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Multipartite entanglement and decoherence

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These are preliminary lecture notes, intended only for distribution to participants
Multipartite entanglement and decoherence

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Nonlinear Dynamics in Quantum Systems

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Entanglement, a central resource of quantum information processing

Key challenges

- Coherent evolution over sufficiently long time scales
- Controlled entanglement of a large number of qubits/subsystems
- How long does this special fuel “entanglement” last, under realistic conditions?
- Scalability – how do size and coherence requirements compete?

[The Economist, Quantum Dreams, 10/3/2001]
Open, multipartite systems

Quantum information processing with
- photons
- atoms, molecules
- trapped atoms or ions
faces residual coupling to uncontrolled degrees of freedom
- electromagnetic vacuum
- atomic continua
- nonresonantly coupled states, i.e., decoherence

We know little about decoherence in many degrees of freedom systems, let alone about the entanglement between these different degrees of freedom in the presence of decoherence.

[Arndt, Hornberger & Zeilinger, Physics World 2005]
What we want, and what we need

**Wanted!** “Life time” of entanglement for large quantum registers – need

- . . . efficient entanglement detector
- . . . efficient entanglement quantifier
- . . . description of entanglement dynamics

experiments: life times of atomic Bell states  [Roos et al., PRL 2004]
Entanglement measures – pure states

Any pure state $|\Psi\rangle$ which cannot be written as a product $|\phi\rangle \otimes |\chi\rangle$, i.e.,

$$|\Psi\rangle \neq |\phi\rangle \otimes |\chi\rangle$$

is nonseparable or entangled.

A useful measure of entanglement is provided by the concurrence

[Rungta, Buzek, Caves, Hillery, Milburn, PRA 2001]

$$c(\Psi) = \sqrt{2(1 - \text{tr}\rho_r^2)} ,$$

which vanishes precisely for separable states, for which

$$\text{tr}\rho_r^2 = 1 .$$

Further quantifiers are, e.g., Schmidt coefficients, von Neumann entropy, etc . . .
Entanglement measures – mixed states

Given an arbitrary (generally mixed) quantum state \( \rho \), with a pure state decomposition

\[
\rho = \sum_j p_j |\Psi_j \rangle \langle \Psi_j | ,
\]

one might be tempted to generalize pure state concurrence,

\[
c(\rho) \rightarrow \sum_j p_j c(\Psi_j) .
\]

However, this is **WRONG**, since the decomposition (1) is **NOT UNIQUE**!
Need to **optimize over all possible decompositions of \( \rho \):**

\[
c(\rho) = \inf_{\{p_j, \Psi_j\}} \sum_j p_j c(\Psi_j) .
\]

Unfavourable scaling of optimization space! Explicit solution only available for pairs of qubits  [Wootters, PRL 1998].
Reformulating concurrence

An efficient quantifier for entanglement is derived by rewriting pure state concurrence as

\[ c(\Psi) = \sqrt{\langle \Psi \rangle \otimes \langle \Psi | A | \Psi \rangle \otimes | \Psi \rangle} , \]

where \( A \) acts on two copies of the given state \(| \Psi \rangle\).

\( A \sim P^{(1)} \otimes P^{(2)} \) projects on the antisymmetric subspaces of the underlying factor spaces. Thus, \( c \) vanishes for states which are invariant under exchanges of the individual copies. Possible interpretation in terms of suitable measurement(s) on two copies.
Mixed state concurrence in higher dimensions

For mixed states of bi- or multipartite states of arbitrary finite dimension we obtain

\[ c(\rho) = \inf_{\{p_j, \Psi_j\}} \sum_j \sqrt{\langle \Psi_j | \otimes \langle \Psi_j | A | \Psi_j \rangle \otimes | \Psi_j \rangle}, \]

which can be immediately generalized to multipartite systems, by generalizing the representation of \( A \).

This provides the desired tool for our assessment of the crucial scaling properties!
Explicit evaluations

The (numerical) evaluation of the infimum

\[ c(\rho) = \inf_{\{p_j, \Psi_j\}} \sum_j \sqrt{\langle \Psi_j | \otimes \langle \Psi_j | A | \Psi_j \rangle \otimes | \Psi_j \rangle} \]  

(2)

provides an upper bound of \( c(\rho) \) . . . we need lower bounds!

