Some Aspects of Mean Field Dynamo Theory

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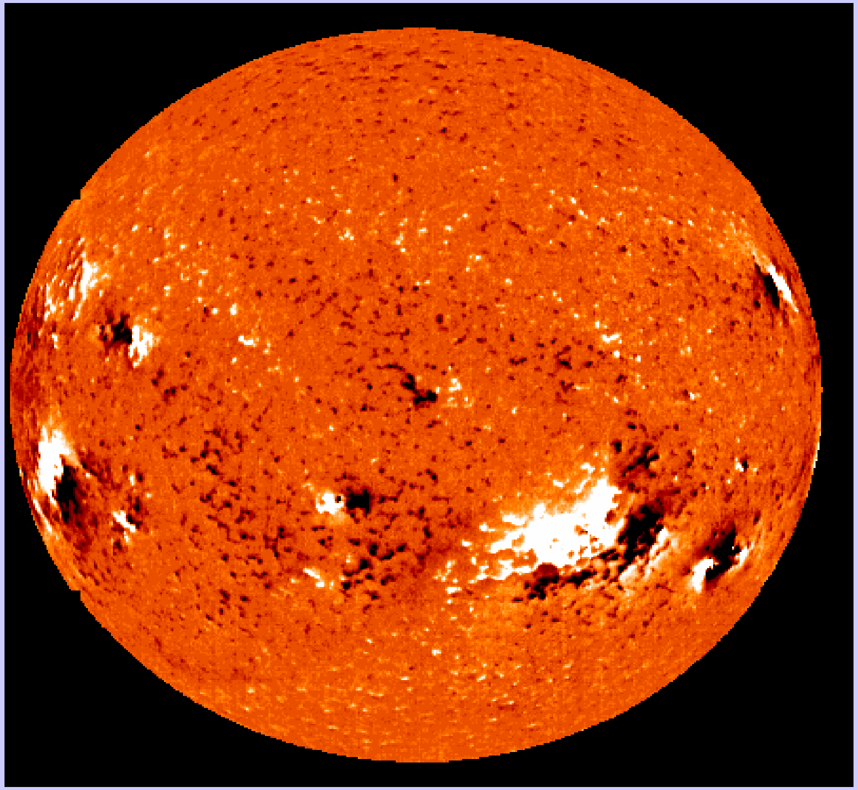
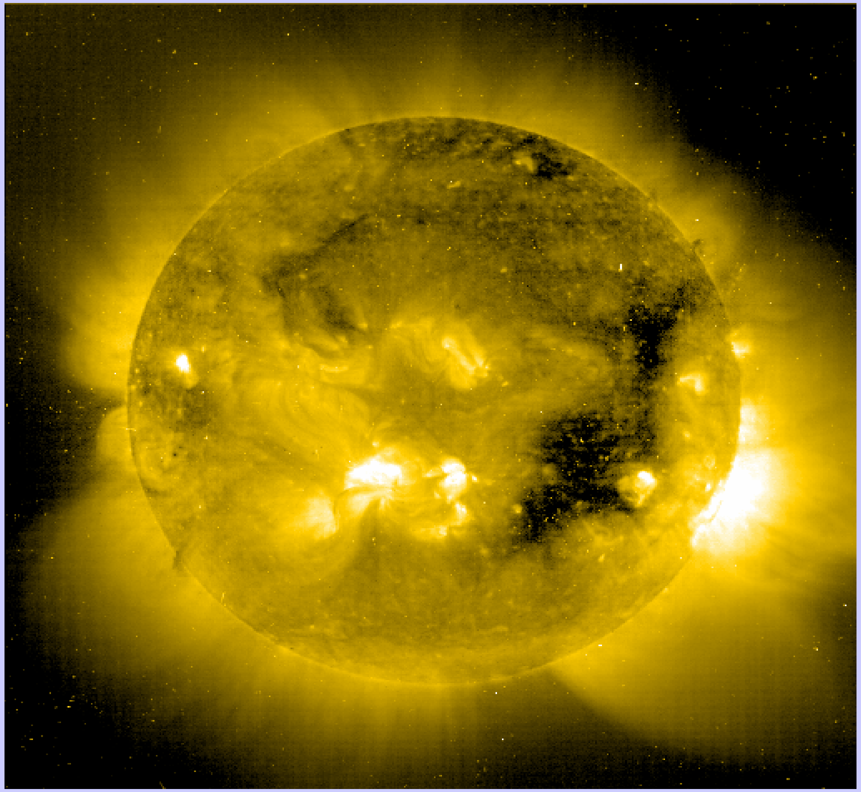
The Sun's Global Magnetic Field

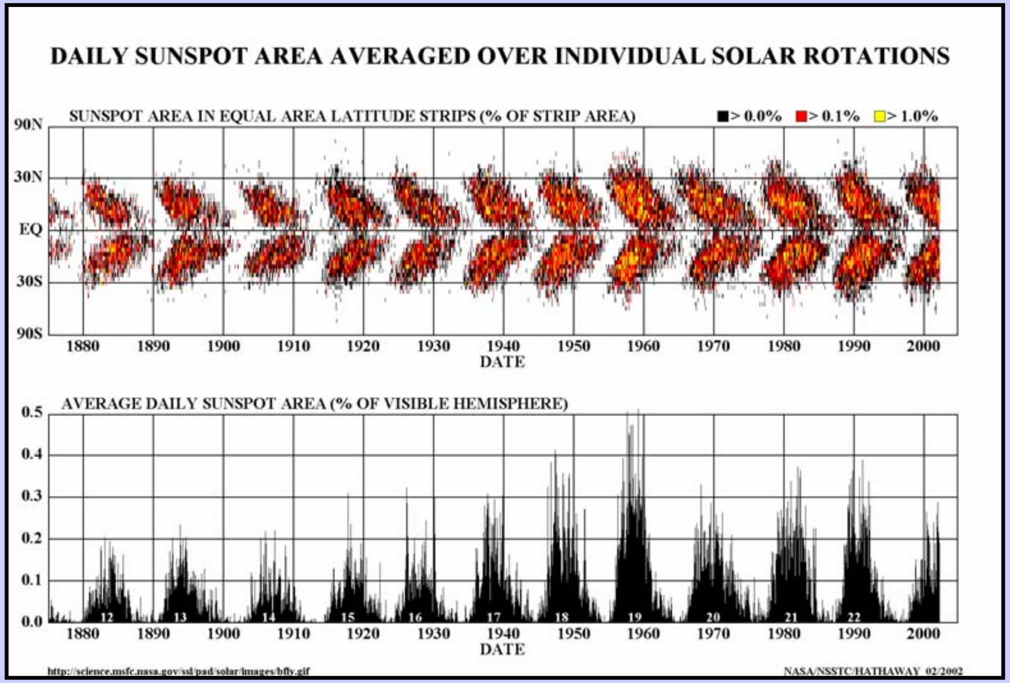


Ca II emission



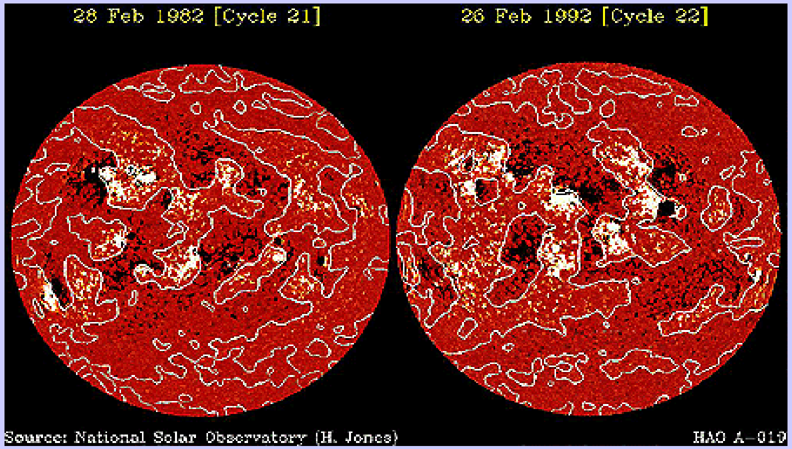
Extreme ultra violet

Temporal variation of sunspots



Number of sunspots varies cyclically with an approximately 11 year cycle.

