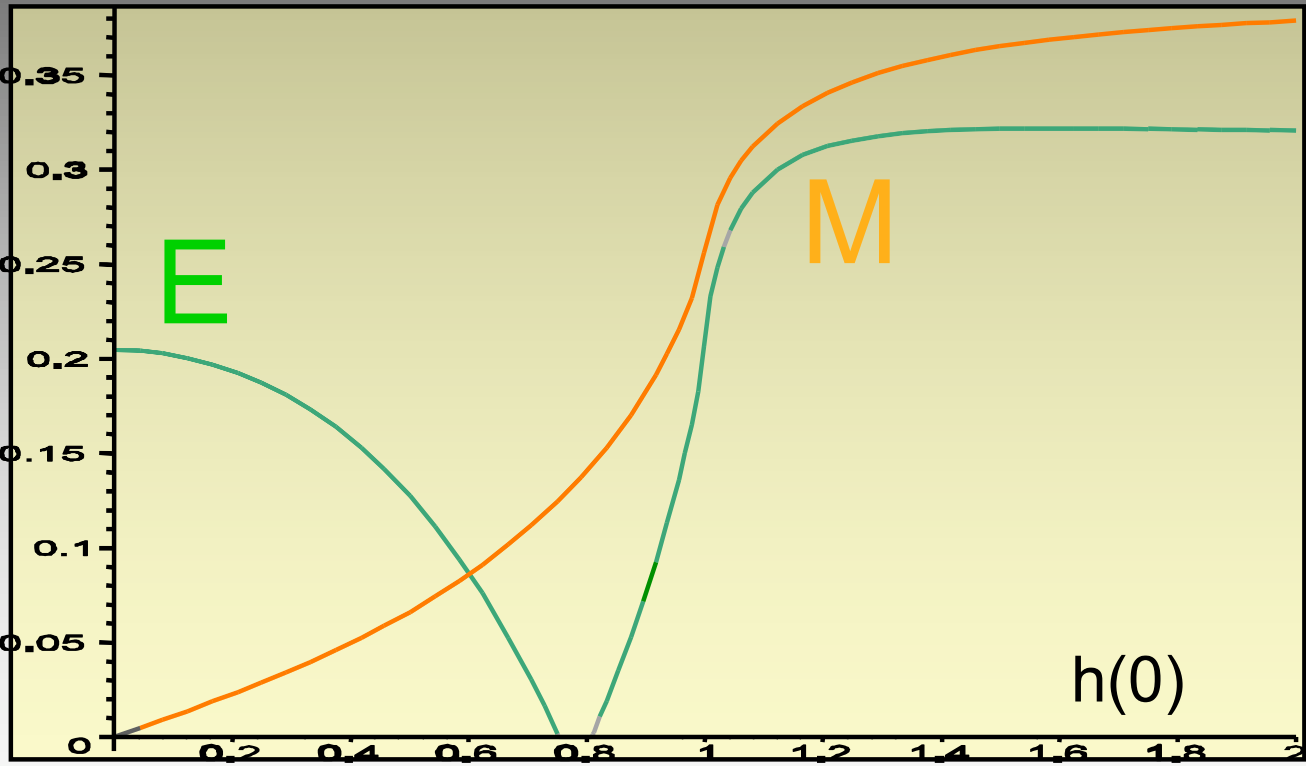
Entanglement jumps to **0.157** for **t>0.**

Entanglement (Pink) and magnetization (Red) of the evolved state: Entanglement converges to **0.18** for large time.



Dynamical phase transition of an infinite spin chain (quant-ph/0505006)

For a fixed time, entanglement (green) shows **critical behavior** w.r.t the initial transverse field (In figure: **t=1, γ= 0.5)**



Character of DPT is **different** for short and long evolution times.

