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SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS

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Sasaki-Einstein geometry and the AdS/CFT correspondence

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The AdS/CFT correspondence motivates the study of Sasaki-Einstein manifolds and related geometry. In particular, to any SE 5-manifold there is associated a dual superconformal field theory. The study of this correspondence has lead to new results in the past two years, and in particular has brought together various areas of physics and mathematics.

In this set of lectures I will focus on some aspects concerning mainly the <u>geometry</u> side.

The topics covered by these lectures should be:

- introduction / motivation
- toric geometry
- Sasaki-Einstein geometry
- superconformal quivers & a-maximisation
- "Z-minimisation" (i.e. volume minimisation)

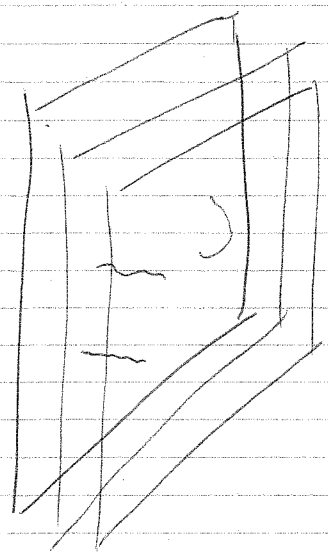
Some related topics will not be discussed for lack of time (and expertise!), but are by no means less interesting! Among these, are for instance "dimer models", deformations, supersymmetry breaking ("cascades"), etc.

A list of useful <u>references</u> is the following:

- Morrison + Plesser hep-th/9810201
- Ana Cannas da Silva, "Symplectic toric manifolds" - lecture notes
- David Cox, "What is a toric variety?" lecture notes.
- D.M. + Sparks hep-th/0411238
- S. Franco et al hep-th/0505211
- D.M., Sparks, Yau hep-th/0503183, 0603021
- Intriligator + Wecht hep-th/0304128

Introduction / motivation

Gauge theories emerge as the dynamics of open strings attached to D-branes:



N D3-branes placed in $\mathbb{R}^{3,1}$

$\downarrow$

SU(N) $\mathcal{N}=4$ SYM in 3+1 dimensions.
(This theory is exactly conformal: β-function $\equiv 0$)

This preserves $\frac{1}{2}\cdot 32$ supersymmetries of Type IIB.

D-branes back-react on the geometry, curving the space-time:

$$ds^2 = H^{-1/2}(r)\, ds^2(\mathbb{R}^{1,3}) + H^{1/2}(r)\underbrace{\left(dr^2 + r^2 ds^2(S^5)\right)}_{\mathbb{R}^6}$$

$$H(r) = 1 + \frac{L^4}{r^4} \qquad L^4 = g_s N \alpha'^2$$

In the "near-horizon" limit, $r\to 0$, the geometry becomes exactly $AdS_5 \times S^5$.

This is a solution of Type IIB Supergravity, preserving all the 32 supersymmetries:

$$ds^2 = ds^2(AdS_5) + ds^2(S^5)$$

$$F^5_{RR} = N\left[\mathrm{vol}(AdS_5) + \mathrm{vol}(S^5)\right]$$

Maldacena conjecture:

the large N limit (and $g_s N \gg 1$) of $\mathcal{N}=4$ SYM

$$\Big\updownarrow \text{ dual}$$

type IIB string theory in the $AdS_5 \times S^5$ geometry

First matching: symmetries

$$\underbrace{SO(2,4)}_{\text{Iso}(AdS_5)} \times \underbrace{SO(6)}_{\text{Iso}(S^5)}$$

$SO(2,4)$ → conformal group in 3+1 dim

$SO(6)$ → $\simeq SU(4)$: symmetry that rotates supercharges of $\mathcal{N}=4$ SYM → R-symmetry

operators in SYM $\longleftrightarrow$ fluctuations in geometry

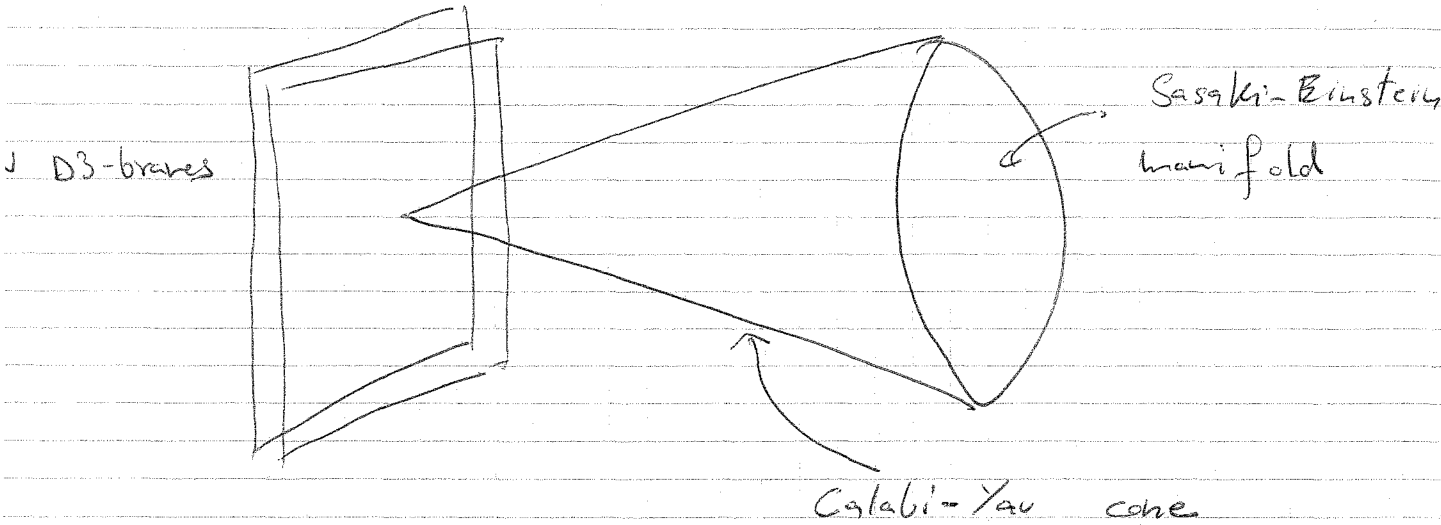
anomalous dimensions $\longleftrightarrow$ energies of KK spectrum

generating function (of correlation functions) $\longleftrightarrow$ on-shell supergravity action (as a function of boundary conditions)

QFT $\longleftrightarrow$ GEOMETRY

Now, we want to construct conformal field theories with <u>less supersymmetry</u>.

$\to$ Put N D3-branes in $\mathbb{R}^{1,3} \times$ Calabi-Yau



CY: Kähler + Ricci-flat metric

$\Uparrow$

$\mathcal{N}=1$ supersymmetry

$$ds^2_{CY} = dr^2 + r^2 ds^2(L)$$

Geometry after back-reaction: $AdS_5 \times L$

AdS/CFT: $AdS_5 \times L \xleftrightarrow{\text{dual}} \mathcal{N}=1$ SCFT

E.g. CY = orbifold $\mathbb{C}^3/\Gamma$ $\qquad \Gamma \subset SU(3) \Rightarrow \mathcal{N}=1$ susy

We will be concerned with much more general singularities than these ones (a large class is given by toric singularities)

L is a Sasaki-Einstein manifold!

$\Rightarrow$ This motivates the study of SE geometry.

To flash out most of the concepts that will be treated in these lectures, let's consider the example of the conifold (very quickly!)

Algebraic definition : $XY = ZW \subset \mathbb{C}^4$

Ricci-flat metric : $ds^2 = dr^2 + r^2 ds^2(T^{1,1})$

$T^{1,1}$ is a (coset) manifold that for our purposes can be tought as a $U(1)$ bundle over $S^2 \times S^2$

$$\begin{array}{ccc} U(1) & \rightarrow & T^{1,1} \\ & & \downarrow \\ & & S^2 \times S^2 \end{array}$$

(notice, is Kähler-Einstein!)

Isometry (is the one naturally guessed) : $SU(2) \times SU(2) \times U(1)$

Topology : $S^2 \times S^3$

$\mathrm{Vol}(T^{1,1}) = \frac{27}{16 \cdot 4} \pi^3$

The gauge theory dual can be inferred by collecting various information from the geometry:

$AdS_5 \Rightarrow$ conformal fixed point

8 supersymmetries $\Rightarrow$ $\mathcal{N} = 1$ SCFT

$U(1)$ isometry $\Rightarrow$ $U(1)$ R-symmetry

$SU(2) \times SU(2) \Rightarrow$ global flavour symmetry

$\mathrm{vol}(T^{1,1}) \Rightarrow$ central charge $a = \frac{16}{27} = \frac{\pi^3}{4\,\mathrm{vol}}$

(where $\langle T_\mu{}^\mu \rangle = a(\text{curvature invariant})^2 \dots$)

So, what is a gauge theory with these properties?

Hint from algebraic description: $XY = ZW$

"Solve" $\quad X = A_1 B_1 \quad Y = A_2 B_2 \quad Z = A_1 B_2 \quad W = A_2 B_1$

then: $\quad |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0$

modulo $\quad A_i \sim e^{i\theta} A_i \qquad B_i \sim e^{-i\theta} B_i$

is an equivalent description of the conifold (this comes from considering first the 4 variables in $\mathbb{C}^4$, modded by equivalences

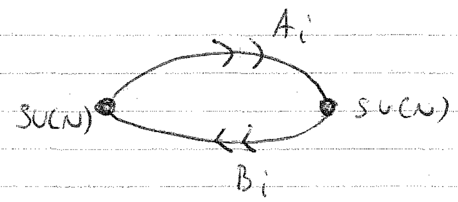
$$A_i \sim \lambda A_i \qquad B_i \sim \lambda^{-1} B_i \qquad \lambda \in \mathbb{C}^*$$

setting then $\lambda = s e^{i\theta}$ the overall modulus s can be fixed as above. In fact these two things are holomorphic and symplectic quotients respectively).

A_i, B_i $i = 1, 2$ are then promoted to matrix-valued BIFUNDAMENTAL CHIRAL FIELDS of the gauge group $SU(N) \times SU(N)$:

	$SU(N) \times SU(N)$	$SU(2) \times SU(2)$	$U(1)_R$
A_i	$(N, \bar{N})$	$(2, 1)$	$1/2$
B_i	$(\bar{N}, N)$	$(1, 2)$	$1/2$

$\Rightarrow$ QUIVER



Need to specify a superpotential (in any $N=1$ theory)

$$W = \epsilon^{ij} \epsilon^{kl} \, \mathrm{Tr}\, A_i B_k A_j B_l$$

Toric geometry

symplectic point of view — algebraic point of view

These two points of view on toric geometry are complementary, and both are useful for applications in the context of AdS/CFT.

In the following, I would like to introduce to you a number of concepts and tools that you will encounter in the recent literature on AdS/CFT. The utility of these concepts of course goes beyond the applications that we want to consider.
In particular I will talk about:
moment maps, polytopes, quotient constructions, cones, fans, toric diagrams, "pq-webs", etc.

We start with a symplectic manifold M, that is a manifold endowed with a closed non-degenerate two-form ω: $d\omega = 0$.