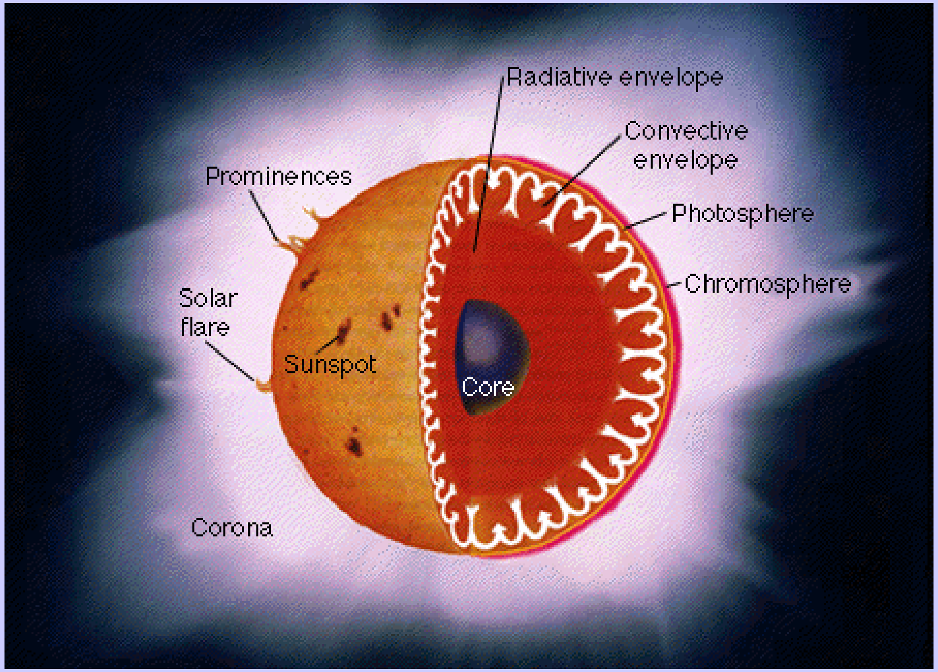
Latitudinal location of spots varies with time – leading to *butterfly diagram.*



Sunspots typically appear as bipolar pairs.
Polarity of sunspots opposite in each hemisphere
Polarity of magnetic field reverses every 11 years.
22 year magnetic cycle.

Known as *Hale's polarity laws*.

The Solar Dynamo



Now almost universally believed that the solar magnetic field is maintained by some sort of *dynamo mechanism,* in which the field is regenerated by inductive motions of the electrically conducting plasma.

The precise site of the dynamo is still a matter of some debate – though is certainly in all, or part, of the convection zone and, possibly, in the region of overshoot into the radiative zone.

Dynamo theory deals with the regeneration of magnetic fields in an electrically conducting fluid or gas – nearly always through the equations of *magnetohydrodynamics* (MHD).

The vast majority of the modelling of astrophysical dynamos has been performed within the framework of mean field electrodynamics.

Kinematic Mean Field Theory

Starting point is the magnetic induction equation of MHD:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},$$

where **B** is the magnetic field, **u** is the fluid velocity and η is the magnetic diffusivity (assumed constant for simplicity).

In dimensionless units:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + Rm^{-1} \nabla^2 \mathbf{B},$$

Assume scale separation between large- and small-scale field and flow:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{U} = \mathbf{U}_0 + \mathbf{u},$$

where **B** and **U** vary on some large length scale *L*, and **u** and **b** vary on a much smaller scale *l*.

$$\langle \mathbf{B} \rangle = \mathbf{B}_0, \quad \langle \mathbf{U} \rangle = \mathbf{U}_0,$$

where averages are taken over some intermediate scale *l* « *a* « *L*.

For simplicity, ignore large-scale flow, for the moment.

Induction equation for mean field:

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times \mathbf{E} + \eta \nabla^2 \mathbf{B}_0,$$

where mean emf is $\mathbf{E} = \langle \mathbf{u} \times \mathbf{b} \rangle.$

This equation is exact, but is only useful if we can relate $\mathbf{E}$ to $\mathbf{B}_0$.

Consider the induction equation for the fluctuating field:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b},$$

where $\mathbf{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle.$

Traditional approach is to assume that the fluctuating field is driven solely by the large-scale magnetic field.

i.e. in the absence of $\boldsymbol{B}_0$ the fluctuating field decays.

i.e. No small-scale dynamo

Under this assumption, the relation between $\mathbf{b}$ and $\mathbf{B}_0$ (and hence between E and $\mathbf{B}_0$) is linear and homogeneous.

Postulate an expansion of the form:

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \ldots$$

where α_{ij} and β_{ijk} are *pseudo*-tensors.

Simplest case is that of isotropic turbulence, for which $\alpha_{ij} = \alpha\delta_{ij}$ and $\beta_{ijk} = \beta\varepsilon_{ijk}$.
Then mean induction equation becomes:

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\alpha \mathbf{B}_0) + (\eta + \beta)\nabla^2 \mathbf{B}_0.$$

α: regenerative term, responsible for large-scale dynamo action.
Since $\mathcal{E}$ is a polar vector whereas ***B*** is an axial vector then α can be non-zero only for turbulence lacking reflexional symmetry (i.e. possessing handedness). The simplest measure of the lack of reflexional symmetry is the helicity of the flow, $\mathbf{u} \cdot \nabla \times \mathbf{u}$.

β: turbulent diffusivity.

Analytic progress possible if we neglect the ***G*** term ("first order smoothing").

This can be done if either the correlation time of the turbulence τ or *Rm* is small.

For the former (assuming isotropy):

$$\alpha = -\frac{\tau}{3}\langle \mathbf{u}.\boldsymbol{\omega} \rangle.$$

Correlations between ***u*** and ***b*** have been replaced by correlations between ***u*** and ***ω.***

For the latter:

$$\alpha = -\frac{\eta}{3}\iint \frac{k^2 F(k,\omega)}{\omega^2 + \eta^2 k^4}\, dk\, d\omega.$$

where *F(k,ω)* is the helicity spectrum function.

These results suggest a clear link between α and helicity.

Mean Field Theory – Applications

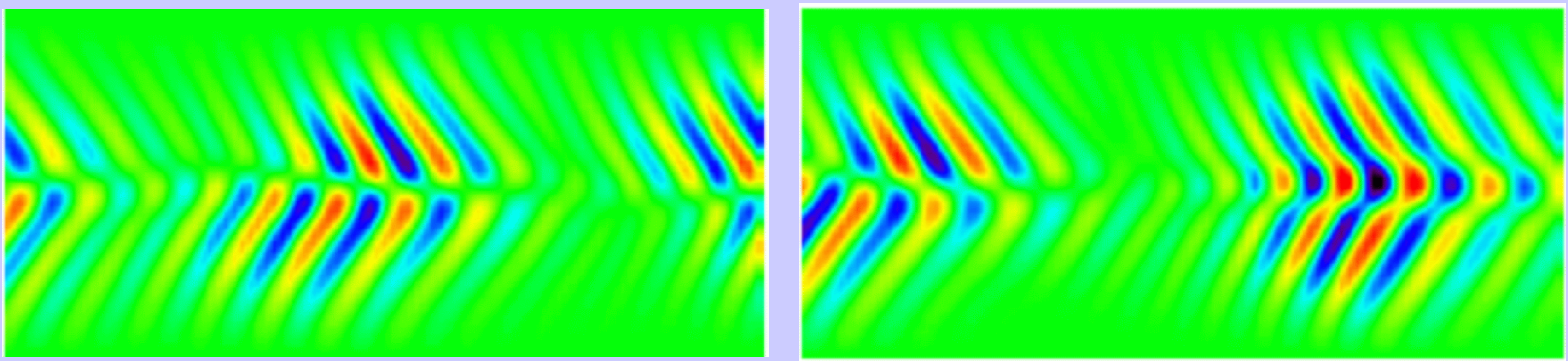
Mean field dynamo theory is very user friendly.

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U}_0 \times \mathbf{B}_0) + \nabla \times (\alpha \mathbf{B}_0) + (\eta + \beta)\nabla^2 \mathbf{B}_0.$$

A dynamo can be thought of as a mechanism for "closing the loop" between poloidal and toroidal fields. Velocity shear (differential rotation) naturally generates toroidal from poloidal field. The α-effect of mean field electrodynamics can complete the cycle and regenerate poloidal from toroidal field.

With a judicial choice of α and β (and differential rotation ω) it is possible to reproduce a whole range of observed astrophysical magnetic fields.

e.g. butterfly diagrams for dipolar and quadrupolar fields:



(Tobias 1996)

Crucial questions

Mean field dynamo models "work well" – and so, at some level, capture what is going on with cosmical magnetic fields.

However, all our ideas come from consideration of flows with either very short correlation times or with very small values of *Rm*.

