Nuclear Structure
(I) Single-particle models

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Overview of nuclear models

- *Ab initio* methods: Description of nuclei starting from the bare nn & nnn interactions.
- Nuclear shell model: Nuclear average potential + (residual) interaction between nucleons.
- Mean-field methods: Nuclear average potential with global parametrisation (+ correlations).
- Phenomenological models: Specific nuclei or properties with local parametrisation.

Nuclear shell model

- Many-body quantum mechanical problem:

$$\hat{H} = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

$$= \underbrace{\sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]}_{\text{mean field}} + \underbrace{\left[\sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l) - \sum_{k=1}^{A} V(\boldsymbol{r}_k) \right]}_{\text{residual interaction}}$$

- Independent-particle assumption. Choose V and neglect residual interaction:

$$\hat{H} \approx \hat{H}_{\text{IP}} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]$$

Independent-particle shell model

- Solution for one particle:

$$\left[\frac{p^2}{2m} + \hat{V}(\boldsymbol{r})\right]\phi_i(\boldsymbol{r}) = E_i\phi_i(\boldsymbol{r})$$

- Solution for many particles:

$$\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \prod_{k=1}^{A} \phi_{i_k}(\boldsymbol{r}_k)$$

$$\hat{H}_{\mathrm{IP}}\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \left(\sum_{k=1}^{A} E_{i_k}\right)\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)$$

Independent-particle shell model

- Anti-symmetric solution for many particles (Slater determinant):

$$\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{i_1}(\boldsymbol{r}_1) & \phi_{i_1}(\boldsymbol{r}_2) & \cdots & \phi_{i_1}(\boldsymbol{r}_A) \\ \phi_{i_2}(\boldsymbol{r}_1) & \phi_{i_2}(\boldsymbol{r}_2) & \cdots & \phi_{i_2}(\boldsymbol{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i_A}(\boldsymbol{r}_1) & \phi_{i_A}(\boldsymbol{r}_2) & \cdots & \phi_{i_A}(\boldsymbol{r}_A) \end{vmatrix}$$

- Example for $A=2$ particles:

$$\Psi_{i_1 i_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{\sqrt{2}} \left[\phi_{i_1}(\boldsymbol{r}_1) \phi_{i_2}(\boldsymbol{r}_2) - \phi_{i_1}(\boldsymbol{r}_2) \phi_{i_2}(\boldsymbol{r}_1) \right]$$

Hartree-Fock approximation

- Vary ϕ_i (*ie* V) to minize the expectation value of H in a Slater determinant:

$$\delta \frac{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)\hat{H}\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A}{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A} = 0$$

- Application requires choice of H. Many global parametrizations (Skyrme, Gogny,…) have been developed.

Poor man's Hartree-Fock

- Choose a simple, analytically solvable V that approximates the microscopic HF potential:

$$\hat{H}_{\mathrm{IP}} = \sum_{k=1}^{A}\left[\frac{p_k^2}{2m} + \frac{m\omega^2}{2}r_k^2 - \zeta\, \boldsymbol{l}_k \cdot \boldsymbol{s}_k - \kappa\, l_k^2\right]$$

- Contains
  - Harmonic oscillator potential with constant ω.
  - Spin-orbit term with strength ζ.
  - Orbit-orbit term with strength κ.
- Adjust ω, ζ and κ to best reproduce HF.

Harmonic oscillator solution

- Energy eigenvalues of the harmonic oscillator:

$$E_{nlj} = \left(N + \tfrac{3}{2}\right)\hbar\omega - \kappa\hbar^2 l(l+1) + \zeta\hbar^2 \begin{cases} -\frac{1}{2}l & j = l + \frac{1}{2} \\ \frac{1}{2}(l+1) & j = l - \frac{1}{2} \end{cases}$$

$N = 2n + l = 0,1,2,\ldots$: oscillator quantum number

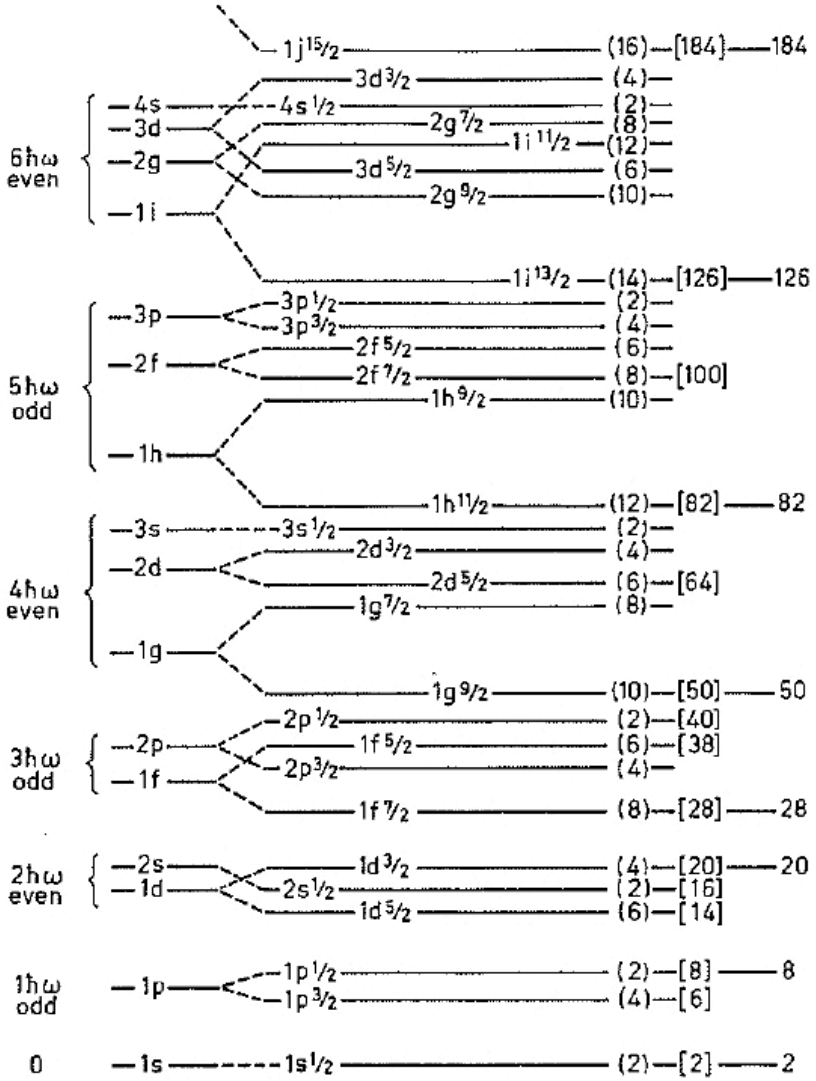
$n = 0,1,2,\ldots$: radial quantum number

$l = N, N-2, \ldots, 1 \text{ or } 0$: orbital angular momentum

$j = l \pm \frac{1}{2}$: total angular momentum

$m_j = -j, -j+1, \ldots, +j$: z projection of j

Energy levels of harmonic oscillator



- Typical parameter values:

$$\hbar\omega \approx 41\, A^{-1/3} \text{ MeV}$$

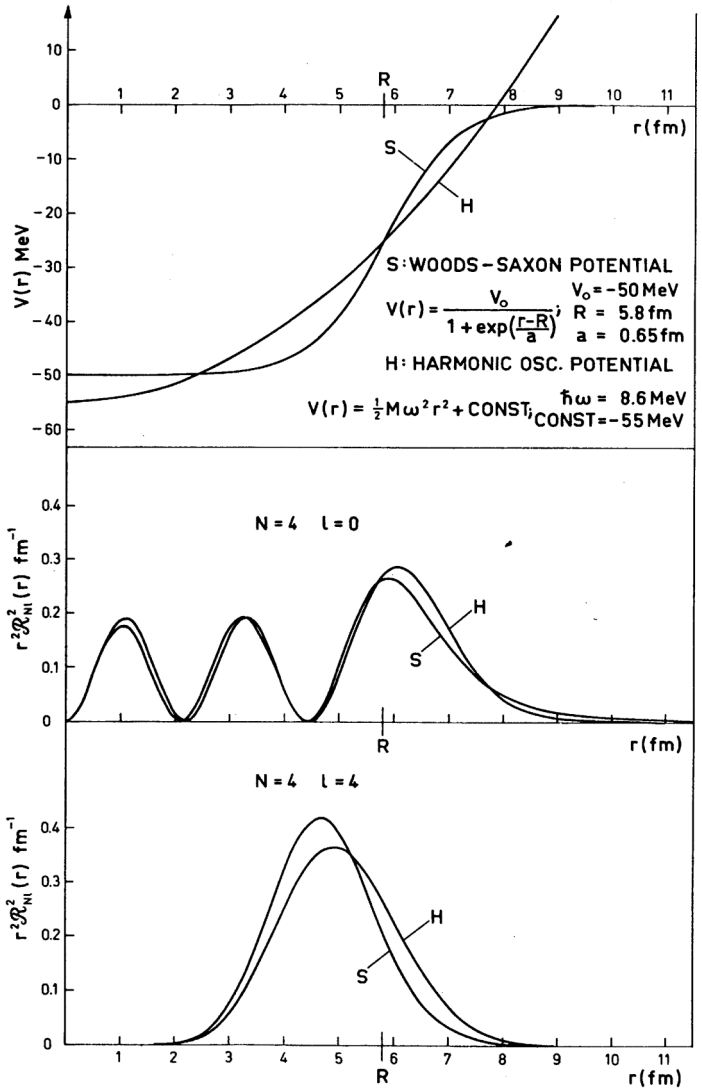
$$\zeta\hbar^2 \approx 20\, A^{-2/3} \text{ MeV}$$

$$\kappa\hbar^2 \approx 0.1 \text{ MeV}$$

$$\therefore b \approx 1.0\, A^{1/6} \text{ fm}$$

- 'Magic' numbers at 2, 8, 20, 28, 50, 82, 126, 184,…

Why an orbit-orbit term?