If we have a $U(1)$ action on this manifold, that preserves the symplectic form ω

$$0 = \mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega$$

↑ infinitesimal vector field that generates the action

we say that the vector is Hamiltonian if we can integrate $d(i_X\omega) = 0 \rightarrow i_X \omega = d\mu$

μ is then called a "Hamiltonian"

A symplectic toric manifold is defined as a symplectic manifold that admits a $T^n = U(1)^n$ Hamiltonian action. There are then n Hamiltonian functions, that define a moment map

$$\mu^i : M \to \mathbb{R}^n$$

$\mathbb{R}^n \leftarrow t_n^*$ is the Lie algebra of the torus T^n

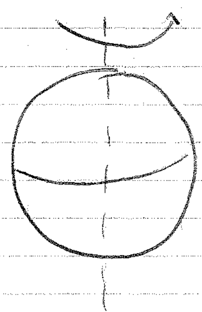
(this is defined up to a constant vector in $\mathbb{R}^n$)

The moment map provides natural coordinates on symplectic toric manifolds, called symplectic coordinates, in which the symplectic form is simply

$$\omega = \sum_{i=1}^{n} d\mu^i \wedge d\phi^i$$

$\phi^i \sim \phi^i + 2\pi$ parameterize the T^n torus.

The simplest example of toric symplectic manifold is the 2-sphere S^2:



$\frac{\partial}{\partial\phi}$ generates $U(1)$ rotation around the vertical axis

$$\omega = d\phi \wedge d\underbrace{\cos\theta}_{\mu} \qquad 0 \leq \phi < 2\pi$$

$$-1 \leq \mu \leq 1$$

$$\mu : S^2 \to [-1, 1] \in \mathbb{R}$$

The image of the moment map is an interval. Notice that the S^2 can be viewed as a circle fibered over $[-1, 1]$, with the circle degenerating to zero-size at the end-points. In fact, these features generalise!

A classic result [Atiyah, Guillemin-Sternberg] states that the image under a moment map of a (compact, connected) symplectic toric manifold is:

- the convex hull of the images of the <u>fixed points</u> of the torus action.

Note:

1) this is useful for constructing in practice $\mu(M)$

2) the theorem applies also for smaller tori T^r $r<n$

3) a generalization of this theorem shows that in conical spaces ("symplectic cones") with a (Hamiltonian) torus action, the image of M under the moment map is convex rational polyhedral cone C

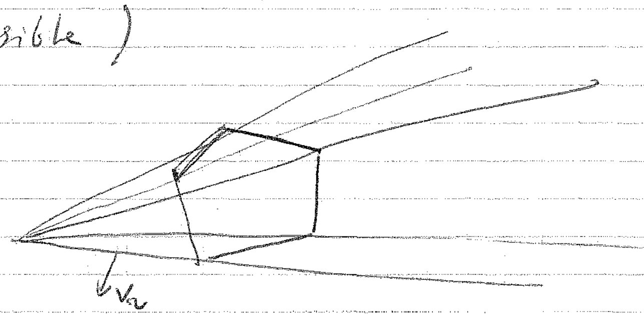
$$\mu : M \to \mathbb{R}^n$$
$$M \mapsto C$$

rational vectors in $\mathbb{R}^n$

$$C = \{ y \in \mathbb{R}^n \mid \langle y, v_a \rangle \leq 0 \ , \ a = 1, \dots, d \}$$

This object is obtained by intersecting a number (d) of planes (orthogonal to the vectors v_a) through the origin (the v_a are taken to be "primitive" - that fixes their lenght, by demanding that are the shortest possible)

E.g. in $\mathbb{R}^3$, $d=5$



To keep things simpler, let us focus on the compact case (the generalization to the non compact case is easy)

The important point is that there is a one-to-one correspondence between symplectic toric manifolds and particular polytopes (a polytope is just the convex hull of a finite number of points)

symplectic toric manifold M $\overset{1-1}{\longleftrightarrow}$ ("Delzant") polytope Δ

"Delzant" polytopes obey by definitions certain properties:

- rationality (the normal to the facets are rational)
  facets → codimension one ($\dim = 2n-1$) hyperplanes bounding the polytope
- simplicity (n edge meet at each vertex)
- smoothness (for each vertex, the corresponding primitive edge vectors v_i $i=1,..,n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$)
  → $a_i v_i$, $a_i \in \mathbb{Z}$ span $\mathbb{Z}^n$

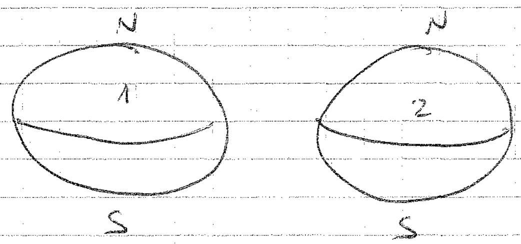
(Incidentally, in the case of polyhedral cones, the last two conditions get replaced by the technical condition of being "good" - we shall not bother here about that condition).

Example : $S^2_1 \times S^2_2$

this is clearly symplectic toric, simply because S^2 was

- T^2 action generated by $\frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2}$
- sympectic form $\omega = \sin\theta_1\, d\theta_1 \wedge d\phi_1 + \sin\theta_2\, d\theta_2 \wedge d\phi_2$
- moment map $\mu = (\cos\theta_1, \cos\theta_2)$

To find out what is the associated polytope, image under the moment map, we can simply identify its vertices, corresponding to the fixed points of the $T^2 = U(1)^2$ action

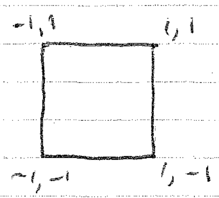


NN : $\theta_1 = 0, \quad \theta_2 = 0 \quad \Rightarrow \quad \mu = (1, 1)$

NS : $\theta_1 = 0, \quad \theta_2 = \pi \quad \Rightarrow \quad \mu = (1, -1)$

SN : $\theta_1 = \pi, \quad \theta_2 = 0 \quad \Rightarrow \quad \mu = (-1, 1)$

SS : $\theta_1 = \pi, \quad \theta_2 = \pi \quad \Rightarrow \quad \mu = (-1, -1)$

we conclude $\mu(S^2 \times S^2) =$  $= \Delta$

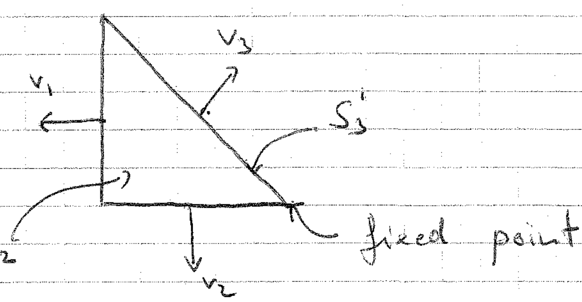
<u>Note</u> : $S^2 \times S^2$ is a fibration of T^2 over Δ, in the interior of Δ, T^2 is everywhere non-zero; over the facets different $U(1) = S^1$ shrink; over the vertices all $U(1)$'s shrink to zero (these are the fixed points by construction).

Each facet in general corresponds to codimension one symplectic submanifolds - in this example it's easy to see that they are just the first S^2 at the N and S poles of the second, and vice-versa.

a-th facet $\leftrightarrow$ v_a $\leftrightarrow$ $\langle v_a, \mu \rangle = b$ symplectic submanifold

As an exercise, the interested person can show that the image of the moment map (moment polytope) of $\mathbb{CP}^2$ is a triangle in $\mathbb{R}^2$

$\mu(\mathbb{CP}^2) =$



• Note: if our symplectic manifold admits an (integrable) complex structure $J_i^{\ j}$, then the symplectic form ω becomes also a Kähler form.

The facets then correspond to Kähler submanifolds. These are calibrated by

$$\sigma = \frac{1}{(n-1)!} \omega^{n-1}$$

recall 1) $\sigma \leq \mathrm{vol}_\xi$ $\forall$ tangent plane $\xi \subset TM$

2) $\sigma = \mathrm{vol}(\Sigma)$ $\Rightarrow$ Σ is calibrated

susy submanifolds!

So, given a toric manifold, we now know how to construct its moment polytope. To explain the converse, I should first explain

symplectic (or Kähler) quotients

(in the physics literature these are also referred to as Gauged Linear Sigma Model construction - Witten)

Start with $\mathbb{C}^d$ with coordinates $Z_1, .., Z_d$

a $U(1)^r$ action on $\mathbb{C}^d$ is specified by an integral charge matrix

$$Q_a^{\ i} \qquad \begin{array}{l} a = 1, .., d \\ i = 1, .., r \end{array}$$

acting as

$$(Z_1, .., Z_d) \sim (e^{i\theta Q_1^i} Z_1, .., e^{i\theta Q_d^i} Z_d) \qquad i = 1, .., r$$

(this is imposing gauge equivalences)

The symplectic (Kähler) quotient $X = \mathbb{C}^d // U(1)^r$ is obtained by imposing the constraints

$$\sum_{a=1}^{d} Q_a^{\ i} |Z_a|^2 = t_i$$

("FI parameters")

(these are the "D-terms" - we shall come back later during the next lectures, to this nomenclature

- The resulting space X has $\dim_{\mathbb{C}}(X) = d - r$
- It is naturally <u>Kähler</u>, with Kähler form induced from the canonical one on $\mathbb{C}^d$
- $\sum_{a=1}^{d} Q_a^i = 0 \quad \forall i \Rightarrow c_1(X) = 0$ i.e. X is "topologically Calabi-Yau" (this is a necessary condition for X to admit a Ricci-flat metric. That would be also sufficient if X was compact, which it isn't).

[One way to understand this condition is as follows: on $\mathbb{C}^d$ there is a nowhere vanishing $(d,0)$-form

$$\Omega = dZ_1 \wedge \dots \wedge dZ_d \xrightarrow[U(1)_a]{} e^{i\theta \sum_{a=1}^{d} Q_a^i} dZ_1 \wedge \dots \wedge dZ_d = \Omega$$

invariant under $U(1)^r$

contracting this with the r vector fields $X_i = Q_a^i \frac{\partial}{\partial Z_a}$ gives an everywhere non-zero holomorphic $(d-r,0)$-form on the quotient space. E.g. consider $\mathcal{L}_{X_a}\Omega = 0 = d(\underbrace{i_{X_a}\Omega}_{\Omega_r}) + i_{X_a} d\Omega$

This is a global section of $\Lambda^{n,0}(X)$, hence it trivialises the canonical bundle.]

- $t_a = 0 \Rightarrow X$ is a cone as $Z_a \to c Z_a$ is a symmetry
- Example: $\mathbb{CP}^n = \mathbb{C}^{n+1} // U(1) \qquad Q = (1, \dots, 1)$

$$\underbrace{|Z_1|^2 + \dots + |Z_{n+1}|^2 = t}_{S^{2n+1}} \qquad (Z_1, \dots, Z_d) \sim e^{i\theta}(Z_1, \dots, Z_d)$$

$$S^{2n+1}/U(1) = \mathbb{CP}^n$$

Alternatively, this can be understood from the equivalence with holomorphic quotient $\mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^* : \{Z_1, \dots, Z_{n+1} \mid (Z_1, \dots, Z_d) \sim (\lambda Z_1, \dots, \lambda Z_d), \lambda \in \mathbb{C}$

We are now ready to address the converse part of the equivalence of toric symplectic manifolds and polytopes. Recall: toric manifold $\xrightarrow{\text{moment map}}$ polytope.