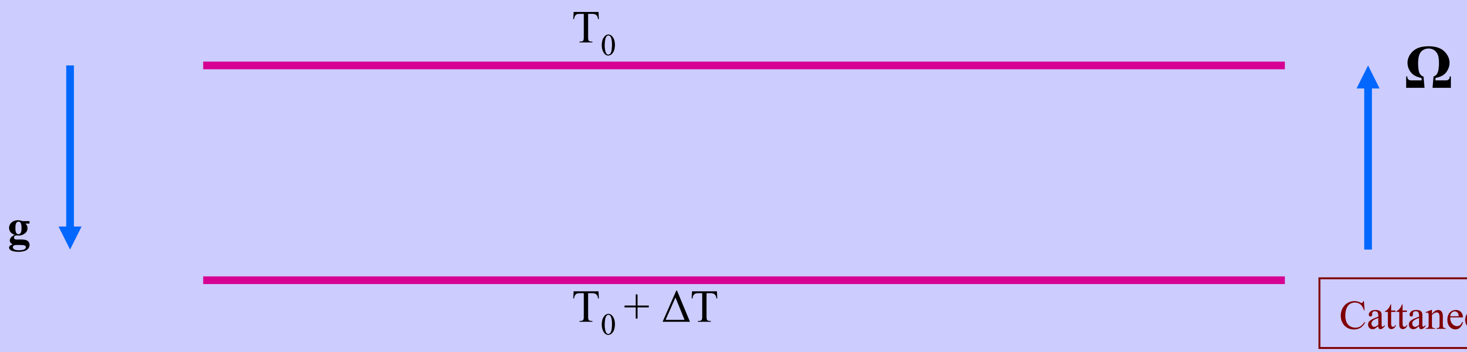
What happens in conventional MHD turbulence with *O(1)* correlation times and *Rm* >> 1?

1. We still do not fully understand the detailed micro-physics underlying the coefficients α, β, etc. – maybe not even in the kinematic regime.
2. What happens when the fluctuating field may exist of its own accord, independent of the mean field?

   What is the spectrum of the magnetic field generated? Is the magnetic energy dominated by the small scale field?
3. What is the role of the Lorentz force on the transport coefficients α and β? How weak must the large-scale field be in order for it to be dynamically insignificant? Dependence on *Rm*?

We shall address some of these via an idealised model.

Rotating turbulent convection



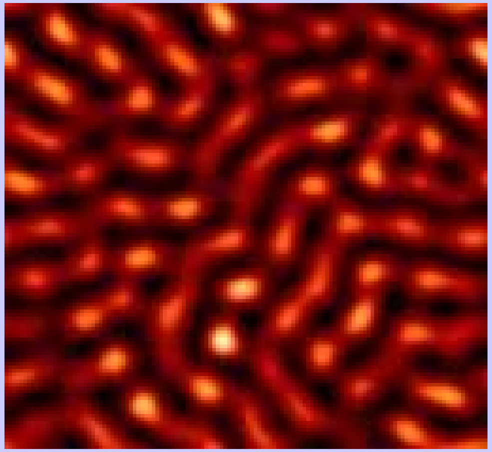
Cattaneo & Hughes 2005

Boussinesq convection.
Boundary conditions: impermeable, stress-free, fixed temperature, perfect electrical conductor.

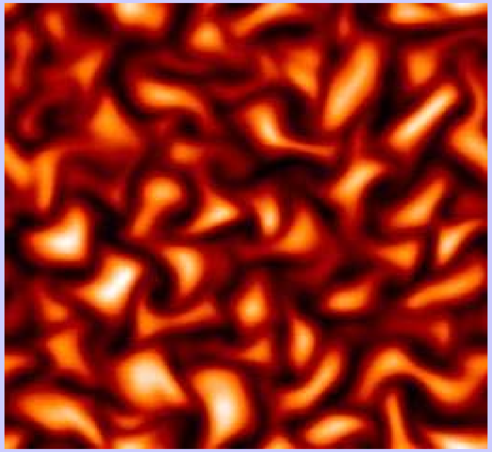
Taylor number, $Ta = 4\Omega^2 d^4/\nu^2 = 5 \times 10^5$,
Prandtl number $Pr = \nu/\kappa = 1$,
Magnetic Prandtl number $Pm = \nu/\eta = 5$.
Critical Rayleigh number = 59 008.

Anti-symmetric helicity distribution $\Longrightarrow$ anti-symmetric α-effect.
Maximum relative helicity ~ 1/3.

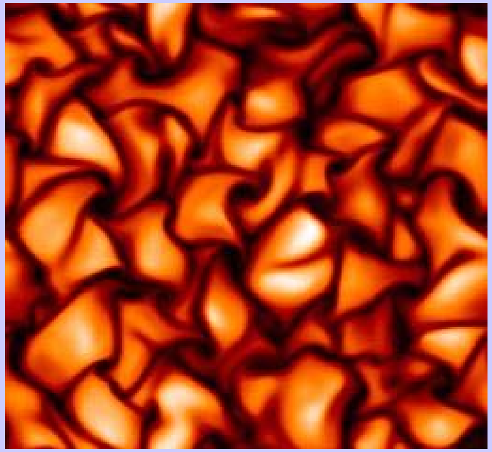
Temperature near upper boundary (5 x 5 x 1 box)



