Neoclassical Effects in the Theory of Magnetic Islands: Neoclassical Tearing Modes and more

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Neoclassical Magnetic Islands

- Neoclassical modes in TFTR, Z. Chang et al. PRL 74, 4663 (1995)

\[
\tau_R \frac{\partial w}{\partial t} = \Delta' + \frac{\beta}{w}
\]

Rutherford growth

\[
w \sim \Delta' t / \tau_R
\]

Bootstrap growth

\[
w \sim \left( \frac{\beta t}{\tau_R} \right)^{1/2}
\]

Saturation for \( \Delta < 0 \)

Beta dependence signatures are critical for NTM identification

FIG. 4. Theory-experiment comparison of saturated magnetic island width. The \( w_{\text{exp}} \) is from Eq. (5). The \( w_{\text{sat}} \) is from Eq. (4). A constant \( k_2 = 1.7 \) has been used for all discharges.
Outline

• Basic island evolution -- extended Rutherford equation
• Finite pressure drive: Bootstrap current
• Stabilization mechanisms:
  Removal of pressure flattening due to finite heat conductivity
  Polarization current
• Other neoclassical effects
  Neoclassical coupling of transverse and longitudinal flows
  Enhanced polarization current due to neoclassical flow damping
• New stabilization mechanism due to parallel dynamics and neoclassical coupling
  Ion sound effects
• Island rotation frequency
Basics of Nonlinear Magnetic Islands Theory

- Effective helical flux function for the rotating island

\[ \psi(x, t) = -\frac{x^2}{2L_s} B_0 + \tilde{\psi}(t) \cos \xi. \]

\( x = r - r_s \) is the distance from the rational surface, \( L_s = qR/S \) is the shear length, \( S = q'r_s/q \), and the helical coordinate \( \xi = m\bar{\theta} - \int^t \omega(t') dt' \), \( \bar{\theta} = \theta - \zeta/q_s \). Magnetic island with half-width \( w^2 = 4L_s\tilde{\psi}/B_0 \).

- Rutherford regime: \( w > \delta_R \). Typical values: \( \delta_R < 0.3 \text{ cm} \) for \( S = 10^5 - 10^8 \), \( w \simeq 1 \text{ cm} \)

- Constant \( \psi \) approximation. Single helicity
Motivation

- Neoclassical Tearing Modes (NTM) are often observed in the ideal MHD stable ($\Delta' < 0$) plasmas – seriously degrade tokamak performance (10-50 %, loss of H-mode and disruptions)

- Modes are driven by finite pressure effects: bootstrap current associated with the perturbation of plasma pressure near/inside the magnetic island –neoclassical effect $\rightarrow$ NTM

- Experimental data seem to suggest that at a small island width the destabilizing effect of the bootstrap current is reduced by some other effect(s) $\rightarrow$ Threshold (*Sauter et al.* PoP 4, 1654 (1997))

  Plasma is metastable with respect to NTM $\rightarrow$ Trigger mechanism, seed island

**Stabilization/driving mechanisms?**


**Bootstrap Current Drive**

- Pressure driven current due to friction between trapped and untrapped particles

\[ J_\parallel = \sqrt{\epsilon} \frac{c}{B_0} \frac{dp}{dr} \]

- Generalized Ohm law

\[ 0 = -en \left( -\nabla \phi - \frac{1}{c} \frac{\partial \psi}{\partial t} \right) - b \cdot \nabla p_e - b \cdot \nabla \cdot \Pi_e + enJ_\parallel / \sigma \]

- Diamagnetic banana current + friction effects

\[ b \cdot \nabla \cdot \Pi_e = n_e n_{\mu_e} V_{\theta e} \]

- Bootstrap current

\[ V_{\theta e} = -\frac{c}{enB_0} \frac{\partial}{\partial r} (p_e + p_i) + \frac{B_0}{B_0} (V_{ze} - V_{zi}) \]

\[ J_b = \langle J_b \rangle \quad \nabla \parallel J_b = 0 \]

Loss of the bootstrap current around the island
Extended Rutherford Equation–Basic Evolution Equation