Now, start with a polytope Δ in $\mathbb{R}^n$ with d facets specified by the primitive normals $\{v_a\}$. Consider the map

$$\mathbb{Z}^d \to \mathbb{Z}^n$$
$$\pi: e_a \to v_a \qquad e_1 = (1, 0, \dots, 0), \text{ etc.}$$

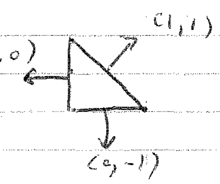
which can be conveniently represented as a $n \times d$ matrix, whose columns are the v_a. The kernel of this map is given by $d-n$ row vectors in $\mathbb{Z}^d$. The matrix assembled with these vectors

$$\{Q_a^{\ i}\} = \ker \pi$$

is then used as charge matrix for a symplectic (or Kähler) quotient $X = \mathbb{C}^d /\!/ U(1)^{d-n}$.
Then we have the non-trivial result (Delzant) that

$$\mu(X) = \Delta$$

- Example: $\mathbb{CP}^2$

Remember: $\Delta =$ 

$$\pi = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \ker(\pi) = Q = (1, 1, 1)$$

$$\mathbb{C}^3 /\!/ U(1)_{(1,1,1)} = \mathbb{CP}^2$$ by the example we just did

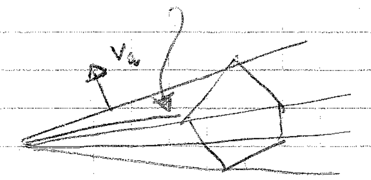
Toric Calabi-Yau singularities

Essentially, the main ideas of the discussion so far extend to non-compact toric varieties, and associated toric singularities.

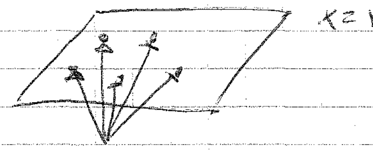
Instead of compact polytopes, we now deal with polyhedral cones, specified by a number of normal vectors to the facets.

The CY condition $c_1(X)=0 \Rightarrow \sum Q_a^i = 0$ for the charges of the Kähler quotient $\Rightarrow$ by an $SL(3,\mathbb{Z})$ transformation the normal vectors can be written

toric divisor $\qquad v_a = (1, w_a) \qquad w_a \in \mathbb{Z}^2$

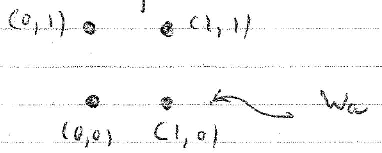


project the normals on the $x=1$ plane



The collection of $w_a \in \mathbb{Z}^2$ is called toric diagram; it contains the full information about the toric singularity

Example: conifold has toric diagram



$$\Rightarrow \Pi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad Ker(\Pi) = (-1, +1, -1, 1)$$

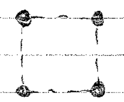
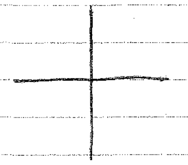
$$\Rightarrow \text{conifold} = \mathbb{C}^4 /\!/ U(1)_{(-1,1,-1,1)} \qquad |Z_1|^2 + |Z_3|^2 - |Z_2|^2 - |Z_4|^2 = 0$$

$$(Z_1, Z_2, Z_3, Z_4) \sim (e^{-i\theta} Z_1, e^{i\theta} Z_2, e^{-i\theta} Z_3, e \ldots$$

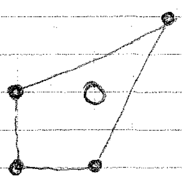
What about the other incarnation of the conifold $(XY=ZW)$? It might be derived easily from here, but I'd like to ... general context: affine toric varieties.

There is a dual version of the toric diagram, that in the physics literature is called pq-web (because sometimes it can be related to webs of (p,q)-fivebranes).

Operationally: points $\to$ faces
edges $\to$ edges

E.g.  $\longrightarrow$ 

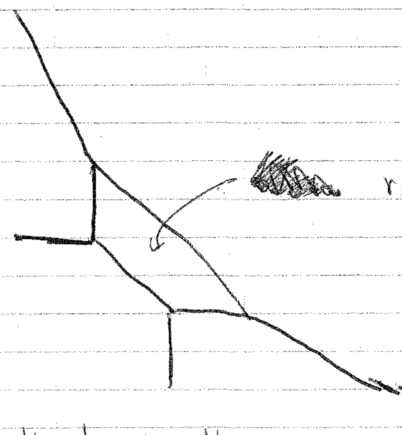
In general, toric diagrams can have internal points

E.g.  these correspond to blow-ups, or (toric) resolutions of the singularity

$\downarrow$

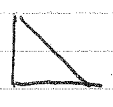
$\mathbb{C}^5 /\!/ U(1)^2$ for generic non-zero FI the singularity is resolved

the pq-web is:



from here it's clear that the pq-web is nothing but the projection on the $x=1$ plane of the polyhedral cone C living in $\mathbb{R}^3$. Thus resolving $\overset{\text{or blowing-up}}{\overbrace{\text{is}}}$ equivalent to "chopping off" the apex of the polyhedral cone

[In fact the same is true for compact polytopes,

e.g. $\mathbb{CP}^2$  $\longrightarrow$  $\mathbb{F}_1 = dP_1$ that is $\mathbb{CP}^2$ with a point replaced by $\mathbb{CP}^1$

Algebraic geometry point of view

So far we have looked at toric manifolds and non-compact CY's from the point of view of the polytopes / polyhedral cones. In particular we have introduced toric diagrams and the Kähler quotient, emphasising the variables $Z_A \in \mathbb{C}^d$ each defining a toric divisor. We will see that these variables are closely related to bi-fundamental fields in the dual gauge theory. To introduce another set of variables, related to gauge invariant operators, we will have to look at toric varieties in the context of algebraic geometry.

An affine variety is defined by polynomial equations

$$\{f_1 = \cdots = f_s = 0\} \subset \mathbb{C}^N$$

$(\mathbb{C}^*)^N$ = N-dimensional algebraic (or complex) torus

A toric variety X of dimension N is defined by

1) $(\mathbb{C}^*)^N$ is a "Zarinski open subset" of X - this means that $(\mathbb{C}^*)^N = X \setminus W$ for some $W \subset X$
$$X \setminus W = \{x \in X \mid x \notin W\}$$

2) the action of $(\mathbb{C}^*)^N$ on itself extends to an action of $(\mathbb{C}^*)^N$ on X

E.g. $\mathbb{CP}^N$ 1) $(\mathbb{C}^*)^N = \mathbb{CP}^N \setminus \{z_1 \cdots z_{N+1} = 0\}$

2) $(\mathbb{C}^*)^N : \mathbb{CP}^N \to \mathbb{CP}^N$

$$(q_1, \dots, q_N) \cdot (z_1, \dots, z_{N+1}) \to (z_1 q_1, \dots, z_N q_N, z_{N+1})$$

Now, let's go back to our polyhedral cones $C^\vee$ (here $C^\vee$ denotes the dual cone of C, which here denotes the cone generated by the v_a).

Given $C^\vee \subset \mathbb{R}^n$, we can consider the associate

Abelian semigroup $S_C = C^\vee \cap \mathbb{Z}^n$

this is finitely generated (Gordan's lemma) that is, there exists a finite number of generators

$m_1, \ldots, m_N \in S_C$ such that every

element of S_C is of the form

$$a_1 m_1 + \cdots + a_N m_N \qquad a_i \in \mathbb{N}$$

The semi-group S_C is nothing else than the set of ~~[illegible]~~ integral points inside $C^\vee$ with group operation being the addition $m + m'$.

Now, introduce the complex variables $w \in (\mathbb{C}^*)^n$ and consider the "characters"

$$w^m : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$$

$$w^m = \prod_{i=1}^n w_i^{m_i}$$

with multiplication rule $\qquad w^m \cdot w^{m'} = w^{m+m'}$

this forms a semi-group algebra $\mathbb{C}[S_C]$ generated by the elements w^{m_a} such that m_a is generator of S.

This is a coordinate ring since you can add and multiply these object.

But coordinate ring of what?

Precisely $\mathbb{C}[S_C]$ is the coordinate ring of toric variety

$$X_C = \mathrm{Spec}_m\, \mathbb{C}[S_C]$$

that is the "maximal spectrum" of $\mathbb{C}[S_C]$

[$\mathrm{Spec}_m A = \{\text{maximal ideals of } A\}$

An ideal I is a subset of a ring $R\ (+, \times)$:

1) $(I, +)$ is a subgroup of $(R, +)$
2) $i \cdot r \in I \quad \forall\ i \in I$ and $r \in R$.

Note, if A is any subset of R then we can define the ideal generated by A as the smallest ideal of R containing A : $\langle A \rangle$.]

In general, there exist suitable binomial functions $f_i \subset \mathbb{C}^N$ such that

"toric ideal"

$$\mathbb{C}[S_C] = \mathbb{C}[z_1, \dots, z_N] / \langle f_1, \dots, f_s \rangle$$

and $\quad X_C = \{ f_1 = 0, \dots, f_s = 0 \} \subset \mathbb{C}^N$

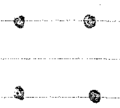
is an affine toric variety.

There is a ring homomorphism : $\mathbb{C}[z_1, \dots, z_N] \to \mathbb{C}[S_C]$ defined by

$$z_i \to w^{m_i} \qquad m_i \text{ generators of } C^\vee$$

- X_C is the smallest variety in which the w^m are defined everywhere.

As a simple example of this construction, let's look again at the conifold. Start with the toric diagram (say):



$$w_a \longrightarrow \text{cone } C\ (v_a) \longrightarrow \text{dual cone } C^\vee$$

generators of $C^\vee$:
$$m_1 = (0,1,0)$$
$$m_2 = (1,0,-1)$$
$$m_3 = (1,-1,0)$$
$$m_4 = (0,0,1)$$

$$m_1 + m_3 = m_2 + m_4 \quad \Rightarrow \quad XY = ZW$$

$$\Rightarrow \quad \text{conifold} = \{XY - ZW = 0\} \subset \mathbb{C}^4$$

ring of holomorphic functions on the conifold is

$$\mathbb{C}[X,Y,Z,W] / \langle XY - ZW \rangle$$

- The bottom line is that integral points inside $C^\vee$ correspond one-to-one to holomorphic functions on the affine toric variety.

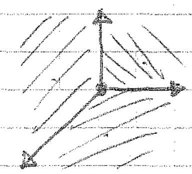
One can consider more general toric varieties by gluing together affine toric varieties (associated to cones). To do this one introduces the concept of fan: a finite collection of cones.

Of course the simplest example of fan is obtained from a cone C, by considering all its faces. The associated toric variety is clearly X_C.

We don't really need fans usually (we always deal with affine variety). However the language of fans lets us also introduce the concept of holomorphic quotient which is related to the Kähler quotient we already discussed.