- Nuclear mean field is close to Woods-Saxon:

$$\hat{V}_{\mathrm{WS}}(r) = \frac{V_0}{1+\exp\dfrac{r-R_0}{a}}$$

- $2n+l=N$ degeneracy is lifted $\Rightarrow E_l < E_{l-2} < \ldots$

Why a spin-orbit term?

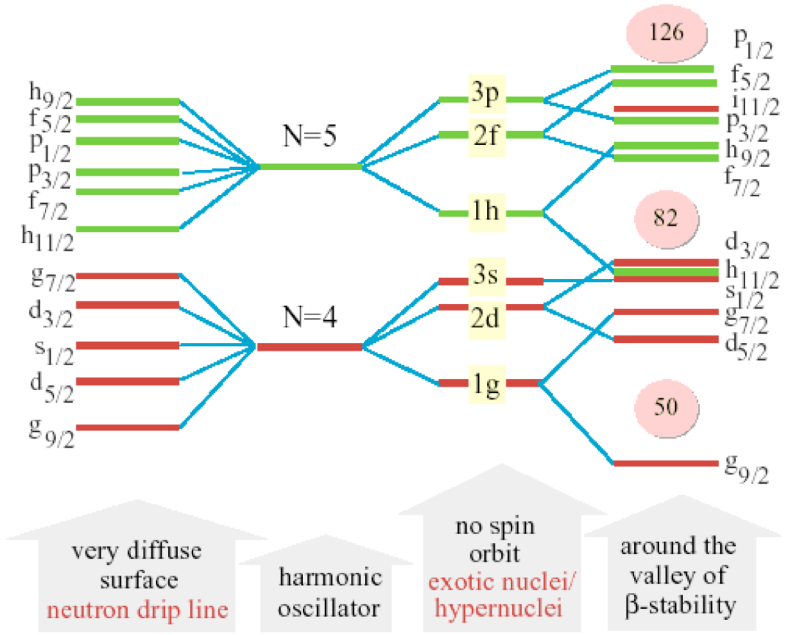
- Relativistic origin (*ie* Dirac equation).
- From general invariance principles:

$$\hat{V}_{\mathrm{SO}} = \zeta(r)\boldsymbol{l}\cdot\boldsymbol{s}, \quad \zeta(r) = \frac{r_0^2}{r}\frac{\partial V}{\partial r}\left[= \zeta \text{ in HO}\right]$$

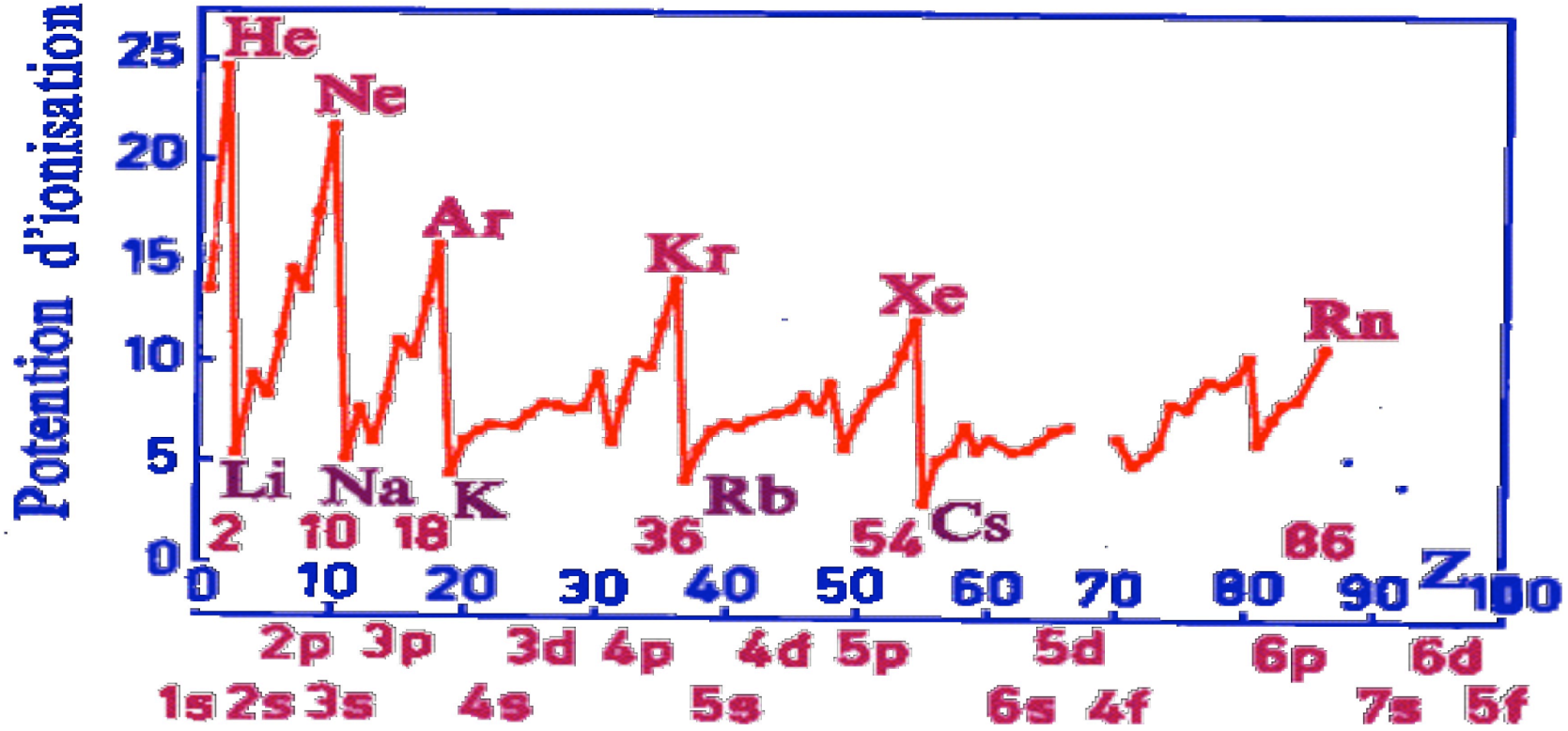
- Spin-orbit term is surface peaked $\Rightarrow$ diminishes for diffuse potentials.

Evidence for shell structure

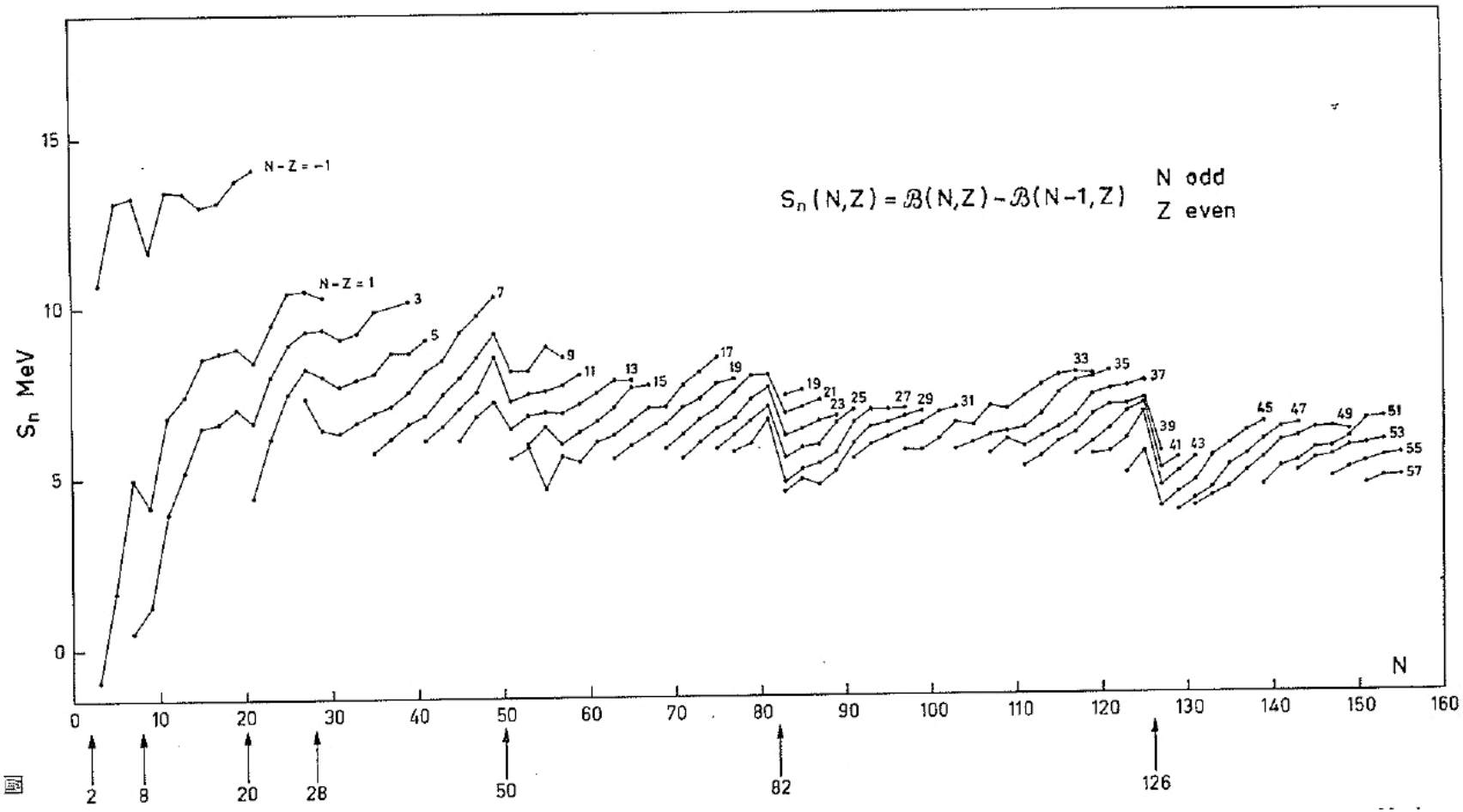
- Evidence for nuclear shell structure from
  - 2^+ in even-even nuclei [E_x, B(E2)].
  - Nucleon-separation energies & nuclear masses.
  - Nuclear level densities.
  - Reaction cross sections.
- *Is nuclear shell structure modified away from the line of stability?*



