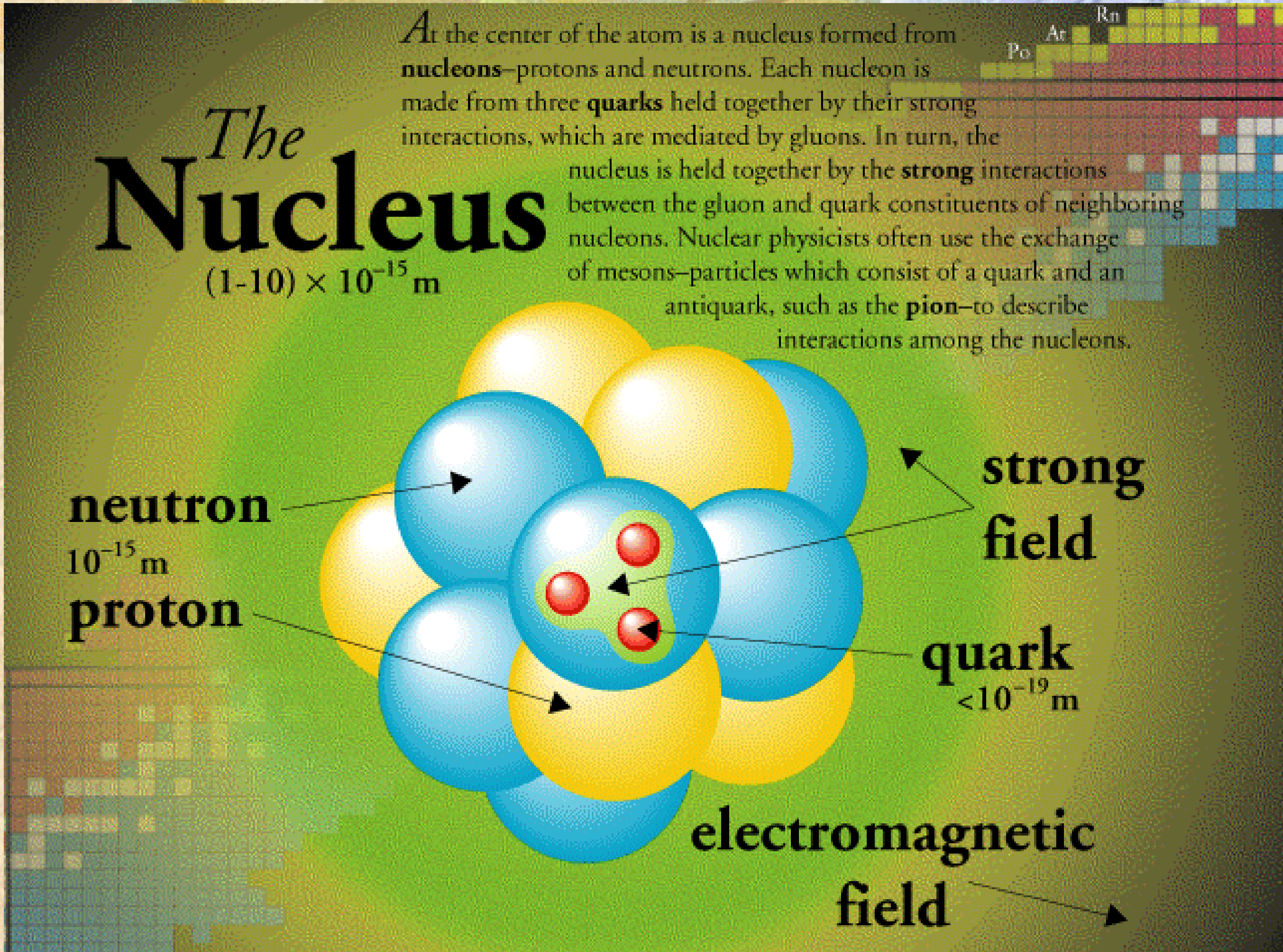
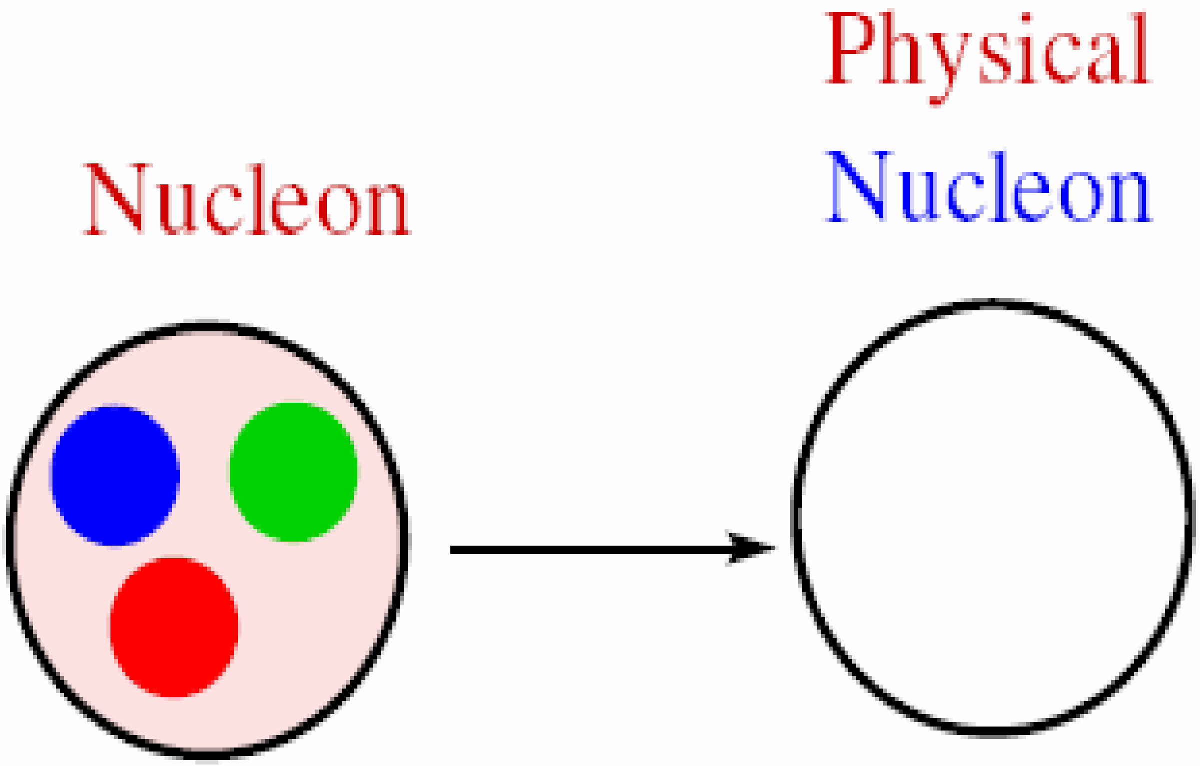
Nuclear Structure

