The Abdus Salam
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Energy Agency

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SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS

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Integrability in AdS/CFT

PART IV

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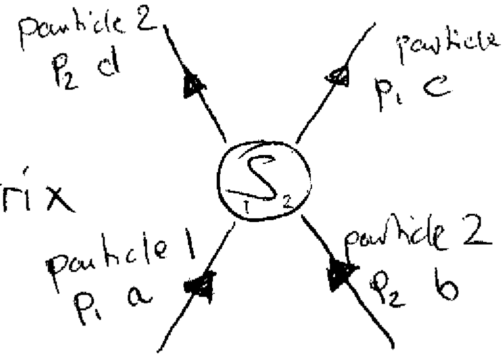
Lecture IV: Formalism and Thermodynamic Limit

- Properties of S-Matrices
- R-Matrix, Monodromy, Transfer Matrix
- Local Commuting Charges
- Algebraic Bethe Ansatz
- Analytic Bethe Ansatz
- Thermodynamic Limit
- Quantum Consistency

Properties of S-Matrices

S-matrix

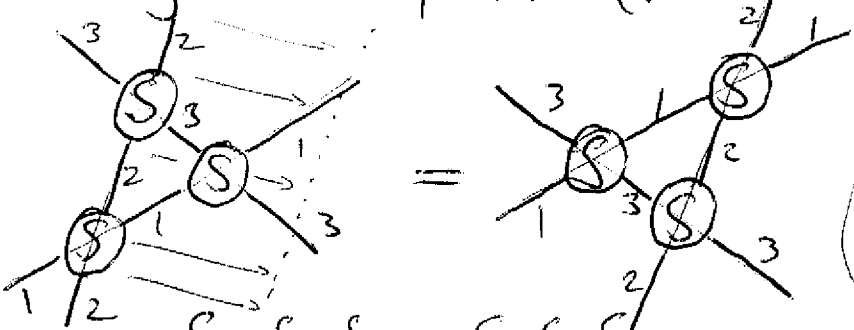


$$S_{12} = S_{ab}^{cd}(p_1, p_2)$$

A particle is sth. that carries momentum p (and possibly flavour a)

Place an S-Matrix where two particles cross.

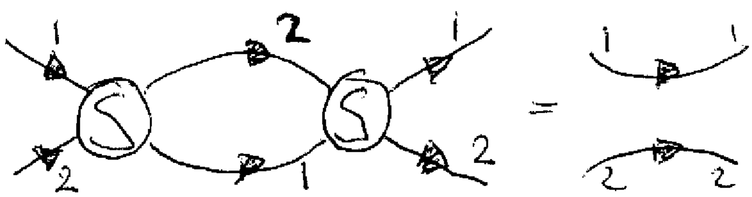
Satisfies Yang-Baxter Equation (Factorisable Scattering)



$$S_{23}\, S_{13}\, S_{12} = S_{12}\, S_{13}\, S_{23}$$

can move one line past a crossing

Satisfies "Unitarity"



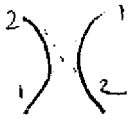
$$S_{21} \cdot S_{12} = I_{12}$$

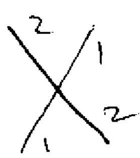
can remove inessential crossings

$\Rightarrow$ Particle lines can be moved at will as long as there is an S-matrix at each crossing.

Some Special Points

for equal types of particles

$$S_{12}(p,p) \stackrel{\downarrow}{=} P_{12} =$$ 

$$S_{12}(0,p) = S_{12}(p,0) = I_{12} =$$ 

Useful diagrammatic tools.

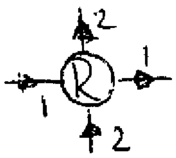
R-Matrix Formalism

Like NBA in reverse. Add one level.

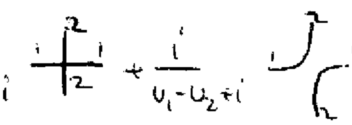
Spin sites become particles (with spectral parameter $u_k = 0$).