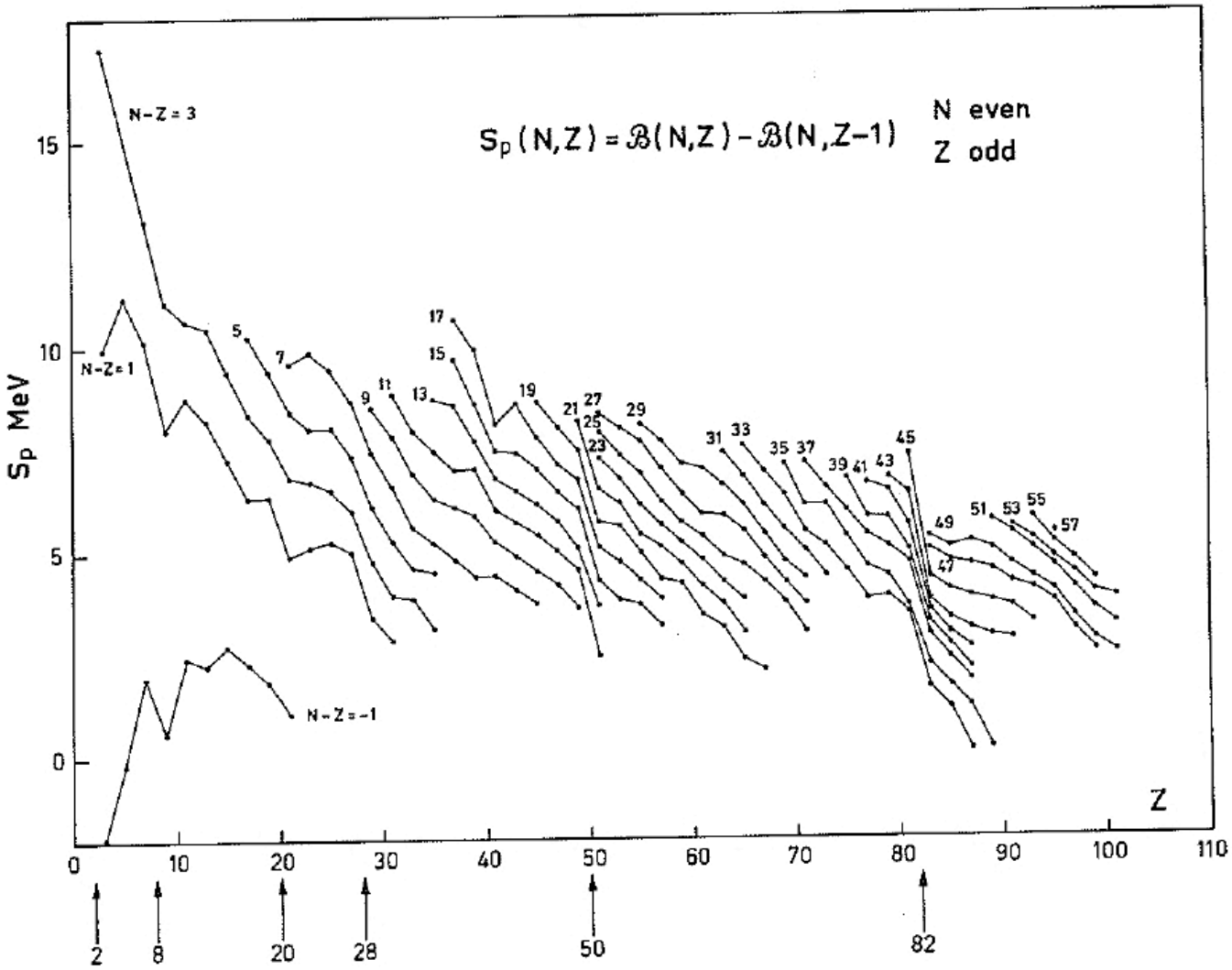
Ionisation potential in atoms



Neutron separation energies



Proton separation energies



Liquid-drop mass formula

- Binding energy of an atomic nucleus:

$$B(N,Z) = a_{\mathrm{vol}}A - a_{\mathrm{sur}}A^{2/3} - a_{\mathrm{cou}}\frac{Z(Z-1)}{A^{1/3}} - a_{\mathrm{sym}}\frac{(N-Z)^2}{A} + a_{\mathrm{pai}}\frac{\delta(N,Z)}{A^{1/2}}$$

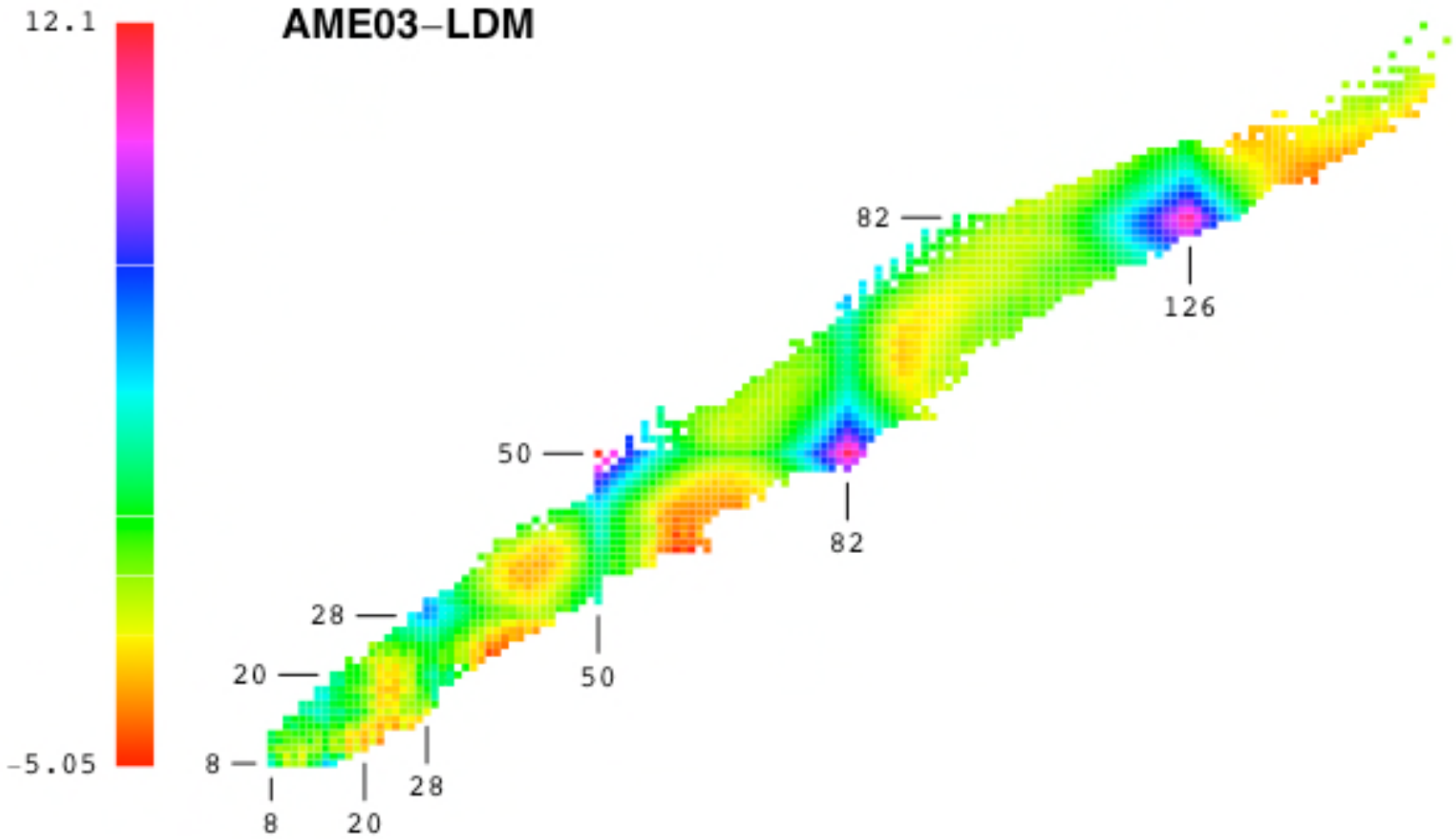
- For 2149 nuclei ($N,Z \geq 8$) in AME03:

  $a_{\mathrm{vol}} \approx 16$, $a_{\mathrm{sur}} \approx 18$, $a_{\mathrm{cou}} \approx 0.71$, $a_{\mathrm{sym}} \approx 23$, $a_{\mathrm{pai}} \approx 13$

  $\Rightarrow \sigma_{\mathrm{rms}} \approx 2.93$ MeV.