THEORY

Y. K. Gambhir

Department of Physics,
Indian Institute of Technology, Bombay


The Nucleus
(1-10) × 10^-15 m
At the center of the atom is a nucleus formed from nucleons–protons and neutrons. Each nucleon is made from three quarks held together by their strong interactions, which are mediated by gluons. In turn, the nucleus is held together by the strong interactions between the gluon and quark constituents of neighboring nucleons. Nuclear physicists often use the exchange of mesons–particles which consist of a quark and an antiquark, such as the pion–to describe interactions among the nucleons.
Rn
At
Po
neutron
10^-15 m
proton
strong field
quark
<10^-19 m
electromagnetic field



Observed Properties: Mass, Charge, Size (0.8 fm), Spin, Mag. mom., ...

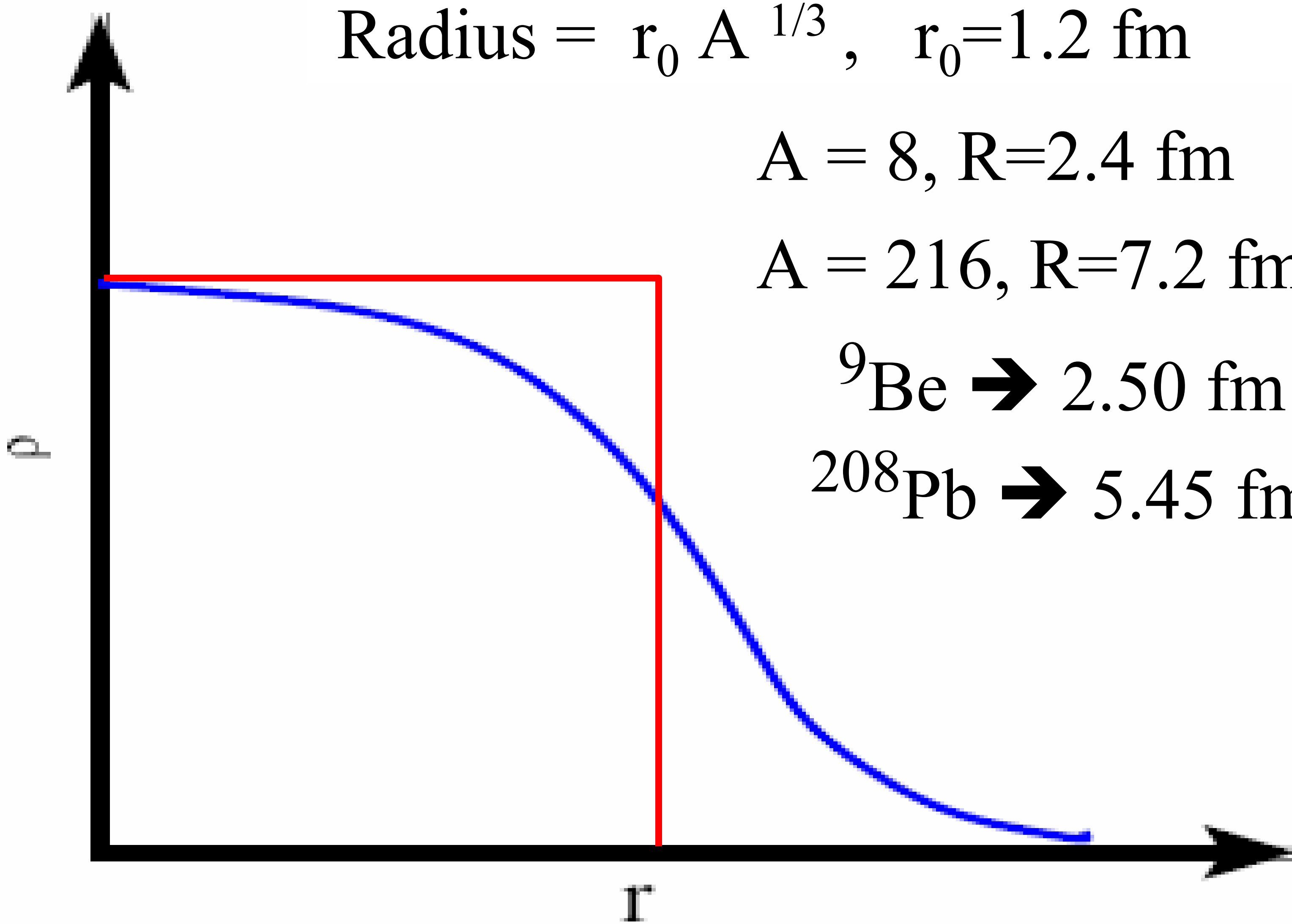