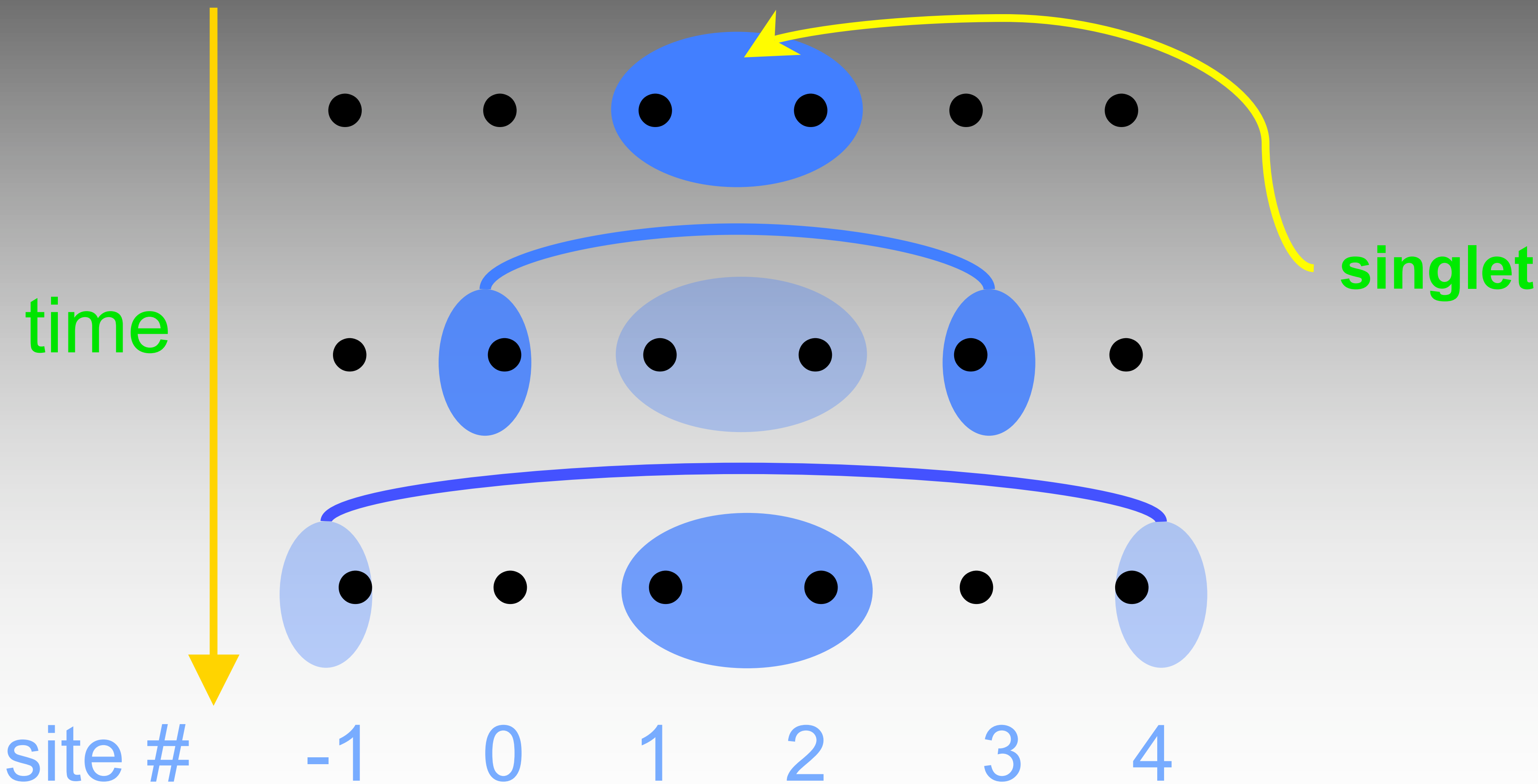
Nonmonotonic behavior of entanglement w.r.t. temperature of initial equilibrium state can be seen **near DPT.**

Propagating entanglement in isotropic and anisotropic XY chain

Amico, Osterloh, Plastina, Fazio, Palma PRA 69, 022304(2004)

Amico, Osterloh J. Phys. A 37, 291(2004)

Singlet Propagation



Loss of Symmetries

Amico, Osterloh, Plastina, Fazio, Palma PRA 69, 022304(2004)

- Only parity symmetry survives

  $\Rightarrow$ more complicated structure of ρ_2

- $$\left\langle S_i^\alpha S_j^\beta \right\rangle = Tr\left[\rho_2 \left(S^\alpha \otimes S^\beta\right)\right] \quad \alpha \neq \beta$$

- $$\left\langle S_i^z - S_j^z \right\rangle \neq 0$$

- $$\left\langle S_i^\alpha S_j^\beta \right\rangle \neq P_{j-i}^{\alpha\beta}$$

No transl. symmetry

Non-Equilibrium Correlations

Amico, Osterloh J. Phys. A 37, 291(2004)

A_l, B_m, C linear in fermionic operators

$$\langle S^\alpha S^\beta \rangle \rightarrow \langle 0 | C e^{iHt} \left\{ \prod_l A_l \prod_m B_m \right\} e^{-iHt} C^+ | 0 \rangle$$