Scattering Matrix for sites is called R-Matrix $R_{kl}(u,v)$

Example: R-Matrix for fundamental spins of $U(N)$:



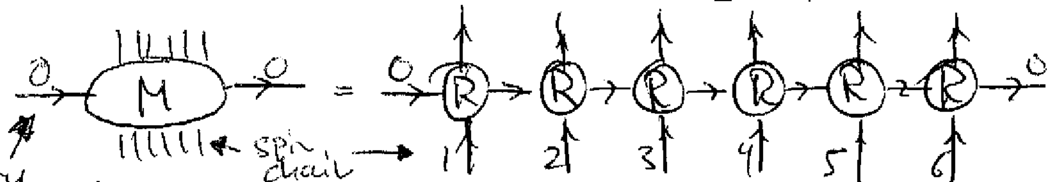
$$= R_{12}(u_1,u_2) = \frac{u_1-u_2}{u_1-u_2+i}\, I_{12} + \frac{i}{u_1-u_2+i}\, P_{12} = \frac{u_1-u_2}{u_1-u_2+i}\ (\ldots) + \frac{i}{u_1-u_2+i}\ (\ldots)$$



R-Matrix is like Lax connection/parallel transport between two sites.

construct Monodromy Matrix

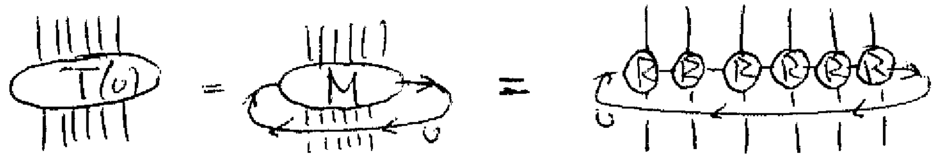
$$M_0(u) = R_{0L}\, R_{0L-1} \cdots R_{02}\, R_{01}$$



Like open Wilson Loop of Lax connection

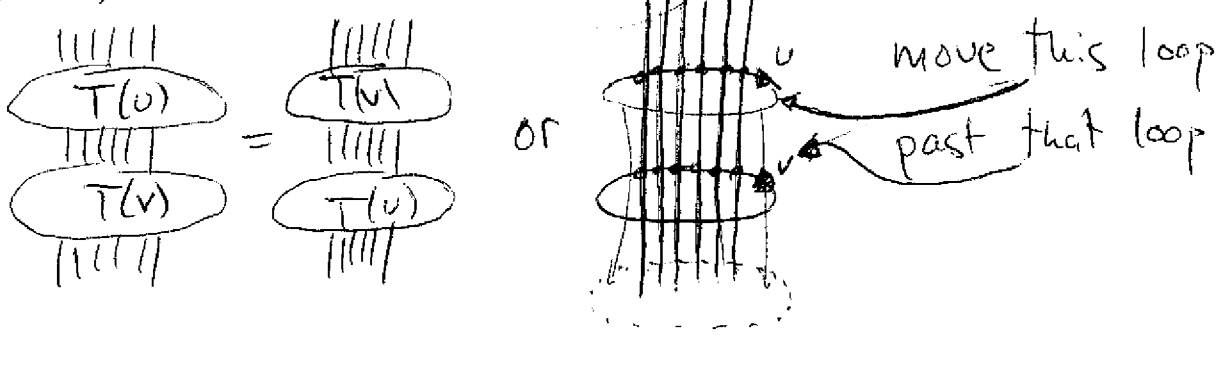
construct Transfer Matrix

$$T(u) = \mathrm{Tr}_0\, M_0(u)$$



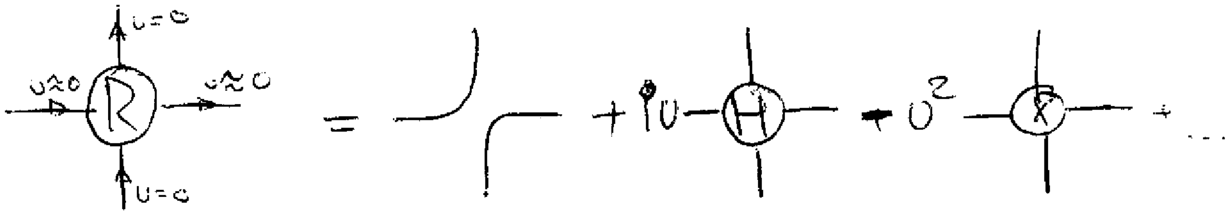
Transfer Matrices Commute at arbitrary spectral parameters u,v