- The nonlinear equations for the evolution of the magnetic island follow from the matching conditions obtained by integration of the Ampere’s law, $4\pi J_\parallel/c = \nabla^2_\perp \psi$, across the nonlinear region

$$\int_{-\pi}^{\pi} d(m\theta) \int_{-\infty}^{\infty} dx J_\parallel \cos \xi = \frac{c}{4} \Delta' \psi$$

- Rutherford equation

$$\tau_R \frac{\partial w}{\partial t} = \frac{\Delta'_c}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S_w} + \ldots$$

$$J_\parallel = \frac{\sigma}{c} \frac{\partial \psi}{\partial t} + \sqrt{\epsilon} \frac{c}{B_\theta} \frac{dp}{dr}$$

- Transition to the linear limit $w \to 0$?

All $m$ mode numbers are unstable? Does not happen in the experiments:
most often $m/n = 3/2, 4/3, 5/4$. →
Threshold mechanism?
Neoclassical Tearing Modes are metastable – Thresholds

- Modification of the bootstrap current for small island width (finite parallel heat conductivity)

\[ \tau_R \frac{\partial w}{\partial t} = \frac{\Delta_c'}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S} w \quad \text{No threshold} \]

\[ \tau_R \frac{\partial w}{\partial t} = \frac{\Delta_c'}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S} w \frac{w}{w_c^2 + w^2} \quad \text{Threshold} \]

- Polarization current threshold

\[ \tau_R \frac{\partial w}{\partial t} = \frac{\Delta_c'}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S} w \frac{w}{w_c^2 + w^2} - \alpha \beta_\theta \frac{1}{w^3} \]

- Magnetic field curvature (Glasser-Green-Johnson) effect is also stabilizing. Usually is small but can be important for small aspect ratio (MAST, R.J. Bitterly et al., PRL 88, 125005-1 (2002), H.Lutjens, J-F Luciani, and X. Garbet, POP 8, 4267 (2002)).

\[ \Delta'_{GGJ} = g_{GGJ} \frac{D_R}{\sqrt{w_c^2 + w^2}} \]
Drift/Inertial, Neoclassical, Curvature, etc Effects

- Quasineutrality equation
  \[ \nabla_\| J_\| + \nabla_\perp \cdot J_\perp = 0 \]

\[ J_\| = \nabla^{-1} \nabla_\perp \cdot J_\perp \]

- Perpendicular current
  \[ J_\perp = \frac{c}{B} b \times \nabla p + \frac{cm_i n_0}{B} b \times \frac{d_0}{dt} V + \frac{c}{B} b \times \nabla \cdot \Pi \]

  Diamagnetic current
  Glasser-Green Johson
  Neoclassical viscosity, enhanced polarization
  Inertia, polarization current
Polarization Current Effects

- Polarization current

\[ J_\parallel = \nabla^{-1} \nabla_\perp \cdot J_\perp \]

\[ J_\perp = \frac{cm_i n_0}{B} \mathbf{b} \times \frac{d_0}{dt} \mathbf{V} \]

- Rutherford equation

\[ \tau_R \frac{\partial w}{\partial t} = \frac{\Delta_\epsilon'}{4} + \sqrt{\epsilon} \frac{\beta_0}{S w} + g \frac{\beta_0}{w} \left( \frac{\rho_s}{w} \right)^2 \frac{\omega (\omega - \omega_{*i})}{\omega_{*e}^2} \]
Neoclassically Enhanced Polarization Current

Coupling of the transverse and longitudinal flows/Neoclassical flow damping

- Current closure equation

\[ \nabla || J || + \nabla \cdot \left( \frac{c m_i n_0}{B} b \times \frac{d_0}{dt} V \right) + \nabla \cdot \left( \frac{c}{B} b \times \nabla \cdot \Pi \right) = 0 \]

\[ \Pi || = \frac{3}{2} \pi || \left( b b - \frac{1}{3} \mathbf{I} \right) \quad \frac{3}{2} \pi || = p_\perp - p || \]