$C^+ | 0 \rangle$ → singlet

- $|0\rangle$ eigenstate → $C^+(t)$
- $|0\rangle$ no eigenstate → $A(t)$, $B(t)$

Wick Theorem → Pfaffians

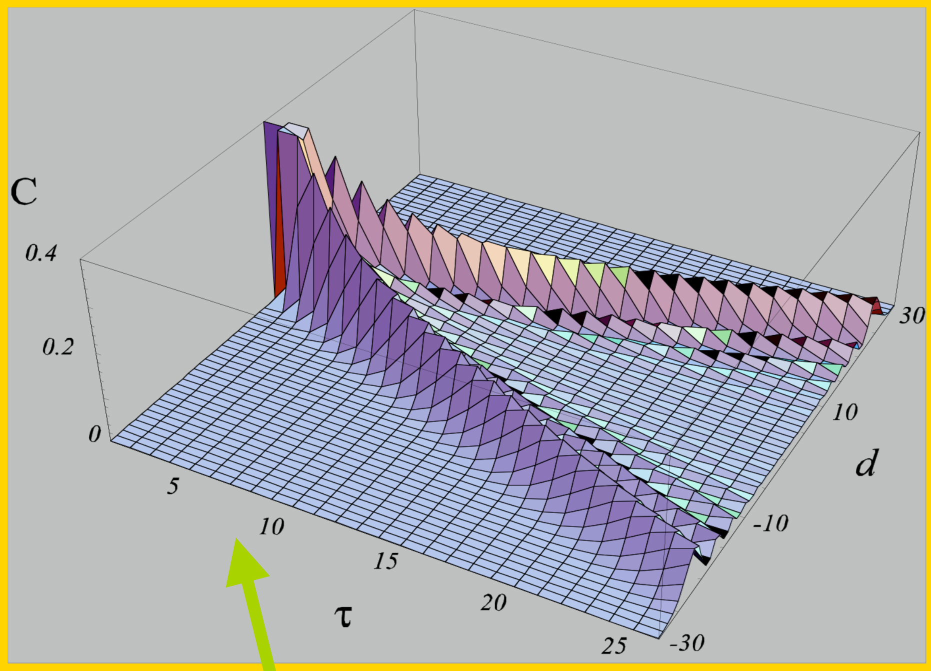
Caianello, Fubini, Nuovo Cimento 9, 1218 (1952)

γ=0 : (isotropic) XY model

- Total S_z conserved
- $|0\rangle$ is eigenstate !

Þ Entanglement measures can be calculated explicitely:

$\rightarrow$ Bessel Functions



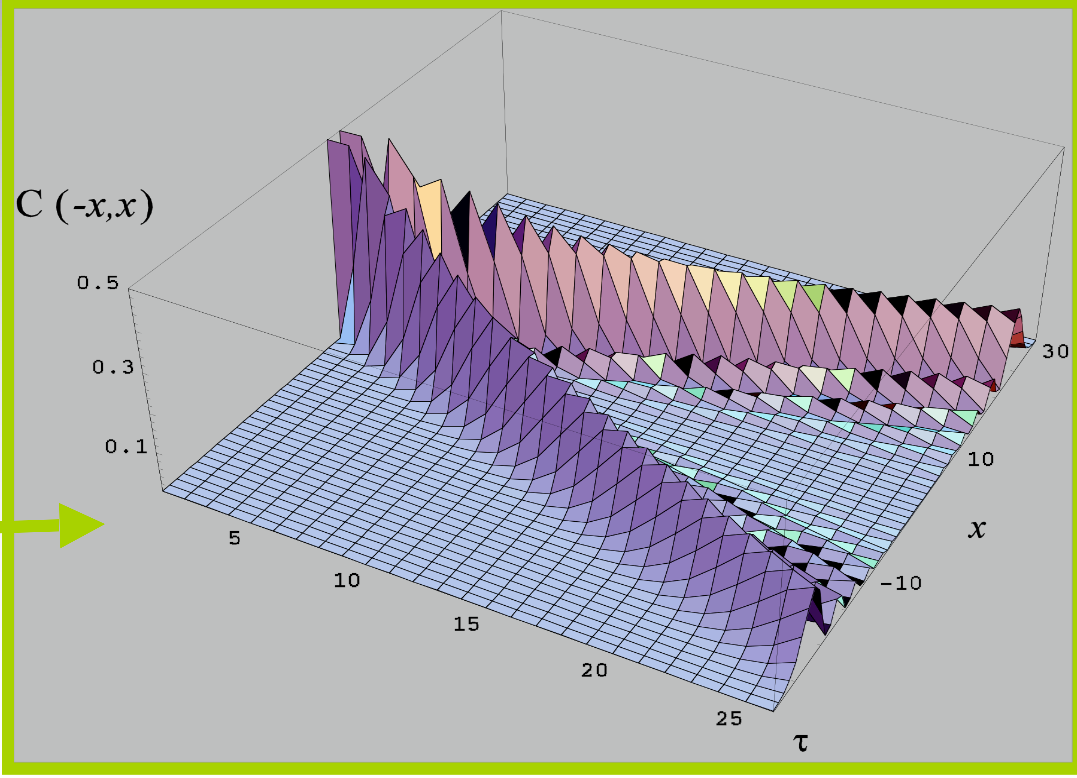
g=0 : XY model

total S_z conserved

Nearest neighbors

EPR-type transport

CKW conjecture ✓



$\gamma \neq 0$: anisotropic XY models

- Spin-flips break Total S_z conservation

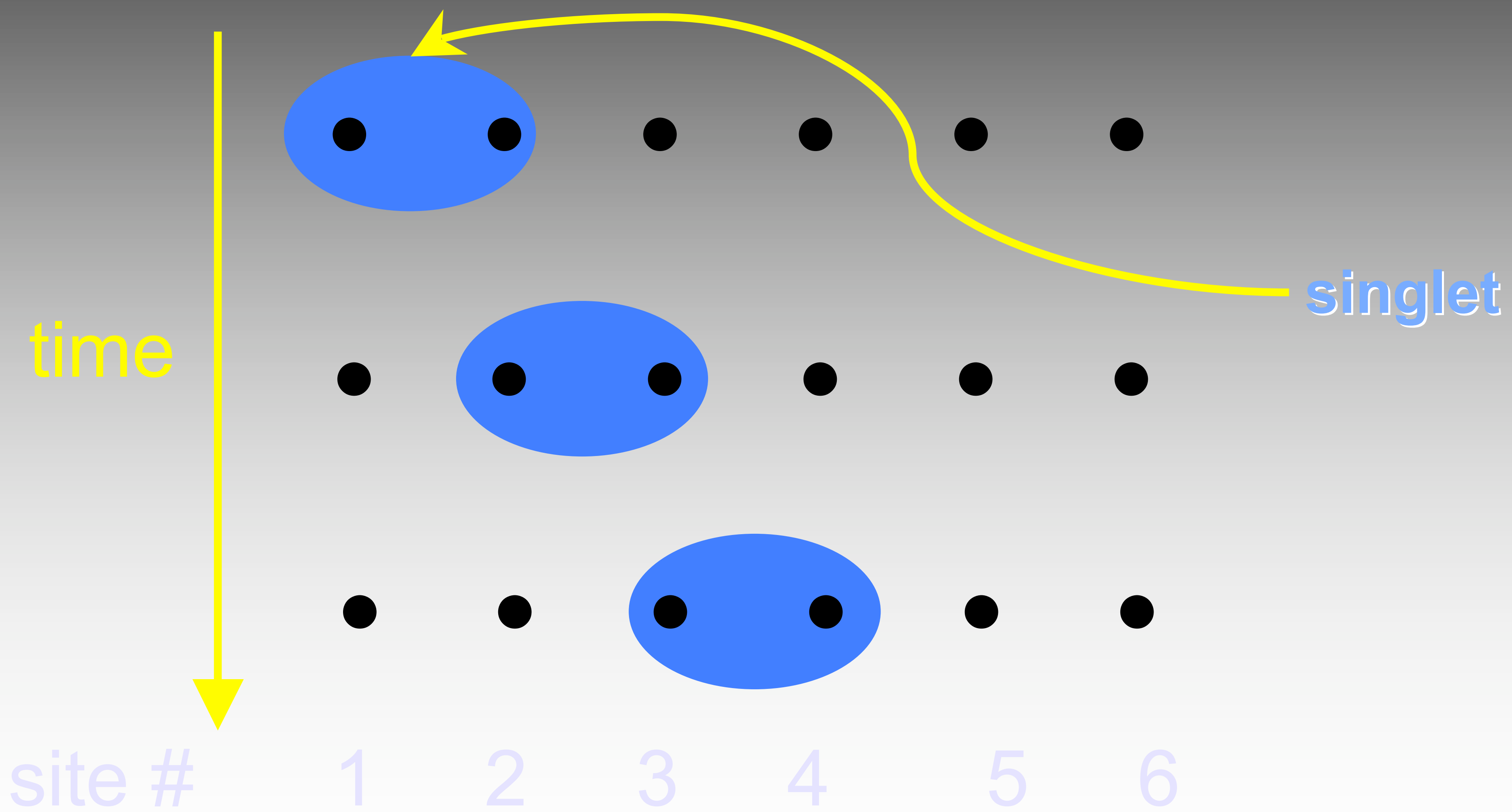
- $|0\rangle$ is NO eigenstate !