Instead of giving more definitions, I will be content with discussing an example, namely $\mathbb{CP}^2$

fan

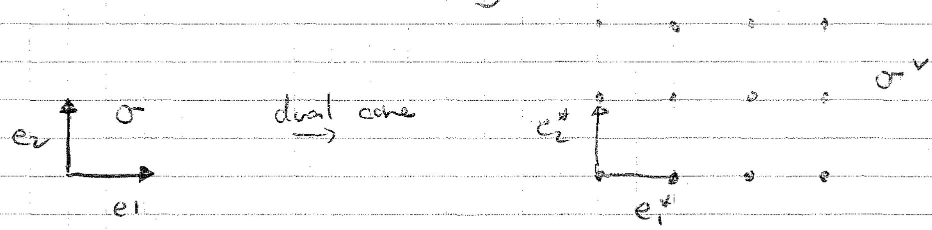


(there are seven cones)

[Notice this fan comes from a polytope. This is indeed a general construction]

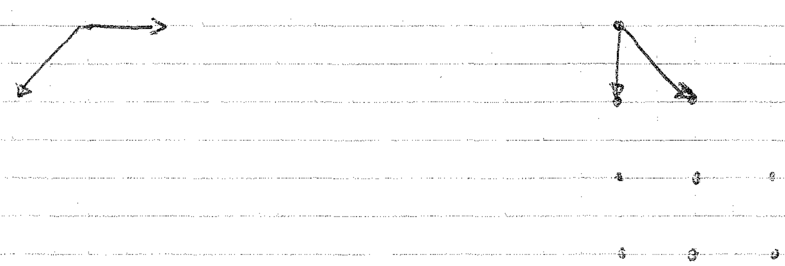
Each 2d cone is locally $\mathbb{C}^2$.

E.g.



$$S_\sigma = \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{Z}^2 \quad \Rightarrow \quad \mathbb{C}[S_\sigma] = \mathbb{C}[X, Y]$$

$$(X, Y) = \left(\frac{w_1}{w_0}, \frac{w_2}{w_0}\right)$$



$$\mathbb{C}[S_\sigma] = \mathbb{C}[Y^{-1}, XY^{-1}] \qquad (Y^{-1}, XY^{-1}) = \left(\frac{w_0}{w_2}, \frac{w_1}{w_2}\right)$$

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X^{-1}, X^{-1}Y] \qquad (X^{-1}, X^{-1}Y) = \left(\frac{w_0}{w_1}, \frac{w_2}{w_1}\right)$$

We see that (w_0, w_1, w_2) can be used as homogeneous coordinates on $\mathbb{CP}^2$.

Any toric variety X_Σ associated to a fan Σ can be represented as a holomorphic quotient. This is a natural generalization of

$$\mathbb{CP}^n \simeq \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*$$

In particular there are suitable d and r such that

$$X_\Sigma = \left(\mathbb{C}^d \setminus F\right) / (\mathbb{C}^*)^r$$

- d is the number of one-dimensional cones in Σ (so if Σ is the fan coming from a single cone, these are the rays generating it, in other words are the normals v_a)
- F is a subset of $\mathbb{C}^d$ that has to be subtracted in order for the quotient to be well defined
- introducing homogeneous coordinates $Z_1, .., Z_d \in \mathbb{C}^d$, the action of $(\mathbb{C}^*)^r$ is

$$(Z_1, .., Z_d) \sim (\lambda^{Q_1^i} Z_1, .., \lambda^{Q_d^i} Z_d) \quad i = 1, .., r$$

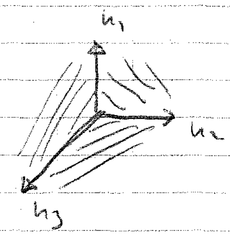
(as in the symplectic quotient)

- $\mathbb{C}[Z_1, .., Z_d]$ is called the homogeneous coordinate ring. It receives a grading by the charges Q^i.

There are various characterizations of F. One (due to Batyrev) is the following:
let $u_i \in \mathbb{Z}^n$ denote the (primitive) edge generators of the one-dimensional cones. A set $\{u_{i_1}, \dots, u_{i_s}\}$ is said primitive if the u_i don't lie in any cone of Σ, but every proper subset does. Then:

$$F = \bigcup_{\{u_{i_1}, \dots, u_{i_s}\} \text{ primitive}} \{Z_{i_1} = 0, \dots, Z_{i_s} = 0\}$$

Going back to our $\mathbb{CP}^2$ example



here the only primitive set is $\{u_1, u_2, u_3\}$

$$\Rightarrow \quad F = \{Z_1 = 0, Z_2 = 0, Z_3 = 0\} \qquad \Rightarrow \quad \mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{0\}) / \mathbb{C}^*$$

This generalises trivially to $\mathbb{CP}^n$. However, in general the subspace F is more complicated.

Sasaki - Einstein geometry

Recall that our original motivation for studying Sasaki-Einstein geometry came from the fact that $AdS_5 \times L$ with L a SE-five manifold is a supersymmetric solution of Type IIB supergravity.

In fact the physical origin, as the "near-horizon" geometry of many D3-branes placed at a CY singularity gives also a precise mathematical definition, i.e. we can regard SE as the "link" of a CY cone.
For practical purposes, we can write

$$ds^2(CY) = dr^2 + r^2 ds^2(L)$$

so that $L = CY \cap \{r=1\}$.

For later purposes, we will relax the "Einstein" condition, and consider more in general the geometry of Sasakian links. **Note**: $AdS_5 \times \underset{\text{Sasakian}}{L}$ is not a solution of Type IIB supergravity!

In general, a Sasakian link can be defined as

$$L = \{\text{Kähler cone}\} \cap \{r=1\}$$

Then L is Einstein $\Leftrightarrow$ cone is Ricci-flat

In order that this can be possible, we will see that we have to require some topological properties of the Kähler cone (i.e. $c_1 = 0$).

Kähler cone:

- it's complex : $\nabla J = 0$
- Kähler : $d\omega = 0$

$\Big\}$ $J^i{}_k\, \omega_{ij} = g_{kj}$ Hermitian metric

- "cone" : $ds^2 = dr^2 + r^2 ds^2(L)$

$\Rightarrow$ there is a homotetic vector (Euler) : $r\frac{\partial}{\partial r}$

We can define the Reeb vector field (this will play a crucial role)

Reeb $\xi = J \cdot \left(r \frac{\partial}{\partial r} \right)$

1) ξ, $r\frac{\partial}{\partial r}$ are holomorphic : $\mathcal{L}_\xi J = \mathcal{L}_{r\frac{\partial}{\partial r}} J = 0$

2) ξ is a Killing vector

<u>Note</u>:

$$\begin{tikzcd}
\xi \arrow[r, leftrightarrow, "\text{complex structure}"] \arrow[d, leftrightarrow] & r\frac{\partial}{\partial r} \arrow[d, leftrightarrow] \\
\text{R-symmetry} \arrow[r, leftrightarrow, "\text{supersymmetry}"'] & \text{dilations}
\end{tikzcd}$$

We can introduce the dual one-form η ("contact 1-form") as

$$\eta = \frac{1}{r^2}\, g(\xi, \cdot) \qquad \left(\text{in components } \eta_\mu = \frac{1}{r^2} g_{\mu\nu} \xi^\nu \right)$$

$$\eta = J\left(\frac{dr}{r}\right) = i(\bar\partial - \partial) \log r$$

The Reeb vector field, ore equivalently, its dual contact one-form allows to introduce a natural split of the Sasakian structure.

$$ds^2_6 = dr^2 + r^2 (ds^2_T + \eta \otimes \eta)$$

$\downarrow$

this is a (transverse) Kähler metric

Moreover, its Kähler form is given by

$$\omega_T = \frac{1}{2} d\eta = i \partial \bar{\partial} \log r$$

the Kähler form on the Kähler cone is instead

$$\omega = r^2 \omega_T + r dr \wedge \eta = \frac{1}{2} d(r^2 \eta)$$

$$\Rightarrow \omega = \frac{1}{2} i \partial \bar{\partial} r^2$$

Note that (ds^2_T, ω_T) is only a "transvere structure" that is, it is not in general a manifold, nor an orbifold. We will come back to this shortly.

Note: $\frac{1}{2} r^2$ is the Kähler potential

(it is a non-trivial function of the complex coordinates!).

Similarly, $\log r$ is a Kähler potential for the transverse Kähler structure.

Finally: ds^2_T Einstein $\Leftrightarrow$ $ds^2(L)$ Einstein

We will also require that the Kähler cone be topologically Calabi-Yau, such that it can admit a Ricci-flat metric.
We do this by requiring that there is a nowhere vanishing holomorphic $(n,0)$ form

$$\Omega \in \Lambda^{n,0}(M)$$

$$d\Omega = 0 \quad \Rightarrow \quad J \text{ is integrable}$$

Note that if M was compact, Ω would be unique up to a constant multiple.
Here instead, $\alpha\Omega$, α holomorphic and nowhere vanishing is admissible.
We fix this additional freedom by requiring that

$$\mathcal{L}_{r\frac{\partial}{\partial r}}\Omega = n\Omega \qquad \left(\mathcal{L}_\xi \Omega = in\Omega \rightarrow \Omega \text{ has charge } n \text{ under the Reeb} \right).$$

This is certainly true for a Calabi-Yau cone metric, and it can be shown that it can be always achieved for any Kähler cone metric.

[In fact, one can show that $\mathcal{L}_{r\frac{\partial}{\partial r}}\Omega = \gamma\Omega$ can always be achieved, and then $\gamma = n$ follows from the variational problem that we will discuss.]

Now Ω is unique up to a constant multiple, and we have, for a suitable real function f:

$$\frac{i^n}{2^n}(-1)^{n(n-1)/2}\,\Omega\wedge\bar{\Omega} = e^f\,\frac{\omega^n}{n!}$$

Now, we can classify Sasakian (-Einstein) structure according to the properties of the Reeb vector ξ, or equivalently, the transverse Kähler structure. [Since ξ is nowhere-vanishing, namely ~~unit~~-norm, its orbits define a "foliation" of L, i.e. a local product structure].

• REGULAR: ξ has closed orbits $\to$ generates $U(1)$ action on L. This is free [i.e. for any two different $g, h \in U(1)$, $g(x) \neq h(x)$ $\forall x \in L$]. Then $L/U(1) = V$ is a manifold and

$$\begin{array}{c} U(1) \to L \\ \downarrow \\ V \end{array}$$

is a principle circle bundle.

• QUASI-REGULAR: orbits still close, but $U(1)$ action is locally free [the isotropy (stabiliser) group of at least one point x is $\Gamma_x = \mathbb{Z}_m \subset U(1)$; $\Gamma_x = \{g \in U(1) \mid g(x) = x\}$]

Then $L/U(1) = V$ is an orbifold and

$$\begin{array}{c} U(1) \to L \\ \downarrow \\ V \end{array}$$

is a principle circle (orbi)-bundle.

• IRREGULAR: orbits of ξ don't close. "$L/U(1)$" doesn't exist. [Orbits of $\xi \sim \mathbb{R}$. Closure of orbits $\sim T^r \subset$ Isom(L) because L is compact. $r = \text{rank}(L)$]

In five dimensions, regular Sasaki-Einstein are a small number: S^5, $T^{1,1}$, $U(1)$ bundles over dP_k $3 \leq k \leq 8$. The explicit metrics are known only for S^5, $T^{1,1}$.

