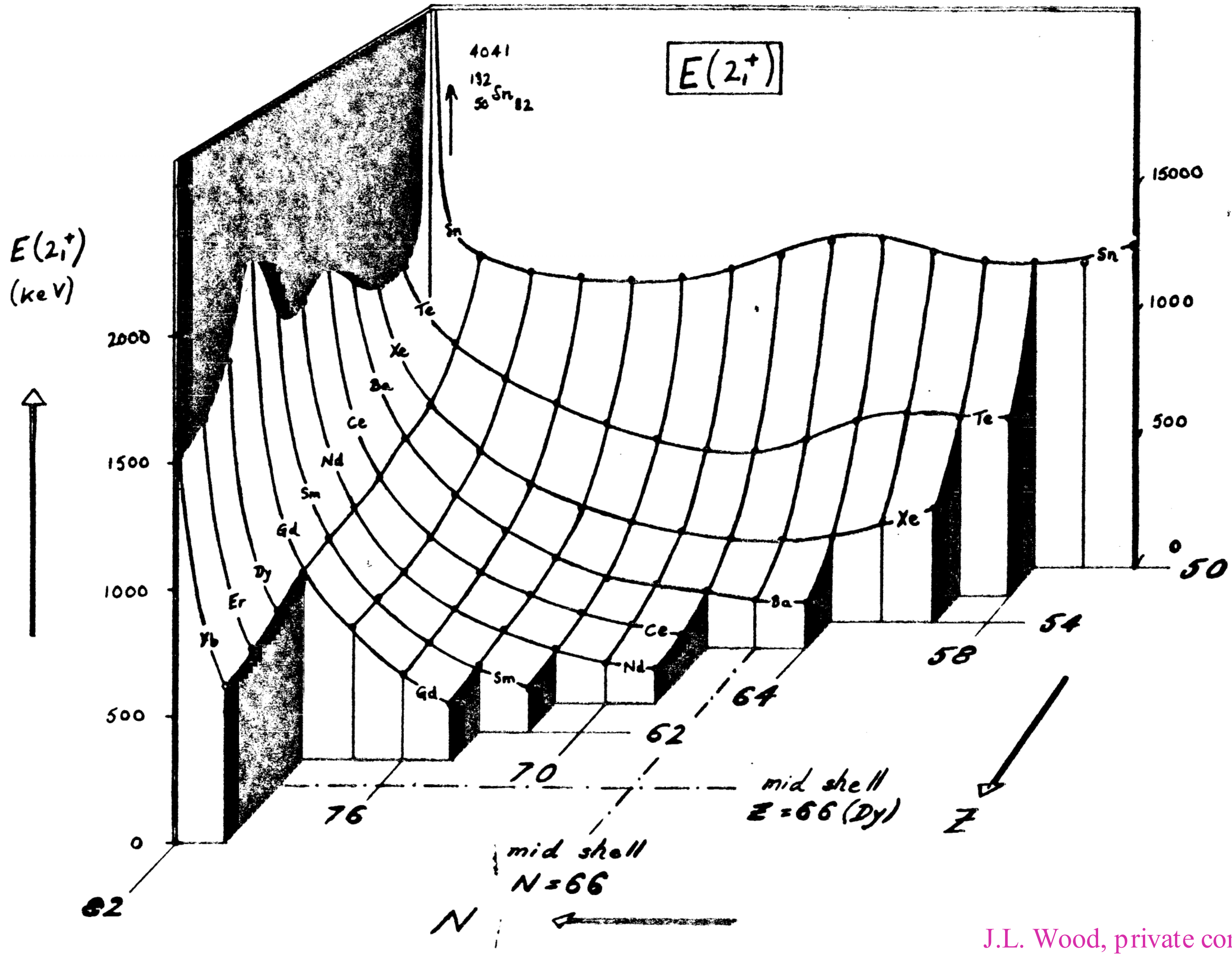
Nuclear Structure
(II) Collective models

P. Van Isacker, GANIL, France

Overview of collective models

- (Rigid) rotor model
- (Harmonic quadrupole) vibrator model
- Liquid-drop model of vibrations and rotations
- Interacting boson model
- Particle-core coupling model
- Nilsson model

Evolution of $E_x(2^+)$



J.L. Wood, private communication

Quantum-mechanical symmetric top

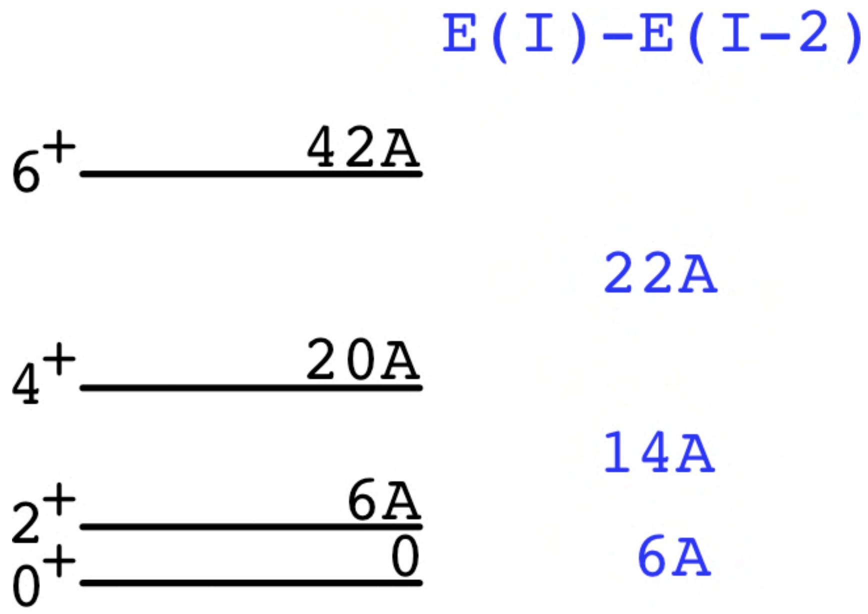
- Energy spectrum:

$$E_{\text{rot}}(I) = \frac{\hbar^2}{2\Im} I(I+1)$$

$$\equiv A\, I(I+1), \quad I = 0,2,4,\ldots$$

- Large deformation $\Rightarrow$ large $\Im \Rightarrow$ low $E_x(2^+)$.
- R_{42} energy ratio:

$$E_{\text{rot}}(4^+)/E_{\text{rot}}(2^+) = 3.333\ldots$$



Rigid rotor model

- Hamiltonian of quantum-mechanical rotor in terms of 'rotational' angular momentum $\boldsymbol{R}$:

$$\hat{H}_{\mathrm{rot}} = \frac{\hbar^2}{2}\left[\frac{R_1^2}{\Im_1} + \frac{R_2^2}{\Im_2} + \frac{R_3^2}{\Im_3}\right] = \frac{\hbar^2}{2}\sum_{i=1}^{3}\frac{R_i^2}{\Im_i}$$

- Nuclei have an additional intrinsic part H_{intr} with 'intrinsic' angular momentum $\boldsymbol{J}$.
- The total angular momentum is $\boldsymbol{I}=\boldsymbol{R}+\boldsymbol{J}$.

Rigid axially symmetric rotor

- For $\Im_1 = \Im_2 = \Im \neq \Im_3$ the rotor hamiltonian is

$$\hat{H}_{\text{rot}} = \sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} (I_i - J_i)^2 = \underbrace{\sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} I_i^2}_{\hat{H}'_{\text{rot}}} - \underbrace{\sum_{i=1}^{3} \frac{\hbar^2}{\Im_i} I_i J_i}_{\text{Coriolis}} + \underbrace{\sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} J_i^2}_{\text{intrinsic}}$$

- Eigenvalues of H'_{rot}:

$$E'_{KI} = \frac{\hbar^2}{2\Im} I(I+1) + \frac{\hbar^2}{2} \left(\frac{1}{\Im_3} - \frac{1}{\Im} \right) K^2$$

- Eigenvectors $|KIM\rangle$ of H'_{rot} satisfy:

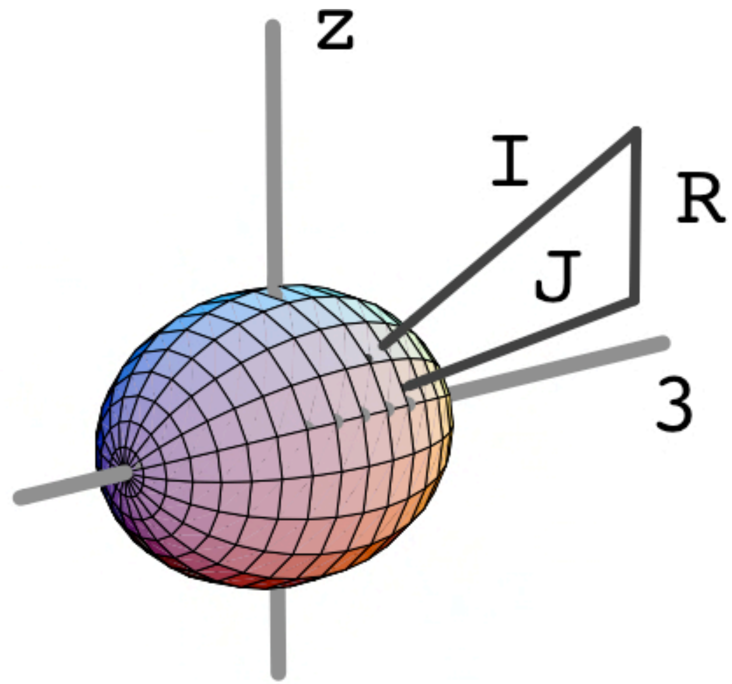
$$I^2 |KIM\rangle = I(I+1) |KIM\rangle,$$

$$I_z |KIM\rangle = M |KIM\rangle, \quad I_3 |KIM\rangle = K |KIM\rangle$$

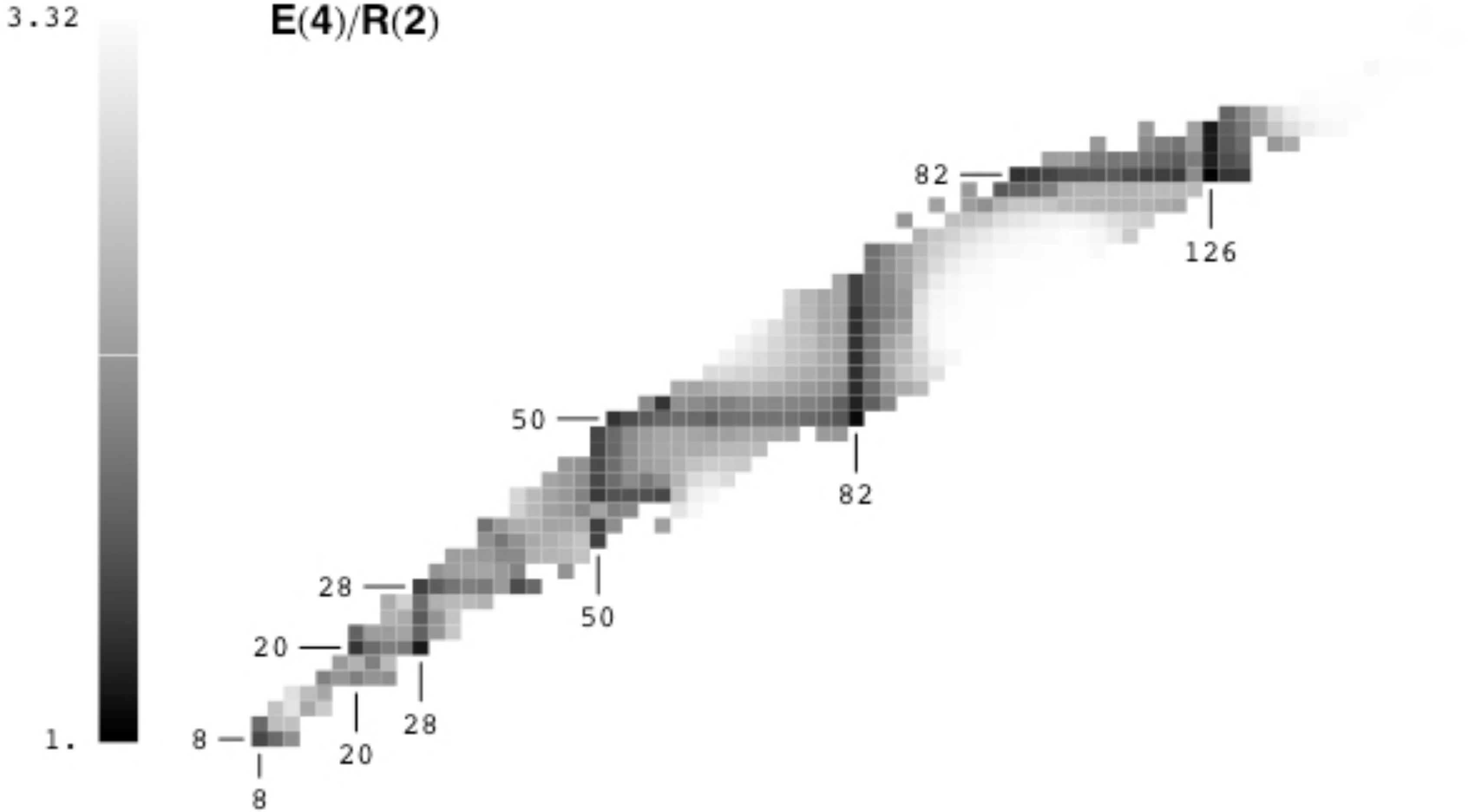
Ground-state band of an axial rotor

- The ground-state spin of even-even nuclei is I=0. Hence K=0 for ground-state band:

$$E_I = \frac{\hbar^2}{2\Im} I(I+1)$$



The ratio R_{42}



Electric (quadrupole) properties

- Partial γ-ray half-life:

$$T^{\gamma}_{1/2}(\mathrm{E}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda+1}{\lambda[(2\lambda+1)!!]^2} \left(\frac{E_\gamma}{\hbar c} \right)^{2\lambda+1} B(\mathrm{E}\lambda) \right\}^{-1}$$