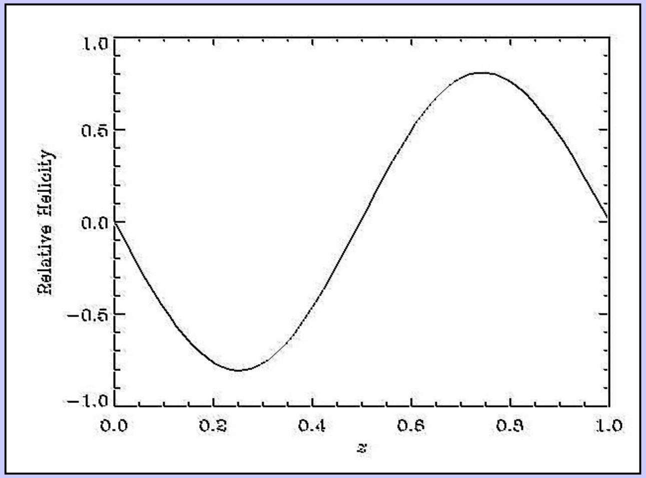
Ra = 70,000

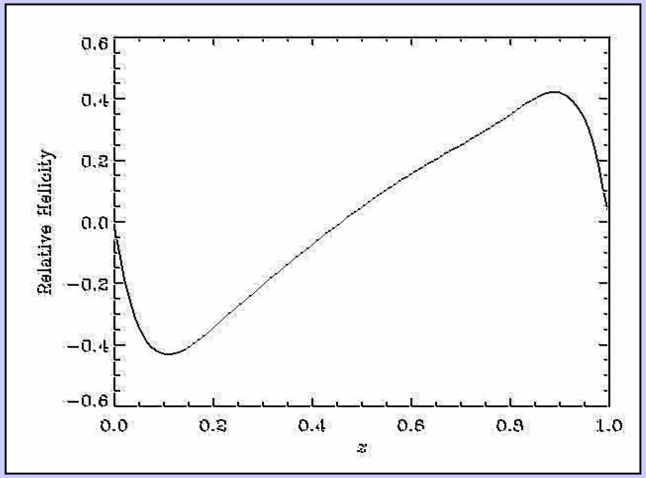


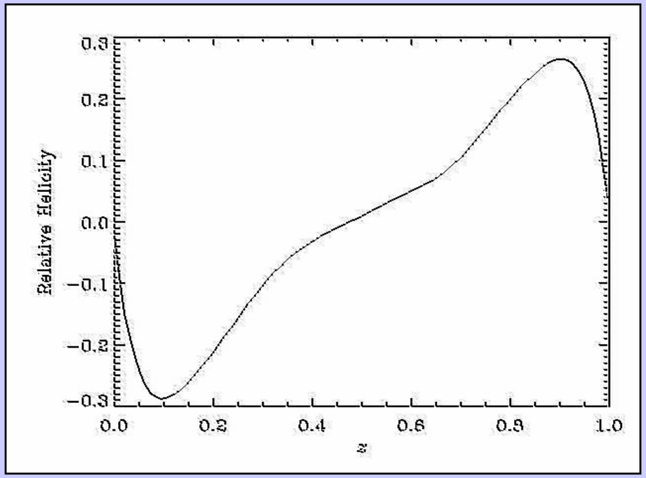
Ra = 150,000



Ra = 500,000

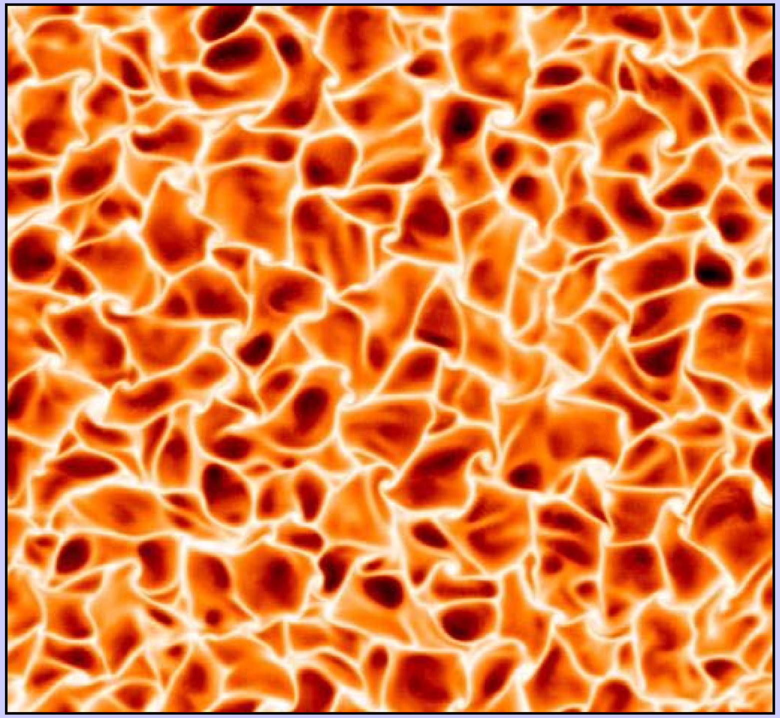


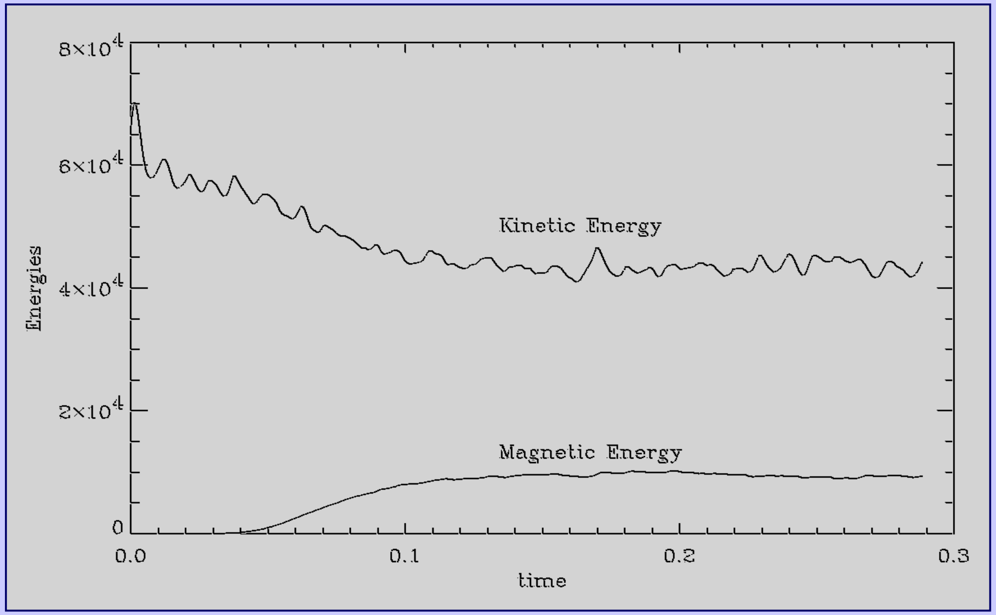




Relative Helicity $= \dfrac{\langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle}{\langle |\mathbf{u}|^2 \rangle^{1/2} \langle |\nabla \times \mathbf{u}|^2 \rangle^{1/2}}$

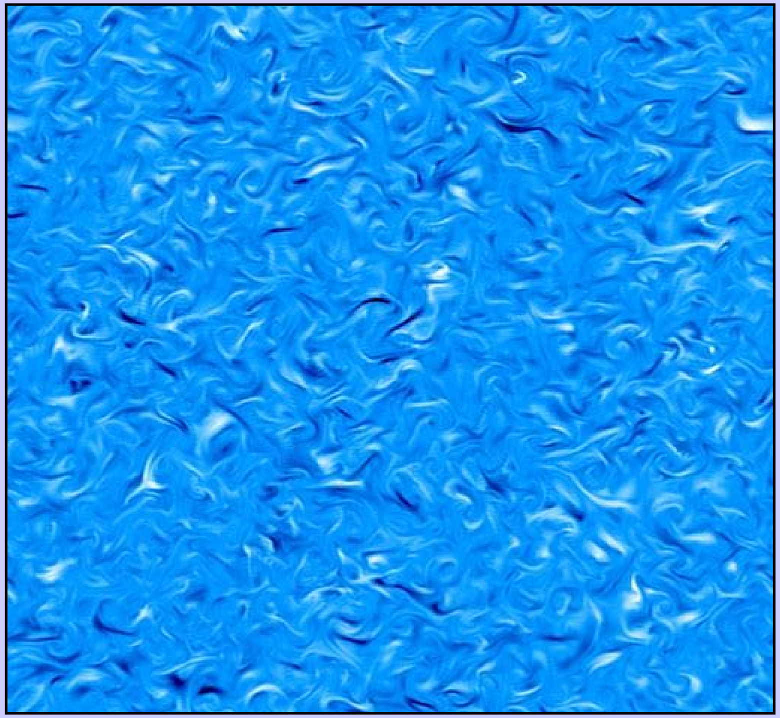
A Potentially Large-Scale Dynamo Driven by Rotating Convection

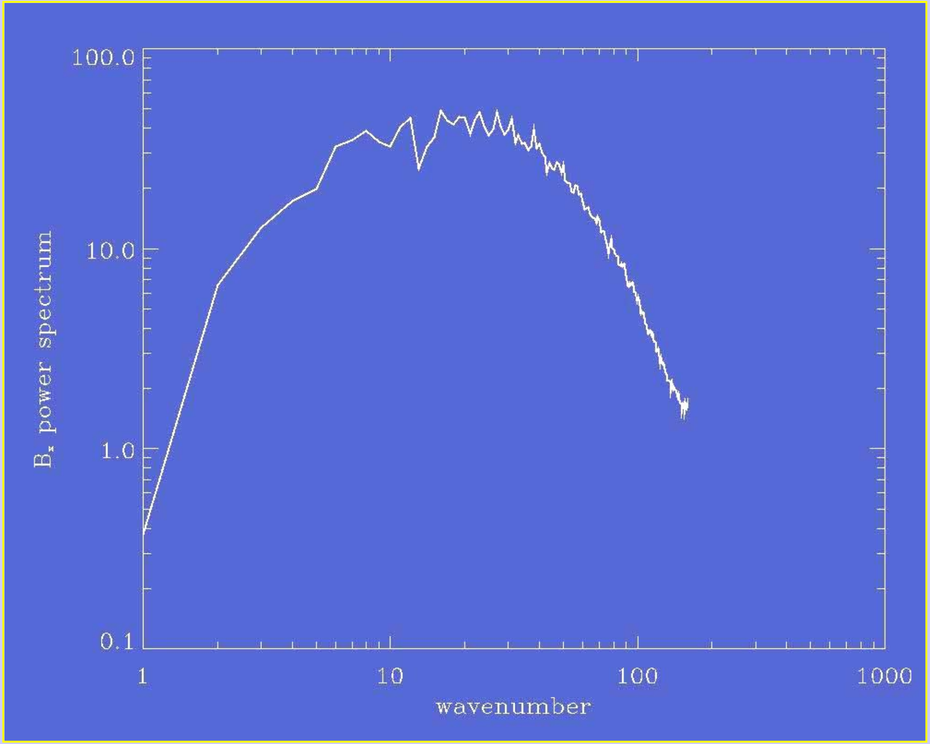




Growth of magnetic energy takes place on an *advective* (i.e. fast) timescale.

$Ra = 10^6$
Box size: 10 x 10 x 1,
Resolution: 512 x 512 x 97
Snapshot of temperature.
No imposed mean magnetic field.



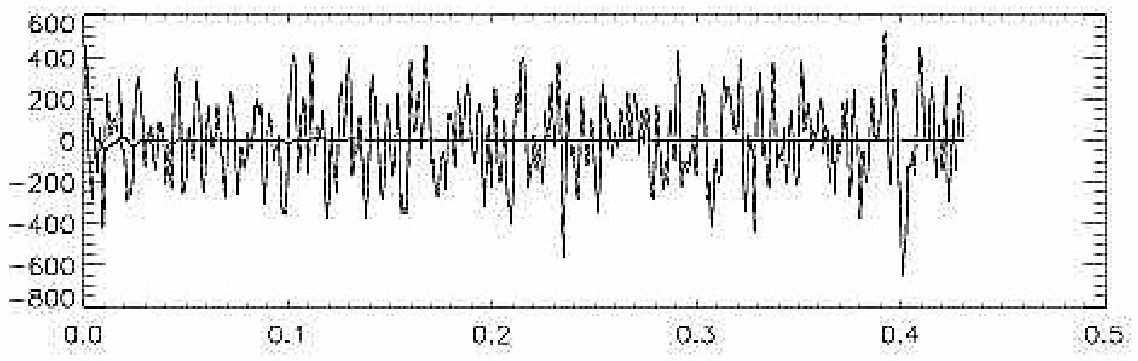


B_x

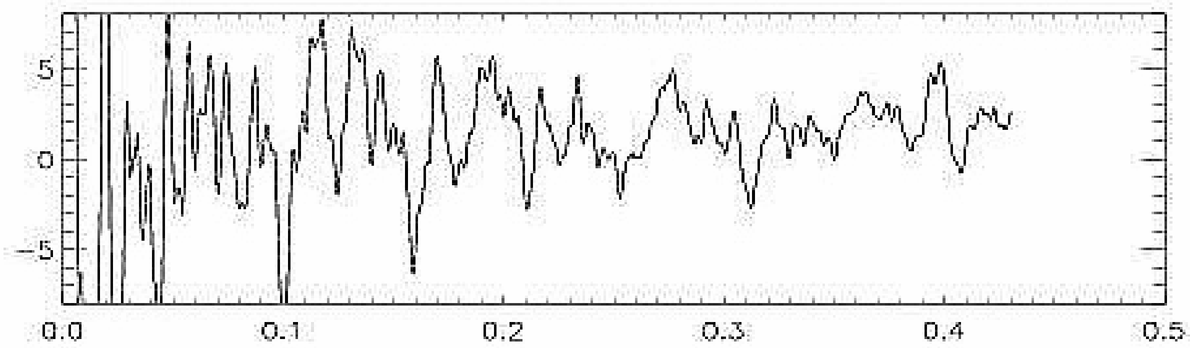
No evidence of significant energy in the large scales – either in the kinematic eigenfunction or in the subsequent nonlinear evolution.