- Neoclassical current

\[ \nabla \cdot J_{nc} \equiv \nabla \cdot \left( \frac{c}{B^2} \mathbf{B} \times \nabla \cdot \Pi \right) = \frac{c}{B \theta} \frac{\partial}{\partial x} \langle b \cdot \nabla \cdot \Pi \rangle \]

- Divergence of the transverse current is related to the component of the parallel force

\[ m_i n_0 \frac{d}{dt} V || = -\nabla || p - \langle b \cdot \nabla \cdot \Pi \rangle \theta \]

\[ \nabla \cdot J_{nc} = \frac{c}{B \theta} m_i n_0 \frac{\partial}{\partial x} \frac{d_0}{dt} V || + \frac{c}{B \theta} \frac{\partial}{\partial x} \nabla || p \]

Enhanced inertia, replaces the standard polarization current
Neoclassical Flow Damping

- Neoclassical force

\[
\langle b \cdot \nabla \cdot \Pi \rangle = -\frac{3 \varepsilon}{2q} \langle \pi_\parallel \frac{1}{r_s} \frac{\partial}{\partial \theta} \nabla_\perp \ln B \rangle = m_i n_0 \chi_\theta V_\theta
\]

\[
\chi_\theta = \frac{q^2}{\varepsilon^{1/2}} \left( \frac{d_0}{dt} + \frac{\nu_i}{\varepsilon} \right) \quad V_\theta = V_v + \frac{\varepsilon}{q} V_\parallel \quad \dot{V}_v = \frac{c}{B_0} \frac{\partial \phi}{\partial x}
\]

Resulting equation for the parallel flow velocity is

\[
\frac{d_0}{dt} V_\parallel = -q \varepsilon^{1/2} \left( \frac{d_0}{dt} + \frac{\nu_i}{\varepsilon} \right) \left( V_y + \frac{\varepsilon}{q} V_\parallel \right) - \frac{1}{m_i n_0} \nabla_\parallel p
\]
Neoclassical Flow Damping II

- Large collisional frequency: $\nu_i/\varepsilon \gg d_0/dt$

$$V_{\parallel}^{(0)} = -\frac{q}{\varepsilon} V_y$$

$$\frac{d_0}{dt} V_{\parallel}^{(0)} = -q\varepsilon^{1/2} \frac{d_0}{dt} V_y - \varepsilon^{1/2} \nu_i V_{\parallel}^{(1)} - \frac{1}{m_i n_0} \nabla_{\parallel p}$$

$$\langle b \cdot \nabla \cdot \Pi \rangle = q\varepsilon^{1/2} \frac{d_0}{dt} V_y + \varepsilon^{1/2} \nu_i V_{\parallel}^{(1)}$$

- Low collisional frequency: $\nu_i/\varepsilon \ll d_0/dt$

Zero order

$$\frac{d_0}{dt} V_{\parallel}^{(0)} = -q\varepsilon^{1/2} \frac{d_0}{dt} V_y$$

$$V_{\parallel}^{(0)} = -q\varepsilon^{1/2} V_y$$

$$\langle b \cdot \nabla \cdot \Pi \rangle = q\varepsilon^{1/2} \frac{d_0}{dt} V_y + q \frac{\nu_i}{\varepsilon^{1/2}} V_y$$

Neoclassical polarization
Neoclassically Enhanced Polarization Current II

- Neoclassical current

\[ \nabla \cdot J_{nc} = \frac{c}{B_\theta} m_i n_0 \frac{\partial}{\partial x} \frac{d_\theta}{dt} V_\parallel + \frac{c}{B_\theta \partial x} \nabla p_\parallel \]

- From the radial momentum balance

\[ V_\parallel \approx V_\zeta = V_\theta \frac{B_\zeta}{B_\theta} + \frac{c}{\epsilon n_0 B_\theta} E_r - \frac{c}{B_\theta \partial r} \frac{\partial p}{\partial r} \quad V_\theta = k \frac{cT'}{eB} \]