- Electric quadrupole transitions:

$$B(\mathrm{E2}; I_\mathrm{i} \to I_\mathrm{f}) = \frac{1}{2I_\mathrm{i}+1} \sum_{M_\mathrm{i}} \sum_{M_\mathrm{f}\mu} \left| \left\langle I_\mathrm{f} M_\mathrm{f} \left| \sum_{k=1}^{A} e_k r_k^2 Y_{2\mu}(\theta_k, \varphi_k) \right| I_\mathrm{i} M_\mathrm{i} \right\rangle \right|^2$$

- Electric quadrupole moments:

$$eQ(I) = \left\langle IM = I \left| \sqrt{\frac{16\pi}{5}} \sum_{k=1}^{A} e_k r_k^2 Y_{20}(\theta_k, \varphi_k) \right| IM = I \right\rangle$$

Magnetic (dipole) properties

- Partial γ-ray half-life:

$$T^{\gamma}_{1/2}(\mathrm{M}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda+1}{\lambda[(2\lambda+1)!!]^2} \left(\frac{E_\gamma}{\hbar c}\right)^{2\lambda+1} B(\mathrm{M}\lambda) \right\}^{-1}$$

- Magnetic dipole transitions:

$$B(\mathrm{M1}; I_\mathrm{i} \rightarrow I_\mathrm{f}) = \frac{1}{2I_\mathrm{i}+1} \sum_{M_\mathrm{i}} \sum_{M_\mathrm{f}\mu} \left| \langle I_\mathrm{f} M_\mathrm{f} | \sum_{k=1}^{A} \left(g_k^l l_{k,\mu} + g_k^s s_{k,\mu} \right) | I_\mathrm{i} M_\mathrm{i} \rangle \right|^2$$

- Magnetic dipole moments:

$$\mu(I) = \langle IM = I | \sum_{k=1}^{A} \left(g_k^l l_{k,z} + g_k^s s_{k,z} \right) | IM = I \rangle$$

E2 properties of rotational nuclei

- *Intra*-band E2 transitions:

$$B(\mathrm{E2}; KI_\mathrm{i} \rightarrow KI_\mathrm{f}) = \frac{5}{16\pi} \langle I_\mathrm{i} K\, 20 | I_\mathrm{f} K \rangle^2 e^2 Q_0(K)^2$$

- E2 moments:

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

- $Q_0(K)$ is the 'intrinsic' quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(\boldsymbol{r}') r^2 \left(3\cos^2\theta' - 1\right) d\boldsymbol{r}', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

E2 properties of ground-state bands

- For the ground state (usually $K=I$):

$$Q(K=I)=\frac{I(2I-1)}{(I+1)(2I+3)}Q_0(K)$$

- For the gsb in even-even nuclei ($K=0$):

$$B(\mathrm{E2};I\to I-2)=\frac{15}{32\pi}\frac{I(I-1)}{(2I-1)(2I+1)}e^2Q_0^2$$

$$Q(I)=-\frac{I}{2I+3}Q_0$$

$$\Rightarrow\left|eQ(2_1^+)\right|=\frac{2}{7}\sqrt{16\pi\cdot B(\mathrm{E2};2_1^+\to 0_1^+)}$$

Generalized intensity relations

- Mixing of *K* arises from
  - Dependence of Q_0 on ***I*** (stretching)
  - Coriolis interaction
  - Triaxiality
- Generalized *intra*- and *inter*-band matrix elements (*eg* E2):

$$\frac{\sqrt{B(\mathrm{E2};K_\mathrm{i}I_\mathrm{i}\to K_\mathrm{f}I_\mathrm{f})}}{\langle I_\mathrm{i}K_\mathrm{i}\;2K_\mathrm{f}-K_\mathrm{i}|I_\mathrm{f}K_\mathrm{f}\rangle} = M_0 + M_1\Delta + M_2\Delta^2 + \cdots$$

with $\Delta = I_\mathrm{f}(I_\mathrm{f}+1) - I_\mathrm{i}(I_\mathrm{i}+1)$

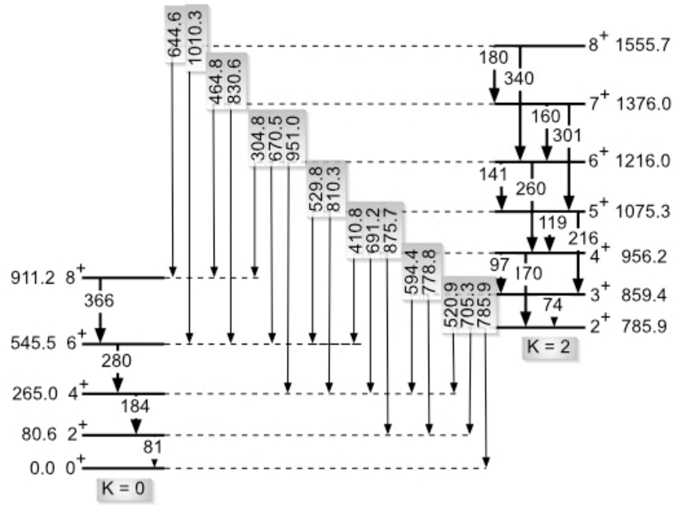
Inter-band E2 transitions

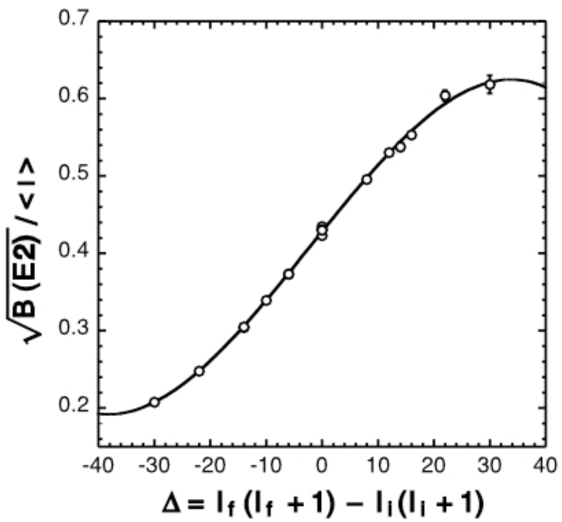
- Example of $\gamma \to g$ transitions in ^{166}Er:

$$\frac{\sqrt{B(\mathrm{E2}; I_\gamma \to I_\mathrm{g})}}{\langle I_\gamma 2\, 2 - 2 | I_\mathrm{g} 0 \rangle}$$

$$= M_0 + M_1 \Delta + M_2 \Delta^2 + \cdots$$

$$\Delta = I_\mathrm{g}(I_\mathrm{g}+1) - I_\gamma(I_\gamma+1)$$





W.D. Kulp *et al.*, Phys. Rev. C **73** (2006) 014308

Modes of nuclear vibration

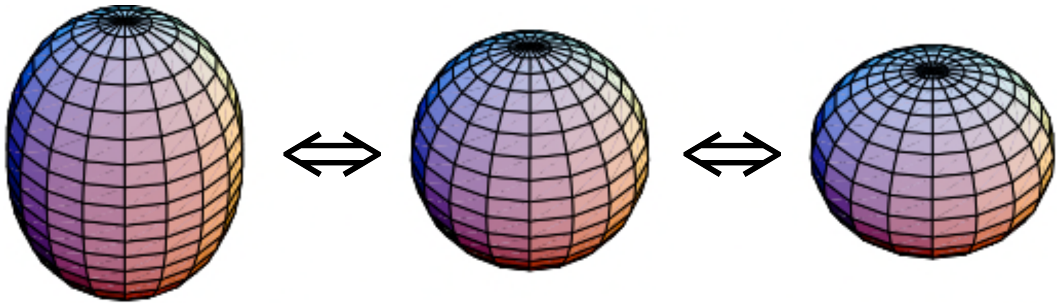
- Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.
- Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:
  - Spherical equilibrium shape
  - Spheroidal equilibrium shape

Vibrations about a spherical shape

- Vibrations are characterized by a multipole quantum number λ in surface parametrization:

$$R(\theta,\varphi) = R_0\left(1 + \sum_{\lambda}\sum_{\mu=-\lambda}^{+\lambda}\alpha_{\lambda\mu}Y^{*}_{\lambda\mu}(\theta,\varphi)\right)$$

  - λ=0: compression (high energy)
  - λ=1: translation (not an intrinsic excitation)
  - λ=2: quadrupole vibration



Properties of spherical vibrations

- Energy spectrum:

$$E_{\mathrm{vib}}(n) = \left(n + \tfrac{5}{2}\right)\hbar\omega,\ n = 0,1\ldots$$

- R_{42} energy ratio:

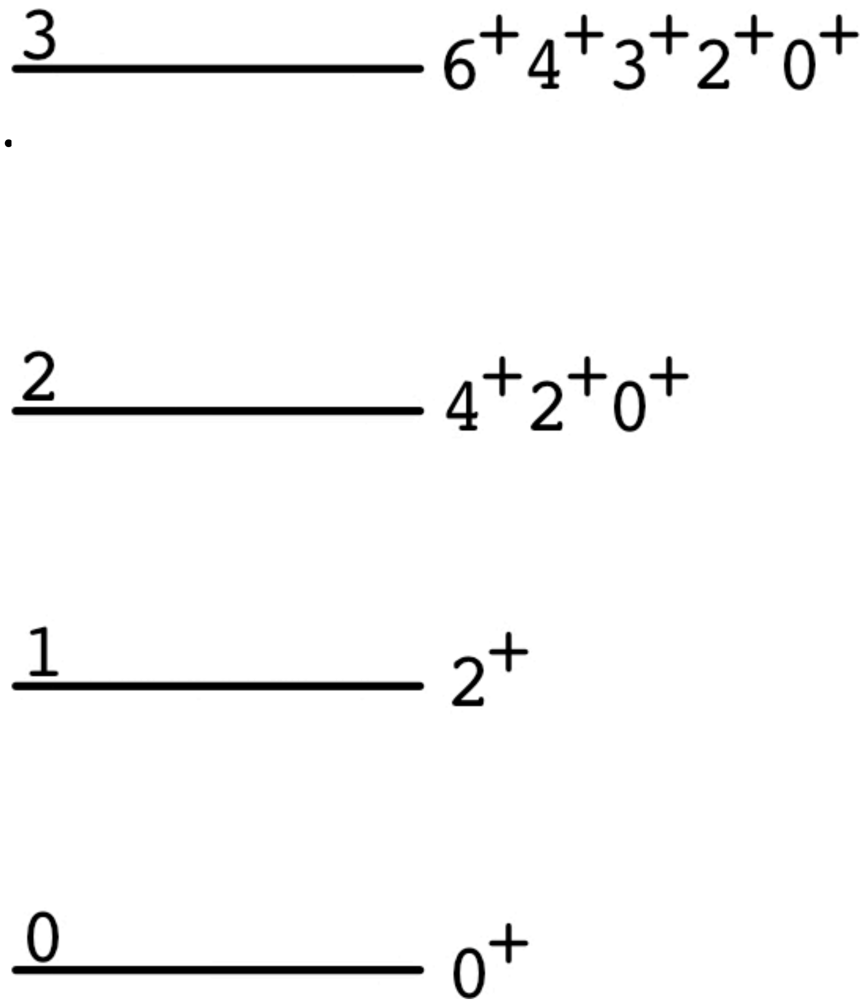
$$E_{\mathrm{vib}}(4^+)/E_{\mathrm{vib}}(2^+) = 2$$

- E2 transitions:

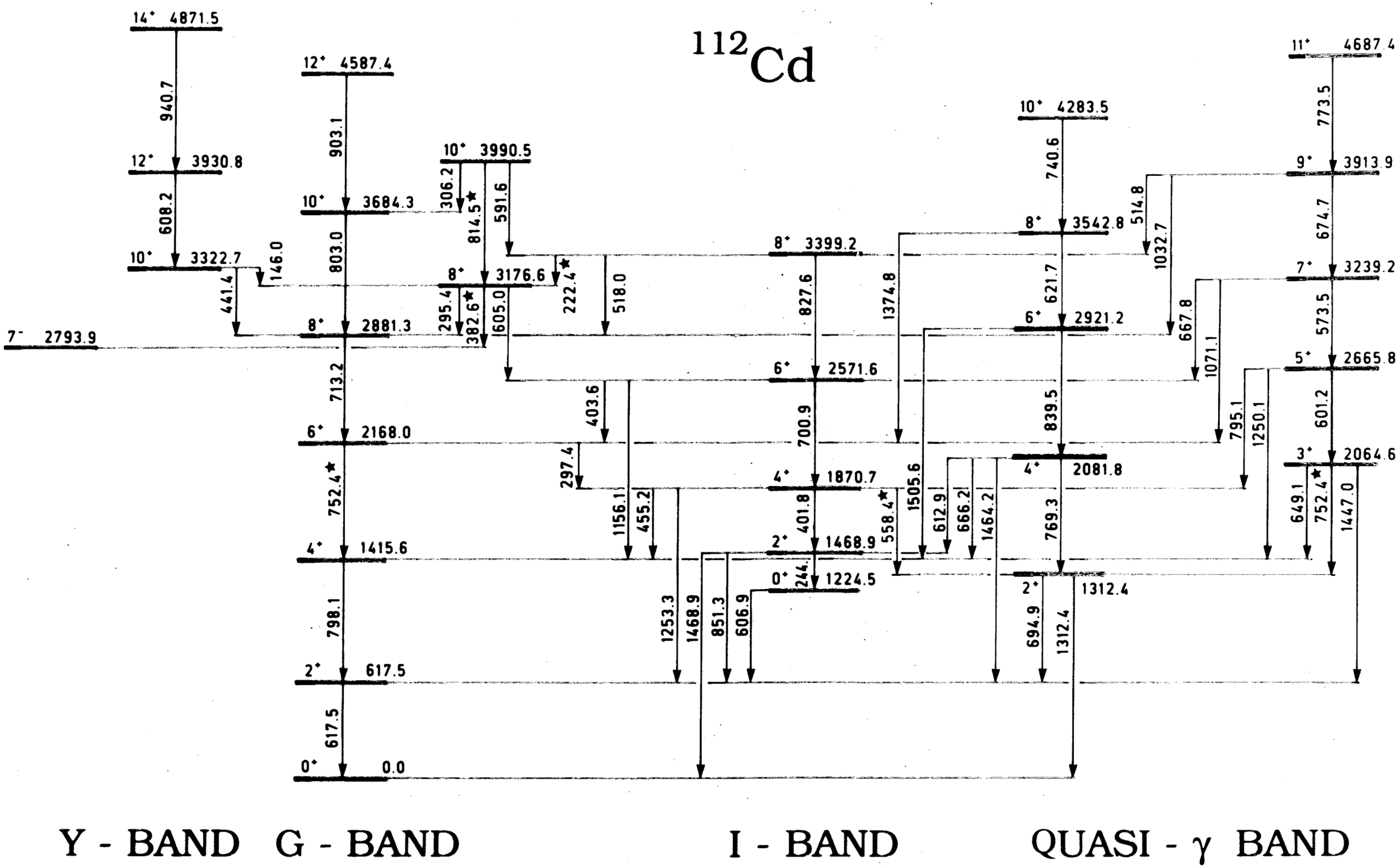
$$B(\mathrm{E2};2_1^+ \to 0_1^+) = \alpha^2$$

$$B(\mathrm{E2};2_2^+ \to 0_1^+) = 0$$

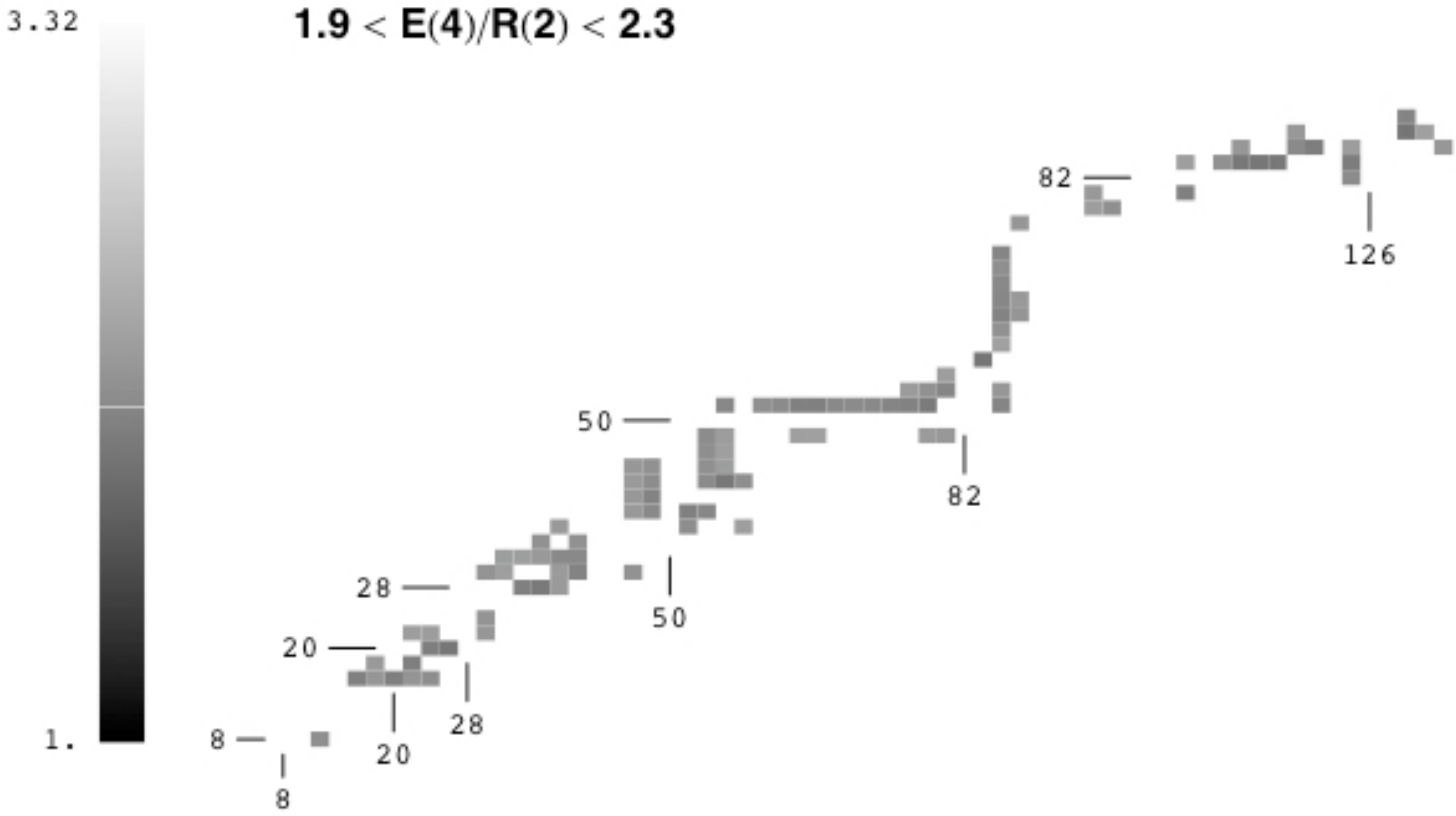
$$B(\mathrm{E2};n = 2 \to n = 1) = 2\alpha^2$$