Radius = $r_0 A^{1/3}$, r_0=1.2 fm

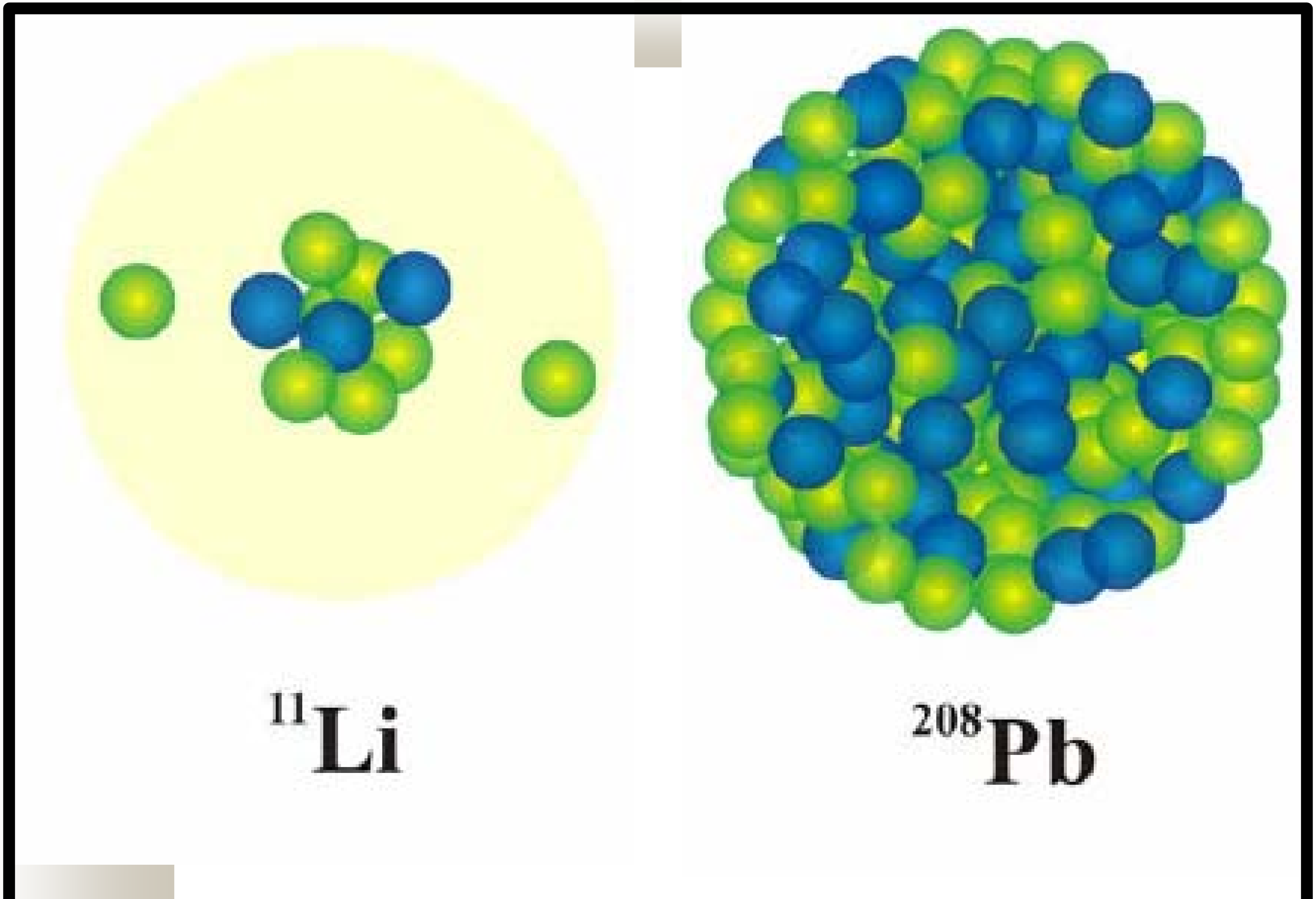
A = 8, R=2.4 fm

A = 216, R=7.2 fm

^{9}Be → 2.50 fm

^{208}Pb → 5.45 fm





11Li
208Pb

$$\frac{\text{Nucl. Den}}{\text{Atom. Den}} = \frac{\text{Atom. Vol}}{\text{Nucl. Vol}} = \frac{(0.8\ A^{1/3})^3}{(1.2\ A^{1/3})^3} \approx 10^{14}$$

VERY VERY DENSE MATTER

$$\frac{\text{A- nu. Vol.}}{\text{Nucl. Vol.}} = \frac{A.\,(4\pi/3)\,(0.8)^3}{(4\pi/3)\,(1.2\ A^{1/3})^3} = \frac{8}{27} \approx 30\%$$

Most of Nucl. Vol. is Empty

N-N int.:

V. Strong, Net Attractive

Short range, State Dep.

Non - Central

- **Nucleus: A (N+Z) – Body Problem**

$$\mathcal{H}\Psi_\lambda = \left[\sum_i \frac{-\hbar^2}{2m_i}\nabla_i^2 + \sum_{i>k} V_{ik}\right]\Psi_\lambda = E\Psi_\lambda$$

Can Not be Solved:

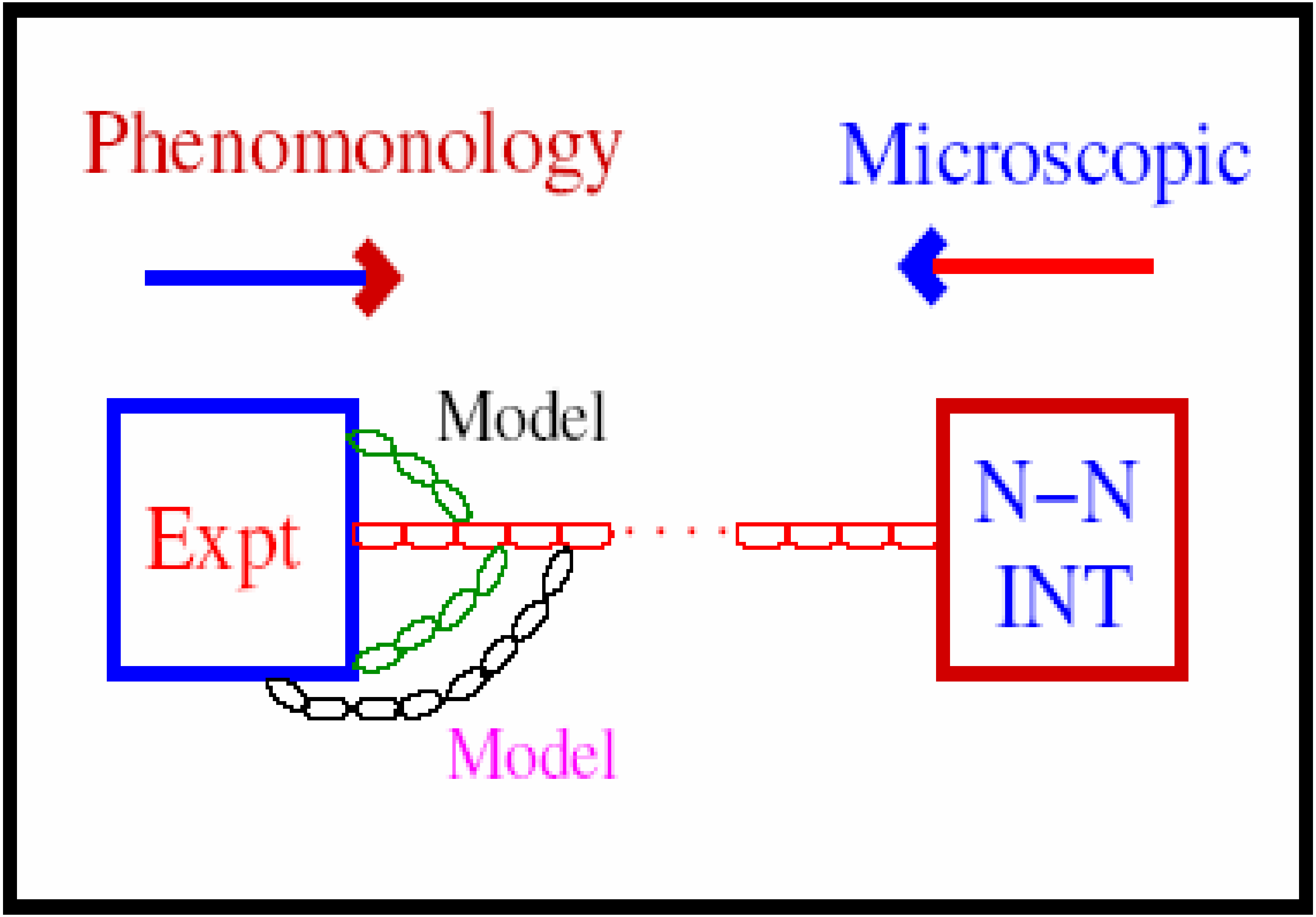
Difficulties:

- **Mathematical**
- **Two-Body Interaction (in the Nucleus)**

Approximate Methods:

Models Developed:

Many Models Exists



Phenomonology
Microscopic
Model
Expt
N–N
INT
Model

Mean Field Concept:

$$\mathcal{H} = \sum_i \frac{-\hbar^2}{2m_i}\nabla_i^2 + \sum_{i>k} V_{ik}$$

$$= \sum_i \underbrace{\left(\frac{-\hbar^2}{2m_i}\nabla_i^2 + \mathcal{O}_i\right)}_{\mathcal{H}_o^i} + \underbrace{\left(\sum_{i>k} V_{ik} - \sum_i \mathcal{O}_i\right)}_{h_I}$$

$$= \sum_i \mathcal{H}_o^i + h_I = \mathcal{H}_o + h_I$$

Advantage

is

Freedom to Choose $\mathcal{O}_i$

Choose $\mathcal{O}_i \to h_I$ Zero (Minimum)

Mean Field Helps to reduce

A-Body Problem $\longrightarrow$ One Body Problem

Phenomenological (Shell Model,…)
Microscopic (BBHF)

Phenomenological

- **H.O. + l.s**

$$O_i = \frac{1}{2} m\omega^2 r^2 + \alpha_{ls}\, \hat{l} \cdot \hat{s}$$

$$\psi_{nljm_j} = R_{nl}(r) \left[Y_l \bigotimes \chi_{1/2} \right]_{jm}$$

Plan

- **Mean Field Concept**
- **Shell Model**
- **Magic Nuclei : TDA - RPA**
- **Open Shell Nuclei**

a. Configuration Mixing

b. Truncations: Seniority, BPA ….

c. BCS – Quasiparticle Method

d. HF, HFB, PHF, PHFB

Plan

NO CORE

- **ab intio Shell Model**
- **DDHF – Skyrme Type Interaction**
- **RMF – Rel. Mean Field**

Schrodinger Equation

$$\mathcal{H}\Psi_\lambda = \left[\sum_i \frac{-\hbar^2}{2m_i}\nabla_i^2 + \sum_{i>k} V_{ik}\right]\Psi_\lambda = E\Psi_\lambda$$

Bound State Problem

Basis Expansion Method

Basis Expansion Method

$$\mathcal{H}\Psi_\alpha = E_\alpha\Psi_\alpha \; : \; \mathcal{H} = \mathcal{H}_o + \mathcal{V}$$

$$\mathcal{H}_o\Phi_I^\alpha = e_I\Phi_I^\alpha$$

$$\Psi_\alpha = \sum_I x_\alpha^I \Phi_I^\alpha$$

$$\sum_I \left[e_I\delta_{IK} + \langle\Phi_K^\alpha|\mathcal{V}|\Phi_i^\alpha\rangle - E_\alpha\delta_{IK}\right] x_\alpha^I = 0$$

$$\mathcal{H}_{IK} = e_I\delta_{IK} + \langle\Phi_K^\alpha|\mathcal{V}|\Phi_I^\alpha\rangle$$

$$\left|\Psi^{v}_{\alpha J^{\pi} M}\right\rangle = \sum_{I} \chi^{v}_{\alpha J^{\pi} M}(I) \left|\Phi^{I}_{\alpha J^{\pi} M}\right\rangle$$

$$\mathcal{H} \left|\Psi^{v}_{\alpha J^{\pi} M}\right\rangle = E_{\alpha J^{\pi} M} \left|\Psi^{v}_{\alpha J^{\pi} M}\right\rangle$$