Þ Calculation of correlation functions gets much more involved

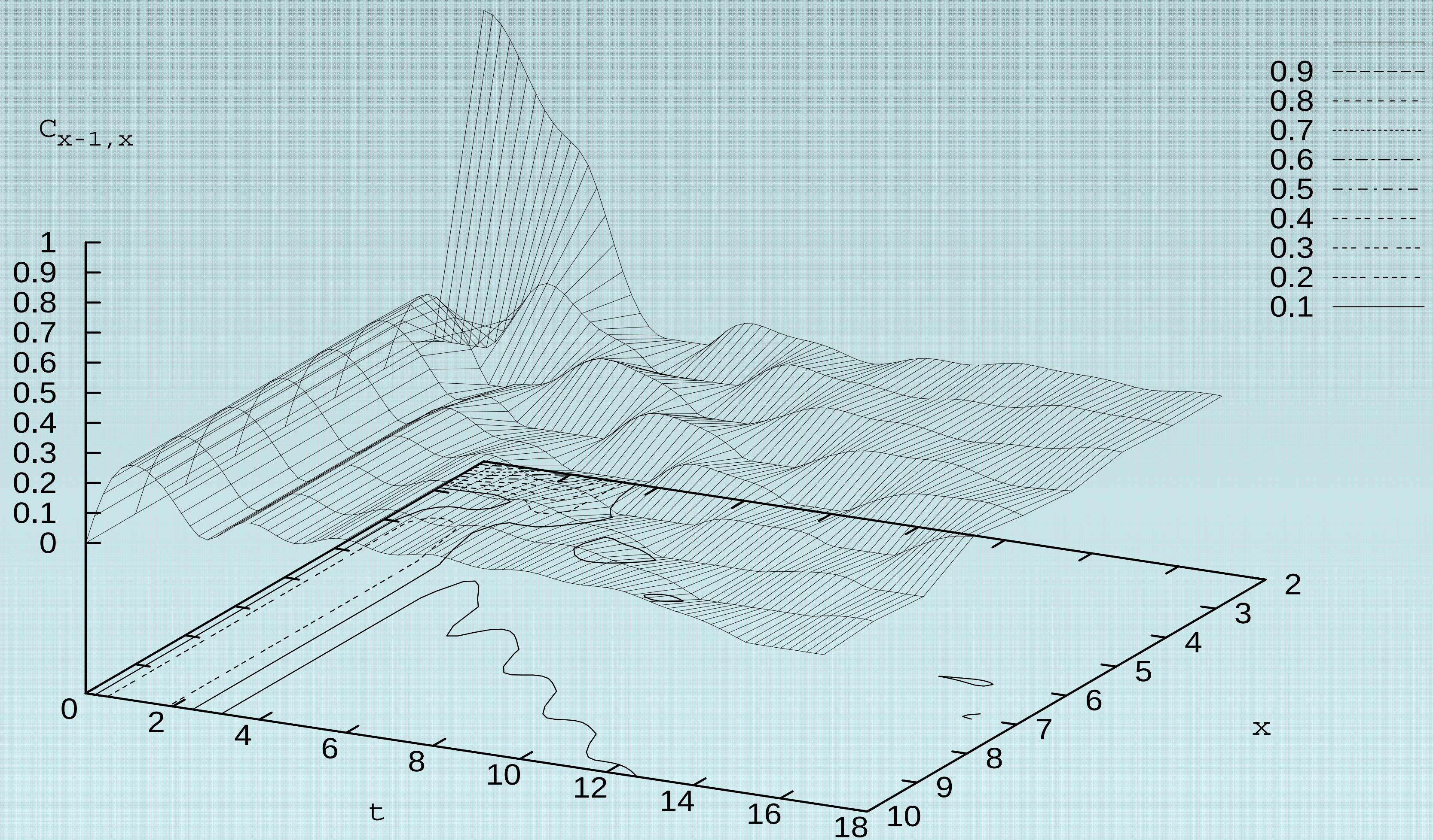
Þ only at γ=1 at λ=1 Bessel Functions

appear as matrix elements

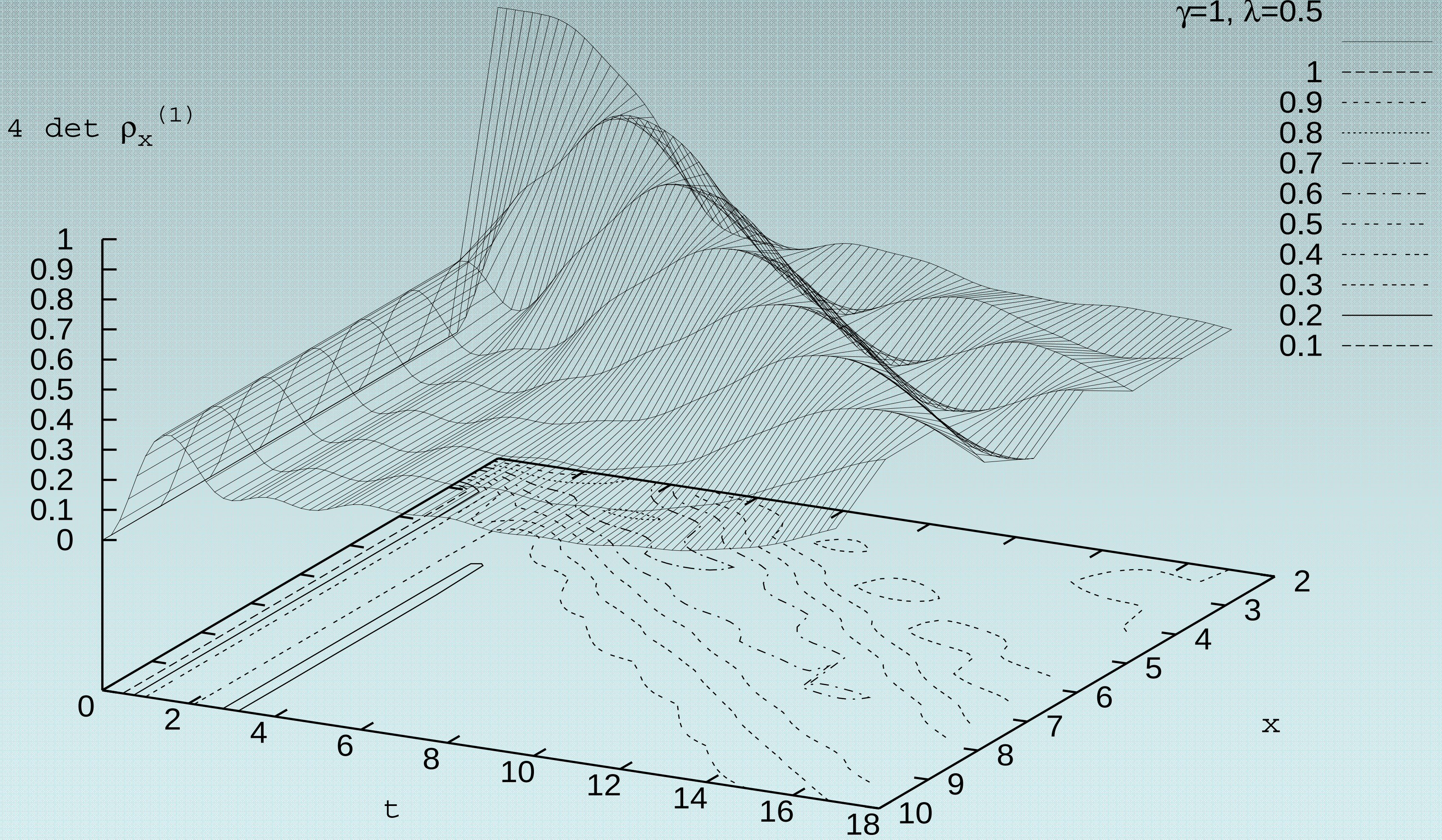
Singlet Propagation



C_1 for γ=1, λ=0.5



4 det ρ_x for γ=1, λ=0.5



Quantum information in disordered ultracod gases and complex systems?

W. Dür, L. Hartmann, M. Hein, M. Lewenstein,
and H.J. Briegel, quant-ph/0407075
A. Sen (De), U. Sen, V. Ahufinger, H.J. Briegel, A. Sanpera
and M. Lewenstein, quant-ph/0507XXX
V. Ahufinger, M. Pons, A. Sanpera, C. Wunderlich,
and M. Lewenstein, quant-ph/0507XXX

Quantum information with complex systems?

1. **Can one generate entanglement in trapped ion systems of Ising spin chains with long range couplings?** **YES!**

- We prepare the system in the product state $\psi(0) = |+>|+>|+>...$, where $|+>$ is an eigenstate of σ_x
- We then engineer the couplings and apply for certain time, so that $\psi(t) = \exp(-iH_{Ising}t)\psi(0)$
- We plot entanglement .....

2. **Can one generate entanglement in atomic spin glass (short range Edwards – Anderson model)** **YES!**