Example of ^{112}Cd

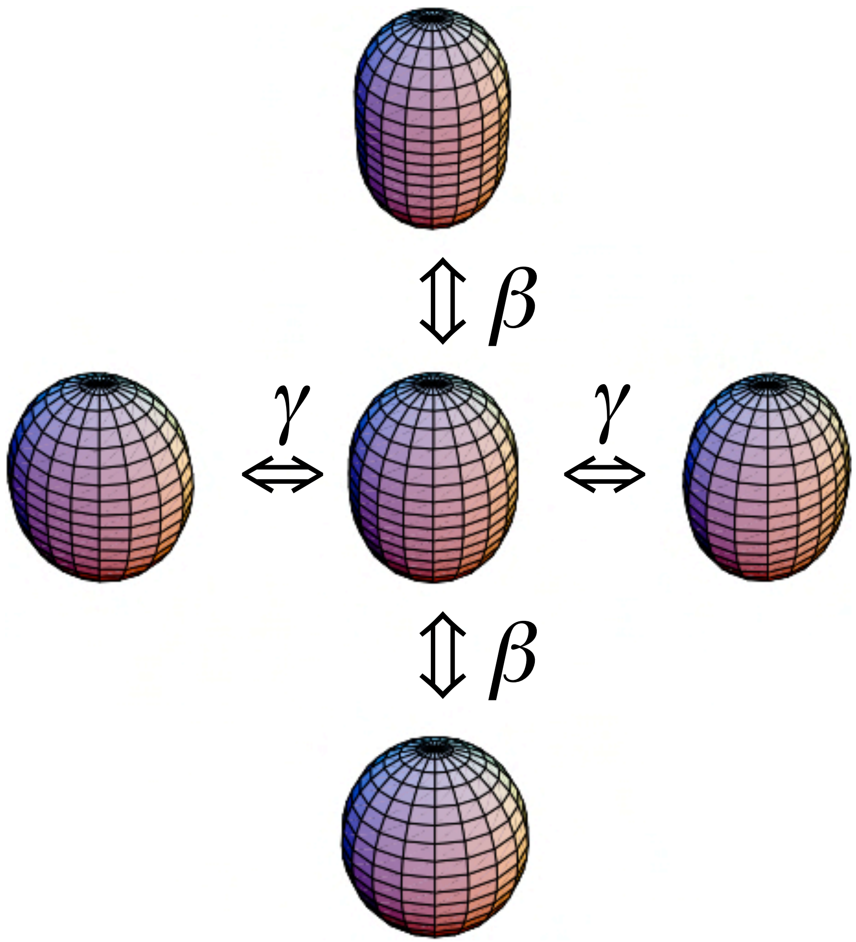


Possible vibrational nuclei from R_{42}

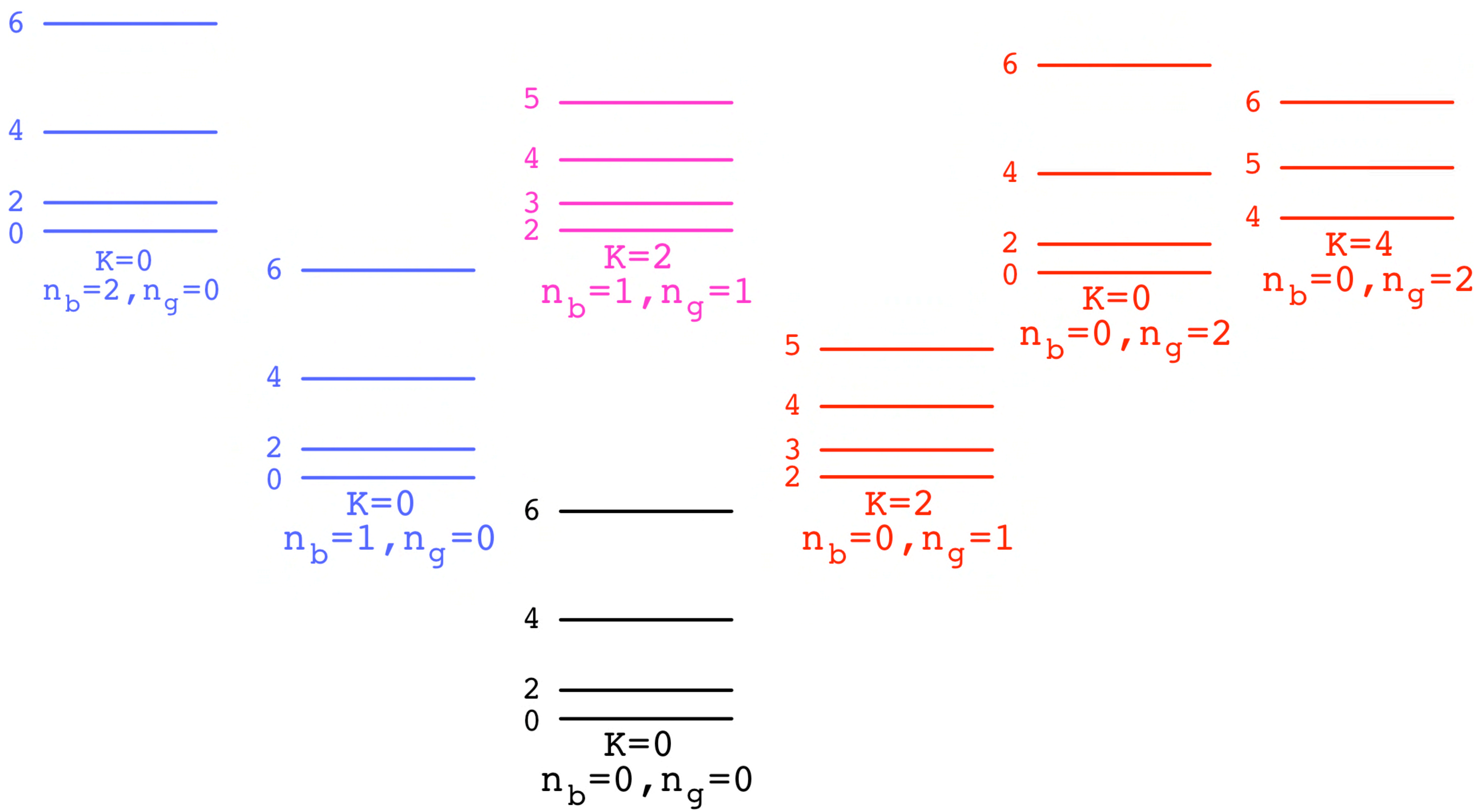


Vibrations about a spheroidal shape

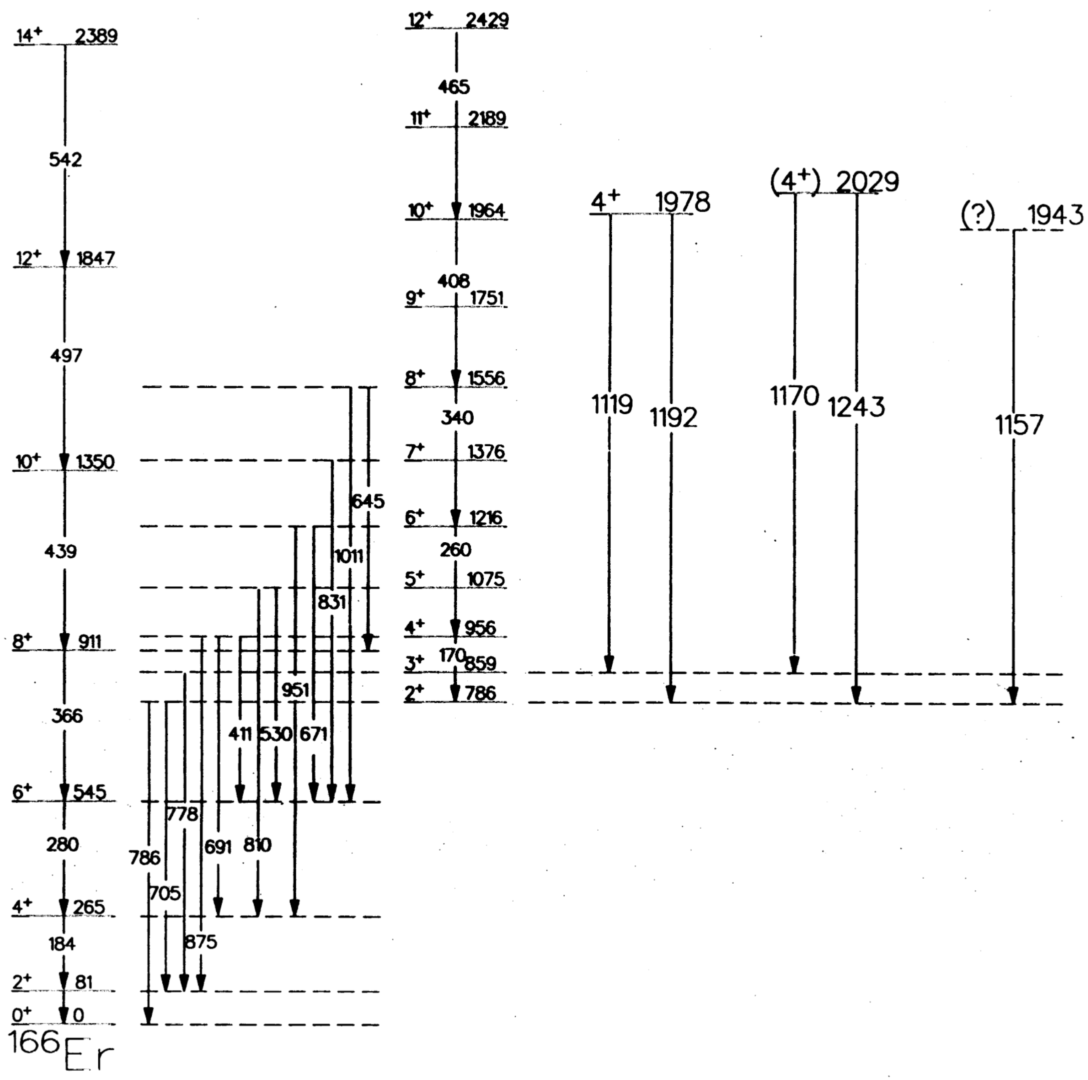
- The vibration of a shape with axial symmetry is characterized by $a_{\lambda\nu}$.
- Quadrupole oscillations
  - ν=0: along the axis of symmetry (β)
  - ν=±1: spurious rotation
  - ν=±2: perpendicular to axis of symmetry (γ)



Spectrum of spheroidal vibrations



Example of ^{166}Er



Rigid triaxial rotor

- Triaxial rotor hamiltonian $\Im_1 \neq \Im_2 \neq \Im_3$:

$$\hat{H}'_{\mathrm{rot}} = \sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\Im} I^2 + \frac{\hbar^2}{2\Im_f} I_3^2}_{\hat{H}'_{\mathrm{axial}}} + \underbrace{\frac{\hbar^2}{2\Im_g}\left(I_+^2 + I_-^2\right)}_{\hat{H}'_{\mathrm{mix}}}$$

$$\frac{1}{\Im} = \frac{1}{2}\left(\frac{1}{\Im_1} + \frac{1}{\Im_2}\right), \quad \frac{1}{\Im_f} = \frac{1}{\Im_3} - \frac{1}{\Im}, \quad \frac{1}{\Im_g} = \frac{1}{4}\left(\frac{1}{\Im_1} - \frac{1}{\Im_2}\right)$$

- H'_{mix} non-diagonal in axial basis $|KIM\rangle \Rightarrow K$ is *not* a conserved quantum number

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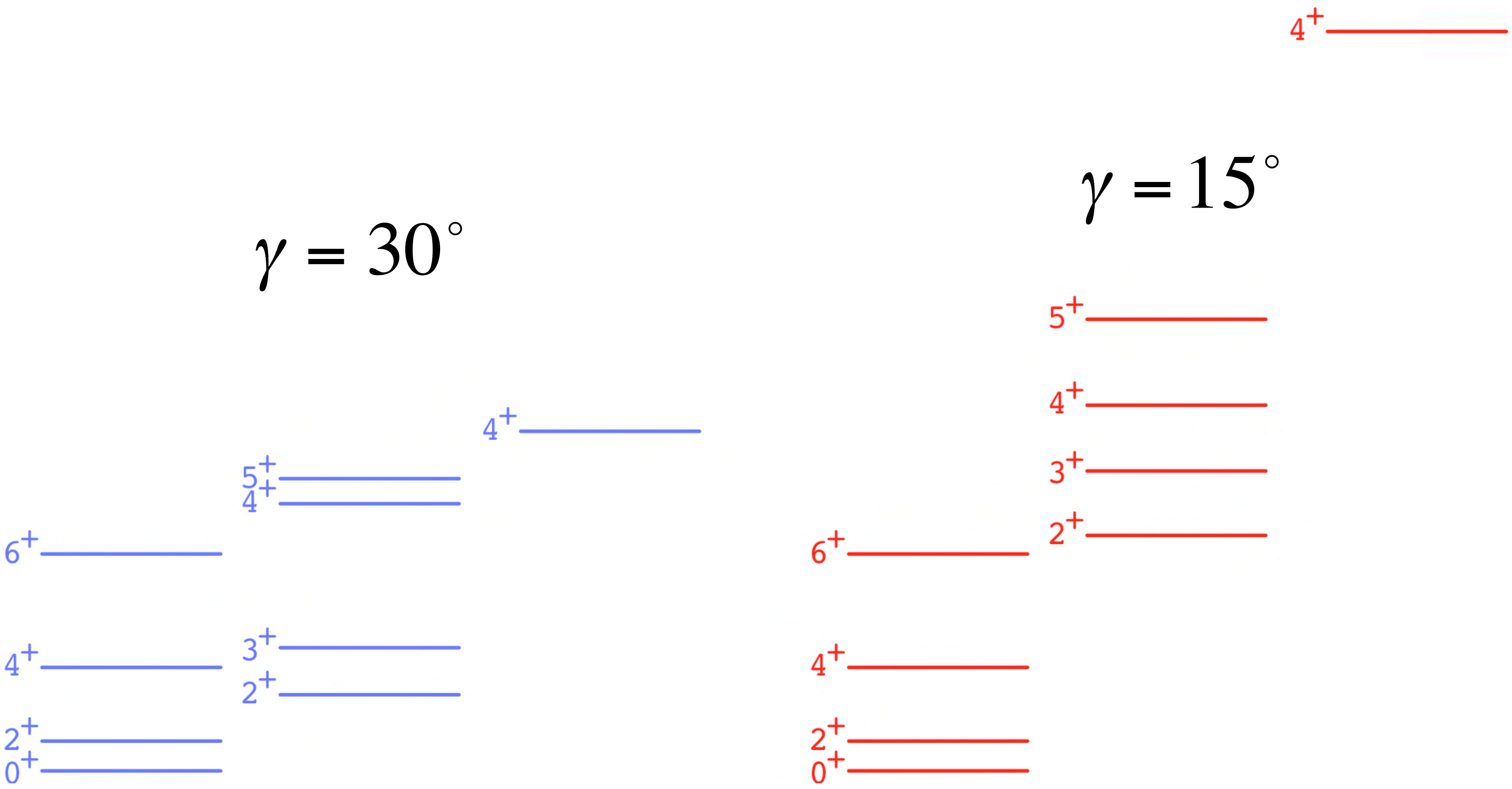
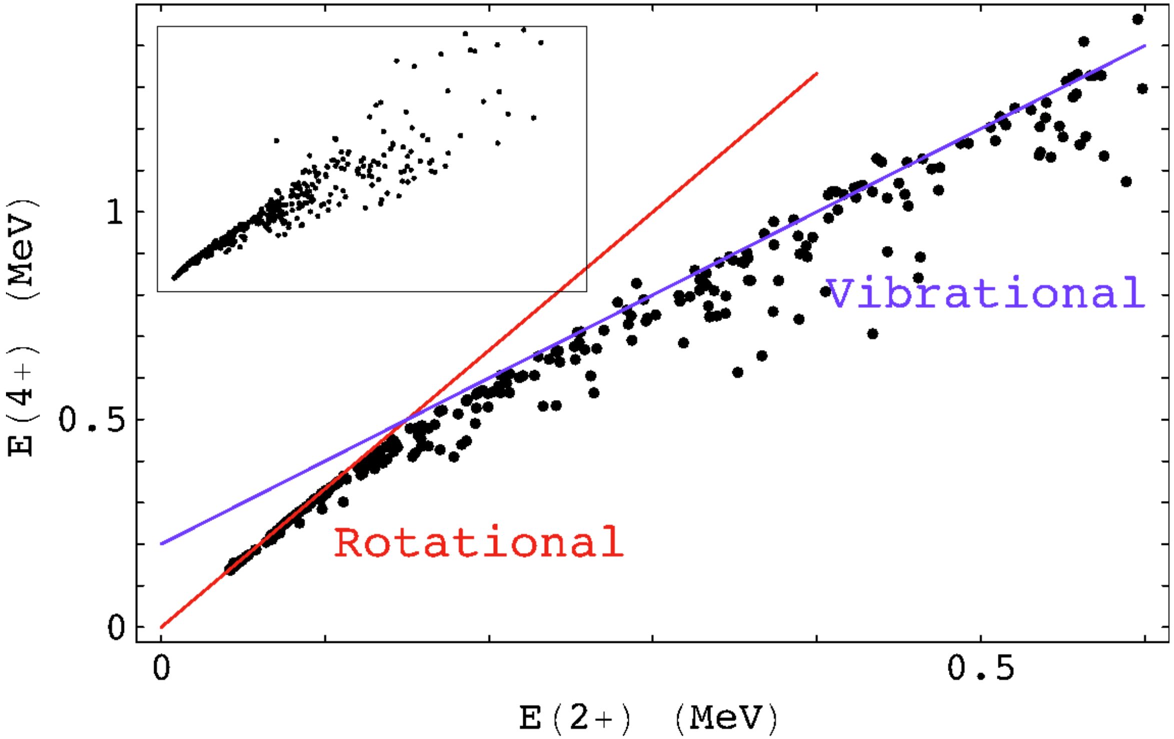
Rigid triaxial rotor spectra



Tri-partite classification of nuclei

- Empirical evidence for seniority-type, vibrational- and rotational-like nuclei:



- Need for model of *vibrational* nuclei.