For irregular, there are "implicit" (i.e. existence arguments) metrics due to Boyer + Galicki + et al. For irregular, $Y^{p,q}$ where the first examples. Also disprove a conjecture! Now there are a few more examples in $d \geq 5$, all coming from physics

Let's now discuss Killing spinors and their uses.
Recall that, after all, the relevance of SE arises because they preserve supersymmetry in Type IIB, after reducing the killing spinor equations on AdS_5, with only $F_5 \neq 0$, we have

$$\nabla_\mu \theta = \frac{i}{2}\gamma_\mu \theta \quad \text{on SE} \quad \text{, i.e. } \mu = 1,2,3,4,5$$

Since we are studying the more general class of Sasakian manifolds, let's step back and look at spinors on them.
On a Kähler manifold we always have "spinors" Ψ satisfying

$$\nabla_M \Psi - \frac{i}{2} A_M \Psi = 0$$

[" " is due to the fact that Ψ are sections of the spin^c bundle $\mathcal{V} = S \otimes K^{-1/2}$

complex spinor bundle; $K = \Lambda^{n,0}(M)$ canonical line bundle

in fact $\mathcal{V} \simeq \Lambda^{0,*}(M) \simeq \mathcal{V}^+ \oplus \mathcal{V}^- \simeq \Lambda^{0,\text{even}}(M) \oplus \Lambda^{0,\text{odd}}(M)$

$-\frac{1}{2}A$ is the connection one-form on $K^{1/2}$, since

A is the connection one-form on K : $dA = \rho$

Ricci-form

$c_1 = \frac{[\rho]}{2\pi}$

By "reducing" along dr, we get the equation obeyed by spinors on Sasakian links

$\theta = \Psi|_{r=1}$
$$\nabla_\mu \theta - \frac{i}{2}\gamma_\mu \theta - \frac{1}{2} A_\mu \theta = 0$$

These killing spinors can be used to construct differential forms as bi-linears. In particular

$\eta_\mu = \bar{\theta}\gamma_\mu\theta$ is the contact one-form

then one can define the Reeb as $\xi^\mu = g^{\mu\nu}_L \eta_\nu$

$\nabla^L_{(\mu}\eta_{\nu)} = 0 \Rightarrow \xi^\mu$ is Killing

also $d\eta = -2i\bar{\theta}\gamma_{(2)}\theta = 2\omega_T$ ↳ transverse kähler form

and finally we can define an $(n,0)$-form on M

$$K = \bar{\Psi}^c \gamma_{(n)} \Psi \quad \rightarrow \quad dK = iA \wedge K$$

Now $K = e^{-f/2}\Omega \qquad (\rho = i\partial\bar{\partial} f) \qquad A = \frac{1}{2}d^c f$

$$\Rightarrow \quad d\Omega = \frac{1}{2}(df + d^c f)\wedge\Omega = \partial f \wedge \Omega = 0 .$$

Before making finally contact with the toric geometry that I discussed previously, let me introduce another crucial object, that is "supersymmetric sub-manifold". In fact, these play an important role in the relation to the gauge theory, as we shall see.

These are submanifolds on which D3 branes can be wrapped, while preserving supersymmetry, which means that the killing spinors of the background solution are compatible with the K-symmetry of a D3 brane

$\Gamma_K \epsilon = \epsilon \qquad \Gamma_K \sim \Gamma_{0123}$ is a projector

A more intrinsically geometric characterization of supersymmetric submanifolds can be given in terms of divisors of the Kähler cone. Recall these are complex co-dimension one submanifolds that can be characterised in terms of the kähler form ω

$$\omega = \frac{1}{2} r^2 d\eta + r dr \wedge \eta$$

by considering the calibrating form $\sigma = \frac{1}{(n-1)!}\omega^{n-1}$

calibration $\begin{cases} 1) & \sigma \leq \mathrm{vol}|_\xi \quad \forall \text{ tangent plane } \xi \subset TM \\ 2 & \sigma = \mathrm{vol}(D) \Rightarrow D \text{ is calibrated} \end{cases}$

Focus in $n=3$. The divisors are $\dim_{\mathbb{C}} = 2$ submanifolds calibrated by $\frac{1}{2}\omega\wedge\omega = \frac{1}{2}\left(\frac{1}{4} r^4 d\eta\wedge d\eta + r^3 dr\wedge\eta\wedge d\eta\right)$.
These are cones ove real $\dim = 3$ submanifolds:

$$D_i = C(\underbrace{\Sigma_i}_{\text{supersymmetric 3-submanifolds}})$$

Note: the number of these submanifolds can be arbitrarily large in a Sasakian manifolds with fixed topology. In particular (assuming L is simply connected), in $n=3$, if

$$H_3(L;\mathbb{Z}) = \mathbb{Z}^{b_3} \qquad \Rightarrow \qquad L \simeq b_3 \# (S^2\times S^3)$$

(Smale's theorem)

\# divisors / susy submanifolds is typycally larger than b_3 [in fact, one can show with a calculation in equivarian cohomology, that $d = b_3 + r$]. We'll see an example in a second, in the toric context.

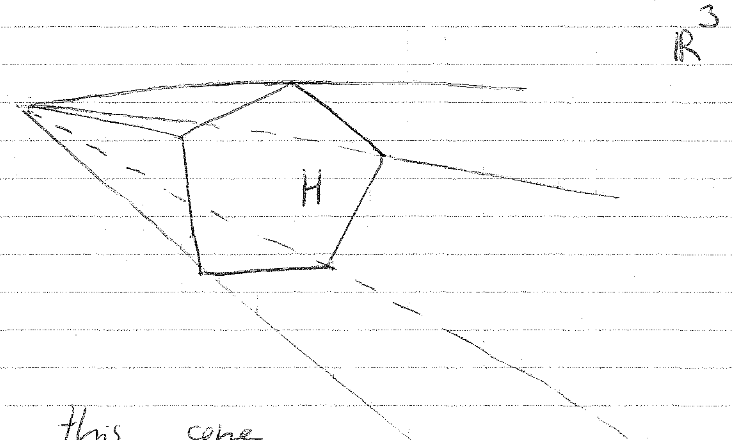
If the Kähler cone is toric $\Rightarrow$ Sasakian L is toric.

We can take this as a the definition of "toric Sasakian".
Of course if the cone is CY $\Rightarrow$ toric SE manifold.

In particular, the $U(1)^3$ symmetry descends to the Sasakian link, and the Reeb

$$\xi = \sum_{i=1}^{3} b_i \frac{\partial}{\partial \phi_i} \in U(1)^3$$

Recall there is a (strictly convex, rational) polyhedral cone C which is the image under its moment map of the Kähler cone:



The equation $\{r=1\}$ cuts this cone with a hyperplane H to form a finite polytope.
The Sasakian link can then be viewed as

$$\begin{array}{ccc} U(1)^3 & \to & L \\ & & \downarrow \\ & & H \end{array}$$

remember: facets $\leftrightarrow$ divisors $\longrightarrow$ susy submanifolds Σ.
These are all

$$\begin{array}{ccl} T^2 \to & \Sigma_i & \\ & \downarrow & \\ & I_i & \to \text{ ith edge of } H \end{array}$$

and hence are all Lens spaces, with topology $S^3/\mathbb{Z}_m$.
We will compute m shortly.

I will briefly recall some basic facts about $N=1$ supersymmetric gauge theories and in particular quiver gauge theories.

Field content:

- vector (Yang-Mills) multiplet: A_μ, λ, D
- chiral (matter) multiplets: X_i, ψ_i, F_i

Recall, it's convenient to use superspace notation:

x^μ, θ^α ↳ Grassmann coordinates (anti-commuting)

superfield $Y(x, \theta, \bar\theta)$

- chiral superfield: $\bar D_{\dot\alpha} \Phi = 0$ $\qquad \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + \frac{i}{2}\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

$$\Downarrow \qquad \{Q_\alpha, \bar Q_{\dot\alpha}\} = -\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$\Phi = X + \theta\psi + \theta^2 F$$

- vector superfield: (in the Wess-Zumino gauge)

$$V^a = -\theta\sigma^\mu\bar\theta A^a_\mu + i\sqrt{2}\,\theta^2\bar\theta\bar\lambda^a - i\sqrt{2}\,\bar\theta^2\theta\lambda^a + \theta^2\bar\theta^2 D^a$$

→ field strenght super field: (chiral) $\qquad W_\alpha = \bar D^2\left(e^{-V}(D_\alpha e^V)\right)$

[in WZ gauge: $W^a_\alpha = -i\sqrt{2}\lambda^a_\alpha + \theta_\alpha D^a - i\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta F^a_{\mu\nu} + \sqrt{2}\theta^2\sigma^\mu_{\alpha\dot\beta}D_\mu\bar\lambda^{\dot\beta}$]

- The chiral (super) fields also transform in some representation of the gauge group G.

The complete Lagrangian is then:

$$S = \int_{\text{chiral}} \mathrm{tr}(W^\alpha W_\alpha) + \int_{\text{chiral}} W(\Phi) + \text{complex conjugate}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad$$

$$d^4x\, d^2\theta \qquad\qquad + \int_{\text{non-chiral}} \bar{\Phi} e^V \Phi$$

$$\downarrow$$

$$d^4x\, d^2\theta\, d^2\bar{\theta}$$

$W(\Phi)$ is the superpotential $\rightarrow$ must be holomorphic.

Expanding this in components, in the WZ gauge

$$L = \mathrm{tr}\left\{ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + D_\mu \bar{X}_i D^\mu X_i + \text{fermions} + V \right\}$$

The potential V contains two positive-definite terms:

$$V = \left(\frac{1}{2g^2}\right) D^a D_a + \bar{F}^i F_i \qquad \bar{F}_i = -\frac{\partial W}{\partial X_i}$$

$$D^a = -g^2 \bar{X}^i T^a X_i$$

(classical) supersymmetric vacua $\Rightarrow$ $V = 0$

$$\Rightarrow \begin{cases} D\text{- flatness conditions} \\ F\text{- flatness conditions} \end{cases}$$

Solutions to these equations, up to gauge transformations $\Rightarrow$ classical moduli space of vacu

Result: classical moduli space is always an algebraic variety, parametrized by the set of gauge-invariant, holomorphic polynomials

Now, we define a quiver gauge theory:

- gauge group $G = \underbrace{U(N_1) \times \cdots \times U(N_\chi)}_{\chi \text{ times}}$

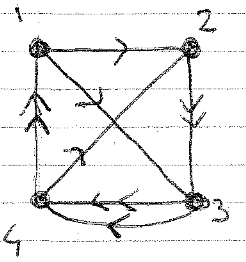
- chiral fields $X_i \to$ bi-fundamental reps of G

$\Rightarrow$ encode this information into a "quiver diagram"

$\bullet = U(N_i)$ gauge group

$i \longrightarrow j$ = bi-fundamental $X_{ij} \in (N_i, \bar{N}_j)$ of $U_i(N_i) \times U_j(N_j)$

E.g.



here X_{41}^α, X_{23}^α, X_{34}^α transform as doublets of a flavour $SU(2)_F$ group

Note:

- at each node the number of arrows outgoing and incoming is the same. This is required by gauge anomaly cancellation

- a necessary condition for the moduli space of vacua to be a toric variety is that all gauge groups have same rank $N_i = N$.

One must also specify a superpotential W!

[In the example this is

$$W = \mathrm{tr}\left[\epsilon_{\alpha\beta} X_{34}^{\alpha} X_{41}^{\beta} X_{13} - \epsilon_{\alpha\beta} X_{34}^{\alpha} X_{23}^{\beta} X_{42} + \epsilon_{\alpha\beta} X_{12} X_{34}^{3} X_{41}^{\alpha} X_{23}^{\beta} \right]$$]

- There is another condition that one has to impose for the moduli space to be toric: each field enters in W once with + sign and once with −

We will consider only such cases from now on.