C.F. von Weizsäcker, Z. Phys. **96** (1935) 431
H.A. Bethe & R.F. Bacher, Rev. Mod. Phys. **8** (1936) 82

Deviations from LDM



Modified liquid-drop formula

- Add surface, Wigner and 'shell' corrections:

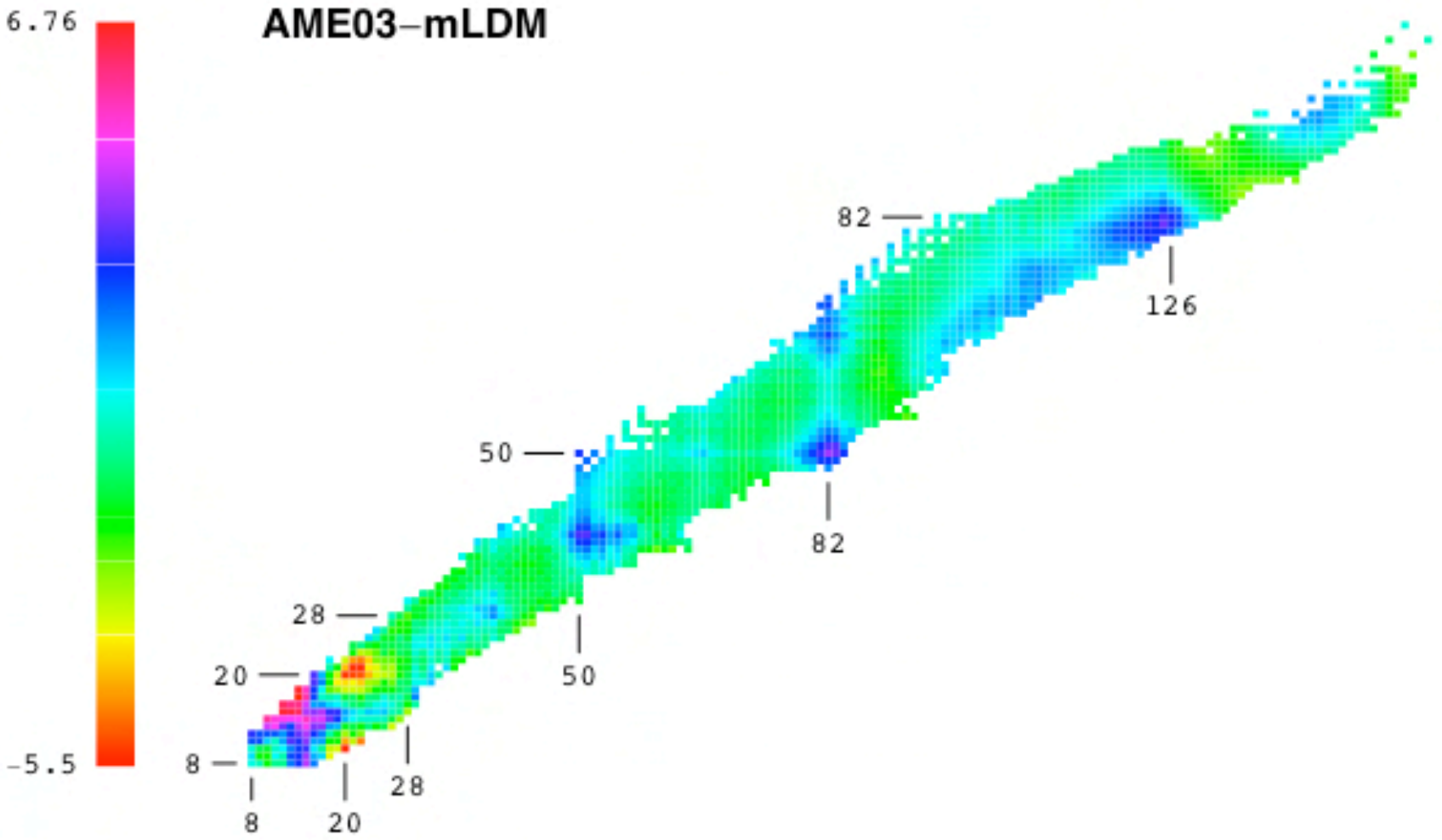
$$B(N,Z) = a_{\mathrm{vol}}A - a_{\mathrm{sur}}A^{2/3} - a_{\mathrm{cou}}\frac{Z(Z-1)}{A^{1/3}} - a_{\mathrm{vsym}}\frac{4T(T+r)}{A}$$
$$+ a_{\mathrm{ssym}}\frac{4T(T+r)}{A^{4/3}} + a_{\mathrm{pai}}\frac{\delta(N,Z)}{A^{1/2}} - a_{\mathrm{f}}F_{\mathrm{max}} + a_{\mathrm{ff}}F_{\mathrm{max}}^2$$

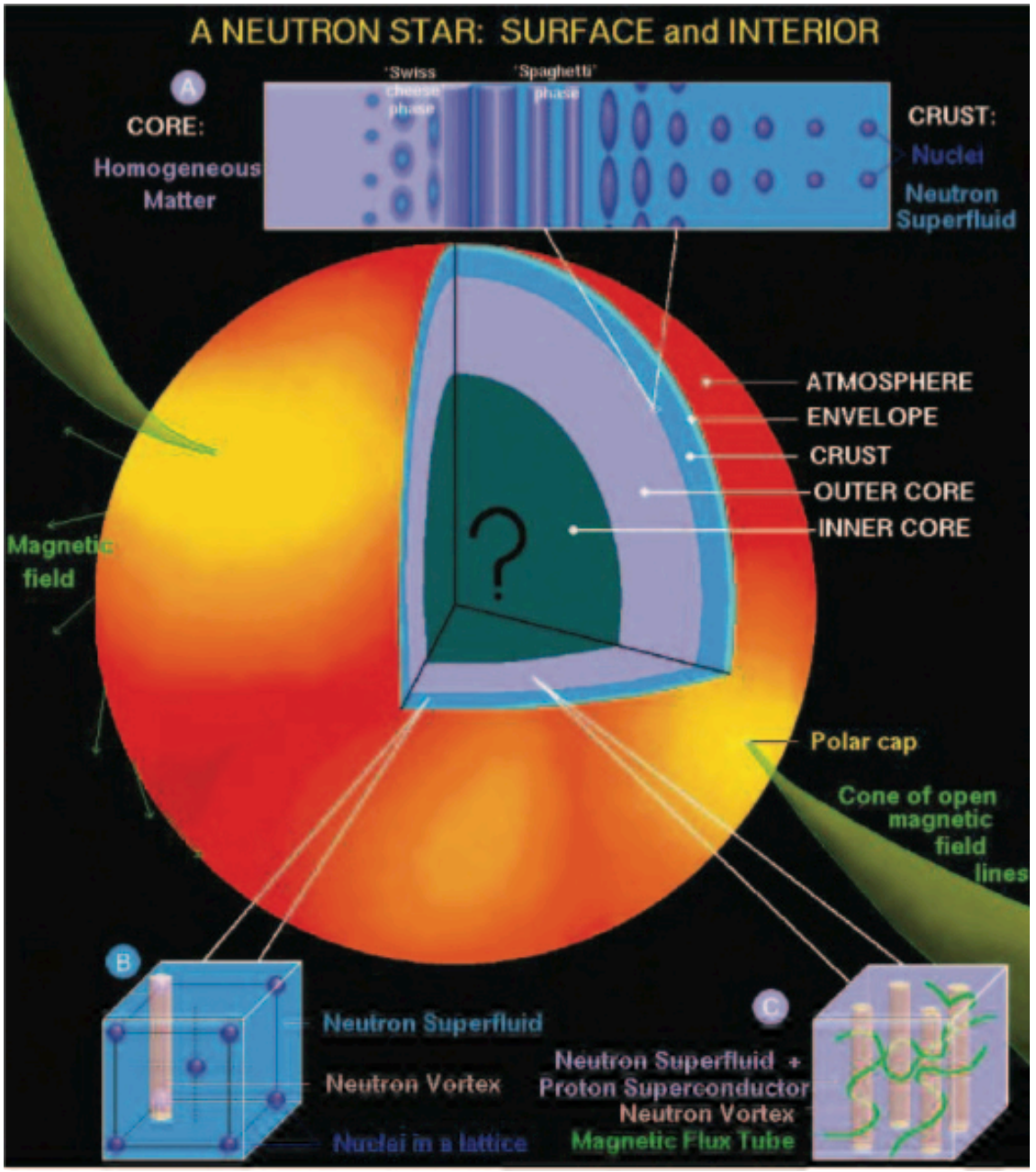
- For 2149 nuclei ($N,Z \geq 8$) in AME03:

  $a_{\mathrm{vol}}\approx 16$, $a_{\mathrm{sur}}\approx 18$, $a_{\mathrm{cou}}\approx 0.72$, $a_{\mathrm{vsym}}\approx 32$, $a_{\mathrm{ssym}}\approx 79$,
  $a_{\mathrm{pai}}\approx 12$, $a_{\mathrm{f}}\approx 0.14$, $a_{\mathrm{ff}}\approx 0.0049$, $r\approx 2.5$