- Extended Rutherford equation

\[ \frac{\tau_R}{\partial t} = \frac{\Delta c}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S_w} + g \frac{\beta_\theta}{\omega \rho_s w} \left( \frac{\rho_s}{w} \right)^2 \frac{\omega (\omega - \omega_{*i})}{\omega_{*e}^2} + g_{\text{neo}} \frac{\beta_\theta}{w} \left( \frac{\rho_s}{w} \right)^2 \frac{\omega (\omega - k \omega_{*i})}{\omega_{*e}^2} + \frac{\text{standard inertia}}{\text{Neoclassically enhanced inertia}} \]

\[ g_{\text{neo}} = \begin{cases} \frac{q^2}{\epsilon^2} & \nu_i \gg \epsilon \omega \\ \frac{q^2}{\sqrt{\epsilon}} & \nu_i \ll \epsilon \omega \\ \text{Smolyakov et al., PoP 2, 1581 (1995)} \\ \text{Wilson et al. PoP 3, 248 (1996)} \end{cases} \]

\[ g_{\text{neo}} \]

depends on collisionality regime and may have further dependence on frequency, Mikhailovskii PPCF 2001
Neoclassically Enhanced Polarization Current

Coupling of the transverse and longitudinal flows/Neoclassical flow damping

- Current closure equation
  \[ \nabla_{\parallel} J_{\parallel} + \nabla_{\perp} \cdot \left( \frac{cm_{i}n_{0}}{B} b \times \frac{d_{0}}{dt} V \right) + \nabla \cdot \left( \frac{c}{B} b \times \nabla \cdot \Pi \right) = 0 \]
  \[ \Pi_{\parallel} = \frac{3}{2} \pi_{\parallel} (bb - \frac{1}{3} I) \]
  \[ \frac{3}{2} \pi_{\parallel} = p_{\perp} - p_{\parallel} \]

- Neoclassical current
  \[ \nabla \cdot J_{nc} \equiv \nabla \cdot \left( \frac{c}{B^2} B \times \nabla \cdot \Pi \right) = \frac{c}{B_\theta} \frac{\partial}{\partial x} \langle b \cdot \nabla \cdot \Pi \rangle \]

- Divergence of the transverse current is related to the component of the parallel force
  \[ m_{i}n_{0} \frac{d}{dt} V_{\parallel} = -\nabla_{\parallel} p - \langle b \cdot \nabla \cdot \Pi \rangle_\theta \]
  \[ \nabla \cdot J_{nc} = \frac{c}{B_\theta} m_{i}n_{0} \frac{\partial}{\partial x} \frac{d_{0}}{dt} V_{\parallel} + \frac{c}{B_\theta} \frac{\partial}{\partial x} \nabla_{\parallel} p \]

Enhanced inertia, replaces the standard polarization current
Finite Ion Orbits

- Cold ions case: $T_e \gg T_i$. Plasma pressure is $p \approx T_e n_e$, and the electron density is determined from the quasineutrality condition $n_e = n_i$ so that $p = T_e n_i$. The ion density is not a function of magnetic flux surface due to the ion inertial drift off the surface

$$n_i = \frac{c^2}{v_A^2} \frac{1}{4\pi e} \nabla_\perp^2 \phi = n_0 \rho_s^2 \nabla_\perp^2 \frac{e\phi}{T_e}$$

For finite $\rho_i$

$p = p(\psi) \rightarrow \nabla_\parallel p = 0$

$\rho_i \rightarrow 0$

- Extended Rutherford equation

$$\tau_R \frac{\partial w}{\partial t} = \frac{\Delta_e'}{4} + \sqrt{\epsilon} \frac{\beta_\theta}{S_w} + g_1 \frac{\beta_\theta}{w} \left( \frac{\rho_s}{w} \right)^2 \frac{\omega(\omega - \omega_{\ast i})}{\omega_{\ast e}^2} + g_{\text{neo}} \frac{\beta_\theta}{w} \left( \frac{\rho_s}{w} \right)^2 \frac{\omega(\omega - k\omega_{\ast i})}{\omega_{\ast e}^2}$$

$$+ g_2 \frac{\beta_\theta \rho_s^2}{w^2} \frac{\omega}{\omega_{\ast}}$$

$g_{\text{neo}} = \left\{ \begin{array}{ll} \frac{q^2}{\epsilon^2} & \nu_i \gg \epsilon \omega \\ \frac{q^2}{\sqrt{\epsilon}} & \nu_i \ll \epsilon \omega \end{array} \right\}$