$$[T(u), T(v)] = 0$$

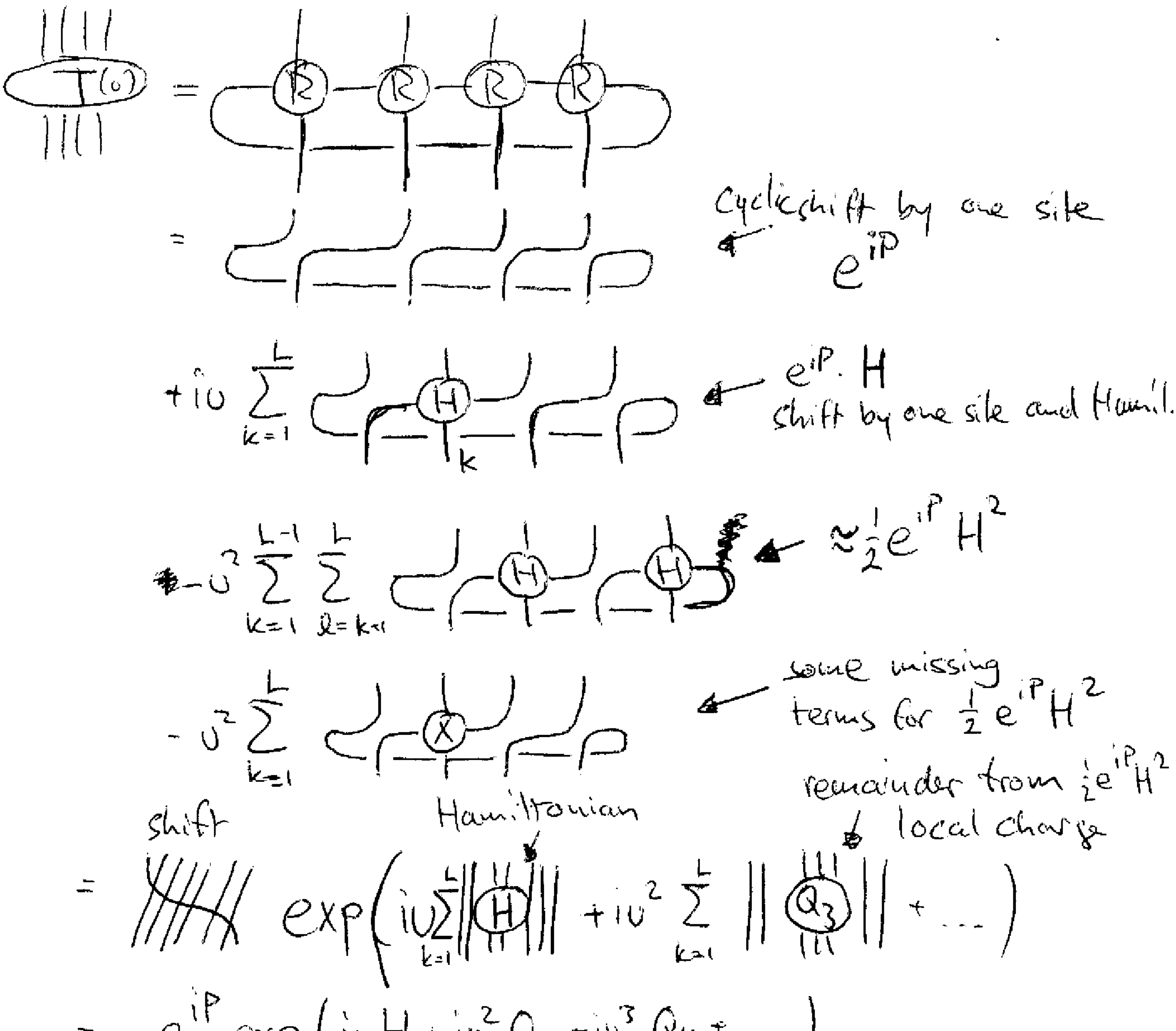


Local Charges

Consider the R-Matrix around $u=0$



Expand Transfer Matrix around $u=0$



$$= e^{iP} \exp\left(iuH + iu^2 Q_3 + iu^3 Q_4 + \dots \right)$$

← $H = Q_2$ ← expansion gives local charges.