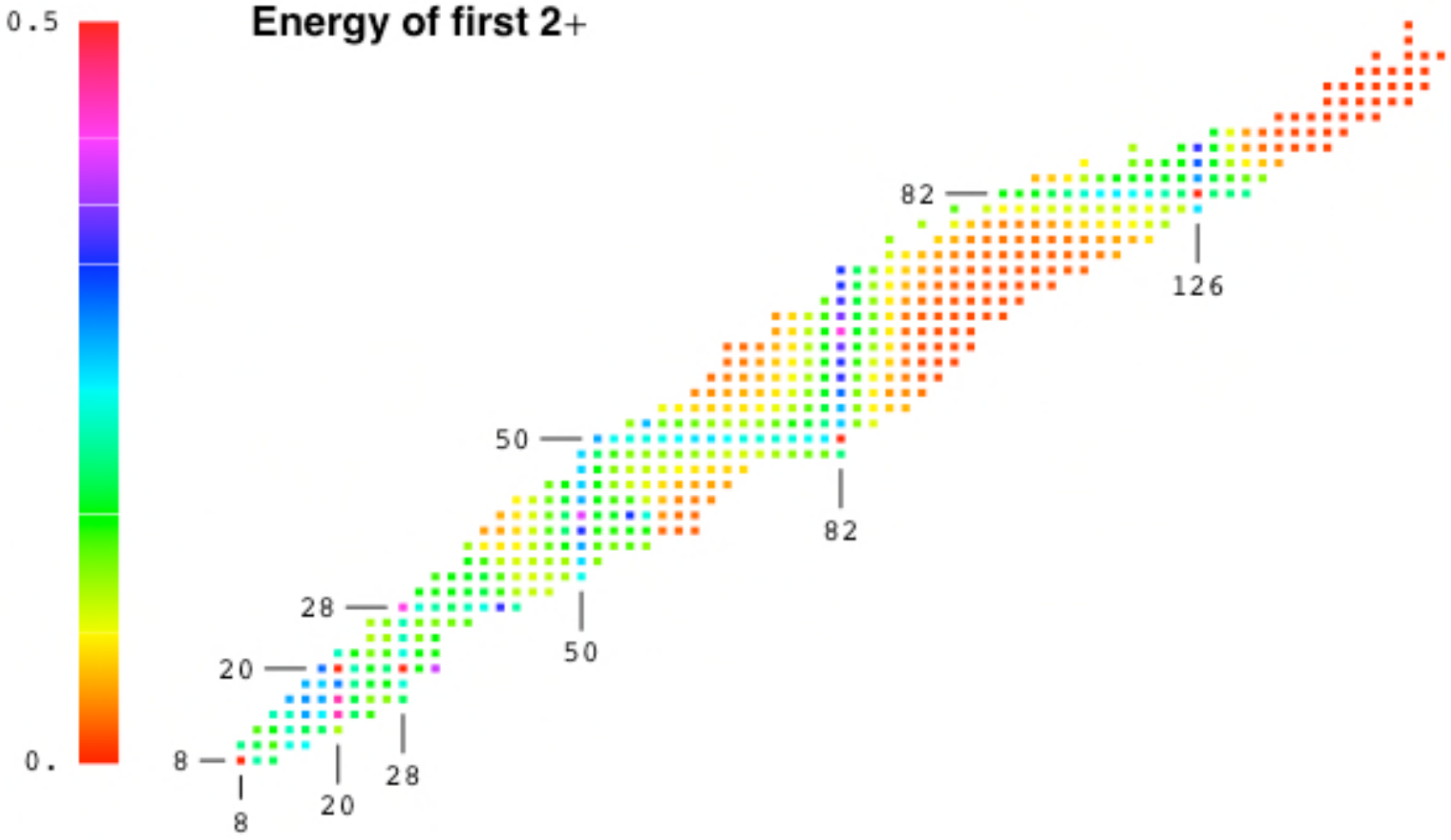
  $\Rightarrow \sigma_{\mathrm{rms}}\approx 1.28$ MeV.

Deviations from modified LDM




A NEUTRON STAR: SURFACE and INTERIOR
A
CORE:
Homogeneous Matter
'Swiss cheese' phase
'Spaghetti' phase
CRUST:
Nuclei
Neutron Superfluid
ATMOSPHERE
ENVELOPE
CRUST
OUTER CORE
INNER CORE
?
Magnetic field
Polar cap
Cone of open magnetic field lines
B
Neutron Superfluid
Neutron Vortex
Nuclei in a lattice
C
Neutron Superfluid + Proton Superconductor
Neutron Vortex
Magnetic Flux Tube

Shell structure from $E_x(2_1)$



Evidence for IP shell model

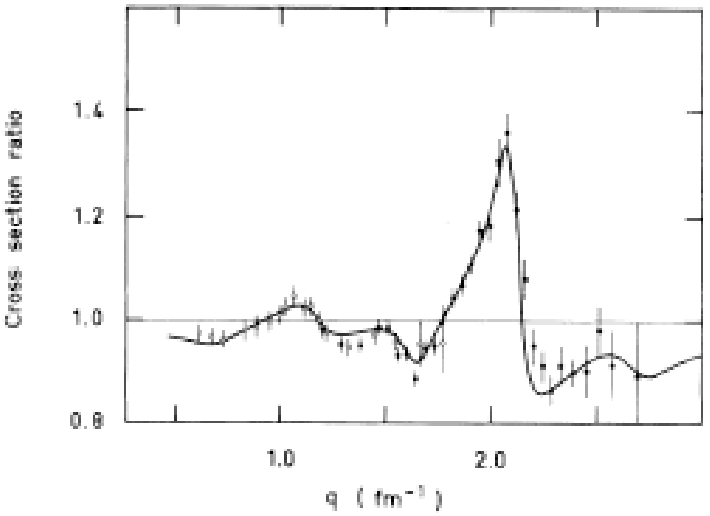
Z	Isotope	Observed J^π	Shell model nlj
3	^{9}Li	$(3/2^-)$	$1p_{3/2}$
5	^{13}B	$3/2^-$	$1p_{3/2}$
7	^{17}N	$1/2^-$	$1p_{1/2}$
9	^{21}F	$5/2^+$	$1d_{5/2}$
11	^{25}Na	$5/2^+$	$1d_{5/2}$
13	^{29}Al	$5/2^+$	$1d_{5/2}$
15	^{33}P	$1/2^+$	$2s_{1/2}$
17	^{37}Cl	$3/2^+$	$1d_{3/2}$
19	^{41}K	$3/2^+$	$1d_{3/2}$
21	^{45}Sc	$7/2^-$	$1f_{7/2}$
23	^{49}Va	$7/2^-$	$1f_{7/2}$
25	^{53}Mn	$7/2^-$	$1f_{7/2}$
27	^{57}Co	$7/2^-$	$1f_{7/2}$
29	^{61}Cu	$3/2^-$	$2p_{3/2}$
31	^{65}Ga	$3/2^-$	$2p_{3/2}$
33	^{69}As	$(5/2^-)$	$1f_{5/2}$
35	^{73}Br	$(3/2^-)$	$1f_{5/2}$

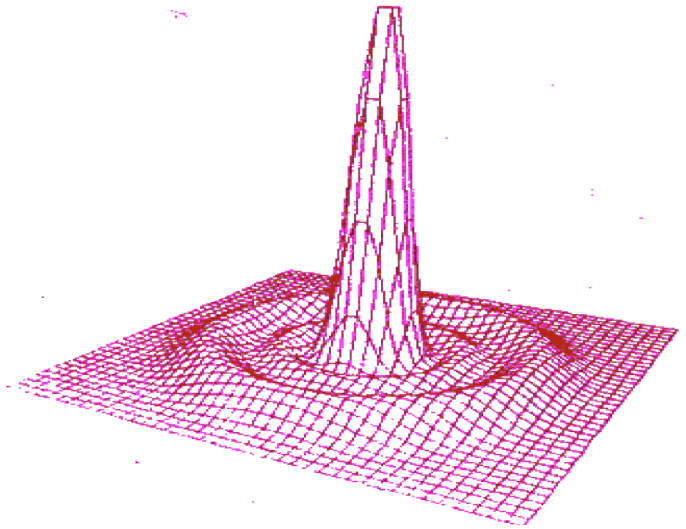
- Ground-state spins and parities of nuclei:

$$\left.\begin{array}{r} j \text{ in } \phi_{nljm_j} \Rightarrow J \\ l \text{ in } \phi_{nljm_j} \Rightarrow (-)^l = \pi \end{array}\right\} \Rightarrow J^\pi$$