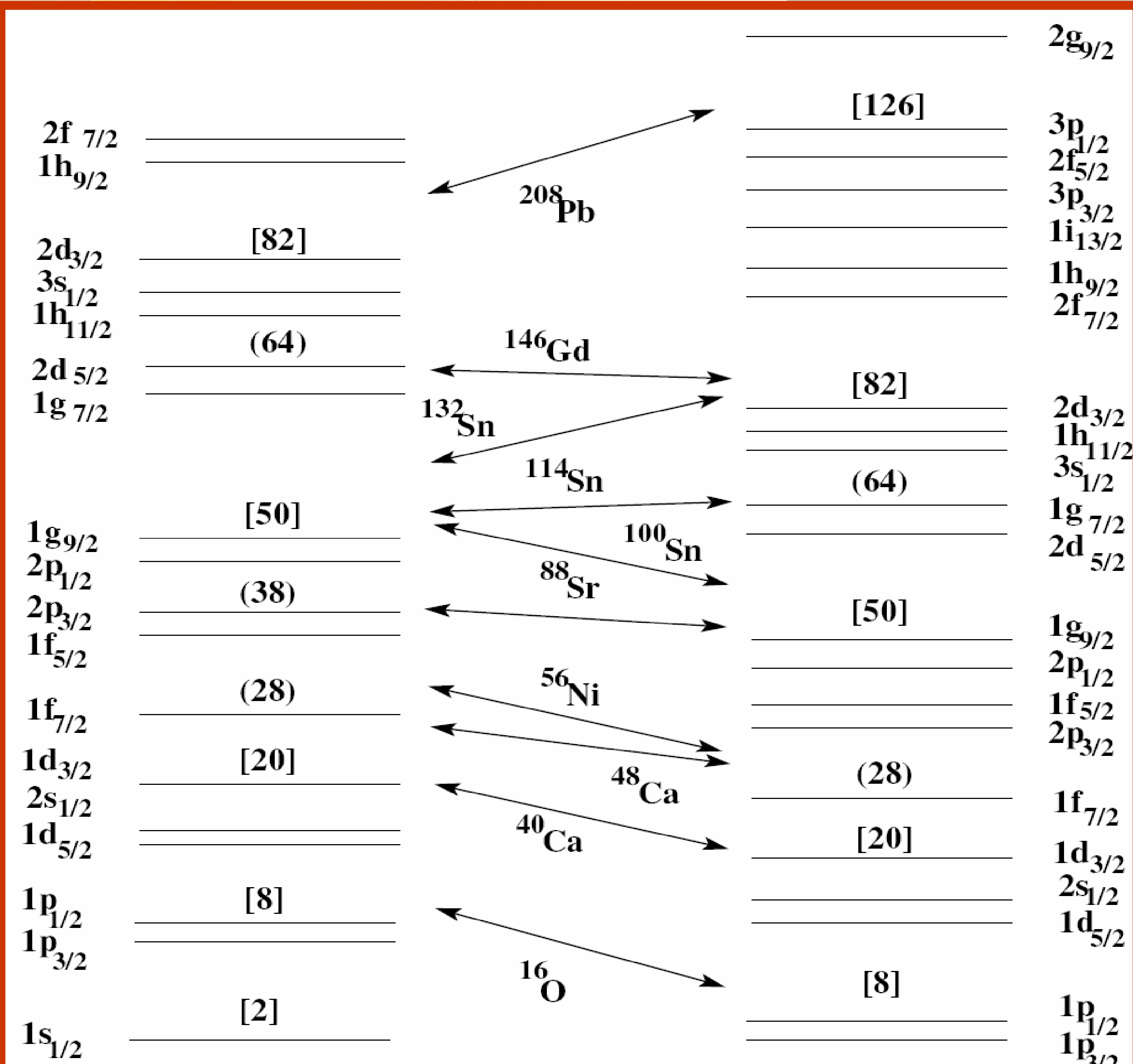
Step I: Choice of Basis (Mean Field)

Step II: Construction of Φ_I - A Nucleons

Unperturbed Energies ε_I

Step III: Setting of Ham. Matrix $\mathcal{H}$

Step IV: Diagonalization of $\mathcal{H}$



2f 7/2
1h9/2
2d3/2
3s1/2
1h11/2
2d 5/2
1g 7/2
1g9/2
2p1/2
2p3/2
1f5/2
1f7/2
1d3/2
2s1/2
1d5/2
1p1/2
1p3/2
1s1/2
[82]
(64)
[50]
(38)
(28)
[20]
[8]
[2]
208Pb
146Gd
132Sn
114Sn
100Sn
88Sr
56Ni
48Ca
40Ca
16O
[126]
[82]
(64)
[50]
(28)
[20]
[8]
2g9/2
3p1/2
2f5/2
3p3/2
1i13/2
1h9/2
2f7/2
2d3/2
1h11/2
3s1/2
1g 7/2
2d 5/2
1g9/2
2p1/2
1f5/2
2p3/2
1f7/2
1d3/2
2s1/2
1d5/2
1p1/2

Step I: Choose Core, Valence Level, s.p. Energy $\boldsymbol{\varepsilon_I}$

ε_I: Expt., Calculated or Parameters

$\hbar\omega = 41A^{-1/3}$

Step II: Orthonormal Basis Set Φ_I

Group Theoretical Method

Step III: Setting up Hamiltonian Matrix

Requires Two –Body Matrix Elements

Realistic, Phenomenological, Empirical

Step IV: Diagonalization of H-Matrix

Repeat for each J^{π}

Hamiltonian:

$$\sum_{\alpha} \varepsilon_{\alpha} C_{\alpha}^{\dagger} C_{\alpha} + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \gamma\delta \rangle C_{\alpha}^{\dagger} C_{\beta}^{\dagger} C_{\delta} C_{\gamma}$$

$$\begin{aligned} \langle \alpha\beta | V | \gamma\delta \rangle &= -\langle \beta\alpha | V | \gamma\delta \rangle \\ &= -\langle \alpha\beta | V | \delta\gamma \rangle \\ &= \langle \beta\alpha | V | \delta\gamma \rangle \end{aligned}$$