Local charges commute $[Q_k, Q_l] = 0$ b/c $[T(u), T(v)] = 0$

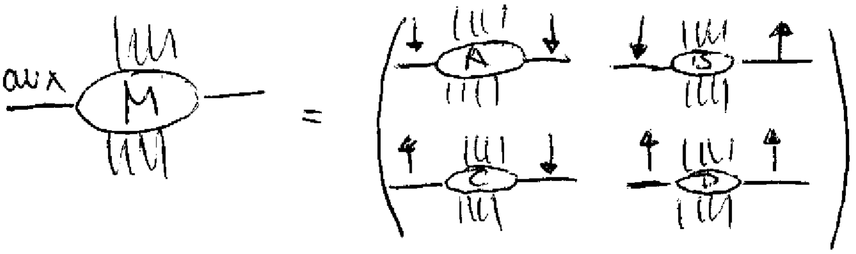
Integrability!

Algebraic Bethe Ansatz

Use R-Matrix Formalism to construct (periodic) Eigenstates

Start with Ferromagnetic Vacuum $|0\rangle = |\downarrow\downarrow\downarrow\downarrow\ldots\downarrow\rangle$ (finite, per.)

Monodromy is Matrix acting on auxiliary spin (and on spin chain)



$$= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

A, B, C, D obey certain commutation algebra (Yang-Baxter-Eq.)

States: $B(u)$ has the same quantum numbers as spin flip

Ansatz: $|u_1, u_2, \ldots, u_K\rangle = B(u_1)\,B(u_2)\ldots B(u_K)\,|0\rangle$

Using algebra we can show that $|u_1, \ldots, u_K\rangle$ is an Eigenstate of $T(u) = A(u) + D(u)$ for all $u \in \mathbb{C}$ if $\{u_j\}$ satisfy the Bethe Equations

Eigenvalue of $T_{op}(u)$ on $|u_1, \ldots, u_K\rangle$ satisfying BE:

$$T_{ev}(u) = \prod_{k=1}^{K} \frac{u - u_k - i/2}{u - u_k + i/2} + \left(\frac{u}{u+i}\right)^{L} \prod_{j=1}^{K} \frac{u - u_k + 3i/2}{u - u_k + i/2}$$

Analytic Bethe Ansatz, Baxter Equation

Eigenvalue of Transfer Matrix

$$T_{ev}(u) = \prod_{j=1}^{K} \frac{u-u_j-i/2}{u-u_j+i/2} + \left(\frac{u}{u+i}\right)^L \prod_{j=1}^{K} \frac{u-u_j+3i/2}{u-u_j+i/2}$$

Compare to definition of Transfer Matrix

$$T_{op}(u) = \mathrm{Tr}_0\, R_{0L}(u,0) \ldots R_{01}(u,0) \quad \text{with } R_{12}(u,v) = \frac{u-v}{u-v+i} I_{12} + \frac{i}{u-v+i} P_{12}$$

Observe $T_{op}(u)$ has L-fold pole at $u=-i$

but $T_{ev}(u)$ has apparent poles at $u = u_k - \frac{i}{2}$ in addition.

What went wrong? Nothing:

$$T_{ev}(u_k - \tfrac{i}{2} + \varepsilon) = \frac{-i}{\varepsilon} \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k-u_j+i}{u_k-u_j} + \frac{i}{\varepsilon}\left(\frac{u_k-i/2}{u_k+i/2}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k-u_j+i}{u_k-u_j} + O(\varepsilon^0)$$

$$= -\frac{i}{\varepsilon} \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k-u_j+i}{u_k-u_j} \left(1 - \left(\frac{u_k-i/2}{u_k+i/2}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k-u_j+i}{u_k-u_j-i}\right) + O(\varepsilon^0)$$

Bethe Equations ensure absence of dynamical poles $u_k - \frac{i}{2}$

Eigenstate $\Leftrightarrow$ $T_{ev}(u)$ is analytic function with L-fold pole at $u=-i$