Validity of SM wave functions

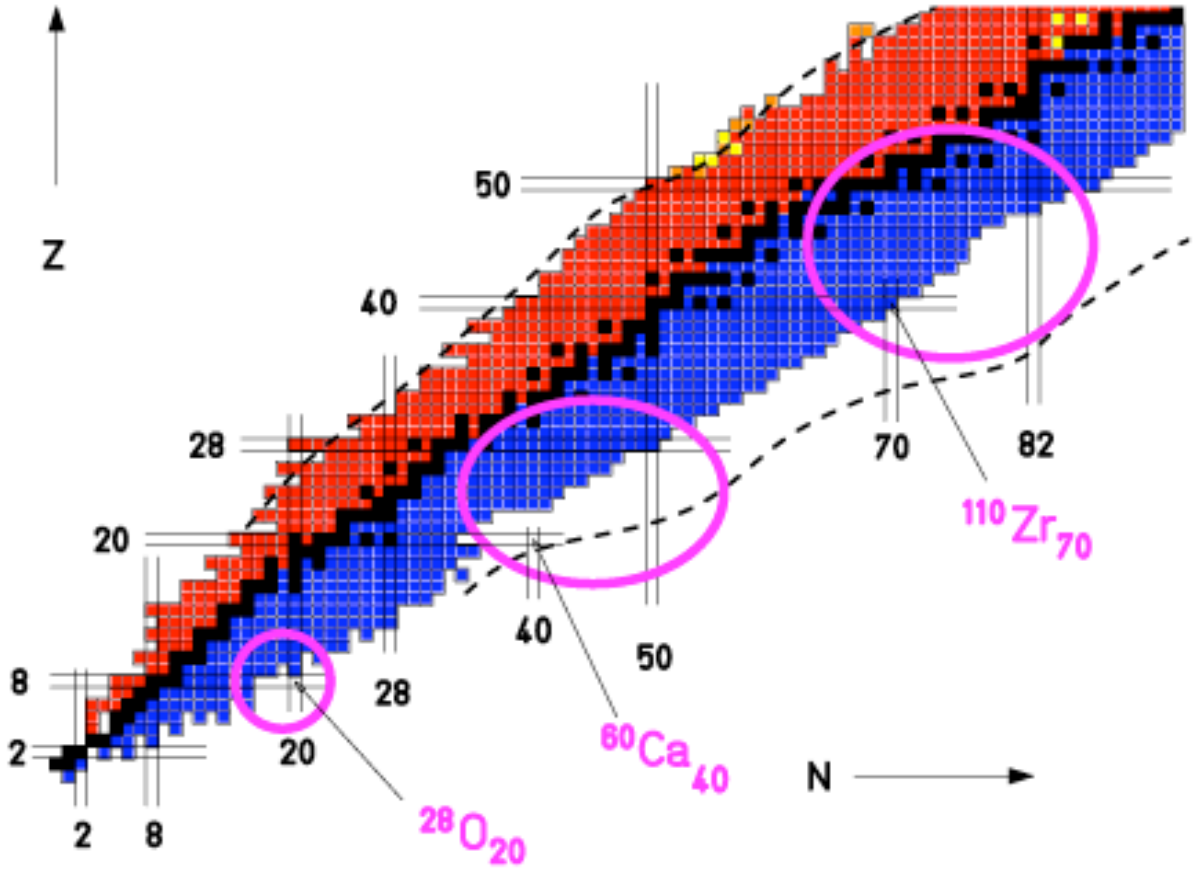
- Example: Elastic electron scattering on ^{206}Pb and ^{205}Tl, differing by a 3*s* proton.
- Measured ratio agrees with shell-model prediction for 3*s* orbit.

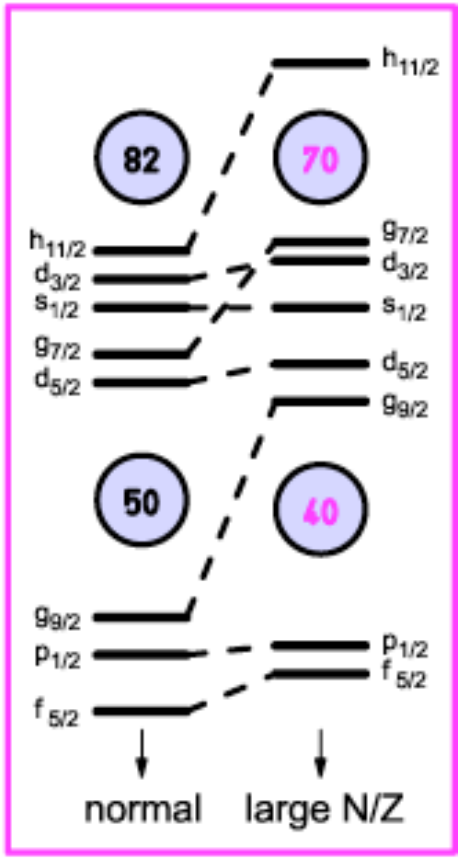




J.M. Cavedon *et al.*, Phys. Rev. Lett. **49** (1982) 978

Variable shell structure





Beyond Hartree-Fock

- Hartree-Fock-Bogoliubov (HFB): Includes pairing correlations in mean-field treatment.
- Tamm-Dancoff approximation (TDA):
  - Ground state: closed-shell HF configuration
  - Excited states: mixed 1p-1h configurations
- Random-phase approximation (RPA): Correlations in the ground state by treating it on the same footing as 1p-1h excitations.

Nuclear shell model

- The full shell-model hamiltonian:

$$\hat{H} = \sum_{k=1}^{A}\left[\frac{p_k^2}{2m} + \hat{V}(\boldsymbol{r}_k)\right] + \sum_{k<l}^{A}\hat{V}_{\mathrm{RI}}(\boldsymbol{r}_k,\boldsymbol{r}_l)$$

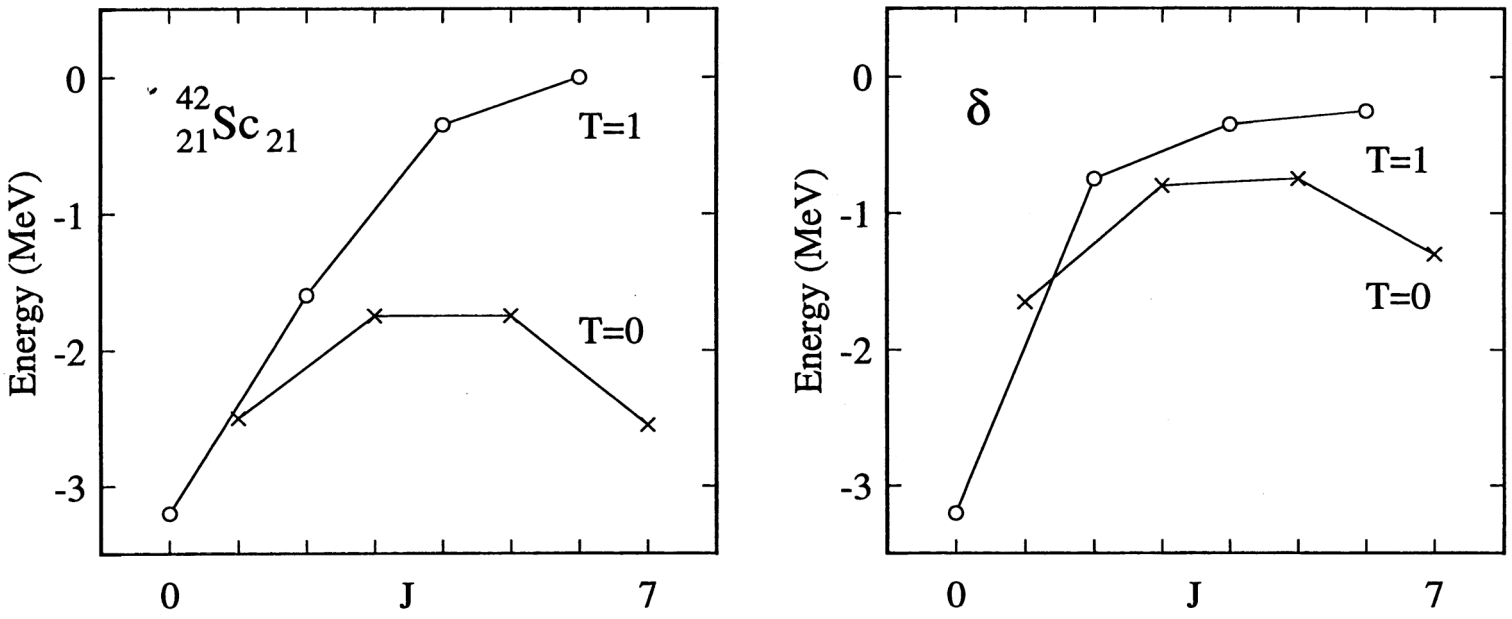
- Valence nucleons: Neutrons or protons that are in excess of the last, completely filled shell.
- Usual approximation: Consider the residual interaction V_{RI} among valence nucleons only.
- Sometimes: Include selected core excitations ('intruder' states).

Residual shell-model interaction

- Four approaches:
  - Effective: Derive from free nn interaction taking account of the nuclear medium.
  - Empirical: Adjust matrix elements of residual interaction to data. Examples: *p*, *sd* and *pf* shells.
  - Effective-empirical: Effective interaction with some adjusted (monopole) matrix elements.
  - Schematic: Assume a simple spatial form and calculate its matrix elements in a harmonic-oscillator basis. Example: δ interaction.