Picture entirely consistent with an extremely feeble α-effect.

Healthy small-scale dynamo; feeble large-scale dynamo.



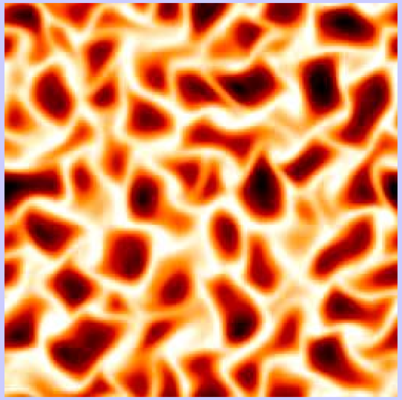
α and its cumulative average versus time.
Imposed horizontal field of strength $B_0 = 10$.



Enlargement of the above.

Turbulent α effect with no small-scale dynamo

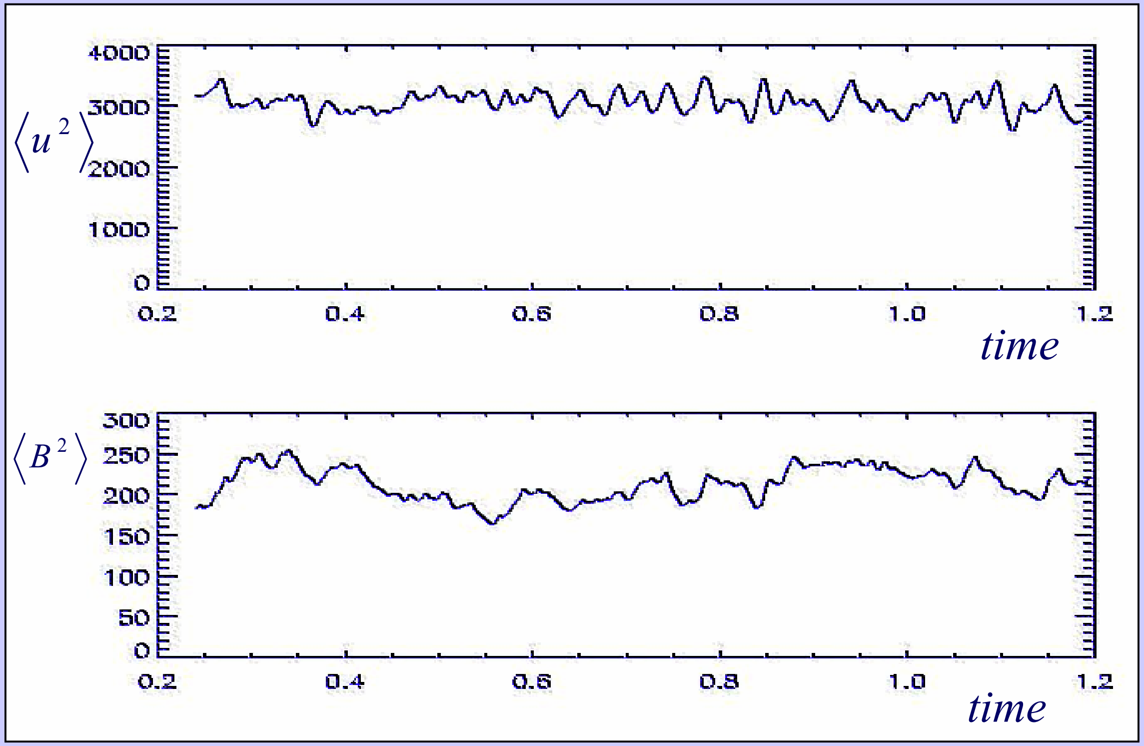
Ra = 150 000
Temperature on a horizontal slice close to the upper boundary.



Ra = 150,000.
No dynamo at this Rayleigh number – but still an α-effect.

Mean field of unit magnitude imposed in *x*-direction.

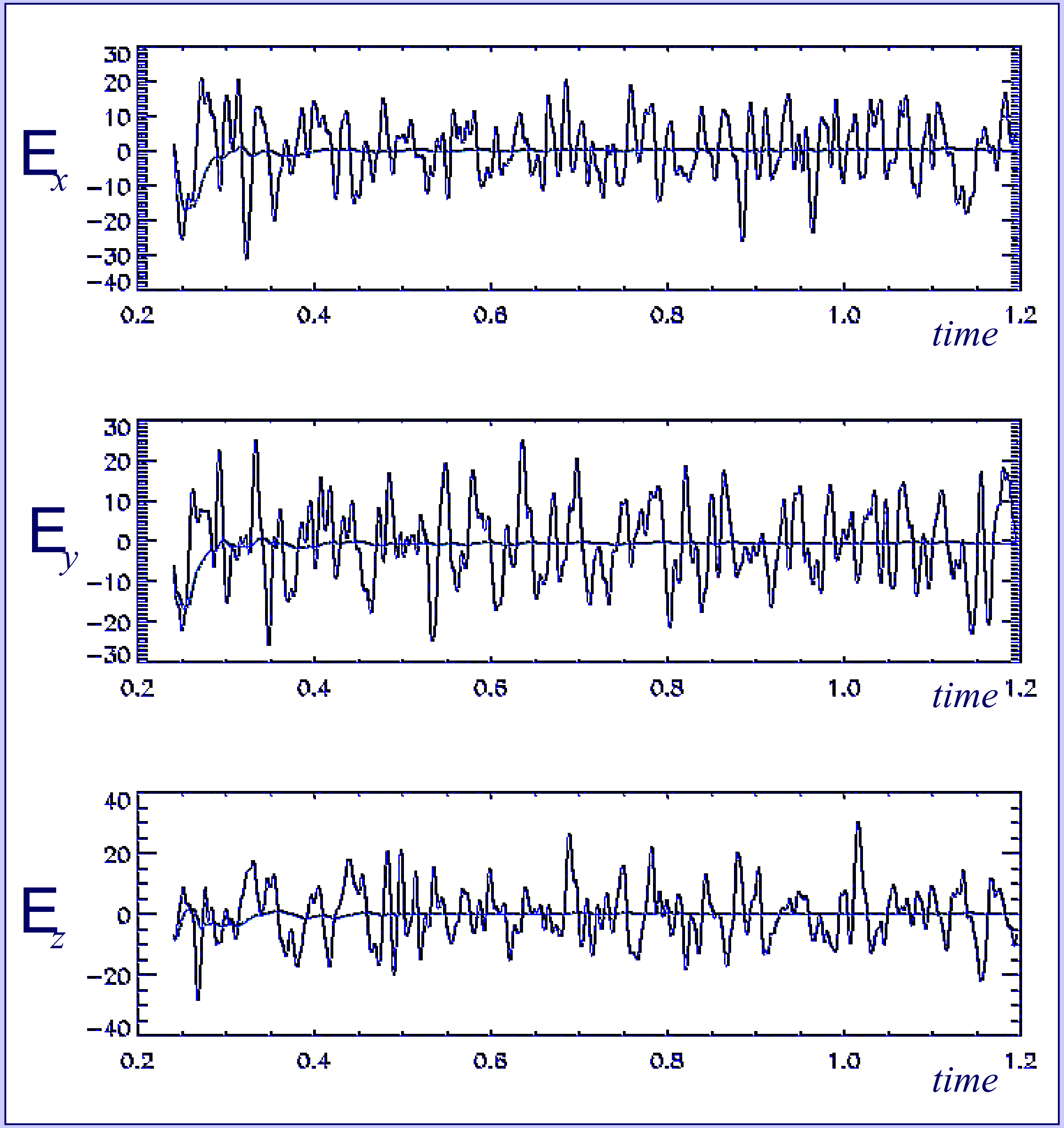
Self-consistent dynamo action sets in at *Ra* ≈ 180,000.



e.m.f. and time-average of e.m.f.

Ra = 150,000
Imposed B_x = 1.