N.V. Zamfir *et al.*, Phys. Rev. Lett. **72** (1994) 3480

Interacting boson model

- Describe the nucleus as a system of N interacting s and d bosons. Hamiltonian:

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

- Justification from
  - Shell model: s and d bosons are associated with S and D fermion (*Cooper*) pairs.
  - Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

Dimensions

- Assume Ω available 1-fermion states. Number of *n*-fermion states is $\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega-n)!}$
- Assume Ω available 1-boson states. Number of *n*-boson states is $\binom{\Omega+n-1}{n} = \frac{(\Omega+n-1)!}{n!(\Omega-1)!}$
- Example: $^{162}Dy_{96}$ with 14 neutrons (Ω=44) and 16 protons (Ω=32) ($^{132}Sn_{82}$ inert core).
  - SM dimension: $\sim 7 \cdot 10^{19}$
  - IBM dimension: 15504

Dynamical symmetries

- Boson hamiltonian is of the form

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

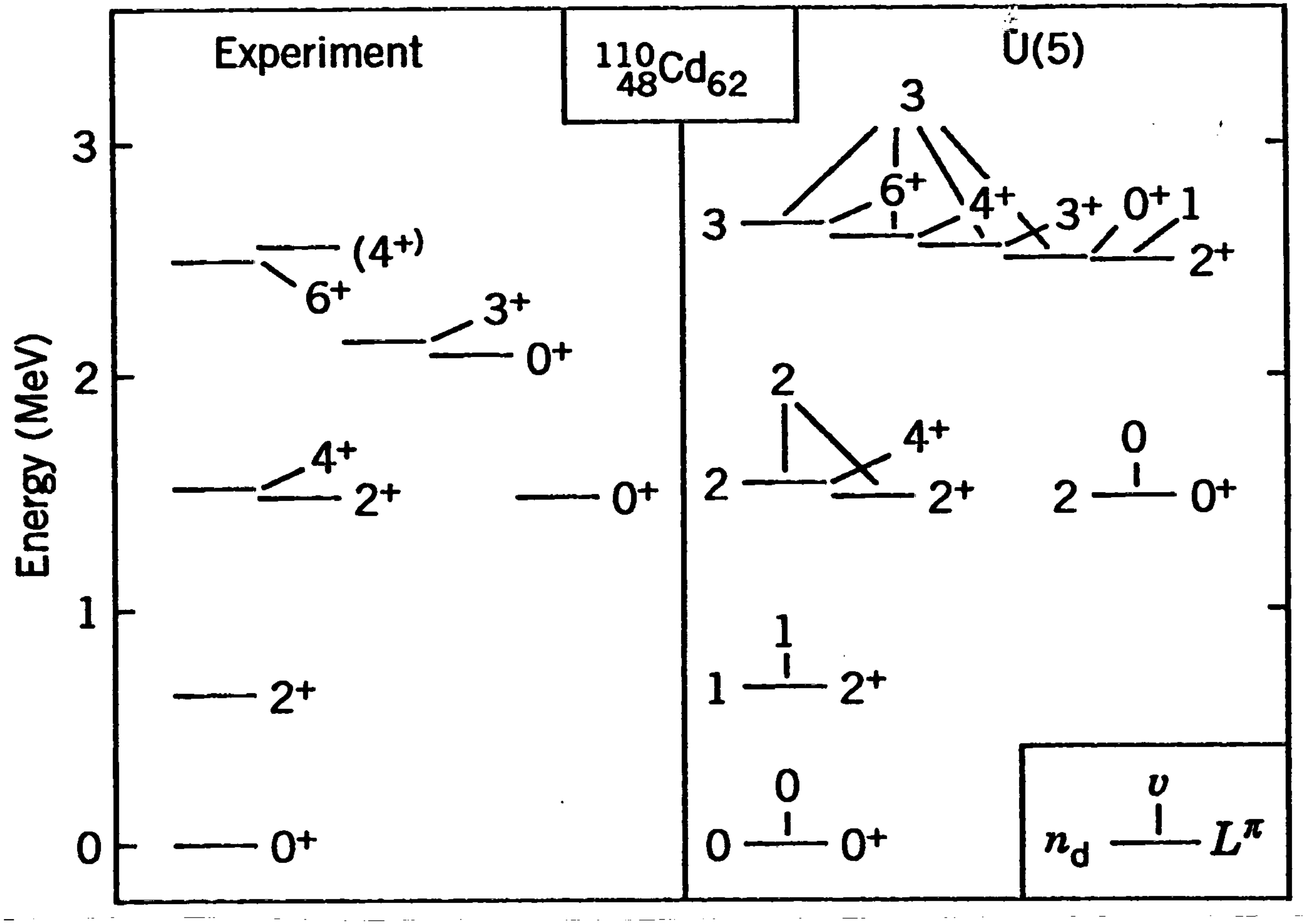
- In general not solvable analytically.
- Three solvable cases with SO(3) symmetry:

$$\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

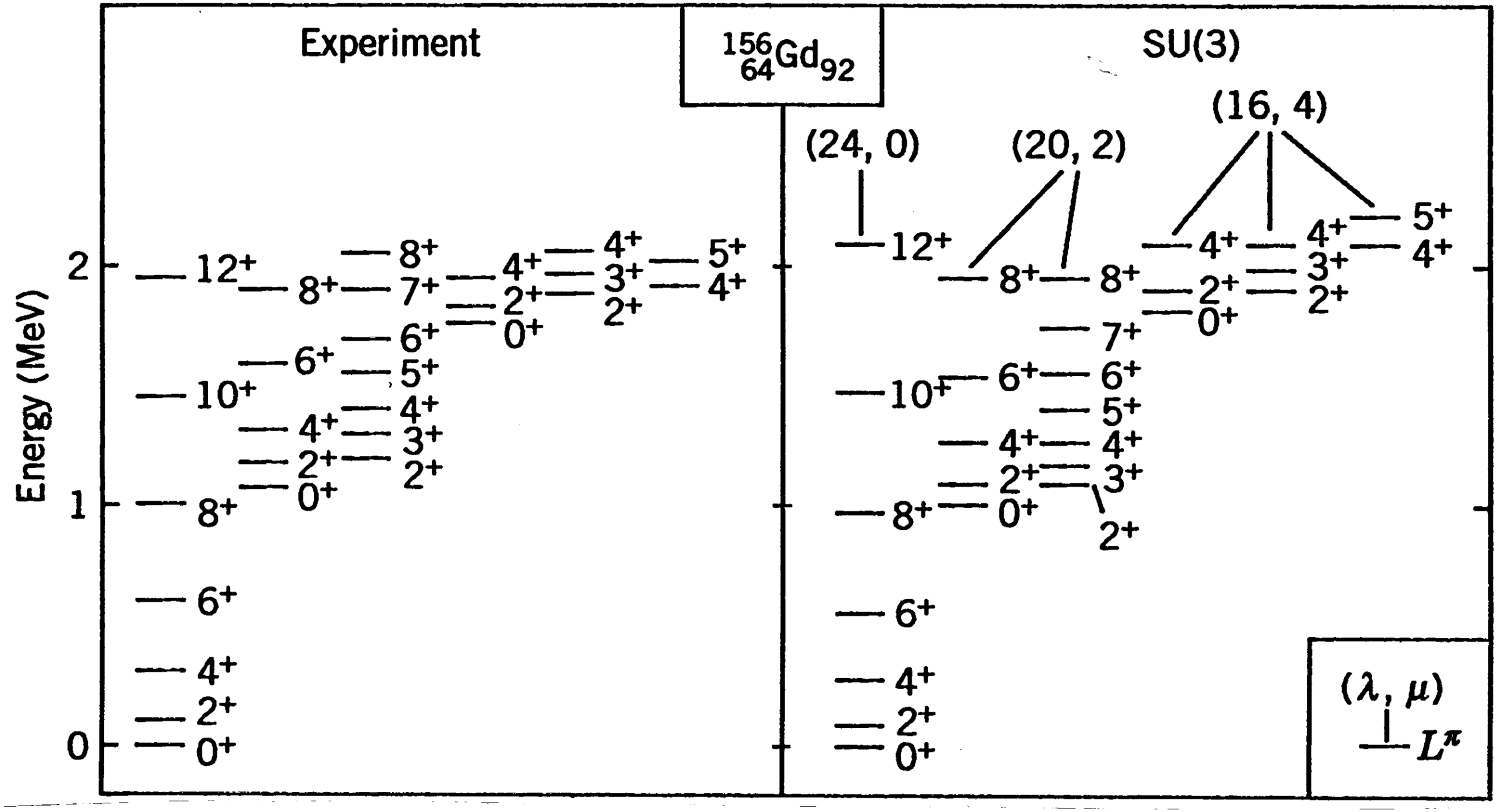
$$\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3)$$

$$\mathrm{U}(6) \supset \mathrm{SO}(6) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