$C^\dagger(C)$: Particle Creation (destruction) Operator

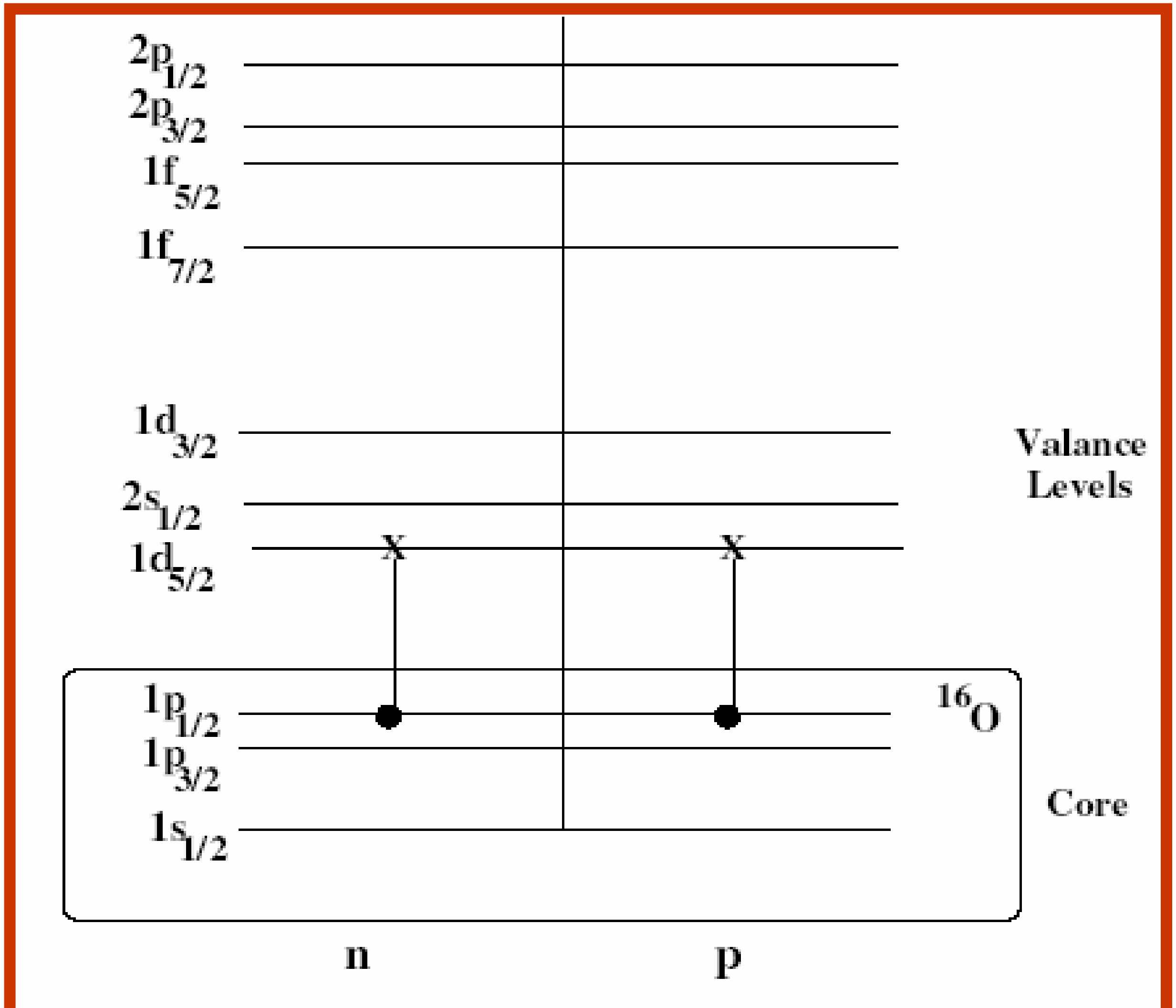
Vacuum Obeys $C_\alpha|0\rangle = 0$

$C^\dagger(C)$ obey anti commutation relations

$$\left\{C_\alpha, C_\beta^\dagger\right\} \equiv C_\alpha C_\beta^\dagger + C_\beta^\dagger C_\alpha = \delta_{\alpha\beta} ,$$

$$\{C_\alpha, C_\beta\} \equiv \left\{C_\alpha^\dagger, C_\beta^\dagger\right\} = 0 .$$

Application to Closed Shell Nuclei



Hole Levels : $h_1, h_2, h_3, \ldots.$

Particle Levels : $p_1, p_2, p_3, \ldots$

- **1p - 1h : Lowest Energy – Excitation**

$$(C_p^\dagger C_h)$$

- **Higher Order (Energy) Excitations**

  2p - 2h, 3p – 3h, …..

Equation of Motion Method

Operator $Q_\alpha^\dagger$ **Obeys:**

$$\iota \frac{\partial Q_\alpha^\dagger}{\partial t} = \left[H, Q_\alpha^\dagger\right] = E_\alpha Q_\alpha^\dagger,$$

$$H Q_\alpha^\dagger |\psi\rangle = (E_\alpha + E) Q_\alpha^\dagger |\psi\rangle$$

$Q_\alpha^\dagger$ (Q_α) acts as step up down Operator

Vaccum (g.s.) is **defined by** $Q_\alpha |\psi_o\rangle = 0 .$

if Set of Operators $a_i^\dagger$ (i= 1, 2, 3, ..., N) Obey

$$\left[H, a_i^\dagger\right] = \sum_{j=1}^{N} M_{ij} a_j^\dagger$$

Step up operator:

$$Q_\alpha^\dagger = \sum_j x_j^\alpha a_j^\dagger$$

$$\sum_j \tilde{M}_{ij} x_j^\alpha = E_\alpha x_i^\alpha$$

We require $\left[H, C_p^\dagger C_h\right]$ and its HC

It Contains Two Terms:

$$\sum_\alpha \epsilon_\alpha \left[C_\alpha^\dagger C_\alpha, C_p^\dagger C_h\right]$$

$$\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|\mathcal{V}|\gamma\delta\rangle \left[C_\alpha^\dagger C_\beta^\dagger C_\delta C_\gamma, C_p^\dagger C_h\right]$$

Use

$$[A, BC] = -[BC, A] = [A, B]C + B[A, C]$$
$$= \{A, B\}C - B\{A, C\}$$

and

$$C_p|0\rangle = C_h^\dagger|0\rangle = 0$$

Where

$$\{A, B\} = AB + BA$$

Notice

$$\left[C^\dagger_\alpha C_\alpha, C^\dagger_p C_h\right] = \delta_{\alpha p} C^\dagger_\alpha C_h - \delta_{\alpha h} C^\dagger_p C_\alpha$$

$$\left[C^\dagger_\alpha C^\dagger_\beta, C^\dagger_p C_h\right] = \left(1 - P(\alpha \leftrightarrow \beta)\right) C^\dagger_p C^\dagger_\alpha \delta_{\beta h}$$