- We apply the same procedure as above, but engineer the SG couplings and apply for certain time, so that $\psi(t) = \exp(-iH_{SG}t)\psi(0)$

Can quantum information be processed in atomic complex systems?

???

Long range Ising spin models

- Hamiltonian:

$$H_{lr} = (1/N)\sum J_{ij}\, \sigma_i^z \sigma_j^z$$

N: total no of spins

$J_{ij} = \sum_\mu \xi_\mu^{\,i}\, \xi_\mu^{\,j} / (\lambda_\mu)^2$ (for harmonically confined ions $J_{ij} \sim 1/|r_i - r_j|^3$)

Proposed realization:

Trapped ions: F. Mintert and C. Wunderlich, Phys. Rev. Lett. 87, 257904 (2001); D. Porras and J.I. Cirac, Phys. Rev. Lett. 92, 207901 (2004); V. Ahufinger et al. (2005).

Dipolar gases: for a review see M.A. Baranov, Ł. Dobrek, K. Góral, L. Santos, and M. Lewenstein, Phys. Scripta T102, 74 (2002); see also contributions of the T. Pfau group (experiments, Stuttgart), Hannover group (Baranov, Santos, Lewenstein), G.V. Shlyapnikov group, K. Rzążewski group (Warsaw), L. You group (Atlanta)…; in optical lattices see K. Góral, L. Santos, and M. Lewenstein, Phys. Rev. Lett. 88,170406 (2003).