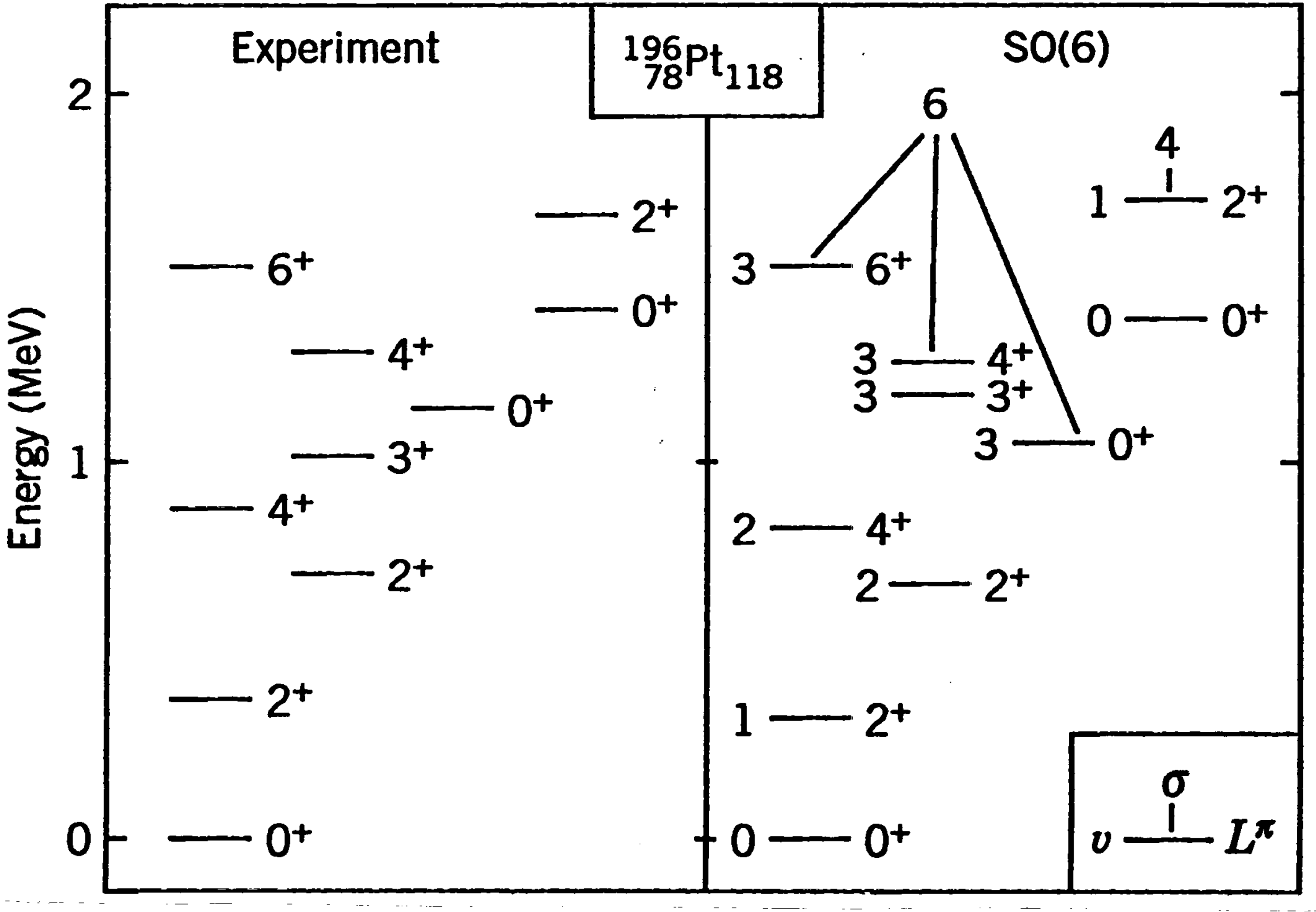
U(5) vibrational limit: $^{110}Cd_{62}$



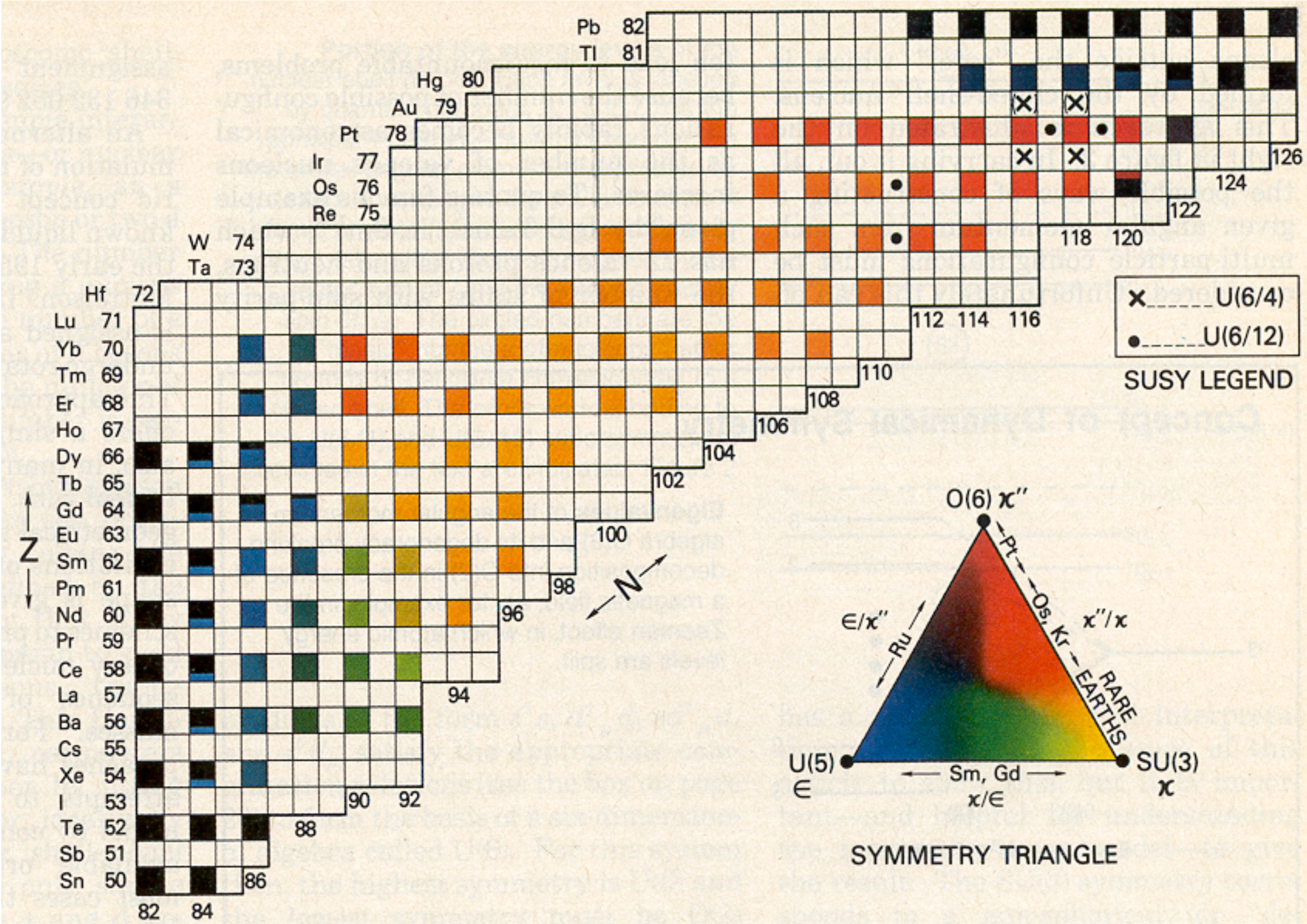
SU(3) rotational limit: $^{156}Gd_{92}$



SO(6) γ-unstable limit: $^{196}Pt_{118}$



Applications of IBM

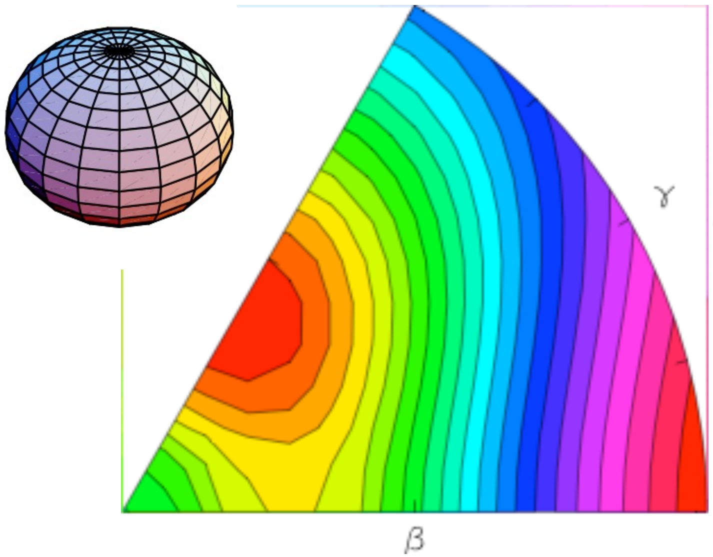


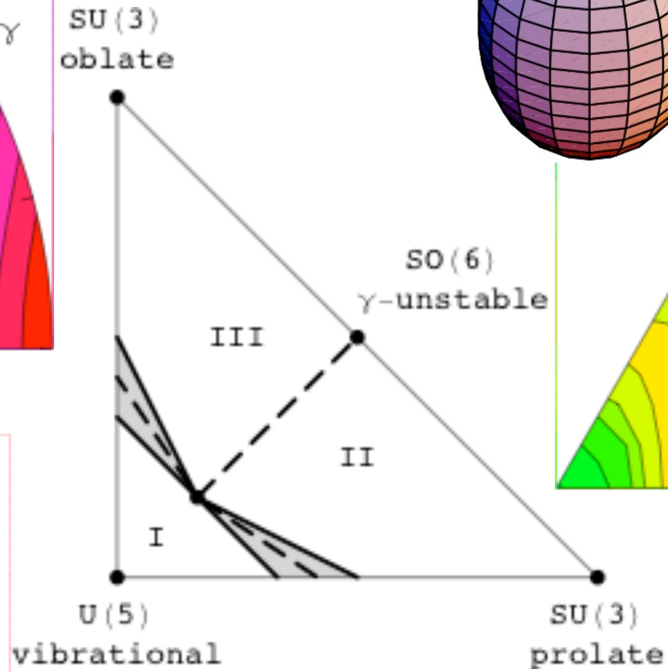
Classical limit of IBM

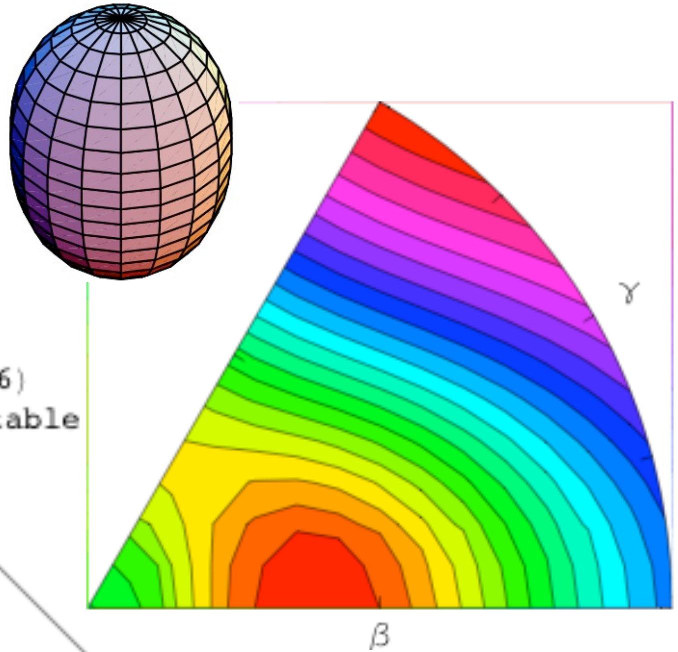
- For large boson number N the minimum of $V(\beta,\gamma)=\langle N;\beta\gamma|H|N;\beta\gamma\rangle$ approaches the exact ground-state energy:

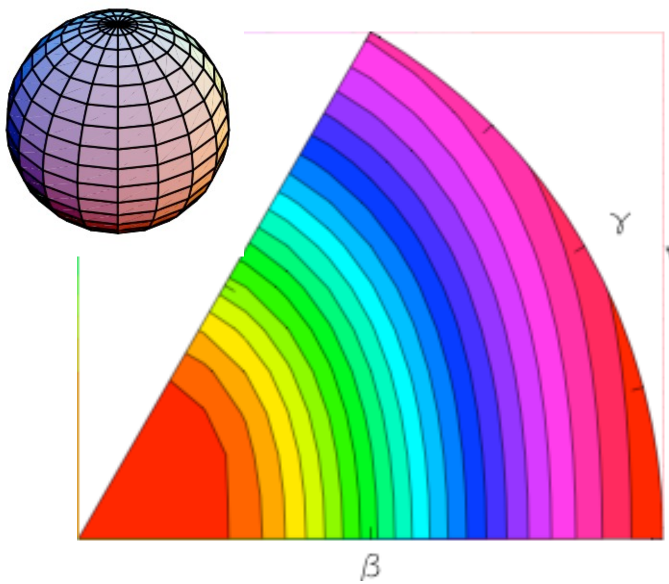
$$V(\beta,\gamma) \propto \begin{cases} \mathrm{U(5)}: & \dfrac{\beta^2}{1+\beta^2} \\ \mathrm{SU(3)}: & \dfrac{\beta^4 - 4\sqrt{2}\beta^3\cos 3\gamma + 8\beta^2}{8\left(1+\beta^2\right)^2} \\ \mathrm{SO(6)}: & \left(\dfrac{1-\beta^2}{1+\beta^2}\right)^2 \end{cases}$$