Imposed field extremely weak – kinematic regime.



Cumulative time average of the e.m.f.

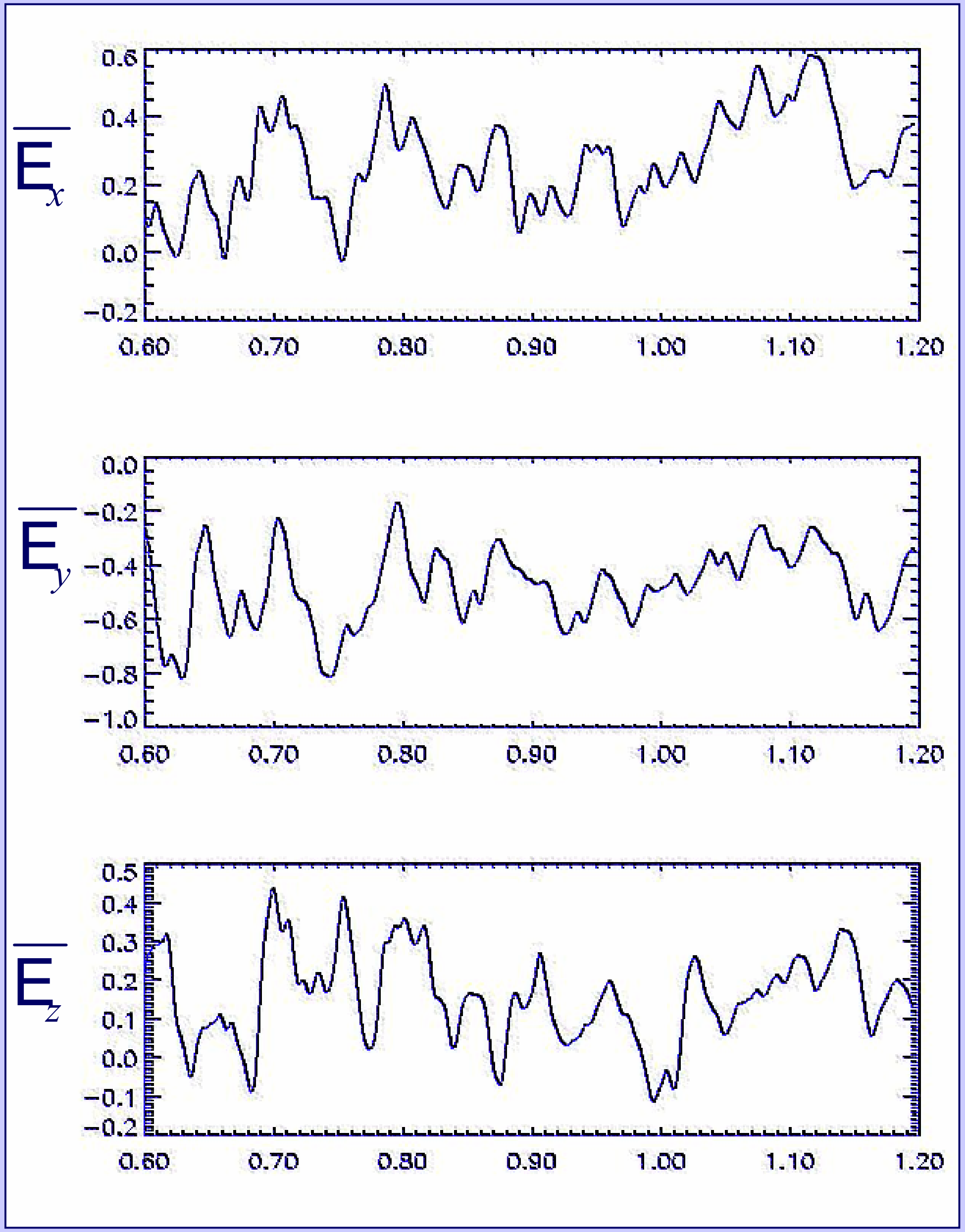
Not fantastic convergence.

α – the ratio of e.m.f. to applied magnetic field – is very small.

At first sight this appears to be consistent with the idea of α-effect suppression.

However, the field here is too weak for this.

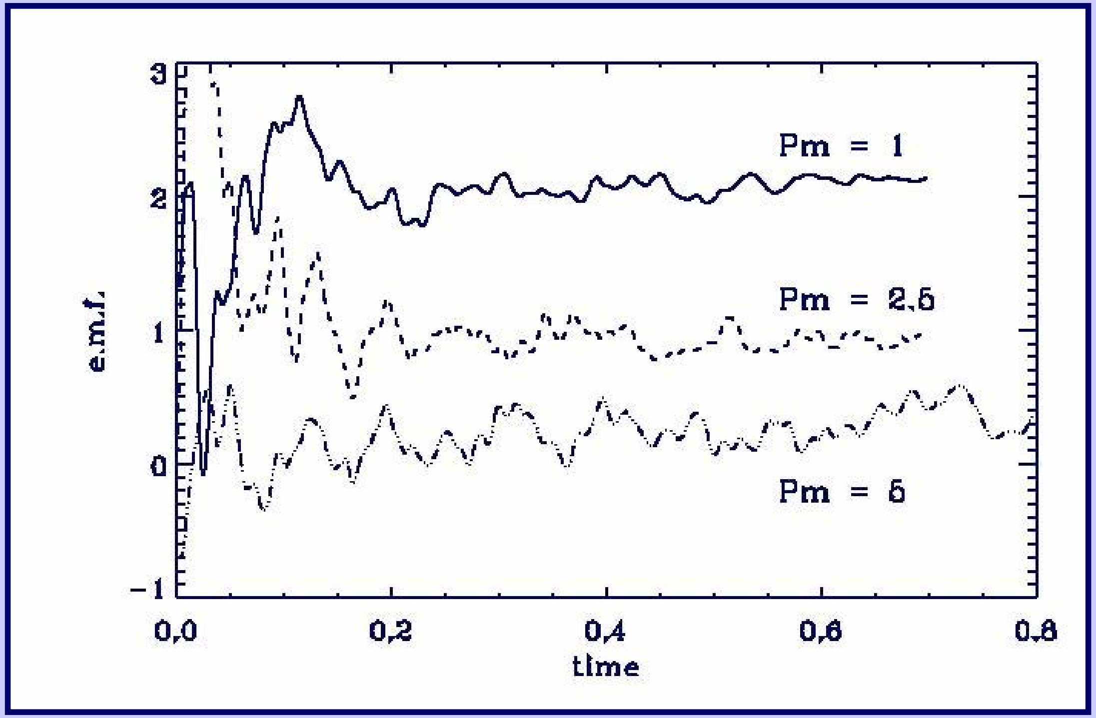
Thus it appears that the α-effect here is *not* turbulent (i.e. fast), but diffusive (i.e. slow).



<u>Changing *Pm*</u>

The α-effect here is inversely proportional to *Pm* (i.e. proportional to η).

It is therefore *not* turbulent (i.e. fast), but diffusive (i.e. slow).



Relation to the work of Jones & Roberts (2000)

Similar model – dynamo driven by rotating Boussinesq convection – but with the following differences:

- Infinite Prandtl number
- Different boundary conditions
  - (i) No-slip velocity conditions
  - (ii) Magnetic field matches onto a potential field.
- Smaller box size

Jones & Roberts work with the Ekman number E and a modified Rayleigh number Ra_Ω

$$Ta = E^{-2}, \qquad Ra = \frac{Ra_\Omega}{E}.$$

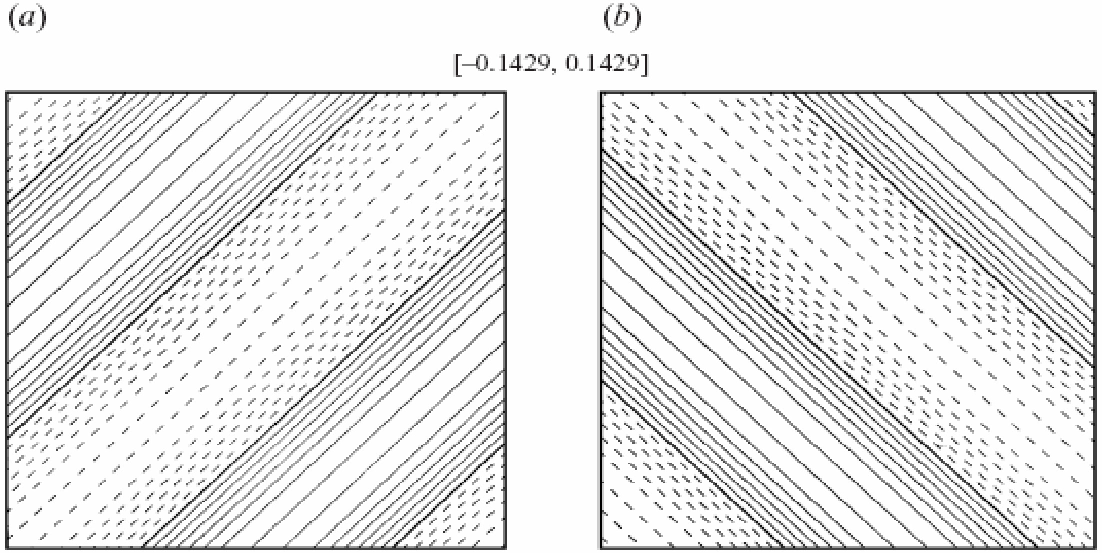


FIGURE 1. Convection without magnetic field at $Ra = 95$, $E = 0.001$ in the (x, y)-plane at $z = 0$ with $\alpha = \beta = 2\pi$. Contours of temperature at intervals of 0.02 for (*a*) the $(1, -1)$ roll, (*b*) the $(1, 1)$ roll. In this and subsequent similar figures dashed contours have negative values, the bold contour is the zero level. The values in brackets denote the minimum and maximum values attained by the contoured quantity.