Schematic short-range interaction

- Delta interaction in harmonic-oscillator basis:
- Example of $^{42}Sc_{21}$ (1 neutron + 1 proton):



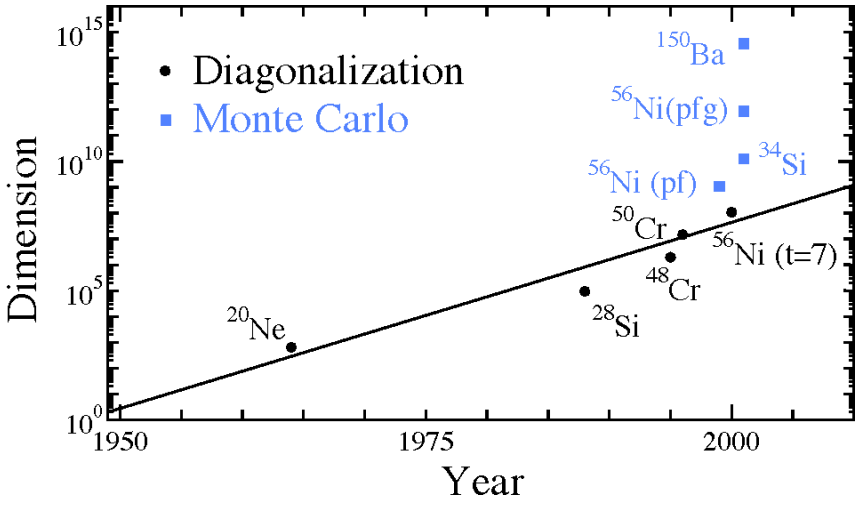
Large-scale shell model

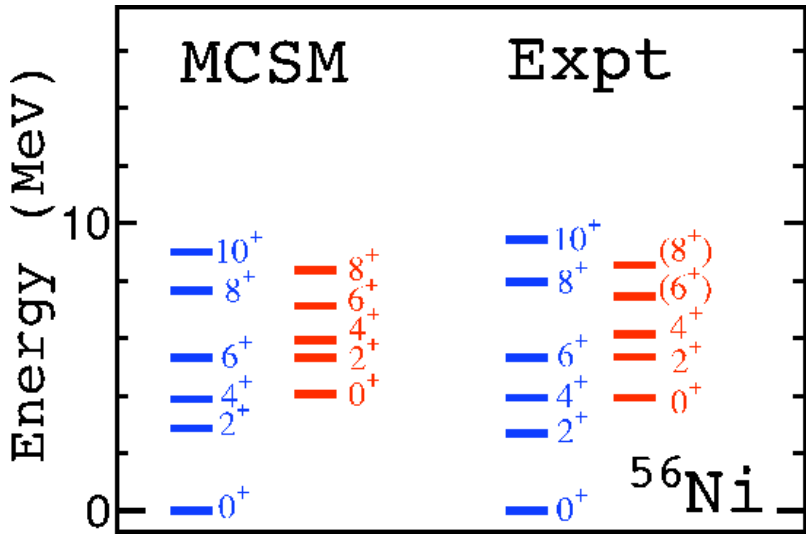
- Large Hilbert spaces:

$$\left\langle \Psi_{i'_1 i'_2 \ldots i'_A} \left| \sum_{k<l}^{n} \hat{V}_{\mathrm{RI}}(\boldsymbol{r}_k, \boldsymbol{r}_l) \right| \Psi_{i_1 i_2 \ldots i_A} \right\rangle$$

  - Diagonalisation : ~10^9.
  - Monte Carlo : ~10^{15}.
  - DMRG : ~10^{120} (?).

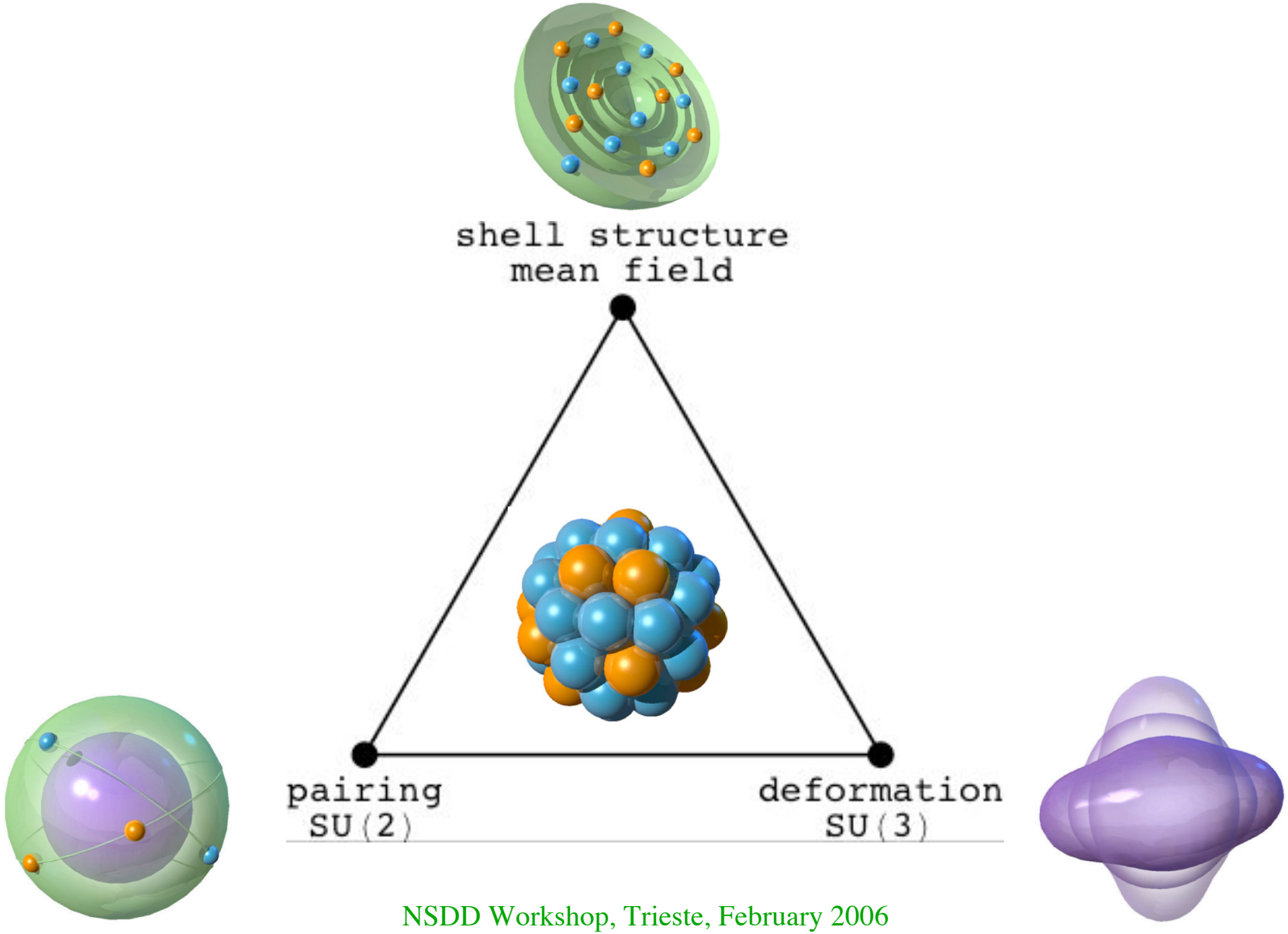
- Example : 8n + 8p in $pfg_{9/2}$ (^{56}Ni).





M. Honma *et al.*, Phys. Rev. C **69** (2004) 034335

The three faces of the shell model

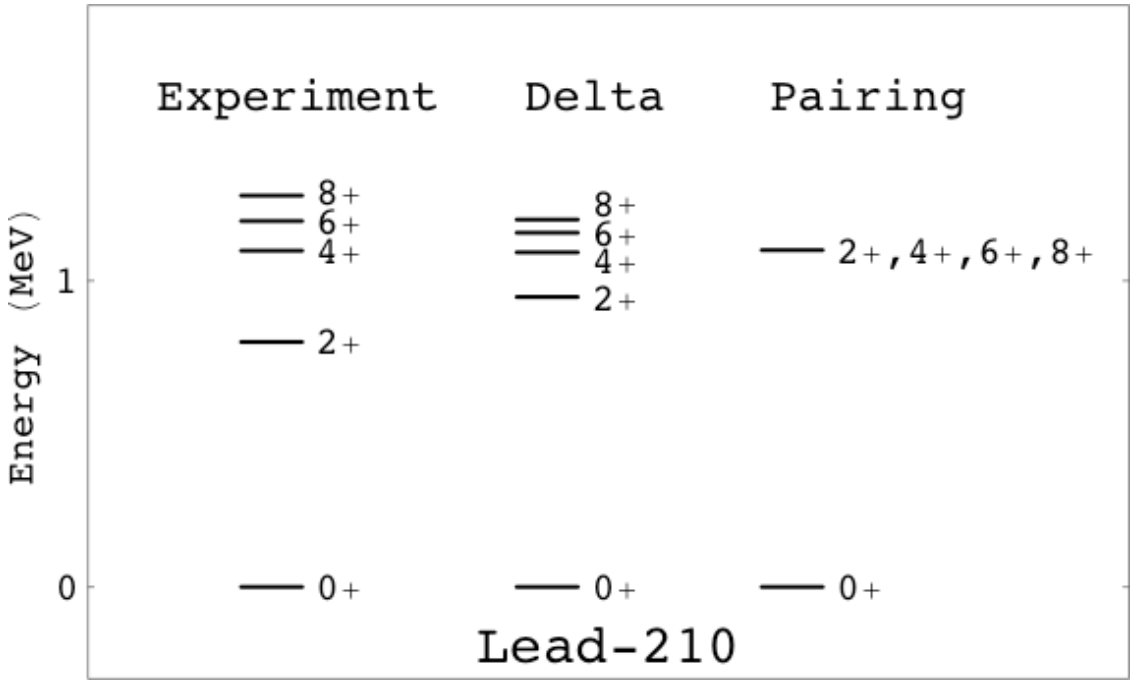


Racah's SU(2) pairing model

- Assume pairing interaction in a single-*j* shell:

$$\left\langle j^2 JM_J \left| \hat{V}_{\text{pairing}}(\boldsymbol{r}_1,\boldsymbol{r}_2) \right| j^2 JM_J \right\rangle = \begin{cases} -\frac{1}{2}(2j+1)g_0, & J=0 \\ 0, & J\neq 0 \end{cases}$$

- Spectrum ^{210}Pb:



Solution of the pairing hamiltonian

- Analytic solution of pairing hamiltonian for identical nucleons in a single-j shell:

$$\left\langle j^n \upsilon J \left| \sum_{1 \le k < l}^{n} \hat{V}_{\text{pairing}}(\boldsymbol{r}_k, \boldsymbol{r}_l) \right| j^n \upsilon J \right\rangle = -g_0 \tfrac{1}{4}(n-\upsilon)(2j-n-\upsilon+3)$$

- Seniority υ (number of nucleons not in pairs coupled to J=0) is a good quantum number.
- Correlated ground-state solution (*cf.* BCS).