Phase diagram of IBM



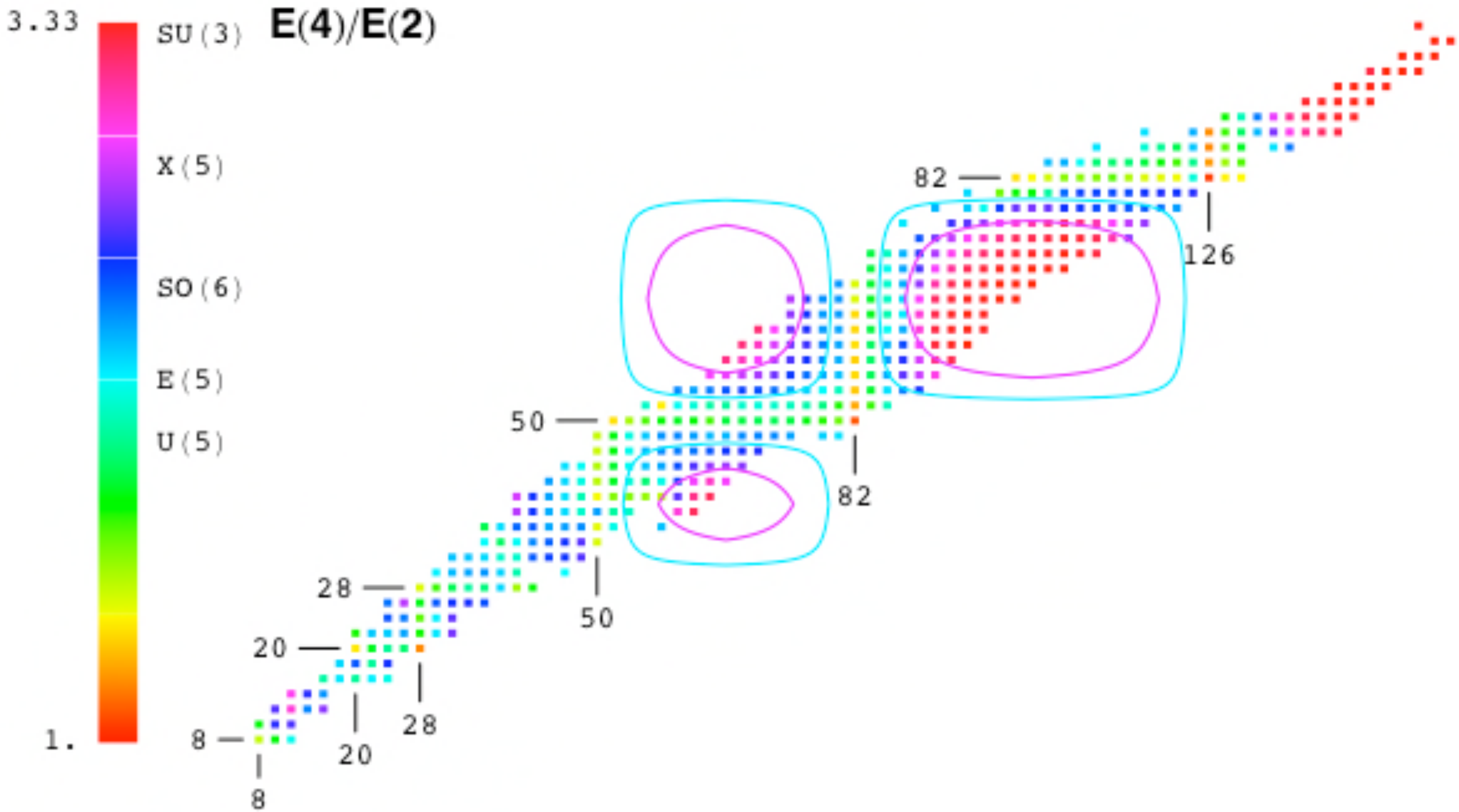






J. Jolie *et al.* , Phys. Rev. Lett. **87** (2001) 162501.

The ratio R_{42}

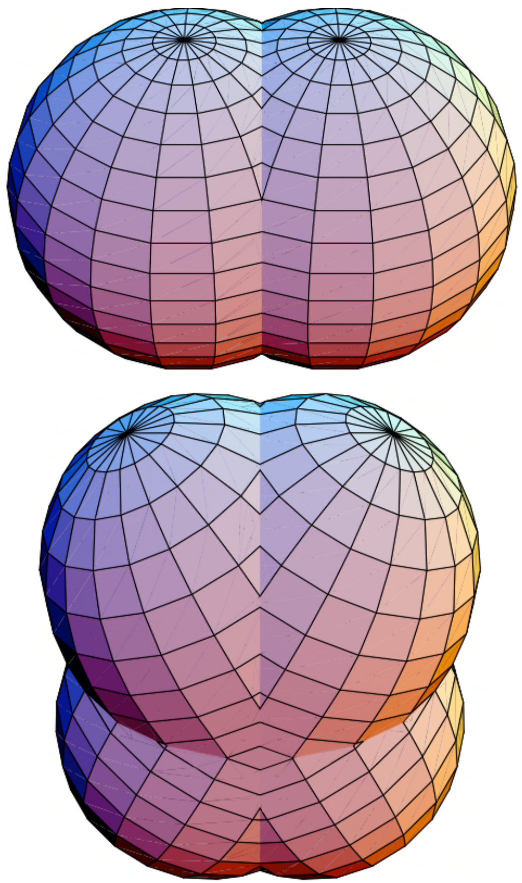


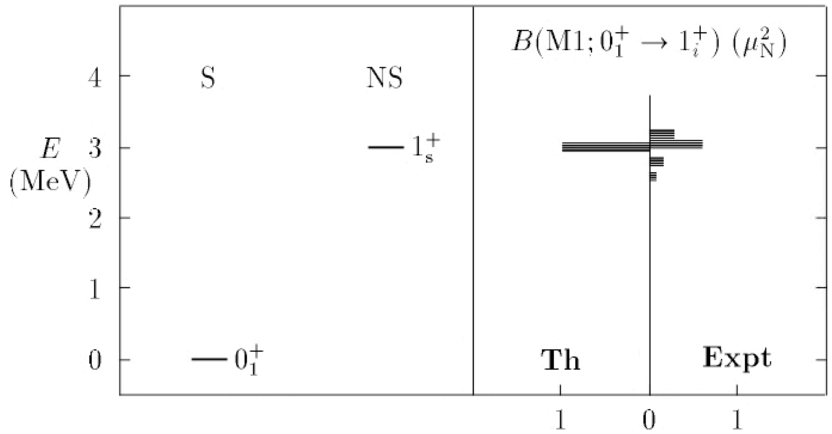
Extensions of IBM

- Neutron and proton degrees freedom (IBM-2):
  - *F*-spin multiplets ($N_\nu+N_\pi$=constant)
  - Scissors excitations
- Fermion degrees of freedom (IBFM):
  - Odd-mass nuclei
  - Supersymmetry (doublets & quartets)
- Other boson degrees of freedom:
  - Isospin T=0 & T=1 pairs (IBM-3 & IBM-4)
  - Higher multipole (g,…) pairs

Scissors mode

- Collective displacement modes between neutrons and protons:
  - Linear displacement (giant dipole resonance): R_ν-$R_\pi \Rightarrow$ E1 excitation.
  - Angular displacement (scissors resonance): L_ν-$L_\pi \Rightarrow$ M1 excitation.





Bosons + fermions

- Odd-mass nuclei are fermions.
- Describe an odd-mass nucleus as N bosons + 1 fermion mutually interacting. Hamiltonian:

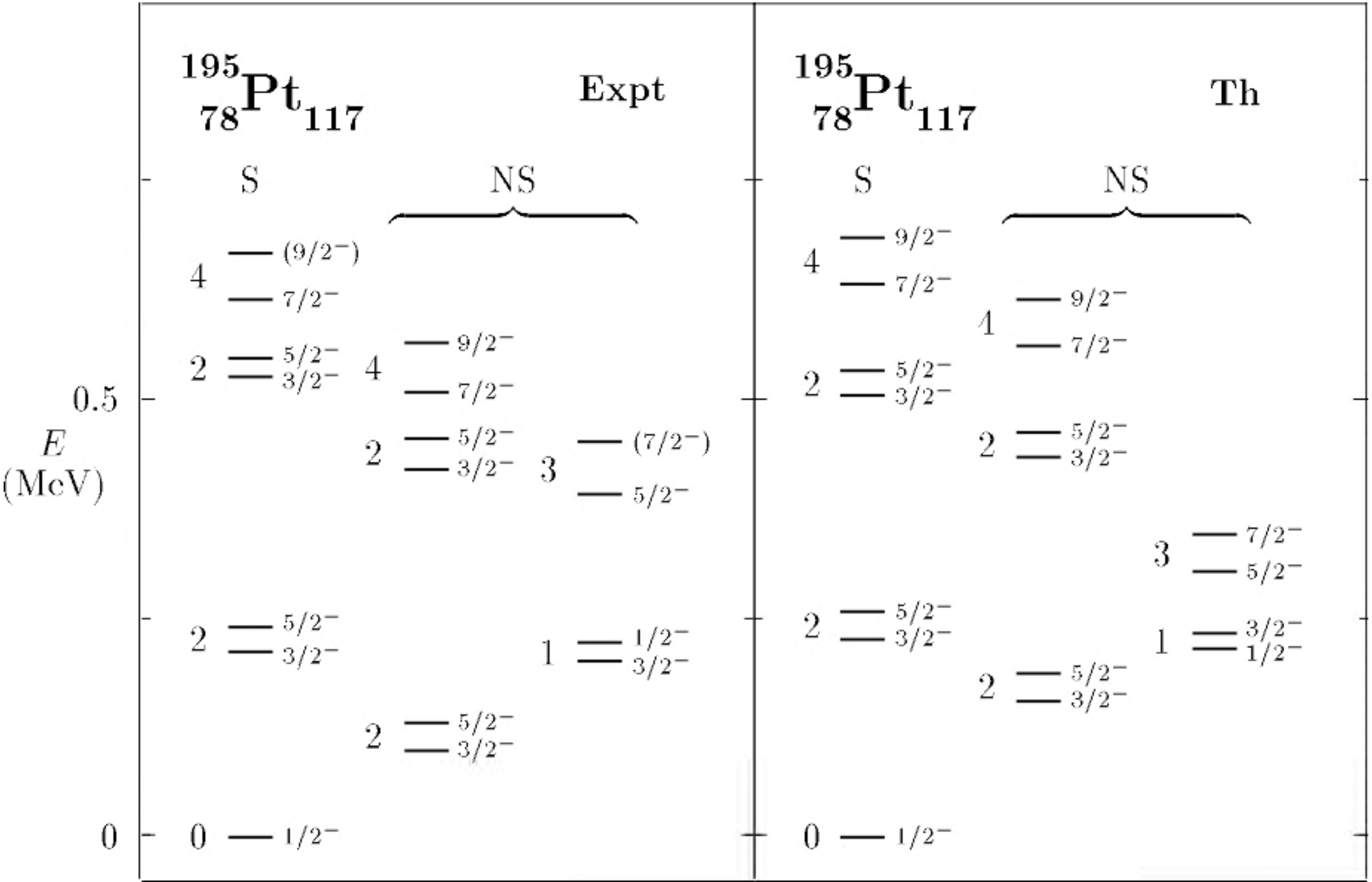
$$\hat{H}_{\mathrm{IBFM}} = \hat{H}_{\mathrm{IBM}} + \sum_{j=1}^{\Omega} \bar{\varepsilon}_j \hat{a}_j^+ \hat{a}_j + \sum_{i_1 i_2 = 1}^{6} \sum_{j_1 j_2 = 1}^{\Omega} \bar{\upsilon}_{i_1 j_1 i_2 j_2} \hat{b}_{i_1}^+ \hat{a}_{j_1}^+ \hat{b}_{i_2} \hat{a}_{j_2}$$

- Algebra:

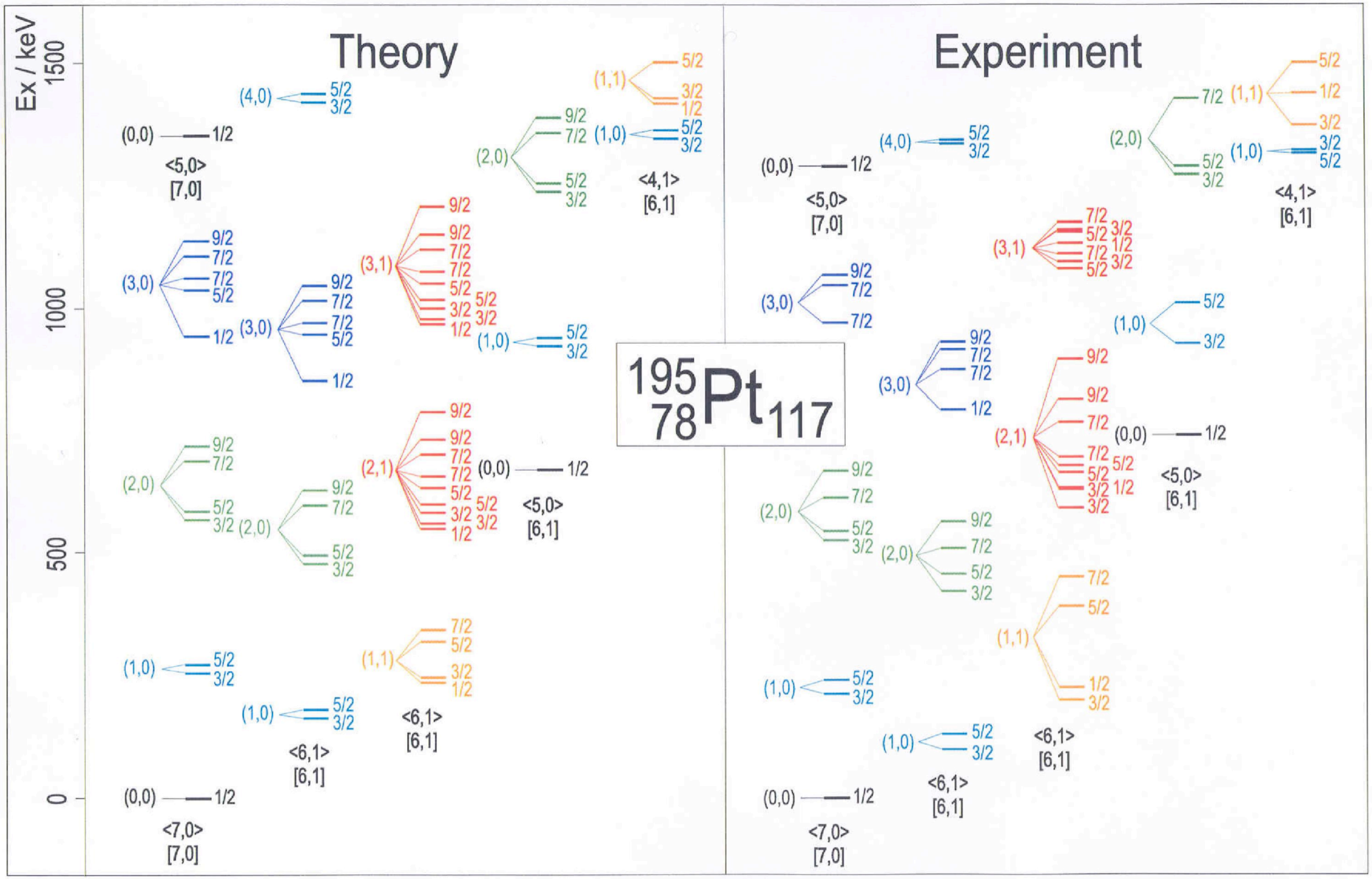
$$\mathrm{U}(6) \oplus \mathrm{U}(\Omega) = \left\{ \begin{matrix} \hat{b}_{i_1}^+ \hat{b}_{i_2} & \\ & \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{matrix} \right\}$$

- Many-body problem is solved analytically for certain energies ε and interactions υ.