- Finite banana width

$$n_b = n_0 \rho_s^2 \frac{q^2}{\epsilon} \sqrt{\epsilon} \nabla_\perp^2 \frac{e\phi}{T_e}$$

$$g_2 \sim 1 \rightarrow g_2 \sim \frac{q^2}{\sqrt{\epsilon}}$$

Finite orbit effect provides threshold of the same order as polarization current!
Effect of a Finite Heat Conductivity Along the Magnetic Field

- Finite parallel heat conductivity results in the variations of plasma pressure along the perturbed magnetic surface
  \[ p \neq p(\psi) \quad \nabla_{\parallel} p \neq 0 \]
  \[ w_c^4 = \frac{\chi_{\perp}}{\chi_{\parallel}} \frac{L_s^2}{k_b^2} \]

- Pressure gradient across the magnetic island is partially restored
  \[ \tau_R \frac{\partial w}{\partial t} = \frac{\Delta_c'}{4} + c_1 \sqrt{c} \frac{1}{S} \frac{\beta_0}{w_c^2} w - c_2 \frac{1}{S w \sqrt{1 + (w/w_c)^2}} \]

- At the threshold Ware pinch term is comparable or larger than the standard bootstrap current (driving) term!
Ion Sound Effects

- Ion sound effects are known to stabilize drift-tearing modes in linear regimes. Bussac et al. PRL 40, 1500 (1978). There are indications that these are also stabilizing nonlinearly, however may be destabilizing for large $\omega_*$, M. Ottaviani, et al., PRL 93 (2004)

- Basic equations

\[-enE_\parallel - Te_\nabla_\parallel n = 0.\]

\[\frac{d_0}{dt} \left( n_i - n_0 \rho_s^2 \nabla_\perp^2 \frac{e \phi}{T_e} \right) + \nabla_\parallel (n_0 V_\parallel_i) = 0\]

\[-\frac{c^2}{4\pi v_A^2} \frac{d_0}{dt} \nabla_\perp^2 \phi + \nabla_\parallel J_\parallel = 0.\]

\[\frac{d_0}{dt} n_e + \nabla_\parallel (n_0 V_\parallel_e) = 0\]

\[n_0 m_i \frac{d_0}{dt} V_\parallel = -T \nabla_\parallel n.\]

- Coupling of the Alfvén and ion sound modes

\[(k_\parallel^2 v_A^2 - \omega^2) \left( 1 - \frac{k_\parallel^2 v_s^2}{\omega^2} \right) + k_\parallel^2 v_A^2 k_\perp^2 \rho_s^2 = 0\]
Stabilizing ion sound, but $\Omega^*$

- Extended Rutherford equation taking into account the ion sound effects, *Smolyakov et al., PPCF, 46 (2004)*

$$\Delta' + \frac{g_1 \omega^2 L_s^2}{w^3 k_\theta^2 v_A^2} - \frac{g_2 c_s^2}{w v_A^2} = 0$$

- Caveat: $\Omega^*$ dependence has been omitted.

$$1 \sim \frac{k_{||}^2 c_s^2}{\omega^2}, \quad k_{||} = -k_\theta w / L_s$$

- Coupling of the Alfvén and ion sound modes

$$(k_{||}^2 v_A^2 - \omega^2) \left(1 - \frac{k_{||}^2 v_A^2}{\omega^2}\right) + k_{||}^2 v_A^2 k_{\perp}^2 \rho_s^2 = 0$$
Stabilization Mechanisms


- Removal of pressure flattening due to finite $\chi_\parallel/\chi_\perp$, (Fitzpatrick PoP 2, 825 (1995); Gorelenkov, Zakharov PoP 3, 3379 (1996))

- Other stabilizing neoclassical effects/ion sound effects?
Island Rotation Frequency