One can construct two basic types of operators:

- (mesonic) gauge invariant operators by multiplying bi-fundamentals viewed as $N\times N$ matrices, and tracing at the end: $O = \mathrm{tr}(X_{12} X_{23} \cdots X_{n1})$

$$\updownarrow$$

loops in the quiver

- di-baryonic operators, by taking $N\times N$ determinants

$$B[X] = \epsilon^{\alpha_1 \cdots \alpha_N} \epsilon_{\beta_1 \cdots \beta_N} X_{\alpha_1}{}^{\beta_1} \cdots X_{\alpha_N}{}^{\beta_N}$$

The set of GIO operators, modulo F-term relations defines the chiral ring of the theory.

[We will see an example of this later, maybe.]

The moduli space of toric quiver gauge theory is captured by Higgsing $U(N) \to U(1)^N$ each gauge group factor. One then considers $N=1$, with $N>1$ giving N (symmetrised) copies of the $N=1$ case.

$\Rightarrow U(1)^{\chi}$ theory

with charges summarised by the incidence matrix

$$I = \begin{array}{c} \\ \text{gauge groups} \end{array} \left\{ \begin{array}{c|cccc} & \multicolumn{4}{c}{\text{bi-fundamentals}} \\ & X_{12} & X_{13} & X_{42} & \cdots \\ 1 & -1 & -1 & 0 & \cdots \\ 2 & 1 & 0 & 1 & \cdots \\ 3 & 0 & 1 & 0 & \cdots \\ 4 & 0 & 0 & -1 & \cdots \end{array} \right) \quad \leftarrow \text{each row sums to zero (arrow conservation)}$$

each column contains a $+1$, a -1 and zeros

Classical moduli space:

$$\begin{cases} \text{D-terms} & I \cdot |\vec{X}|^2 = 0 \quad (\text{or FI terms}) \\ \text{F-terms} & dW = 0 \\ \text{modulo gauge transformations.} \end{cases}$$

(Equivalently, one can consider complex gauge transformations, and no D-terms $\to$ algebraic variety. Then $dW=0$ cuts an hypersurface in it, which is still an algebraic affine variety)

The procedure for finding the resulting moduli space has been algorithmised (Forward Algorithm).
It involves manipulating (large) matrices, and per se, it's not very illuminating to describe in detail.

The main idea is that F-terms can be reduced to D-terms type of relations, by introducing some auxiliary variables p_α $\alpha = 1, \dots, c$ with c a number to be determined in the algorithm.

Then $\quad p_\alpha \in \mathbb{C}^c$

$$\Rightarrow \qquad \mathbb{C}^c // U(1)^{c-3}$$

is a Kähler quotient which gives as a result a $\dim_{\mathbb{C}} = 3$ affine toric variety.

- The $U(1)^{c-3}$ charges sum up to zero $\to$ the quotient is in fact Calabi-Yau (check)

- From the point of view of AdS/CFT, the quiver gauge theory can flow to a non-trivial IR fixed point only if there exist a Sasaki-Einstein metric on the link! In fact, the existence of these metrics is still an open problem in general.

Superconformal field theories and a-maximisation

So far I have only talked about supersymmetric field theories, however theories with $AdS_5 \times SE$ dual must be also conformal field theories.

Microscopic (Lagrangian) description in the UV
[Yang-Mills fields, chiral fields, et]

$\downarrow$

In the IR the theory flows to a conformal fixed point of the renormalization group.
$\Rightarrow$ Effective theory
$\rightarrow$ superconformal group and algebra:

- conformal algebra: $\underbrace{M_{\mu\nu},\ P_\mu}_{\text{Poincaré}},\ D,\ K_\mu$

$D \rightarrow$ dilations $x^\mu \to \lambda x^\mu$

$K_\mu \rightarrow$ "special conformal" transformations $x^\mu \to \dfrac{x^\mu + a^\mu x^2}{1+2x^\nu a_\nu + a^2 x^2}$

algebra isomorphic to $SO(4,2)$
$\downarrow$
isom of AdS_5

- susy extension : $Q,\ \bar{Q},\ S,\ \bar{S},\ R$

$R \rightarrow$ R-symmetry

$[K, Q] \sim S$ $\qquad$ $\{Q, S\} \sim M + D + R$

in $\mathcal{N}=1$, $D=4$, the R-symmetry is a $U(1)_R$

$\Rightarrow$ bosonic part of superconformal algebra $SO(4,2) \times U(1)_R$
[full supergroup is $SU(2,2|1)$].

- Scaling dimension of a scalar (primary) field: $\Delta = 1 + \frac{\gamma}{2}$ $[x \to \lambda x \quad \phi \to \lambda^{\Delta} \phi]$

γ is the anomalous dimension at the fixed point:

$$\left[\gamma = \mu \frac{\partial}{\partial \mu} \log Z \Big|_{g = g^*} \qquad \beta = \mu \frac{\partial}{\partial \mu} g \right]$$

- for chiral primary operators: $\Delta = \frac{3}{2} R$ ← R-symmetry charge

- $N=1$ → exact NSVZ beta-functions:

$$\beta = \frac{1}{8\pi^2} \frac{3N - \sum_i \mu(r_i)(1-\gamma_i)}{1 - \frac{g^2 N}{8\pi^2}}$$

$\mu_i = \frac{1}{2}$ for bi-fundamentals

for each gauge group coupling constant

$\Rightarrow$ at each node $N - \frac{1}{2} N \sum_{i \in \text{node}} (1 - R_i) = 0$

gives a set of constraints on the R-charges. In some simple cases (e.g. the conifold theory) this set of relations (together with the fact that W has R-charge 2 - W is not renormalised!) is enough to determine all R-charges.

- However, in general this is <u>not</u> <u>true</u>!

Problem: the full symmetry group $SO(4,2) \times U(1)_R \times F$ ← $U(1)^n$ may contain additional $U(1)$ factors which may "mix" with $U(1)_R$. How to determine the true $U(1)_R$?

The resolution comes from considering some central charges of the superconformal field theory.

Recall in a 2d CFT $\langle T_\mu{}^\mu \rangle = c\, R$

c is a "central charge" because it appears as a central extension of the Virasoro algebra. It also appears in $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \sim \Pi_{\mu\nu\rho\sigma}\, c$

Ricci scalar of a background metric

In 4d CFT we have:

$$\langle T_\mu{}^\mu \rangle = a\,(\text{Euler}) + c\,(\text{Weyl})$$

quadratic curvature invariants

then in susy CFT we have the remarkable formulae:

$$a = \frac{3}{32}\left[3\,\mathrm{Tr}R^3 - \mathrm{Tr}R\right] \qquad c = \frac{1}{32}\left[9\,\mathrm{Tr}R^3 - 5\,\mathrm{Tr}R\right]$$

(Anselmi et al.)

$\mathrm{Tr}R^{\alpha=1,3}$ are fermionic traces arising from triangle anomalies involving various currents (and possibly $T_{\mu\nu}$) at the vertices

$$\mathrm{Tr}R^\alpha = N_g + \sum_{i\ \text{chiral superfields}} \dim(r_i)\,(\underbrace{R_i - 1}_{})^\alpha$$

$N_g \to \sum_{\text{gauge groups } G} \dim(G)\,(1)^\alpha$

(1): R-charge of gauginos

$R_i - 1$: R-charge of fermionic component of chiral superfield

- Note 't Hooft anomaly matching implies that these kind of triangle anomaly coefficients don't depend on the microscopic constituents of the theory

→ can be evaluated in terms of UV description, and will be still valid for the IR theory, whose microscopic description is not known.

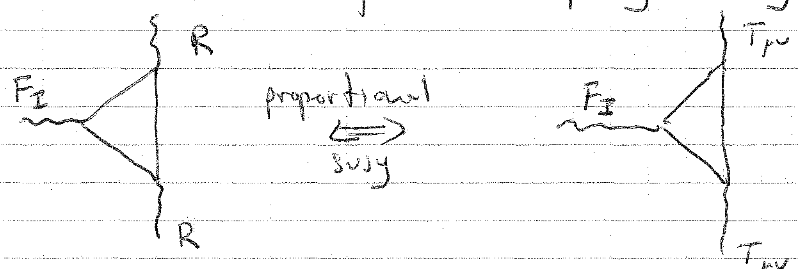
Solution of the problem: (IW) the unique R-symmetry is determined by:

1) $9 \,\mathrm{Tr}\, R^2 F_I = \mathrm{Tr}\, F_I$
2) $\mathrm{Tr}\, R F_I F_J < 0$

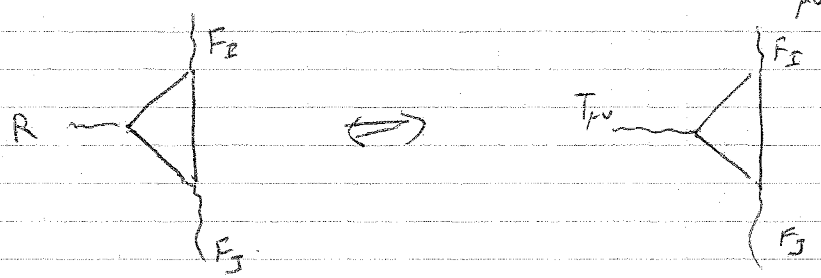
F_I generators of flavour $U(1)$'s

These relations come from supersymmetry: (essentially, $T_{\mu\nu}$ and R are in the same supermultiplet)

1)



2)



⟹ $\mathrm{Tr}\, R F_I F_J \sim -\mathrm{Tr}\, F_I F_J \sim$

$\langle J_\mu^I J_\nu^J \rangle \sim \tau_{IJ}$

unitarity ⟹ ≥ 0

The conditions 1) and 2) can also be interpreted in terms of a simple variational problem.

Define a "trial R-symmetry":

$$R_t = \underbrace{R_0}_{\text{guess for R-symmetry}} + \sum_I s_I \underbrace{F_I}_{\text{flavour generators}} \qquad s_I \in \mathbb{R}$$

$$a_{\text{trial}}(s_I) = \frac{3}{32}\left[3\,\mathrm{Tr}\,R_t^3 - \overbrace{\cancel{\mathrm{Tr}\,R_t}}^{=0 \text{ in quivers}} \right]$$

$$\Rightarrow \quad \frac{\partial a}{\partial s_I} = 0 \quad \Leftrightarrow \quad 9\,\mathrm{Tr}\,R^2 F_I = \mathrm{Tr}\,F_I$$

$$\frac{\partial^2 a}{\partial s_I \partial s_J} < 0 \quad \Leftrightarrow \quad \mathrm{Tr}\,R F_I F_J < 0$$

$a_{\text{trial}}(s_{\max}) = a_{\text{exact}}$ → the central charge is fixed!

<u>Note</u>: any $\mathcal{N}=1$ SCFT in $d=4$ will have R-charges and central charge given by algebraic numbers.

Gauge theory from toric geometry and $Y^{p,q}$

In the following I want to explain how a great deal of information on the gauge theory can be extracted from toric geometry, without explicit knowledge of a Sasaki-Einstein metric on L. This set of ideas will culminate with a detailed description on how to extract the R-charges of the field theory, with a calculation that is the geometric counterpart of a-maximisation.

