Resonant ordering

We now follow Moestam, Dastgeer and Weiland, Phys Plasmas **11**, 4801 (2004) and apply the resonant ordering from: K. Nozaki, T. Taniuti and K. Watanabe, J. Phys. Soc. Japan **46**, 983 (1979). We expand the fields as:

$$f = \sum_n \sum_l \varepsilon^{1+(2/3)n} f_l^{1+(2/3)n}(x,\xi,\tau) e^{il(k_{\|} z + k_y y - \omega t)} + c.c + \sum_n \varepsilon^{4/3+(2/3)n} f_0^{4/3+(2/3)n}(x,\xi,\tau)$$

Here x=x, $\xi = \varepsilon^{2/3}(y - \lambda t)$, $\tau = \varepsilon^{4/3} t$. Here λ is a velocity of the envelope and ε is the small parameter of order $\frac{e\phi}{T}$ which is considered to be of order 10^{-2} in the core. The parameter l is the harmonic number. It is 1 for the linear drift waves and 0 for the flow. We assume a standing wave in the radial (x) direction. This expansion leads to:

Order $\varepsilon^{5/3}$

$$lD_l \phi_l^{5/3} + i\Omega_l \frac{\partial}{\partial \xi} \phi_l^{1} = 0$$

Here the dispersion relation is $D_l = 0$ so only the second term remains. It can be fulfilled only if $\lambda = \frac{\partial \omega}{\partial k}$ thus λ is the group velocity.

Order ε^2

$$U \frac{\partial}{\partial \varsigma} \phi_0^{4/3} = 0$$

Here U is a quantity which vanishes for a particular group velocity $\lambda = \lambda_0$. This velocity is a reference velocity for zonal flows.

To orders $\varepsilon^{7/3}$ and $\varepsilon^{8/3}$ we now obtain the coupled equations for drift waves and flow as:

$\varepsilon^{7/3}$
$$\frac{\partial \phi_1}{\partial t} + iC \frac{\partial^2 \phi_1}{\partial \xi^2} = C_{nl} \phi_0 \phi_1$$

$\varepsilon^{8/3}$
$$\frac{\partial \phi_0}{\partial t} + D \frac{\partial \phi_0}{\partial \xi} = D_{nl} \frac{\partial}{\partial \xi} |\phi_1|^2$$

We introduce the closure resonance: $\nabla\cdot\mathbf{q}=\frac{5}{2}n\mathbf{v}_D\cdot\nabla\delta T+\nabla\cdot\mathbf{q}_{gl}$ (+ conv. diamagn. Part) where the Gyro fluid part is: $\nabla\cdot\mathbf{q}_{gl}=(-\frac{9}{8}+i\frac{9\sqrt{2}}{8})n\mathbf{v}_D\cdot\nabla\delta T$ (Waltz, Dominguez Hammet, Phys. Fluids 92) and allow the gyrofluid part to be turned on or off.

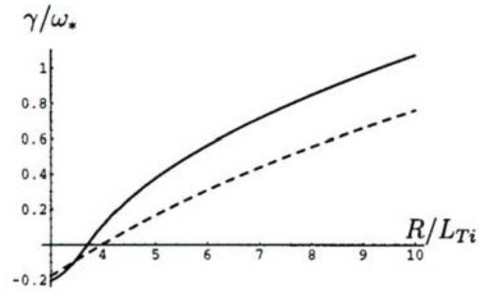

Fig 1a The normalized linear growthrate as a function of temperature gradient for Cyclone base case parameters. The full line is for the reactive model and the dotted is with a simple Gyrofluid resonance included in Eq (8).

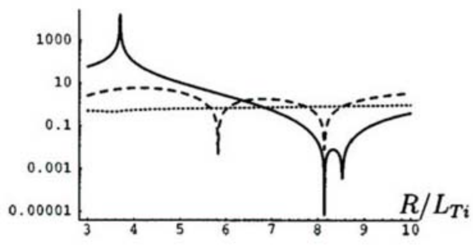

Fig 1b The absolute value of the coupling factor for zonal flow as a function of temperature gradient. The solid line represents the reactive model, the dashed line the model including the Gyro fluid resonance and the dotted line the coefficient of the Reynolds stress.