- Island rotation is determined by dissipation
  - minimum dissipation principle

Dissipation:
- Classical collisions: resistivity and heat conductivity
- Collisionless (Landau damping)
- Perpendicular diffusion density/energy: classical/anomalous
- Perpendicular anomalous viscosity
- Neoclassical flow damping/symmetry breaking
Classical dissipation: parallel resistivity and heat conductivity

\[ \frac{\partial E}{\partial t} = -Q = \int dx d\xi \left( \frac{1}{\sigma} J_{\parallel} - \frac{1}{e} \nabla_{\parallel} T \right) J_{\parallel} \]

\[ \left( \nabla_{\parallel} T \right)^0 = 0 \quad \left( \nabla_{\parallel} T \right)^1 = \frac{1}{\chi_{\parallel}} (...) \]

\[ Q \sim (\omega - \omega_e)(\omega - \omega_i)^2 (\omega - \omega_e (1 - \eta_e / \eta_{cr})) \]

Smolyakov, Sov J Pl Phys 1989
Connor et al; PoP, 2001

\[ \eta_{cr} = \frac{1 + (1 + \alpha)^2 \sigma T / e^2 \chi_{\parallel}}{3(1 + \alpha) \sigma T / 2 e^2 \chi_{\parallel}} \]

\[ \eta_e = \partial \ln T_e / \partial \ln n \]
Classical dissipation: parallel resistivity and heat conductivity

\[
\frac{\partial E}{\partial t} = -Q = \int dx \, d\xi \left( \frac{1}{\sigma} J_{\|} - \frac{1}{e} \nabla_{\|} T \right) J_{\|}
\]

\[
(\nabla_{\|} T)^0 = 0 \quad (\nabla_{\|} T)^1 = \frac{1}{\chi_{\|}} (\ldots)
\]

\[
Q \sim (\omega - \omega_e)(\omega - \omega_i)^2(\omega - \omega_e(1 - \eta_e/\eta_{cr}))
\]

\[
\omega = \omega_e(1 - \eta_e/\eta_{cr})
\]

\[
\eta_{cr} = \frac{1 + (1 + \alpha)^2 \sigma T / e^2 \chi_{\|}}{3(1 + \alpha)\sigma T / 2e^2 \chi_{\|}}
\]

\[
\eta_e = \partial \ln T_e / \partial \ln n
\]

\[
\int dx \, d\xi J^c_{\|} \cos \xi = \Delta_s \vec{\psi}
\]

\[
\int dx \, d\xi J^s_{\|} \sin \xi = \Delta_s \vec{\psi}
\]

\[\Delta_s\] is due to the coupling to external perturbations/wall; otherwise = 0

Smolyakov, Sov J PL Phys 1989
Connor et al; PoP, 2001
Collisional dissipation in toroidal plasma:
mainly collisions at the passing/trapped boundary

\[
\omega = \omega_e \left( 1 + \frac{\eta_e}{4} \right)
\]

Weakly collisional regime, electron dissipation, Wilson et al, 1996

\[
\omega = \omega_e \left( 1 + 0.3\eta_e \right)
\]

\[
\omega = \omega_i
\]

\[
\omega = \omega_e \left( 1 + 2.43\eta_e \right)
\]

\[
\omega = \omega_i
\]

\[
\omega = \omega_e \left( 1 + 0.389\eta_i \right)
\]

\[
\omega = \omega_i
\]

Mikhailovski, Kuvshinov, PPR, 1998

\[
\frac{V_e}{\varepsilon \omega} < 1
\]

Ion dissipation is important for larger collisionality
Neoclassical magnetic damping

Drift waves emission

Anomalous viscosity

Symmetry breaking, neoclassical losses in 3D
Summary

Variety of mechanisms affect the island stability:
neoclassical/bootstrap, polarization/inertial drifts, magnetic field curvature/plasma pressure, parallel heat conductivity, banana orbits, ion-sound effects, …

Each of these has to be carefully evaluated

Critical issues:
Island rotation frequency?
Nonlinear trigger/excitation mechanism
"Cooperative effects" of the error field and neoclassical/bootstrap drive?