Example: $^{195}Pt_{117}$



Example: $^{195}Pt_{117}$ (new data)



Nuclear supersymmetry

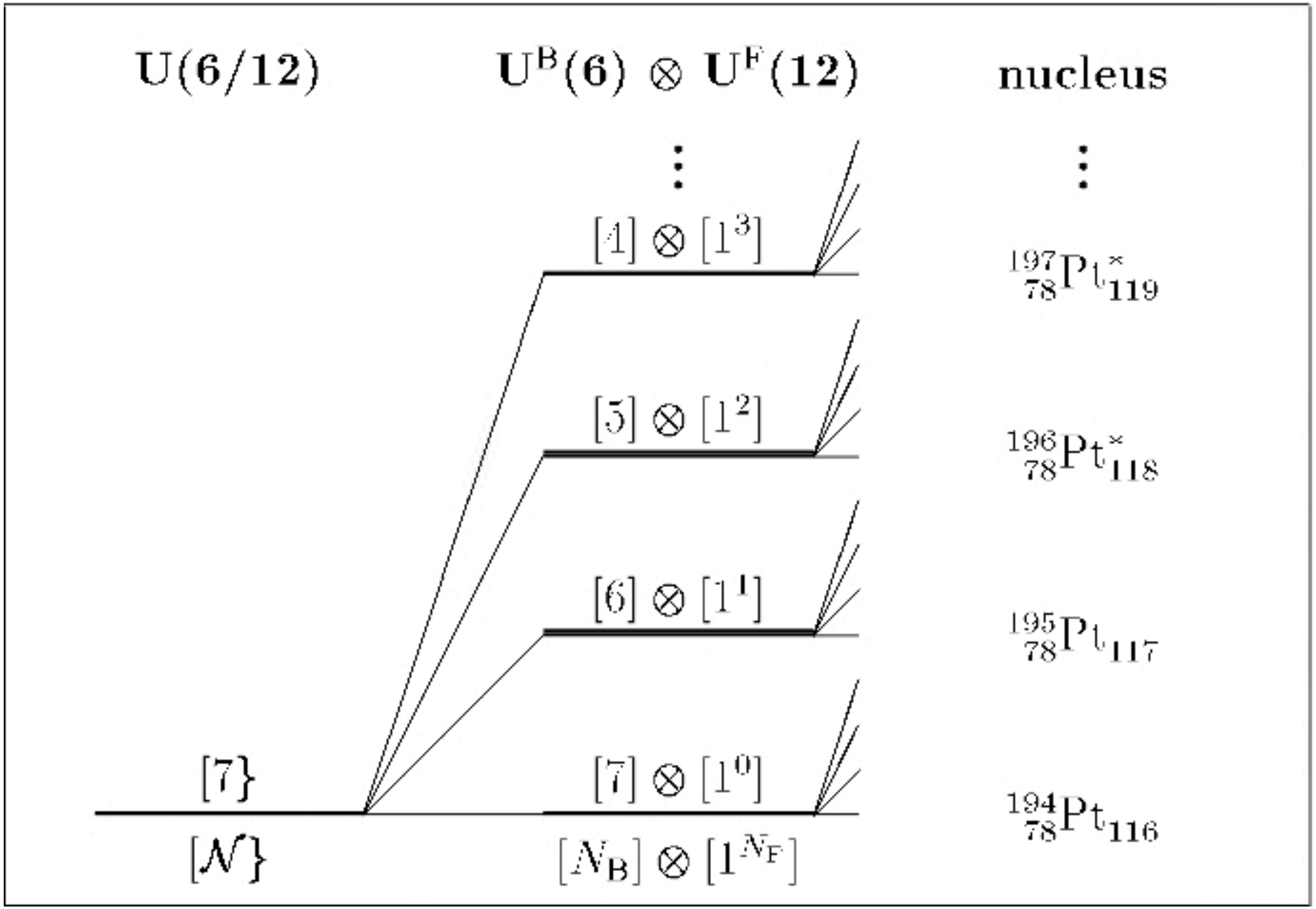
- Up to now: separate description of even-even and odd-mass nuclei with the algebra

$$\mathrm{U}(6) \oplus \mathrm{U}(\Omega) = \begin{Bmatrix} \hat{b}^{+}_{i_1}\hat{b}_{i_2} & \\ & \hat{a}^{+}_{j_1}\hat{a}_{j_2} \end{Bmatrix}$$

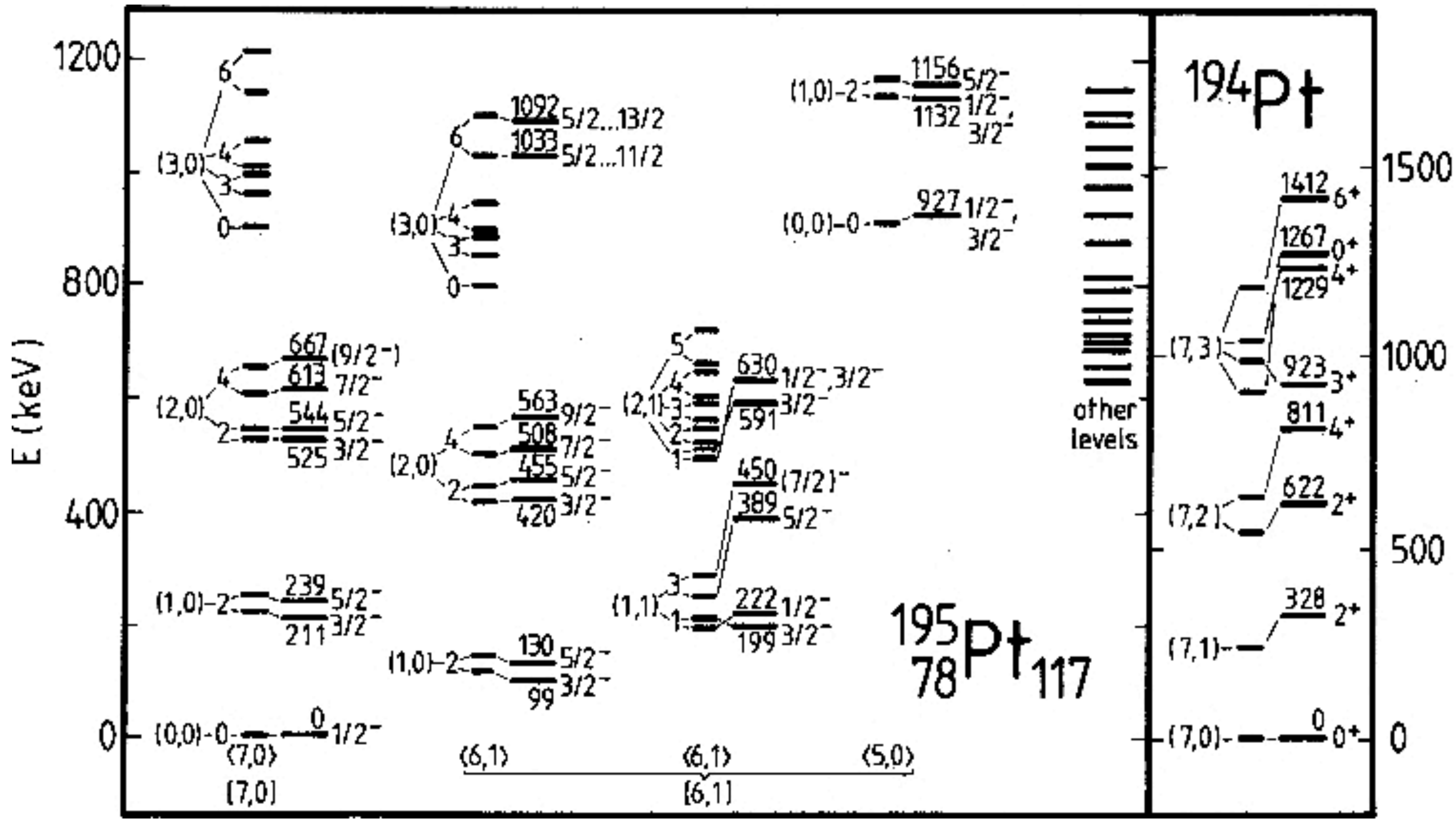
- Simultaneous description of even-even and odd-mass nuclei with the superalgebra

$$\mathrm{U}(6/\Omega) = \begin{Bmatrix} \hat{b}^{+}_{i_1}\hat{b}_{i_2} & \hat{b}^{+}_{i_1}\hat{a}_{j_2} \\ \hat{a}^{+}_{j_1}\hat{b}_{i_2} & \hat{a}^{+}_{j_1}\hat{a}_{j_2} \end{Bmatrix}$$

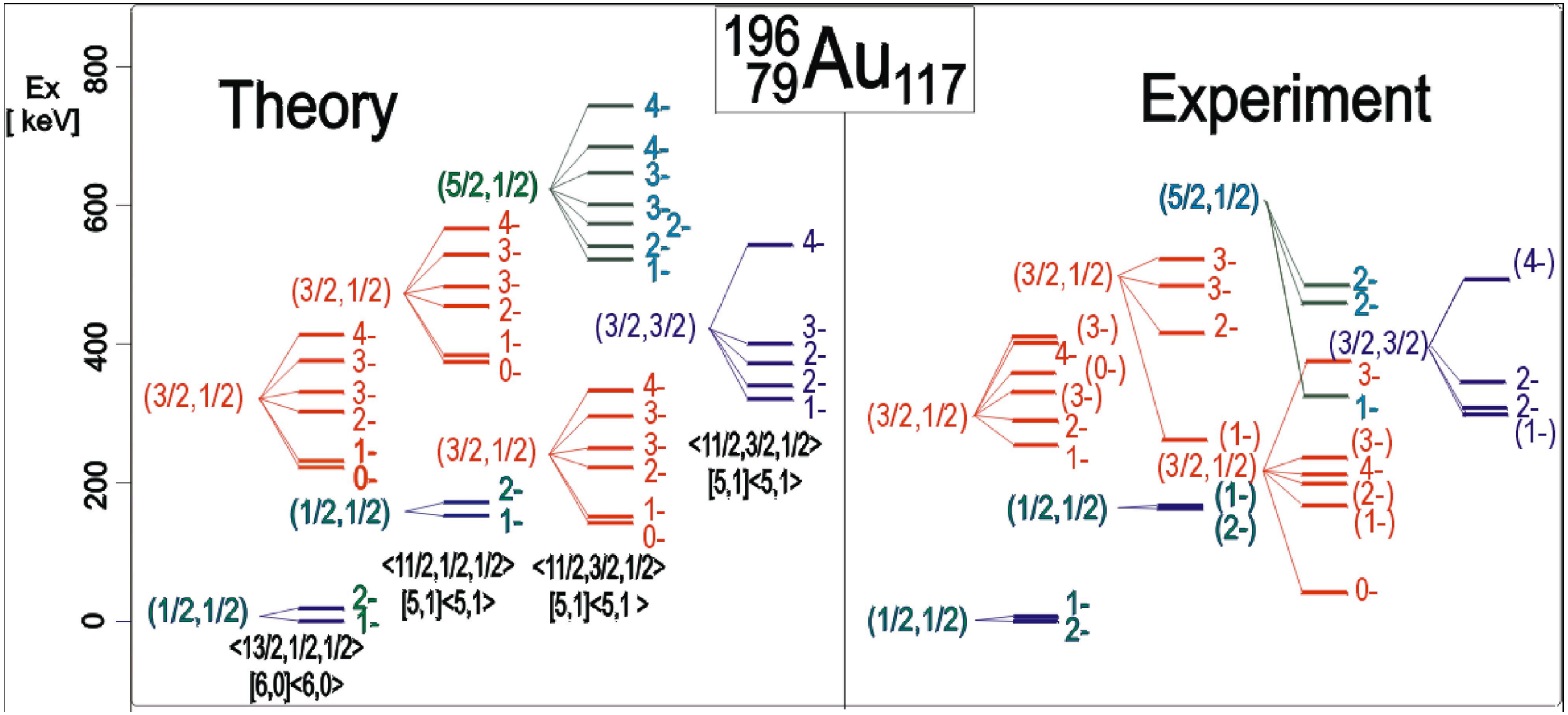
U(6/12) supermultiplet



Example: $^{194}Pt_{116}$ & $^{195}Pt_{117}$



Example: $^{196}Au_{117}$



Bibliography

- A. Bohr and B.R. Mottelson, *Nuclear Structure. I Single-Particle Motion* (Benjamin, 1969).
- A. Bohr and B.R. Mottelson, *Nuclear Structure. II Nuclear Deformations* (Benjamin, 1975).
- R.D. Lawson, *Theory of the Nuclear Shell Model* (Oxford UP, 1980).
- K.L.G. Heyde, *The Nuclear Shell Model* (Springer-Verlag, 1990).
- I. Talmi, *Simple Models of Complex Nuclei* (Harwood, 1993).

Bibliography (continued)

- P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, 1980).
- D.J. Rowe, *Nuclear Collective Motion* (Methuen, 1970).
- D.J. Rowe and J.L. Wood, *Fundamentals of Nuclear Collective Models*, to appear.
- F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge UP, 1987).