Thermodynamic Limit

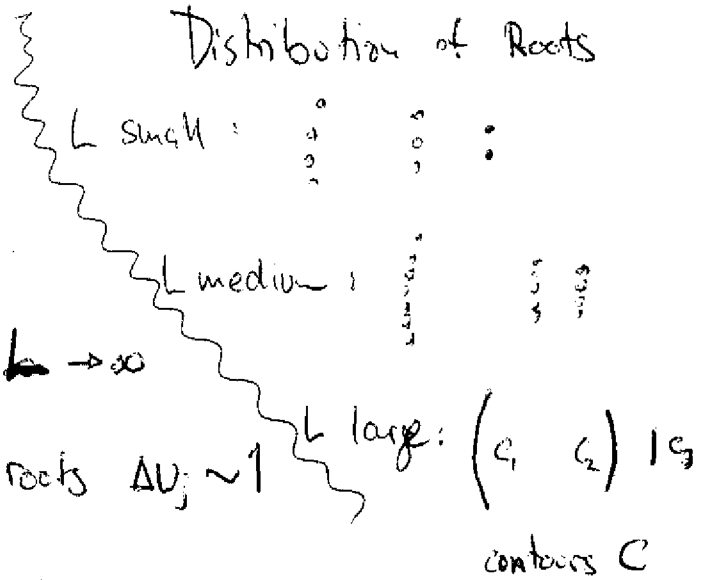
Consider the following Limit:

Length $L \to \infty$

Number of Particles $K \sim L \to \infty$

Bethe Roots, Rapidity $u_j, u \to \infty$

Coherent states, separation of roots $\Delta u_j \sim 1$



Eigenvalues of Transfer Matrix

$$T(u) = \exp\left[\sum_{j=1}^{k} \log \frac{1 - \frac{i/2}{u-u_j}}{1 + \frac{i/2}{u-u_j}}\right] + \exp\left[L \log \frac{1}{1+i/u} + \sum_{j=1}^{k} \frac{1 + \frac{3i/2}{u-u_j}}{1 + \frac{i/2}{u-u_j}}\right]$$

(small) (small) (small) (small)

$$\to \exp\left(-i \sum_{j=1}^{k} \frac{1}{u-u_j}\right) + \exp\left(-\frac{iL}{u} + i \sum_{j=1}^{k} \frac{1}{u-u_j}\right)$$

$$= \exp\left(-\frac{iL}{2u}\right)\left(\exp(iq(u)) + \exp(-iq(u))\right) = 2\cos q(u) \exp\left(\frac{-iL}{2u}\right)$$

with $q(u) = -\frac{L}{2u} + \sum_{j=1}^{k} \frac{1}{u_j - u} \to -\frac{L}{2u} + \int_C \frac{dv\, \rho(v)}{v-u}$

$T(u)$ has essential singularity at $u = -i \approx 0$ $\longleftrightarrow$ $\cos q(u)$ as well ✓

is analytic otherwise (analytic BA) $\longleftrightarrow$ $q(u)$ has branch cuts at contours C. X?

Recover properties of $T(u)$ by requiring

$$q(u) \to \pm q(u) + 2\pi n_K \text{ across a cut } C_K.$$

$\rightsquigarrow$ $q'(u)$ is an algebraic curve (if C has finitely many compt.)

$\rightsquigarrow$ agrees to string spectral curve to some extent.

Quantum Consistency

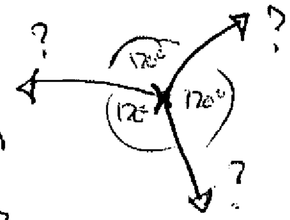
- For classical strings the branch cuts of the spectral curve are artificial, can be deformed at will without affecting physical quantities.
- For the Bethe Ansatz in the thermodynamic limit (NOT: thermodynamic Bethe ansatz! That's sth. else) the branch cuts correspond to distributions of Bethe roots.

In fact: Density of Bethe Roots must be positive

$$dk = du\,\rho(u) \geq 0$$

Fixes path of Branch cuts.

Three possible directions at Branch Points:



Why? $\rho(u) \sim \sqrt{u-u_k}$ $du \sim u-u_k$ $\Rightarrow dk \sim (u-u_k)^{3/2}$ must be positive.

Further condition: NB, Tseytlin, Zarembo '05

$$\int_{C_k} d\log\frac{n+i\rho(u)}{n-i\rho(u)} = 0$$ for all cuts C_k and all integers n

Meaning: Mode number is well-defined for the cut.