Temperature contours for mildly supercritical convection – no field.

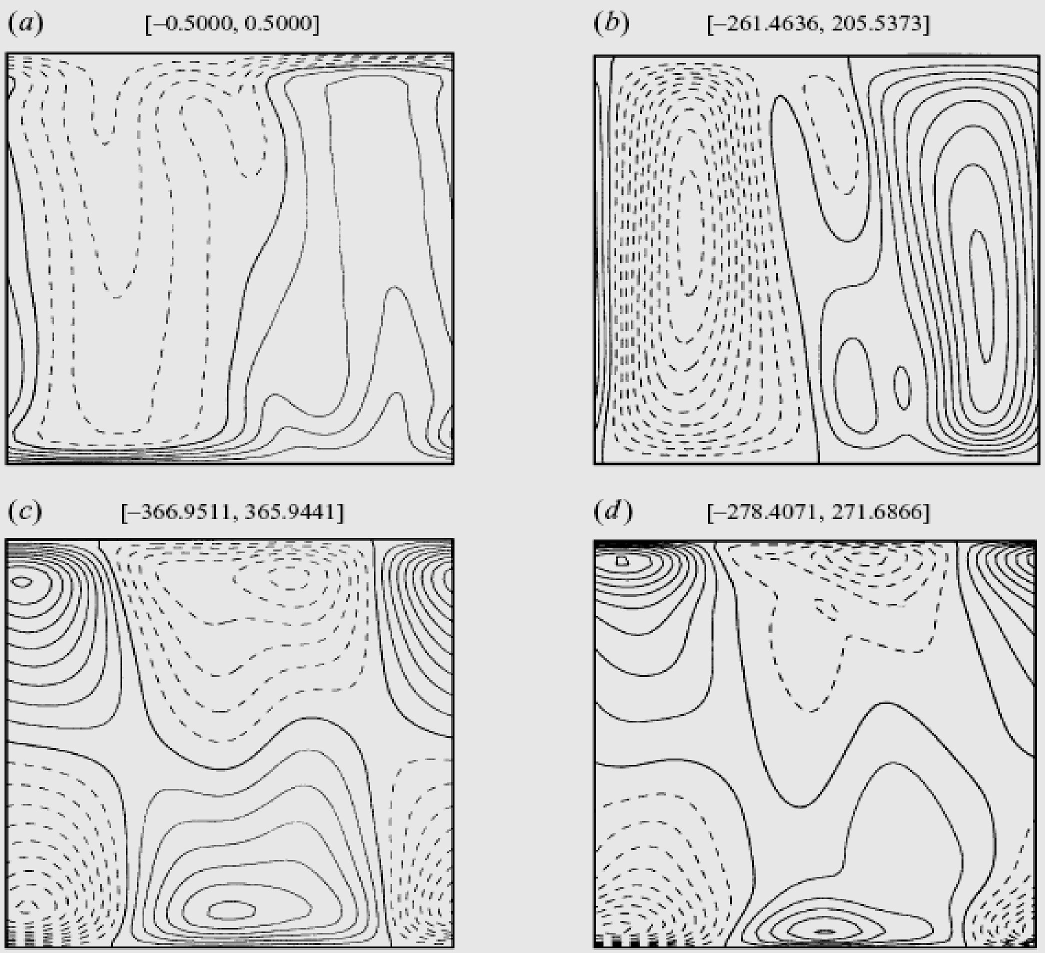
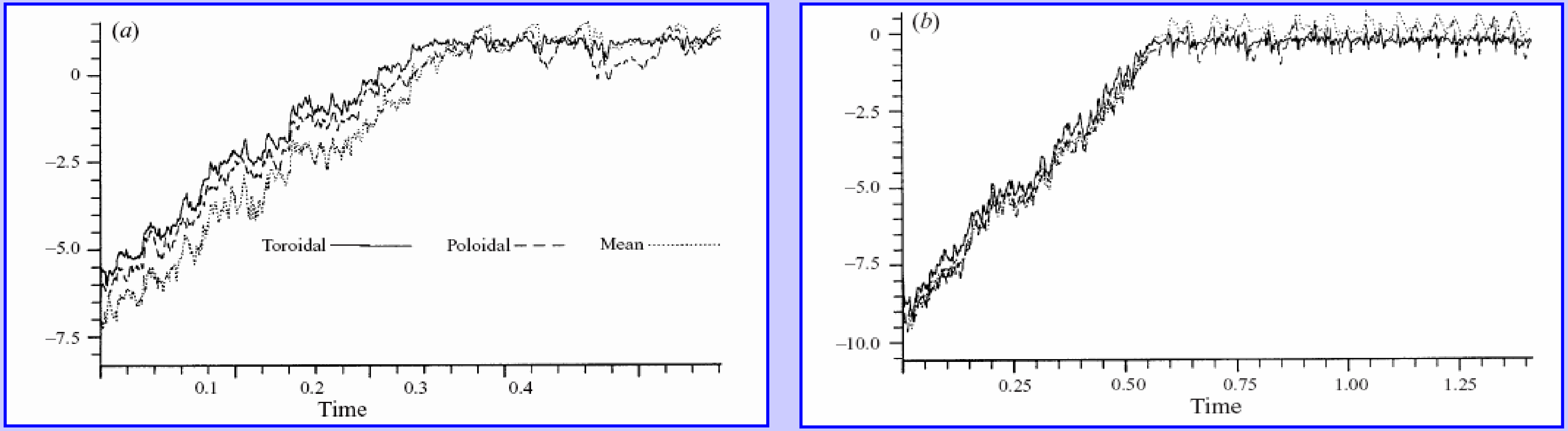


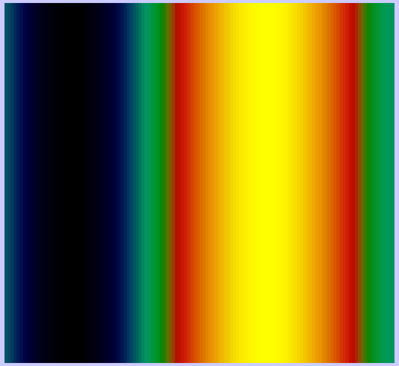
FIGURE 5. A snapshot of convection in the (y, z)-plane for the case $Ra = 500$, $E = 0.001$, $\alpha = \beta = 2\pi$. (a) Temperature at intervals of 0.1; (b) u_z at intervals of 25.0; (c) u_x at intervals of 40.0. (d) u_y at intervals of 30.0.



Magnetic energy *vs* time for (a) Ra_{Ω} = 500, q = 5, E = 0.001
(b) Ra_{Ω} = 1000, q = 1, E = 0.001