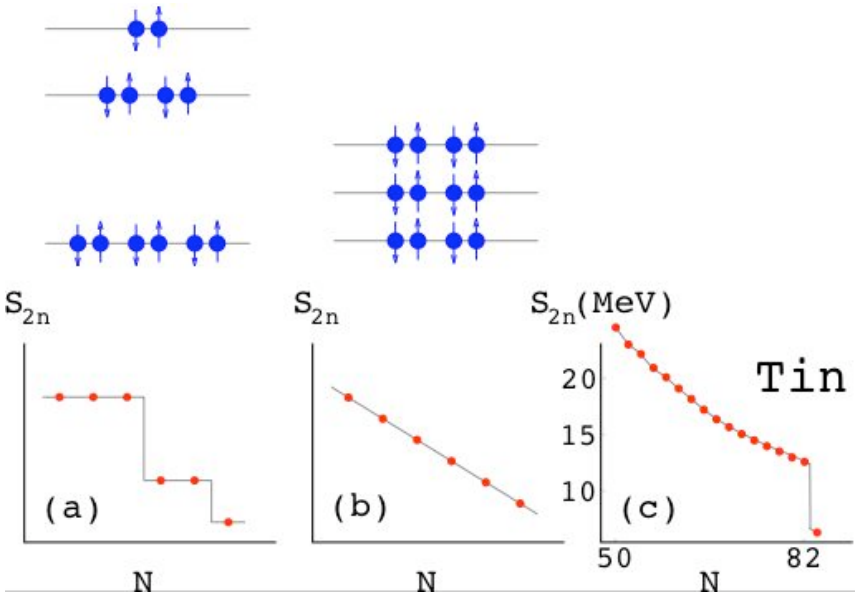
G. Racah, Phys. Rev. **63** (1943) 367

Nuclear superfluidity

- Ground states of pairing hamiltonian have the following *correlated* character:
  - Even-even nucleus (υ=0): $\left(\hat{S}_+\right)^{n/2}|\mathrm{o}\rangle, \quad \hat{S}_+ = \sum_m \hat{a}^+_{m\downarrow}\hat{a}^+_{\bar{m}\uparrow}$
  - Odd-mass nucleus (υ=1): $\hat{a}^+_{m\updownarrow}\left(\hat{S}_+\right)^{n/2}|\mathrm{o}\rangle$
- Nuclear superfluidity leads to
  - Constant energy of first 2^+ in even-even nuclei.
  - Odd-even staggering in masses.
  - Smooth variation of two-nucleon separation energies with nucleon number.
  - Two-particle (2n or 2p) transfer enhancement.

Two-nucleon separation energies

- Two-nucleon separation energies S_{2n}:
  - (a) Shell splitting dominates over interaction.
  - (b) Interaction dominates over shell splitting.
  - (c) S_{2n} in tin isotopes.

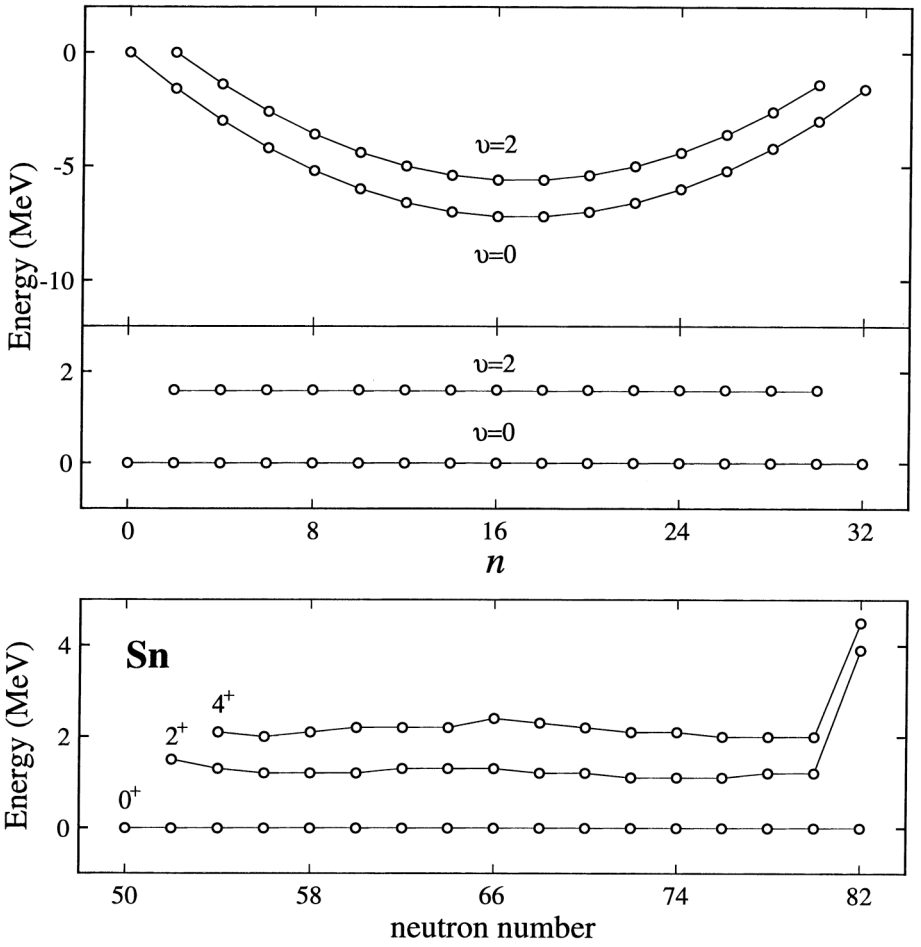


Pairing gap in semi-magic nuclei

- Even-even nuclei:
  - Ground state: $\upsilon=0$.
  - First-excited state: $\upsilon=2$.
  - Pairing produces constant energy gap:

$$E_x\left(2_1^+\right) = \tfrac{1}{2}(2j+1)G$$

- Example of Sn isotopes:

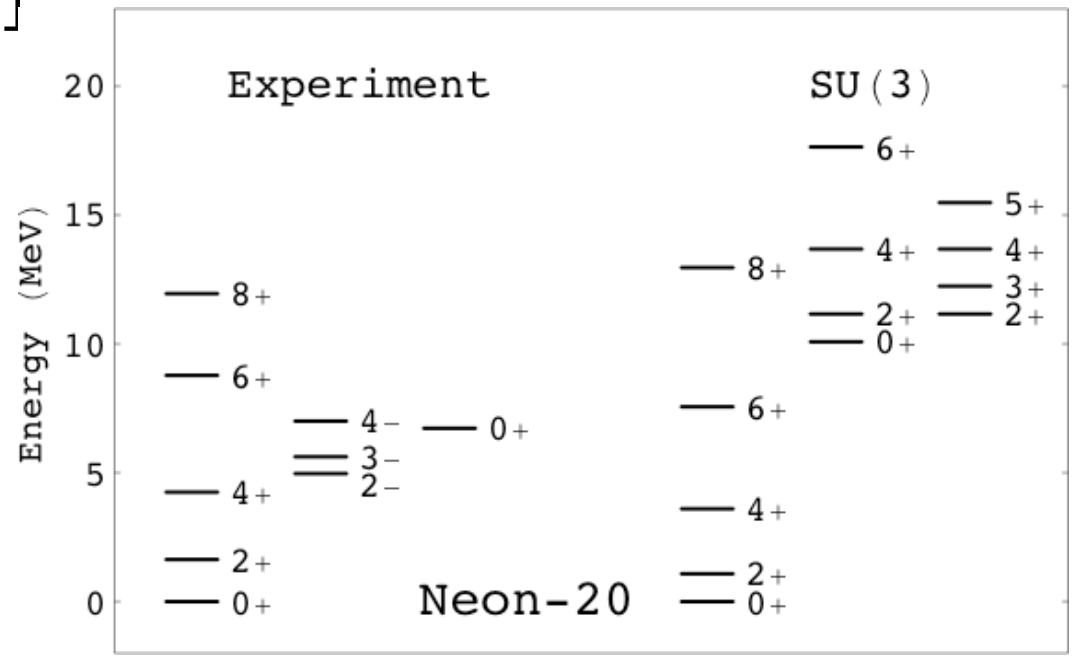


Elliott's SU(3) model of rotation

- Harmonic oscillator mean field (*no* spin-orbit) with residual interaction of quadrupole type:

$$\hat{H} = \sum_{k=1}^{A}\left[\frac{p_k^2}{2m} + \frac{1}{2}m\omega^2 r_k^2\right] - g_2\hat{Q}\cdot\hat{Q},$$

$$\hat{Q}_\mu \propto \sum_{k=1}^{A} r_k^2 Y_{2\mu}(\hat{\boldsymbol{r}}_k) + \sum_{k=1}^{A} p_k^2 Y_{2\mu}(\hat{\boldsymbol{p}}_k)$$



J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562