The algebraic structure of (2) leads to a hierarchy of approximations from below

1. **optimized lower bound** (implies numerical optimization over lower dimensional optimization space)  
   [Mintert, Kuś & A.B., PRL 2004]

2. **algebraic lower bound** (implies diagonalization of a matrix of dimension equal to the maximal rank of \( A \))

3. **quasi pure approximation (qpa)**  
   [Mintert & A.B., PRA 2005]
Quasi pure approximation (qpa)

- implies diagonalization of a matrix only of the dimension of $\rho$! (computationally “cheap”!)

- inspired by state of the art experiments: $\rho$ is initially quasi pure, i.e., it possesses one largely dominant eigenvalue $\lambda_0 \gg \lambda_j$, for all $j \neq 0$.

Nontrivial test case:
nonseparable state with positive partial transpose, parametrized by $a$

qpa turns out to be a lower bound of the concurrence of $\rho$
Dynamics under nonvanishing environment coupling

Three types of dynamics

1. **random** system-environment time evolution with subsequent trace over the environment

\[ H = H_{\text{sys-env}} + H_{\text{sys}} \otimes 1_{\text{env}} + 1_{\text{sys}} \otimes H_{\text{env}} \]

2. entanglement decay due to coupling of subsystems to “private” baths

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H_{\text{sys}}, \rho] + \mathcal{L}\rho = -\frac{i}{\hbar} [H_{\text{sys}}, \rho] + \sum_j \frac{\Gamma_j}{2} \left( 2d_j \rho d_j^\dagger - d_j^\dagger d_j \rho - \rho d_j^\dagger d_j \right) \]

3. entanglement generation vs. decoherence
Random time evolution

- Concurrence for an initially pure, maximally entangled $3 \times 5$ bipartite state $|\Psi_0\rangle = \sum_{j=1}^{3} |j,j\rangle / \sqrt{3}$ under random, non-unitary time evolution;
- $\alpha_{sb}$ – system-environment coupling strength

- dash-double dotted line: von Neumann entropy $S = -\text{tr} \rho_{\text{sys}} \ln \rho_{\text{sys}}$
  (measures mixing)

[Mintert & A.B., PRA 2005]
Entanglement decay of bipartite two-level systems

Initial states $|\Psi_s\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ (left) and $|\Psi_s\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (right)

- coupling to thermal bath with
  - zero temperature (only spontaneous emission; dotted line)
  - finite temperature ($\bar{n} = 0.1$ thermal photon in the environment; dashed)
  - infinite temperature (noisy environment; solid)

- or to dephasing reservoir (only coherence loss; long dashed)

(multi-) exponential decay with finite or infinite separability times

$N$-partite entanglement

We generalize bipartite concurrence $c_2(\Psi) = \sqrt{2(\langle \Psi | \Psi \rangle^2 - \text{tr} \rho_r^2)}$ for $N$-partite systems (with $j$ counting all possible partitions):

$$c_N(\Psi) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2)\langle \Psi | \Psi \rangle^2 - \sum_j \text{tr} \rho_j^2}$$

$c_N(t)$ for $N = 3$

**W**-states (solid lines)

$$|W_N\rangle = (|00\ldots01\rangle + |00\ldots10\rangle + \ldots + |10\ldots00\rangle)/\sqrt{N}$$

**GHZ**-states (dashed lines)

$$|\text{GHZ}_N\rangle = (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}$$

zero temperature (circles), infinite temperature (squares), and dephasing (triangles) environments.
Scaling of the decay rates $\gamma$

\[ \frac{\gamma}{\Gamma} \]

**top:** GHZ – **bottom:** W

- **circles:** zero temperature
- **squares:** infinite temperature
- **triangles:** dephasing

**W-states’ decay rates independent of** $N$ **for zero temperature and dephasing!**

[Carvalho, Mintert & A.B., PRL 2004]
Entanglement generation vs. decoherence

Mølmer-Sørensen scheme to prepare GHZ-like states

\[ |\Psi_N\rangle = \left( |00\ldots0\rangle + e^{i\phi_N} |11\ldots1\rangle \right)/\sqrt{2} \]  

[Mølmer & Sørensen, PRL 1999]

- motional decoherence limits the preparation efficiency
- only \textit{after} the preparation process are internal and vibrational degrees of freedom uncorrelated
Preparation efficiency vs. $t$ and $N$