Define $\bar{P} = 1 - P$

$$
\begin{aligned}
\left[H, C_p^\dagger C_h\right] &= (\epsilon_p - \epsilon_h) C_p^\dagger C_h \\
&- \frac{1}{2} \sum_{\alpha\gamma\delta} \langle \alpha h | V | \gamma\delta \rangle C_\alpha^\dagger C_p^\dagger C_\delta C_\gamma \\
&+ \frac{1}{2} \sum_{\alpha\beta\delta} \langle \alpha\beta | V | p\delta \rangle C_\alpha^\dagger C_\beta^\dagger C_\delta C_h
\end{aligned}
$$

$$C_\alpha^\dagger C_p^\dagger C_\delta C_\gamma \;=\; :C_\alpha^\dagger C_p^\dagger C_\delta C_\gamma:$$

$$+\langle C_\alpha^\dagger C_p^\dagger\rangle :C_\delta C_\gamma: +\langle C_\delta C_\gamma\rangle :C_\alpha^\dagger C_p^\dagger: +\langle C_\alpha^\dagger C_\gamma\rangle :C_p^\dagger C_\delta:$$

$$+\langle C_p^\dagger C_\delta\rangle :C_\alpha^\dagger C_\gamma: -\langle C_p^\dagger C_\gamma\rangle :C_\alpha^\dagger C_\delta: -\langle C_\alpha^\dagger C_\delta\rangle :C_p^\dagger C_\gamma:$$

$$+\langle C_\alpha^\dagger C_p^\dagger\rangle\langle C_\delta C_\gamma\rangle + \langle C_p^\dagger C_\delta\rangle\langle C_\alpha^\dagger C_\gamma\rangle - \langle C_\alpha^\dagger C_\delta\rangle\langle C_p^\dagger C_\gamma\rangle$$

$$
\begin{aligned}
\left[H, C_p^\dagger C_h\right] &= \sum_{p'} \left(\epsilon_p \delta p p' + \sum_{h_1} \langle p' h_1 | \mathcal{V} | p h_1 \rangle \right) C_{p'}^\dagger C_h \\
&- \sum_{h'} \left(\epsilon_h \delta_{hh'} + \sum_{h_1} \langle h h_1 | \mathcal{V} | h' h_1 \rangle \right) C_p^\dagger C_{h'} \\
&+ \sum_{p'h'} \left(\langle h p' | \mathcal{V} | p h' \rangle C_{p'}^\dagger C_{h'} + \langle h h' | \mathcal{V} | p p' \rangle C_{h'}^\dagger C_{p'} \right)
\end{aligned}
$$

$$\left[H, C_p^\dagger C_h\right] = \sum_{p'h'} \left((\tilde{\epsilon}_p - \tilde{\epsilon}_h)\delta_{pp'}\delta_{hh'} \right.$$

$$\left. + \langle hp'|\mathcal{V}|ph'\rangle \right) C_{p'}^\dagger C_{h'} + \sum_{p'h'} \langle hh'|\mathcal{V}|pp'\rangle C_{h'}^\dagger C_{p'}$$

$$= \sum_{p'h'} \left(A(p'h,' ph) C_{p'}^\dagger C_{h'} + B(p'h', ph) C_{h'}^\dagger C_{p'} \right)$$

The Matrices

$$A(p'h', ph) = (\tilde{\epsilon}_p - \tilde{\epsilon}_h)\delta_{pp'}\delta_{hh'} + \langle hp'|\mathcal{V}|ph'\rangle$$

$$B(p'h', ph) = \langle hh'|\mathcal{V}|pp'\rangle$$

In Coupled Representation: J^{Π} T

$$\begin{pmatrix} [H,\Omega^\dagger] \\ [H,\hat{\Omega}] \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \begin{pmatrix} \Omega^\dagger \\ \hat{\Omega} \end{pmatrix}$$

where

$$\hat{\Omega}_{J^\pi M T M_T}(p_i,h_i) = (-1)^{J-M+T-M_T}\Omega_{J^\pi -M T -M_T}(p_i,h_i)$$

$$\begin{aligned} A_{ij}^{J^\pi T} &= (\tilde{\epsilon}_{p_i} - \tilde{\epsilon}_{h_i})\delta_{p_i p_j}\delta_{p_i h_j} + F(p_i h_i p_j h_j J^\pi T) \\ B_{ij}^{J^\pi T} &= (-1)^{j_{p_i}+j_{h_j}+J+T} F(p_i h_i h_j p_j J^\pi T)\ , \end{aligned}$$

Hole Particle Matrix Element

$$F(acdbJ^{\pi}T) =$$

$$\sum_{J'T'}(2J'+1)(2T'+1)W(j_a j_b j_c j_d; J'J)W\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}, T'T\right)$$

$$\times (-1)^{j_a+j_b+j_c+j_d}\langle abJ'T'|\mathcal{V}|dcJ'T'\rangle$$

W is Racah Coefficient

Step Up Operator

$$Q^\dagger = X\Omega^\dagger - Y\hat{\Omega}$$

X , Y are Eigen Vectors of the Matrix

$$\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

With Norm

$$X^2 - Y^2 = 1$$

Both X and Y can be large

Illustration ^{16}O

- **Step I:**

Hole levels (h):	$1p_{3/2}$, $1p_{1/2}$
($\tilde{\epsilon}$) (MeV):	21.8, 15.65 (for neutrons) 18.44, 12.11 (for protons)

Particle levels (p):	$1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$
($\tilde{\epsilon}$) (MeV):	-4.5, -3.27, 0.93 (for neutrons) -0.59, -0.08, 4.65 (for protons)

- **Step II: Construction of h-p Basis**

For $J^\pi = O^-$ $T = 1$ and $T = 0$

$$(1d_{3/2}1p_{3/2}^{-1})_{0^-},\ (2s_{1/2}1p_{1/2}^{-1})_{0^-}$$

For $J^\pi = 1^-$ $T = 1$ and $T = 0$

$$(1d_{5/2}1p_{3/2}^{-1})_{1^-},\ (1d_{3/2}1p_{3/2}^{-1})_{1^-},\ (2s_{1/2}1p_{3/2}^{-1})_{1^-}$$

$$(2s_{1/2}1p_{1/2}^{-1})_{1^-},\ (1d_{3/2}1p_{1/2}^{-1})_{1^-}$$

For 0^- State

1. $\Rightarrow 1d_{3/2}\ 1p_{3/2}^{-1}$
2. $\Rightarrow 2s_{1/2}\ 1p_{1/2}^{-1}$,
3. $\Rightarrow$ hc of 1;
4. $\Rightarrow$ hc of 2;