"Infinite" range Ising spin model

(Lipkin-Meshkov-Glick model)

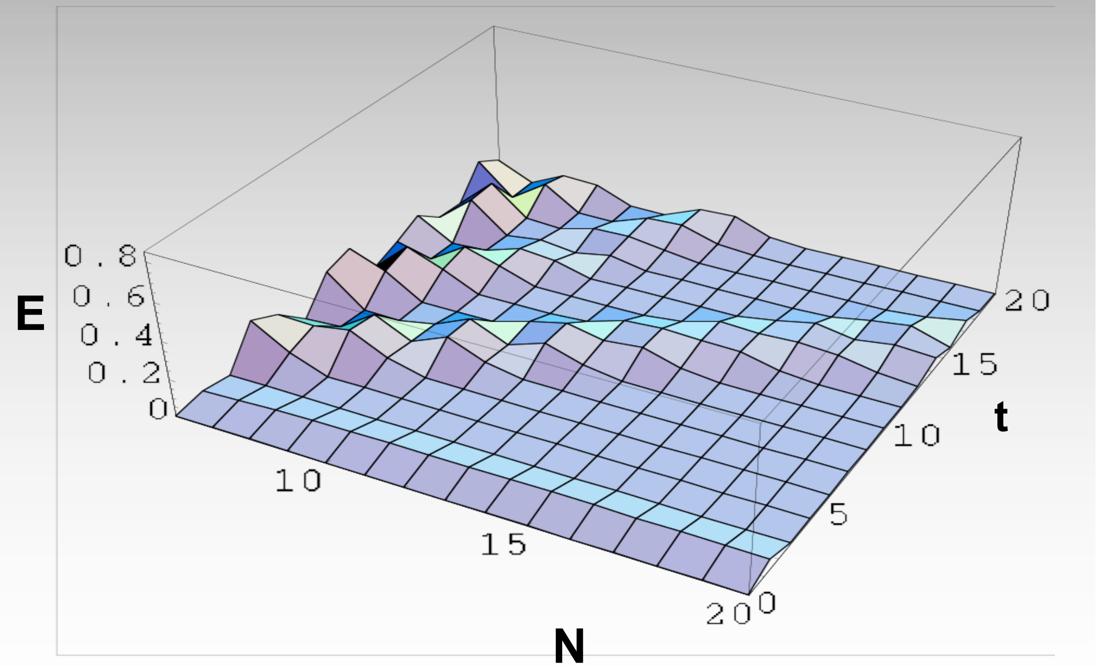
$\lambda_1 = 1$, ξ_1^i = constant for all i and $\lambda_\mu \to \infty$ for $\mu \geq 2$

$H_{lro} = (1/N)(S_z)^2$; $S_z = \sum \sigma_i^z/2$

Initial state: **$|\Psi\rangle = \prod_i |+\rangle_i$**

Possible to **ANALYTICALLY** compute the evolved state and study the dynamics of the entanglement of the evolved states!

Generation of entanglement of bipartite states with respect to time and number of spins. Revival of entanglement happens on the time scale **N** and collapse on the time scale **√N**. Region of separability is much smaller if one considers tripartite entanglement!!!



Hopfield model of Neural Network

$\lambda_\mu = 1$ for all μ

$H_{NN} = (1/N)\sum_{i,j} \sum_\mu \xi_\mu^i \xi_\mu^j \sigma_i^z \sigma_j^z$; $(i, j = 1.. N, \mu = 1..p)$

p: number of "patterns" of the neural network, and are described by random variables $\xi_\mu^i = \pm 1$ with probability ½.

Again it is possible to get the evolved state analytically.

Nearest neighbor entanglement of the evolved state appears and persists for times of order $\sqrt{(N/p)}$. There are repeated revivals in entanglement with period

$\Pi N/2$ for odd p,

ΠN for even p.

Quantum Spin Glass model

- **Edwards - Anderson model (short-range)**

$$H_{E\text{-}A} = -\tfrac{1}{4}\sum J_{ij}\,\sigma_i^z\,\sigma_j^z$$

σ´s are the Pauli matrices.