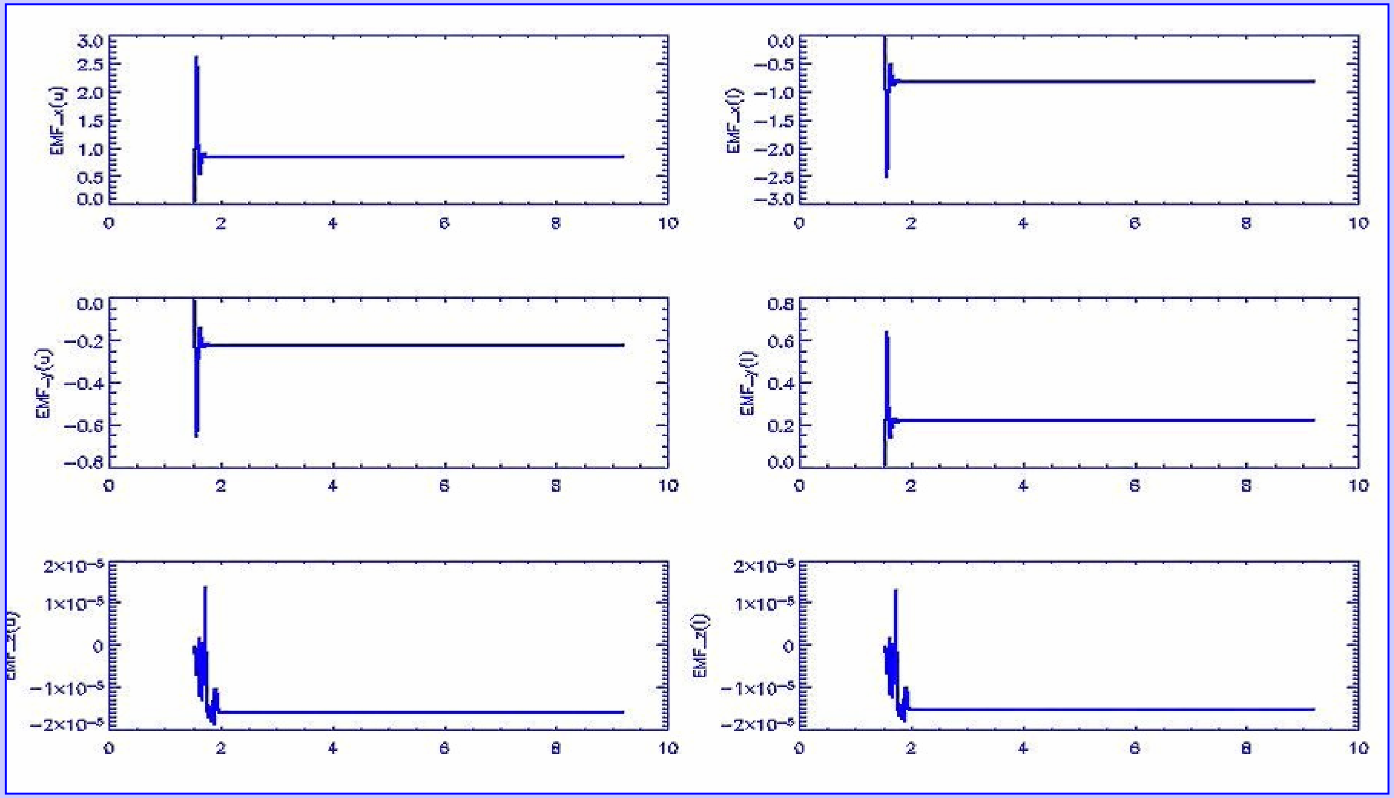
The Influence of Box Size for the Idealised Problem

***Ra* = 80 000**



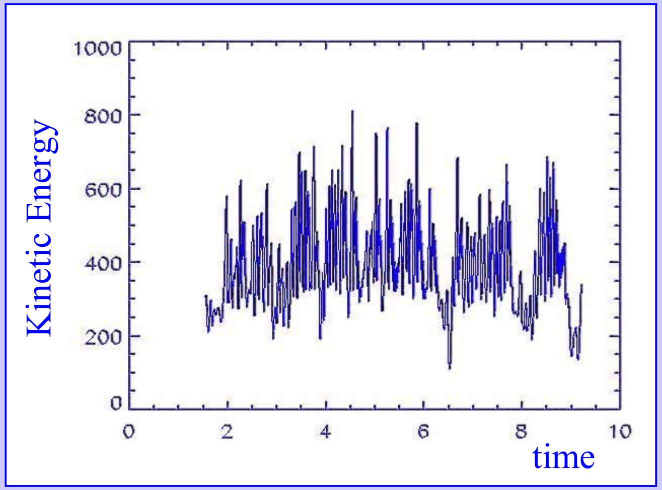
Temperature contours:
aspect ratio = 0.5

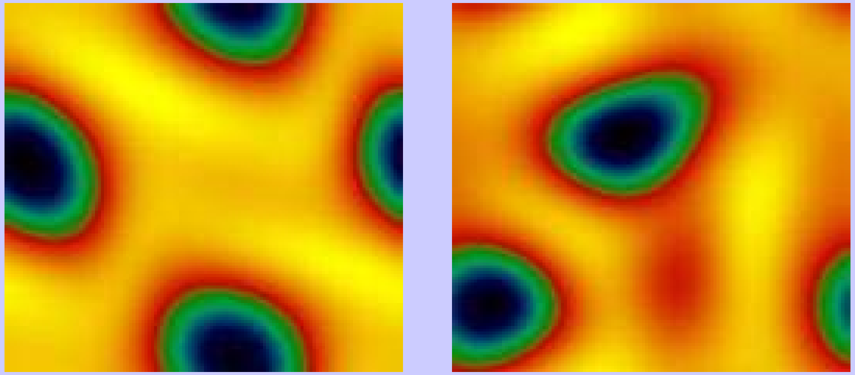
$\langle \mathbf{u}^2 \rangle = 330$



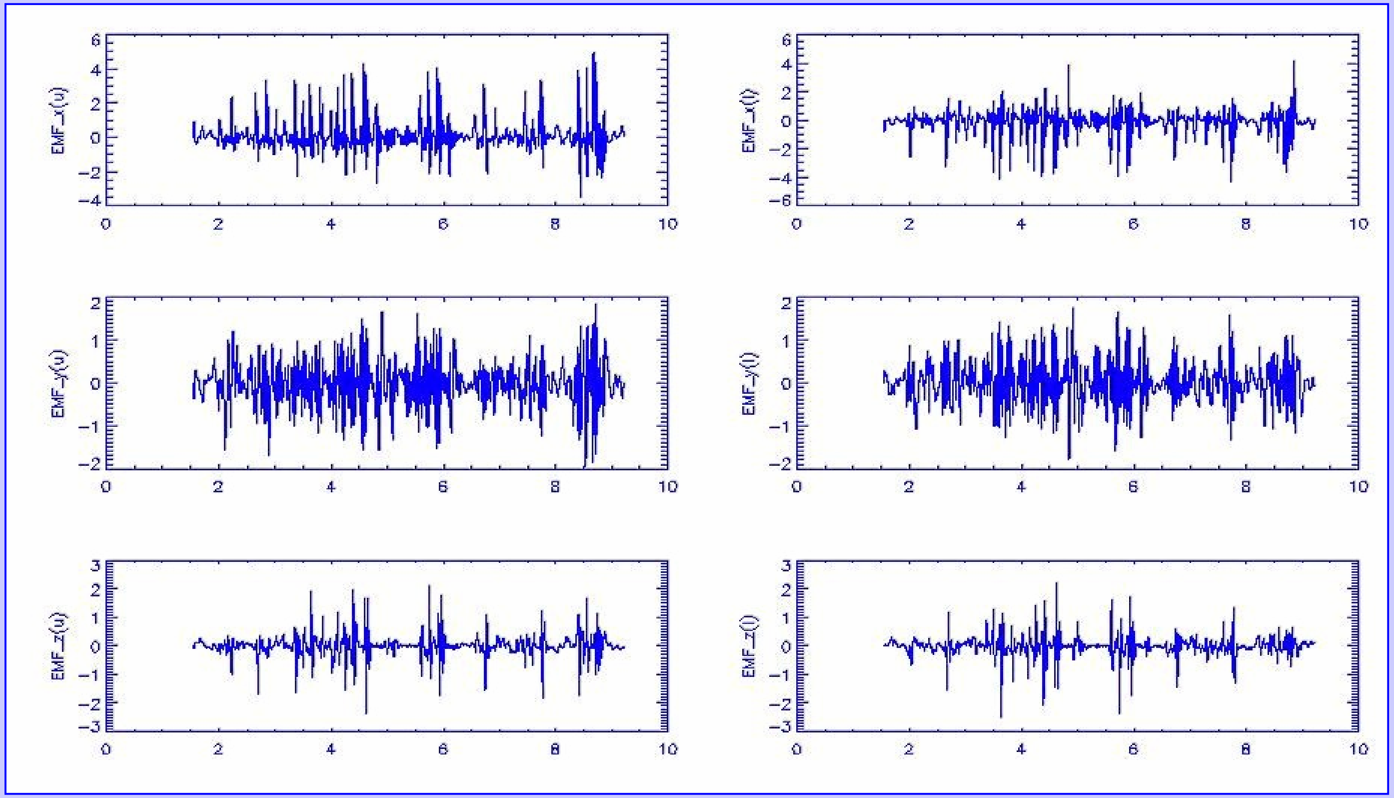
3 components of e.m.f. *vs* time, calculated over upper and lower half-spaces.

$\alpha_{xx} \approx 8.5$





Aspect ratio = 1



$\alpha_{xx} \approx 1.6$

Conclusions

1. Rotating convection is a natural way of producing a helical flow, even at high values of *Ra,* when the flow is turbulent. However, the simple ideas derived for small correlation time or small *Rm* do not carry over to turbulent flows with an *O(1)* value of τ and a high value of *Rm.*

2. The α-effect driven by rotating, "turbulent" convection seems to be
   (a) hard to measure – wildly fluctuating signal in time, even after averaging over many convective cells. Convergence is painfully slow.
   (b) feeble (i.e. diffusive);

3. Given *(a)*, what meaning should we give to the α-effect in this case?