Although the point is precisely to explain that an explicit Sasaki-Einstein metric is not needed, it's fair to say that much of the progress in this area has started with the discovery of the $Y^{p,q}$ Sasaki-Einstein metrics. I shall then devote a little time to describe this side of the story. Roughly, we can draw the following diagram

$$\begin{tikzcd}
Y^{p,q} \text{ metric} \arrow[r, "\text{moment map}"] & \text{toric CY } Y^{p,q} \arrow[r, leftrightarrow] & \begin{array}{c}\text{quotient presentation}\\ \text{algebraic presentation}\end{array} \arrow[d, leftrightarrow] \\
 & & \text{gauge theory}
\end{tikzcd}$$

I should say immediately that there is no constructive technique to obtain the explicit metrics. Rather, one needs some clever ansatz!

The $Y^{p,q}$ metrics ~~where~~ were obtained rather indirectly, by dualising a class of supersymmetric solutions of 11D sugra:

$$\underset{M}{AdS_5 \times_w M_6} \xrightarrow{\text{reduction}} \underset{\text{IIA}}{AdS_5 \times_w (M_4 \times S^1)} \xrightarrow{\text{T-duality}} \underset{\text{IIB}}{AdS_5 \times Y^{p,q}}$$

$$M_6 \downarrow S^2 \times S^2 \times T^2$$

$$ds^2(Y^{p,q}) = \frac{(1-y)}{6}\overbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}^{S^2} + \frac{1}{w(y)v(y)}dy^2 +$$
$$+ \frac{v(y)}{9}(d\psi - \cos\theta\, d\phi)^2 + w(y)(d\alpha + A)^2$$

$$A = \frac{a - 2y + y^2}{6(a-y^2)}(d\psi - \cos\theta\, d\phi) \qquad w(y) = \frac{2(a-y^2)}{1-y}$$

$$v(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}$$

e^{Φ} = constant.

$F^5_{RR} = (1+*)\,\mathrm{vol}(AdS_5)$

Notice the isometry group $SU(2)\times U(1)^2 \supset U(1)^3 \Rightarrow$ "toric"

$$\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi}, \frac{\partial}{\partial\alpha}\right)$$

More precisely, $Y^{p,q}$ is toric, in the sense that the CY cones $ds^2 = dr^2 + r^2 ds^2(Y^{p,q})$ are symplectic toric cones.

To check that the above metric is SE (locally) the best way is to change coordinates:

$$\begin{cases} 6\alpha = -\beta - \psi' \\ \psi = \psi' \end{cases} \Rightarrow \quad ds^2(Y^{p,q}) = ds^2_4 + \eta\otimes\eta$$

$$\eta = \frac{1}{3}\left(d\psi' + \cos\theta\, d\phi\,(y-1) + y\, d\beta\right)$$

$\left(ds^2_4, \frac{1}{2}d\eta\right)$ is Kähler-Einstein (easy exercise)

You may wonder where do p, q come in. From the present point of view these integers arise by requiring the metrics to be globally well-behaved.

The analysis is best performed in the "α-coordinates":

1) first consider the quotient by $\frac{\partial}{\partial \alpha}$, i.e. forget the α-direction

$\to$ $ds^2(B)$ (see previous page) is easily checked to be a smooth metric (not KE!) on a space that is

$$\begin{array}{ccc} & S^2 & \to & B \\ \swarrow & & & \downarrow \\ \theta, \phi & & & S^2 \\ & & & \searrow y, \psi \end{array}$$

topologically $B \approx S^2 \times S^2$ [this is because S^2 bundles over S^2 are classified by $\pi_1(SO(3)) = \mathbb{Z}_2$, and $c_1 = \frac{1}{2\pi}\int_{\text{base } S^2} d(\cos\theta\, d\phi) = 2 = \text{even} \Rightarrow$ bundle is trivial]

2) A is a connection on a $U(1)$ bundle if the periods are rationally related!

$$P_i = \frac{1}{2\pi}\int_{C_i} dA \qquad C_1, C_2 \text{ basis of } H_2(B;\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}$$

$$\frac{P_1}{P_2} = \frac{p}{q} \Rightarrow \text{"quantization" of } a$$

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}$$

[the periodicity of α is then $\alpha \in [0, 2\pi \ell]$

$$\ell = q\left(3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}\right)^{-1} \quad]$$

We can now go over the toric descriptions of $Y^{p,q}$, and essentially forget about the metric.

Introduce symplectic form $\omega = \frac{1}{2} d(r^2 \eta)$ and a basis for an effective $U(1)^3$ action $\left[e_1 = \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \quad (\alpha = \ell \gamma) \right.$

$$\left. e_2 = \frac{\partial}{\partial \phi} - \frac{p-q}{2} \frac{\partial}{\partial \gamma}, \quad e_3 = \frac{\partial}{\partial \gamma} \right]$$

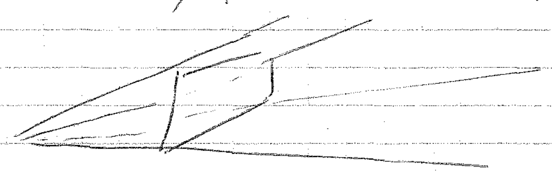
$\rightarrow$ moment map

$$d\mu_i = - e_i \lrcorner\, \omega$$

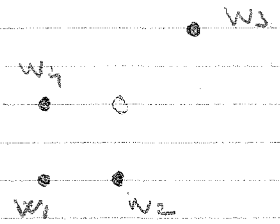
$$\Rightarrow \quad \mu_i = \frac{1}{2} r^2 e_i \lrcorner\, \eta \quad : \quad C(Y^{p,q}) \rightarrow t_3^* \cong \mathbb{R}^3$$

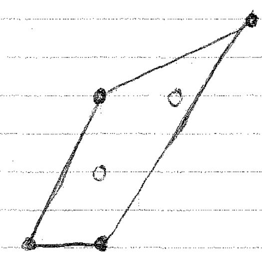
the image of the moment map is a rational polyhedral cone with four facets, specified by the normals (in some SL-related basis):

$$v_1 = [1, 0, 0] \quad v_2 = [1, 1, 0] \quad v_3 = [1, p, p] \quad v_4 = [1, p-q-1, p-q]$$



notice that how p, q show up in the components of the vectors v_a (in fact, this is the "fan"). By forgetting the first component 1, of the vectors $v_a = (1, w_a)$ we obtain the toric diagrams of $Y^{p,q}$ as the "convex hull" of the set of 4 points in $\mathbb{Z}^2$. E.g.

$Y^{2,1}$ 

$Y^{3,1}$ 

etc.

It is now straightforward to obtain the charges for the $U(1)$ action in the Kähler quotient construction

$$\mathcal{V} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & p & p-q-1 \\ 0 & 0 & p & p-q \end{pmatrix} \Rightarrow \ker \mathcal{V} = (-p-q, p, -p+q, p)$$

$$\Rightarrow \quad C(Y^{p,q}) = \mathbb{C}^4 /\!/ U(1) \quad :$$

new meaning of p, q $\longleftarrow$
$$\begin{cases} -(p+q)|Z_1|^2 + p|Z_2|^2 + (-p+q)|Z_3|^2 + p|Z_4|^2 = 0 \\ (Z_1, Z_2, Z_3, Z_4) \sim (e^{iQ_1} Z_1, \dots, e^{iQ_4} Z_4) \end{cases}$$

These $Z_i \in \mathbb{C}$ are the "homogeneous coordinates" we encountered before!

susy submanifold $\Leftrightarrow$ divisor $\Leftrightarrow$ facet F_a $\Leftrightarrow$ normal v_a $\Leftrightarrow$ Z_a

<u>Note:</u> using the explicit metric it is possible to check rather directly that there are four vanishing Killing vectors

$$V_a = \sum_{i=1}^{3} v_a^i \, e_i$$

$\|V_a\|^2 = 0$ at $\quad \theta = 0, \quad \theta = \pi, \quad y = y_1, \quad y = y_2$

$\qquad\qquad\qquad\; N_1 \qquad S_1 \qquad N_2 \qquad S_2$

four susy submanifolds $\quad S^3/\mathbb{Z}_p \quad S^3/\mathbb{Z}_p \quad S^3/\mathbb{Z}_{p+q} \quad S^3/\mathbb{Z}_{p-q}$

Of course in this case we can compute the volumes $\mathrm{vol}(\Sigma_a)$ explicitely from the metric. However, we will see later that these can be computed exactly, even without the metric.

Originally, the dual gauge theories of $Y^{p,q}$ were essentially obtained by guesswork. Now, we will see how a great deal of information can be obtained purely from the toric geometry.

We want to construct a gauge theory whose moduli space is essentially the Calabi-Yau cone $C(Y^{p,q})$. Thus we require for the gauge theory:

1) $\mathcal{N}=1$ supersymmetry
2) quiver with χ gauge group factors $SU(N)^{\chi}$
3) superpotential W such that $dW=0$ "monomial = monomial"

To uncover the matter content of the quiver, we will exploit the existence of dibaryonic operators

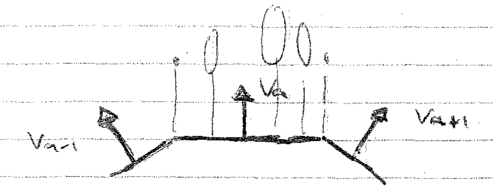
$$X_a \leftrightarrow B[X_a] = \epsilon^{\alpha_1 \dots \alpha_N} \epsilon_{\beta_1 \dots \beta_N} X_{a\,\alpha_1}{}^{\beta_1} \cdots X_{a\,\alpha_N}{}^{\beta_N} \longleftrightarrow \underset{\text{wrapped D3}}{\Sigma_a}$$

thus, to any divisor D_a there is associated a bifundamental field X_a [there will be also "composite" bifundamentals, but their properties are directly inherited from the "toric" ones]

- degeneracies (multiplicities):

if $H^2(\Sigma;\mathbb{Z}) \simeq H_1(\Sigma;\mathbb{Z}) \simeq \pi_1(\Sigma) = \mathbb{Z}_m$

we can have m different flat line bundles on Σ, thus there are m distinct dibaryonic operators (hence bifundamentals)



$\Sigma_a \simeq S^3/\mathbb{Z}_{m_a}$, $m_a = |(v_{a-1}, v_a, v_{a+1})|$

$\Rightarrow$ for $Y^{p,q}$ we have $m_a = p, p, p+q, p-q$

fields: $U_1 \; U_2 \; Y \; Z$

- baryonic charges:

for a toric diagram with d external points, the Sasaki-Einstein link has third homology group:

$$H_3(L;\mathbb{Z}) \simeq \mathbb{Z}^{d-3} \Rightarrow d-3 \text{ homologically distinct 3-cycles } \{C_I\} \quad I=1,\dots,d-3$$

KK reduction of RR C_4: $C_4 = \sum_{I=1}^{d-3} A_I \wedge \mathcal{H}_I$ ↳ dual to C_I: $\int_{C_J} \mathcal{H}_I = \delta_{IJ}$

$\Rightarrow U(1)_B^{d-3}$ global baryonic symmetries

We have the homological relation:

$$[\Sigma_a] = \sum_{I=1}^{d-3} Q_a{}^I C_I \quad \in H_3(L;\mathbb{Z})$$

[explanation: recall $Z_a \to e^{iQ_a^I} Z_a$

$\mathcal{M}_I \in U(1)^{d-3}$ $\qquad \mathcal{L}_a = \bigotimes_{I=1}^{d-3} \mathcal{M}_I^{Q_a^I} \Rightarrow c_1(\mathcal{L}_a) = \sum_{I=1}^{d-3} Q_a{}^I c_1(\mathcal{M}_I)$

Poincaré duality $\Rightarrow$ $[\Sigma_a] = \sum_{I=1}^{d-3} Q_a^I C_I$]

finally: (I-th baryonic charge of field X_a) $= \int_{\Sigma_a} \mathcal{H}_I = Q_a{}^I$

for $Y^{p,q}$ $\quad Y, Z, U_1, U_2$

$\qquad p-q, p+q, -p, -p$

- flavour charges:

$$U(1)^3 \text{ isometry} \rightarrow U_t(1) \times \underbrace{U(1)^2_F}$$

"non-R" flavour symmetry

definition of "non-R": $\mathcal{L}_{V_F}\, \theta = 0$

R-symmetry: $\mathcal{L}_{V_t}\, \theta = i\frac{3}{2}\,\theta$

Note if $U(1)_t$ is an admissible "trial" R-symmetry, then also $U(1)_{t'} = U(1)_t + a\,U_{F_1}(1) + b\,U_{F_2}(1)$, for $\mathcal{L}_{V_{t'}}\theta = i\frac{3}{2}\theta$.