The reactive model gives a much stronger contribution from the energy equation nonlinearity than the Gyro fluid model. We also note that the coupling strength for the reactive model becomes strongly reduced in the regime $R/L_T \geq 6$ which is where the nonlinear upshift ends in both the kinetic Cyclone simulations and in the simulations by Dastgeer et. al. Phys. Plasmas Dec 2002.

We now claim that the nonlinear upshift in the kinetic simulations is caused by the kinetic resonance

Nonlinear gyrokinetic equation from Weiland, Physica Scripta **29**, 234 (1984):

Linear part:

$$(\omega-\omega_D(v_\parallel{}^2, v_\perp{}^2) - k_\parallel v_\parallel)(f^{(1)}{}_{k,\omega} + \frac{q\phi_{k,\omega}}{T} f_0) = (\omega-\omega_\bullet)\frac{q}{T}\phi_{k,\omega} e^{iL_k} J_0(\xi_k) f_0$$

$$L_k = (\vec{v} x \vec{e}_\parallel)\cdot \vec{k}/\Omega_c$$

Using Boltzmann electrons this gives the linear dispersion relation:

$$1+\frac{1}{\tau} = \frac{1}{n_0}\int \frac{\omega-\omega_{*i}}{\varpi} J^2{}_0(\xi_k) f_0 d^3V$$

where

$$\varpi = \omega-\omega_D(v_\parallel{}^2, v_\perp{}^2) - k_\parallel v_\parallel$$

Nonlinear part:

$$f^{(2)}{}_{k,\omega} = \frac{q^2}{T} f_0 \frac{i}{m\varpi\Omega_c}\sum(\vec{k}'\times\vec{k}'')\cdot\vec{e}_\parallel J_0(\xi')e^{iL_k}\phi_{k',\omega'}\phi_{k'',\omega''}\frac{\omega_\bullet''-\omega''}{\varpi''}J_0(\xi'') \qquad \vec{k}'+\vec{k}''=\vec{k}$$

We then get the nonlinear density perturbation

$$\delta n^{(2)}{}_{k,\omega} = i(\frac{e}{T_i})^2 \frac{T_i}{m_i\Omega_{ci}}\int\frac{1}{\varpi}(\vec{k}'\times\vec{k}'')\cdot\vec{e}_\parallel J_0{}^2(\xi')\frac{\omega_{\bullet i}''-\omega''}{\varpi''}J_0{}^2(\xi'')f_0 d^3V\phi_{k',\omega'}\phi_{k'',\omega''}$$

We may divide it into one FLR part and one non-FLR part by adding and subtracting 1 to the Besselfunction. We then get, after expanding the Besselfunction:

$$\delta n^{(2)}{}_{k,\omega} = i(\frac{e}{T_i})^2 \frac{T_i}{m_i\Omega_{ci}} \int \frac{1}{\varpi}(\vec{k}'\times\vec{k}'')\cdot\hat{e}_{\parallel}\ \frac{1}{2}(k''^2 - k'^2)\frac{\omega_{*i}'' - \omega''}{\varpi''} J_0{}^2(\xi'') f_0 d^3V \phi_{k',\omega'}\phi_{k'',\omega''}$$

$$+ i(\frac{e}{T_i})^2 \frac{T_i}{m_i\Omega_{ci}} \int \frac{1}{\varpi}(\vec{k}'\times\vec{k}'')\cdot\hat{e}_{\parallel}\ \frac{\omega_{*i}'' - \omega''}{\varpi''} f_0 d^3V \phi_{k',\omega'}\phi_{k'',\omega''} + \quad (k' \to k'')$$

We note that if we ignore parallel ion motion abd magnetic curvature, $\varpi = \omega$ and the first part gives the nonlinear part of the Hasegava-Mima eq. with FLR correction.