- various coupling strengths to vibrational noise $\Gamma/\nu = 0$ (thick line), $1 \times 10^{-4}$ (full line), $2 \times 10^{-4}$ (dashed), $3 \times 10^{-4}$ (dash-dotted), $4 \times 10^{-4}$ (dotted)

[Turchette et al., PRA 2000]

- **left**: concurrence $c_4$ of four ions vs. preparation time

- **right**: maximum concurrence vs. ion number – nonmonotonous behaviour indicates that environment effects become more detrimental with increasing $N$!
Dynamics revisited

So far

1. Solve master equation

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] + \sum_k \frac{1}{2} \left(2 J_k \rho J^\dagger_k - J^\dagger_k J_k \rho - \rho J^\dagger_k J_k \right),
\]

2. and evaluate entanglement measure $M(\rho(t))$ for all $t$

Now: **Unravel Entanglement!**

[Carvalho, Busse, Brodier, Viviescas, A.B., quant-ph/0510006]

*start with pure state $|\Psi_0\rangle$ and generate stochastic pure state evolution under action of*

- jumps mediated by jump operators $J_k$, with probability $\delta p_k$,
- alternating with nonhermitian free evolution, generated by $H_{\text{eff}} = H_{\text{sys}} - i\hbar \sum_k J^\dagger_k J_k / 2$
Quantum trajectories

... naturally generate a pure state decomposition of $\rho(t)$!

$$\rho(t) \simeq \sum p_i |\Psi^i(t)\rangle \langle \Psi^i(t)| .$$

... but is it optimal (remember the inf above!) ??
Optimal unravelling

• Jump operators which generate the same master equation are **not uniquely defined** – most general form:

\[ L_{k,\pm} = \mu \text{Id} \pm \frac{1}{\sqrt{2}} \sum_i U_{ki} J_i, \]

\( U \) unitary, \( \mu \) complex.

• different jump operators correspond to different measurement prescriptions, i.e. different ways of monitoring the system

• minimize average entanglement \( \overline{M(\delta t)} \) after first time step \( \delta t \)

\[
\overline{M(\delta t)} = (1 - \sum_{k=1}^{N} \delta p_k) M(\Psi_{\delta t}^{N+1}) + \sum_{k=1}^{N} \delta p_k M(\Psi_{\delta t}^{k})
\]
Two qubits – zero and infinite temperature environment

\[ \overline{M} \equiv \sum_i p_i M(\Psi^i(t)) \] measures entanglement!

\[ |\Psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \]

zero temperature (spontaneous emission)

\[ J_1 = \sqrt{\Gamma} \sigma_{-}^{(1)} \text{, } J_2 = \sqrt{\Gamma} \sigma_{-}^{(2)} \]

infinite temperature (inset)

\[ J_{1,3} = \sqrt{\Gamma} \sigma_{\pm}^{(1)} \text{, } J_{2,4} = \sqrt{\Gamma} \sigma_{\pm}^{(2)} \]

optimize

\[ L_{k,\pm} = \frac{\mu \text{ Id} \pm \sum_i U_{ki} J_i}{\sqrt{2}} \]
Three qubits – dephasing environment

tripartite concurrence $C_3(t)$

$|\Psi_0\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$

jump operators for ions $k = 1, 2, 3$

$J_k = \sigma_+^{(k)} \sigma_-^{(k)}$

(here already optimal with coupling operators from master equation)

... works equally well for W states under zero temperature environment coupling
What to take away . . .

!!Scalability of entanglement dynamics!!

! With a suitable reformulation of concurrence, we obtained an efficient quantifier of mixed state entanglement
  • in arbitrary finite dimensions • for bi- as well as $N$-partite systems

! W-states outperform GHZ-states in terms of their robustness against decoherence, with increasing number of entangled qubits

! Our tool box also provides a means to scrutinize the competition between coherent entanglement generation schemes and decoherence

! unravelling entanglement by optimal measurements

! no ambiguity of mixed state entanglement with respect to different unravellings (in contrast to Nha & Carmichael, PRL 2004)
What to work on . . .

? Which observable(s) to measure to obtain a reliable estimate of the entanglement properties? Scaling properties?

? How robust are these entanglement measures against errors in the determination of the density matrix’ elements?

? “Universal” time evolution of entanglement alike the Lindblad form? General classification of arbitrary mixed states? Will unravelling always work?

? Decoherence and entanglement in large/heavy objects with many degrees of freedom?

Literature (selection):