T		E(MeV)	1	2	3	4
	TDA	11.2	0.001	1.000		
	RPA	11.2	0.001	1.000	-0.002	-0.002
	TDA	23.1	1.000	-0.001		
	RPA	23.0	1.000	-0.001	-0.034	-0.002
	TDA	13.7	-0.045	0.999		
	RPA	13.7	-0.048	0.999	-0.012	-0.012
	TDA	25.7	0.999	0.055		
	RPA	25.6	1.000	0.053	-0.040	-0.015

Step III: Two – Body Interaction

M=W=0.15, H=0.4 and B=0.3
Gaussian Shape, Strength = - 40 MeV

Step IV: Diagonalization

Results for $J^{\pi} = 0^{-}$

Both for $T = 1$ **and** $T = 0$

Similar Results for Other States

Open Shell Nuclei

Illustration ^{58}Ni

Step I: Mean Field

Core: $^{56}_{28}\mathrm{Ni}\ (Z=N=28)$

Valence Levels: $2p_{3/2},\ 1f_{5/2},\ 2p_{1/2}$

s. p. Energies : **0.0, 0.78 and 1.08 MeV**

Step II: Orthonormal Basis Set

2 Valence Neutrons

$$\left(1p_{3/2}\right)^2_{J^\pi=0^+\,2^+}\ ;\quad \left(2p_{3/2}\,1f_{5/2}\right)_{J^\pi=1^+,2^+,3^+,4^+}\ ;$$

$$\left(2p_{3/2}2p_{1/2}\right)_{J^\pi=1^+,2^+}\ ;\quad \left(1f_{5/2}\right)^2_{J^\pi=0^+,2^+,4^+}\ ;$$

$$\left(1f_{5/2}\,2p_{1/2}\right)_{J^\pi=2^+,3^+}\ ;\quad \left(2p_{1/2}\right)^2_{J^\pi=0^+}$$

No of Basis States are:

$$0^+(3),\ 1^+(2),\ 2^+(5),\ 3^+(2),\ 4^+(2),$$

Step III: Kuo – Brown Inte. M.E.

Step IV: Diagonalization.

Results for ^{58}Ni, ^{60}Ni, ^{62}Ni and ^{64}Ni.

	J^π	0_1^+	0_2^+	2_1^+	2_2^+	4_1^+	4_2^+
^{58}Ni	EMS	0.0	2.56	1.41	2.86	2.30	
	EXPT.	0.0		1.45	2.78	2.46	
^{60}Ni	EMS	0.0	2.30	1.50	2.20	2.18	
	EXPT.	0.0	2.29	1.33	2.16	2.50	
^{62}Ni	EMS	0.0	2.11	1.56	2.29	2.15	
	EXPT.	0.0	2.05	1.17	2.30	2.34	
	J^π	$1/2_1^-$	$1/2_2^-$	$3/2_1^-$	$3/2_2^-$	$5/2_1^-$	$5/2_2^-$
^{59}Ni	EMS	0.24	1.10	0.0	0.82	0.21	1.47
	EXPT.	0.47	1.32	0.0	0.89	0.34	

Is Nuclear problem solved? NO

Reason : Huge number of Basis Φ:

For ^{112}Sn: 12 neutrons in five s.p. states

$$(2d_{5/2}, 1g_{7/2}, 3s_{1/2}, 2d_{3/2}, 1h_{11/2})$$

The Number of States Φ are for

$J^\pi = 0^+$ is 55,907,

$J^\pi = 2^+$ is 267,720

$J^\pi = 4^+$ is 426,558.

Solution: Truncation Schemes:

Seniority Truncation Scheme

Seniority (ν): No. of Nucleons Left After all Pairs Coupled to J = 0 are Removed.

Even – Even: ν = 0, 2, 4 are OK

Odd - Even : ν = 1, 3, 5 are OK

Seniority Decomposition (in %) ^{61}Ni

State J^π	Energy				
	Theo.	Expt.	ν=1	ν=3	ν=5
$1/2_1^-$	0.02	0.28	96.9	2.9	0.2
$1/2_2^-$	1.02	—	24.1	74.5	1.4
$3/2_1^-$	0.0	0.0	92.4	7.0	0.6
$3/2_2^-$	1.03	0.66	31.2	65.7	3.1
$5/2_1^-$	0.12	0.07	97.1	2.7	0.2
$5/2_2^-$	0.93	0.91	24.3	71.3	0.4
$7/2_1^-$	0.92	1.02	—	94.9	5.1
$9/2_1^-$	1.00	—	—	99.3	0.7

Seniority Decomposition (in %) ^{62}Ni

State J^π	Energy					
	Theo.	Expt.	ν=0	ν=2	ν=4	ν=6
0_1^+	0.0	0.0	99.7	—	0.3	—
0_2^+	2.11	2.05	87.3	—	12.7	—
1_1^+	3.57	—	24.7	70.0	5.3	
2_1^+	1.56	1.17		99.4	0.5	0.1
2_2^+	2.29	2.30		89.1	10.7	0.2
3_1^+	2.84	—		40.6	59.3	0.1
4_1^+	2.15	2.34		92.9	7.0	0.1
4_2^+	2.76	—		41.6	58.3	0.1

Still Problem is Not Solved:
For ^{112}Sn the $\nu=0$ States are 110
While $\nu=2$ states Approach Thousand

Solution:
Quasiparticle (BCS) Theory
Broken Pair Approximation (BPA)

Quasiparticle OR BCS Method

This Takes Into Account the Strong Pairing Part of the Effective Two-Body Interaction.

The Idea is to Go From Particle Picture to Quasiparticle Picture (New Mean Field) Through Bogoliubov or Quasiparticle (qp) Transformation. This Leads in the Lowest approximation, to Independent Quasiparticle Picture - Incorporates the Pairing Interaction.