We now consider the generation of zonal flows. Then we are diving the flow mode $\vec{k}$ with two modes which are almost complex conjugates. We write:

$$\vec{k}' = \vec{k}_1 + \delta\vec{k}, \quad \vec{k}'' = \vec{k}_1 - \delta\vec{k}$$

where $\delta\vec{k}$ is small. Then $\vec{k}'\times\vec{k}'' \approx 2\vec{k}\times\delta\vec{k}$ and

$$\frac{\omega_*'' - \omega''}{\varpi''} \approx \frac{\omega_{1*} - \omega_1}{\varpi_1} + \frac{\partial}{\partial k_\theta}(\frac{\omega_* - \omega}{\varpi})\delta k_\theta$$

and finally

$$\delta n^{(2)}{}_{k,\omega} = -i\rho_s{}^2 \int \frac{1}{\varpi}(\vec{k}_1 \times \delta\vec{k})\cdot\hat{e}_{\parallel}\ 2\vec{k}_1\cdot\delta\vec{k}\,\frac{\omega_{*i}'' - \omega''}{\varpi''} f_0 d^3V \left|\hat{\phi}_{k,\omega}\right|^2 \quad +$$

$$+ i\rho_s{}^2 \int (\vec{k}_1 \times \delta\vec{k})\cdot\hat{e}_{\parallel}\ \frac{2}{\varpi}\left[\frac{\partial}{\partial k_\theta}(\frac{\omega_* - \omega}{\varpi})\right]_{k_1} \delta k_\theta \left|\hat{\phi}_{k,\omega}\right|^2 d^3V$$

Here both parts include the wave-particle resonance. Howeverr, the derivative in the last part increases the sensitivity. Differentiating the linear dispersion relation we obtain:

$$\int \frac{\partial}{\partial\omega}\left(\frac{\omega - \omega_{*i}}{\varpi}\right)\frac{\partial\omega}{\partial k_\theta} J_0{}^2 d^3V + \int \frac{\omega - \omega_{*i}}{\varpi} J_0 k_\theta \rho_i{}^2 d^3V = 0$$

We note that, although the nonlinear parts contain the additional velocity dependent factor $\frac{1}{\varpi}$

The two nonlinear parts will be comparable in this description. The first part is actually the contribution from Reynolds stress with FLR contributions included. The FLR part introduces the temperature perturbation and accordingly the resonance.

If we consider the resonant case, the terms containing the resonance will be larger and, accordingly, the two nonlinear parts will not enter to the same order. This will also require going to higher order in the nonlinear expansion. This can be done by introducing a nonlinear frequency shift in the nonlinear parts as:

$$\delta n^{(2)}{}_{k,\omega} = i\rho_s{}^2 \int \frac{1}{\varpi}(\vec{k}_1 \times \delta\vec{k}) \cdot \hat{e}_{\|} \; 2\vec{k}_1 \cdot \delta\vec{k} \left[\frac{\omega_{\bullet i}'' - \omega''}{\varpi''} + \frac{\partial}{\partial\omega}\left(\frac{\omega_{\bullet i}'' - \omega''}{\varpi''}\right)\delta\omega_{NL} \right] f_0 d^3V \left|\hat{\phi}_{k,\omega}\right|^2 \; +$$

$$+ i\rho_s{}^2 \int (\vec{k}_1 \times \delta\vec{k}) \cdot \hat{e}_{\|} \; \frac{1}{\varpi}\left[\frac{\partial}{\partial k}\frac{\omega_{\bullet} - \omega}{\varpi}\delta k_\theta + \Delta\{\frac{\partial}{\partial\omega}\left(\frac{\omega_{\bullet i} - \omega}{\varpi}\right)\delta\omega_{NL}\} \right]_{k_1} \left|\hat{\phi}_{k,\omega}\right|^2 d^3V$$

Here we can allow the second part, which contain higher derivatives of the resonance, to dominate over the first part i.e. the nonlinearity from the energy equation will dominate even if we include the FLR contribution to the Reynolds stress.

The consequences for the fluid closure of these results were discussed in:

J. Weiland, , S. Dastgeer, R. Moestam, I. Holod and S. Gupta, *Excitation of zonal flows and fluid closure*, **Invited talk**, 13th International Toki Conference on Plasma Physics and Controlled Nuclear Fusion ITC13), Toki, Japan, December 9-12 2003. Paper I-16. Journal of Plasma and Fusion Research (JPFR) SERIES. (Japan) **Vol. 6** p 74-79 (2004)