Apply it to the initial state:

$$|\Psi\rangle = \prod_i |+\rangle_i$$

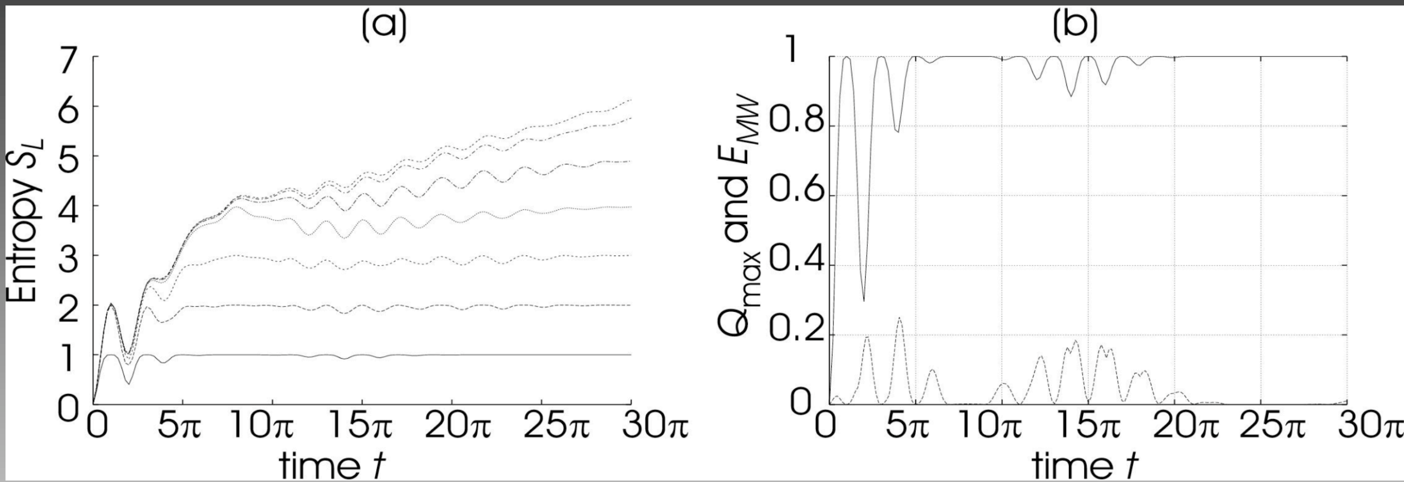
The time dependent density matrix:

$$\rho(t, \{J_{ij}\}) = \exp(-i\, H_{E\text{-}A}\, t)\, |\Psi\rangle \langle\Psi|\, \exp(\, i\, H_{E\text{-}A}\, t)$$

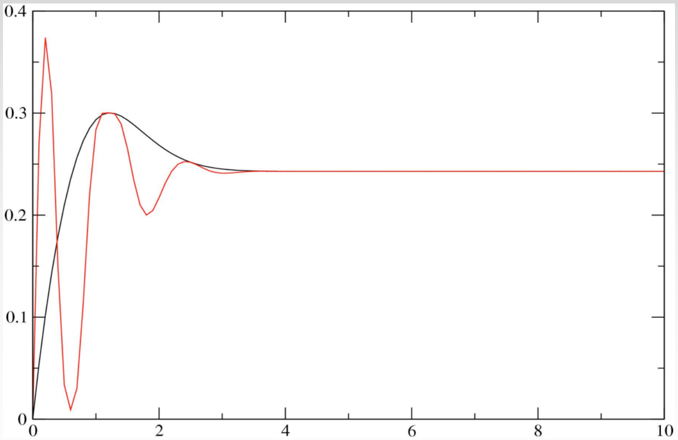
The nearest neighbor two-qubit density matrix:

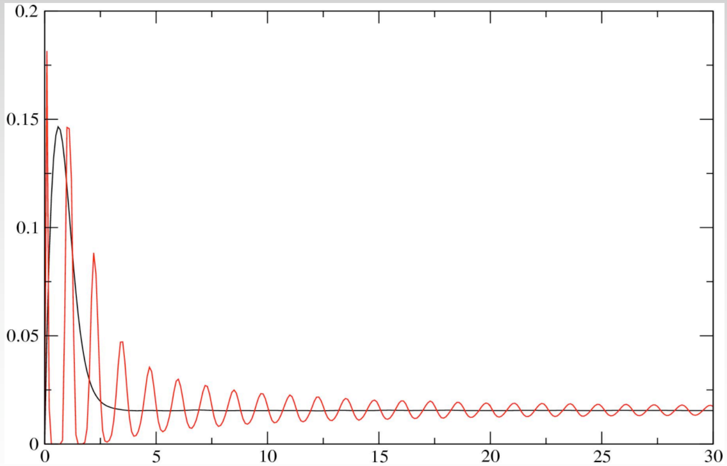
$$\rho_{12}(t) = \mathrm{tr}_{k\neq 1,2}\, (\rho(t, \{J_{ij}\})\,)$$

Dynamics of entaglement



Local entropy growth and entanglement dynamics in the complex system of trapped ions (from W. Dür et al.)





Entanglement dynamics in the E-A spin glass model in 1D and 2D

Remarks

- In 1D and 2D, entanglement behaves similarly.
- In both cases, average state is a **separable** state.
- It is appropriate to perform averaging after calculating entanglement, i.e. quenched averaging.