$e_1 = (1,0,0)$ generates $U(1)_t$

e_2, e_3 generate $U(1)_F$

under the map $\pi: E_a \to v_a$ such that $\ker(\pi) = \mathbb{C}$

$$\mathbb{Z}^d \ni \alpha_2, \alpha_3 \mapsto e_2, e_3$$

[α_i define the circle subgroups of T^d that descend to $U_F(1)^2$ under the quotient construction $\mathbb{C}^d // U(1)^{d-3}$]

$U(1)_{F_i}$ charge of a-th field X_a is α_i^a

e.g. for $Y^{p,q}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & p & p-q-1 \\ 0 & 0 & p & p-q \end{pmatrix} ; \quad (-1,1,0,0) \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$

Y	Z	U_1	U_2
0	-1	1	0

In any case α_i^a is defined only up to linear combinations of Q_2^a.

For $Y^{p,q}$ there is the additional complication that one $U(1) \subset SU(2)$. It's natural to assign $U^{1,2}$ in a doublet of $SU(2)_z$.

- R-charges:

$$R[X_a] = \frac{\pi}{3} \frac{\mathrm{vol}[\Sigma_a]}{\mathrm{vol}[L]}$$

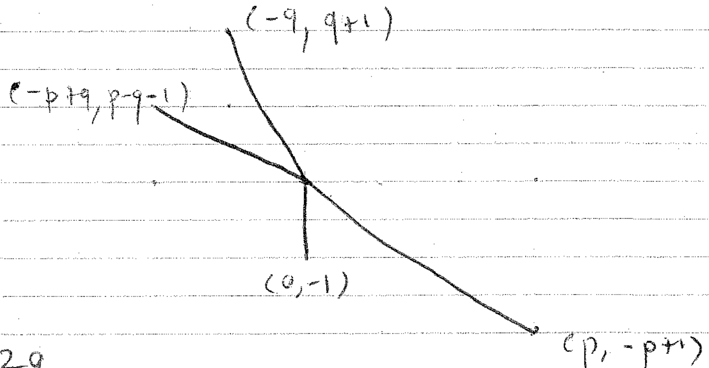
this comes from the fact that the total R-charge of di-baryonic states is $R[B[X_a]] = N R[X_a]$ and R-charge of these is $R = \frac{2}{3}\Delta$, $\Delta = \text{mass} \sim \mathrm{vol}(\Sigma_a)$.

For $Y^{p,q}$ these numbers can be computed just using the metric. However, we will see that these volumes can also be computed without the metric.

$$\# \text{ gauge groups} = \chi = 1 + b_2 + b_4 = 2(\text{area toric diagram}) \overset{Y^{p,q}}{=} 2p$$

$\Rightarrow \quad SU(N)^{2p}$

$$\# \text{ fields} = \frac{1}{2} \sum_{i,j}^{d} \left| \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix} \right|$$ (this formula is essentially conjectured)



$Y^{p,q} \to 4p + 2q$

For the four adjacent legs the terms in the formula are precisely the bifundamental fields X_a with multiplicities:

$U^\alpha, \quad Y, \quad Z$

$(2p) + (p+q) + (p-q) \quad 4p$

$\Sigma_2, \Sigma_4, \quad \Sigma_3, \quad \Sigma_1$

We have to consider two more additional fields,

			mult.	$U(1)_B$	R
$\Sigma_1 \cup \Sigma_2$	$\rightarrow$	V_1	q	$-q-p+p=-q$	R_1+R_2
$\Sigma_4 \cup \Sigma_1$	$\rightarrow$	V_2	q	$-q-p+p=-q$	R_4+R_1

With this prescription, we arrive at an expression for a trial a-function in a general toric quiver:

define: $m_{a,b} = \det\begin{pmatrix} p_b & q_b \\ p_{a-1} & q_{a-1} \end{pmatrix} = (w_{b+1} - w_b) \times (w_a - w_{a-1})$

$$\Rightarrow \quad \frac{32}{9} a = \chi + \frac{1}{2} \sum_{a=1}^{d} \sum_{b=a}^{a-1} m_{a,b} \left[\left(\sum_{c=a}^{b} R_c \right) - 1 \right]^3$$

where the legs are cyclic, i.e. $a = d+1 \sim a = 1$,

This is a function of d "trial" R-charges a priori. Notice, we haven't used the superpotential of the gauge theory, nor it's detailed quiver diagram.

For $Y^{p,q}$ we get:

$$\mathrm{tr}R^3 = 2p + (p-q)(R_Z - 1)^3 + (p+q)(R_Y - 1)^3 + 2p(R_U - 1)^3 + 2q(\overbrace{R_Z + R_U}^{R_V} - 1)^3$$

the local maximum of this function then reproduces precisely the volume of $Y^{p,q}$, using the formula

$$a(Y^{p,q}) = \frac{\pi^3}{4 \cdot \mathrm{vol}(Y^{p,q})}$$

moreover, the volume of the susy submanifolds are reproduced by $R[X_a] = \frac{\pi}{3} \frac{\mathrm{vol}[\Sigma_a]}{\mathrm{vol}[Y^{p,q}]}$

<u>Exercise:</u> pick a toric diagram and find all the charges using the methods described: e.g. $(0,0)$, $(0,1)$, $(1,2)$, $(3,1)$, $(1,0)$.

We can go further, and predict the superpotential as well as the detailed form of the nodes of the quivers. By now there are sophisticated algorithms that allow to obtain this information from a toric geometry (e.g. so-called "Fast Forward Algorithm"). These use the ideas related to dimer models or "brane tilings" - however these will not be discussed in these lectures. Instead I will give a set of constraints that help determining the gauge theory, and in some simple cases suffice.

- "Euler formula" $N_{\text{fields}} = \underbrace{N_{\text{gauge groups}}}_{\chi} + \underbrace{N_W}_{\text{superpotential terms}}$

E.g. $Y^{p,q}$ $N_W = (4p+2q) - 2p = 2(p+q)$ terms

- each superpotential term $\leftrightarrow$ $\underbrace{\bigcup_{a=1}^{d} \Sigma_a}_{R\text{-charge}=2,\ U_B(1)=0,\ U_F(1)=0}$

E.g. $Y^{p,q}$

$$W_{\text{quartic}} = \mathrm{Tr}[Z U_1 Y U_2] \qquad W_{\text{cubic}} = \mathrm{Tr}[V_1 Y U_2]$$
$$W_{\text{cubic}} = \mathrm{Tr}[V_2 Y U_1]$$

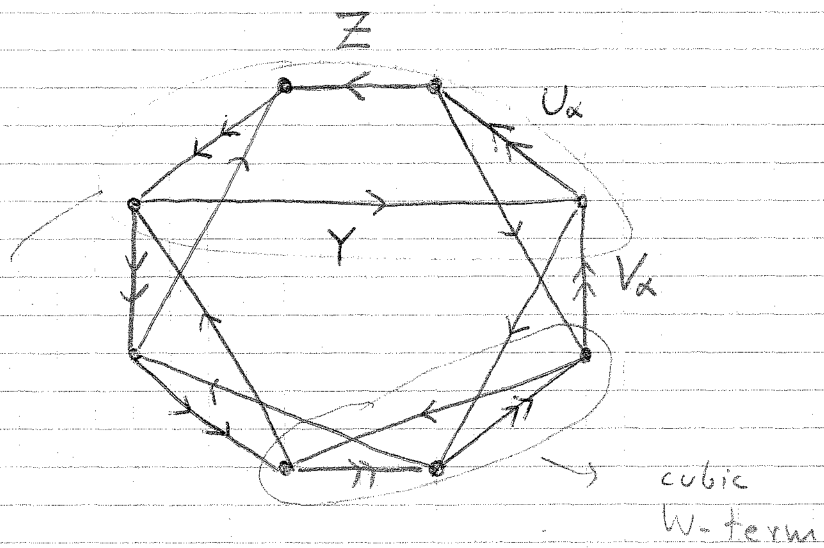
Combining this information with the $SU(2)_F$ symmetry and the fact that each field must appear twice, each time with a different sign its possible to get the superpotential

$$W = \sum^{2q} \epsilon_{\alpha\beta} \mathrm{Tr}[V_\alpha Y U_\beta] + \sum^{p-q} \epsilon_{\alpha\beta} \mathrm{Tr}[Z U_\alpha Y U_\beta]$$

Each superpotential term corresponds to loops in the quiver. Moreover, there are also constraints:

- at each node there are an even number of arrows (from anomaly cancellation)
- the beta-functions at each node vanish: $\sum_{i \in \text{node}} (R_i - 1) + 2 = 0$
- for $Y^{p,q}$, there is an $SU(2)_F$ symmetry to respect

$\underline{\underline{Y^{4,3}}}$



$p-q = 1 \quad Z$

$p+q = 7 \quad Y$

$2p = 8 \quad U_\alpha$

$2q = 6 \quad V_\alpha$

W: 1 quartic + 6 cubic = $7 = p+q$ terms

<u>Note</u>: a generic $Y^{p,q}$ quiver can be obtained with the operation of replacing $V_\alpha \to Z$ and "sewing" together the 2 Y fields in front of it

$p,q \quad p+q$ terms $-2\,(V_\alpha) + 1\,(Z) - 1\,(Y) = p+q-2$

at each step the number of fields decreases by two (this operation can be repeated p times).

In this way a generic $Y^{p,q}$ can be easily obtained starting from $Y^{p,p} = \mathbb{C}^3/\mathbb{Z}_{2p}$ $\leftarrow$ only cubic terms, and down to $Y^{p,0} =$ conifold$/\mathbb{Z}_p$ $\leftarrow$ only quartic terms.

[Note: moving around Z's $\leftrightarrow$ Seiberg duality]

So far, we have emphasised the role of the "homogeneous variables" $z_a \in \mathbb{C}^d$, associated to toric divisors, in constructing the gauge theory.
However, recall that the algebraic geometric description of the toric Calabi-Yau varieties is given in terms of some hypersurface

$$\{f_1 = 0, \dots, f_s = 0\} \subset \mathbb{C}^N$$

related to the semi-group algebra

$$\mathbb{C}[S_C = C^\vee \cap \mathbb{Z}^3] = \frac{\mathbb{C}[z_1, \dots, z_N]}{\langle f_1, \dots, f_s \rangle} \leftarrow \text{toric ideal}$$

given by the characters $w^m = \prod_{i=1}^{3} w_i^{m_i}$ $\quad m \in S_C$

It turns out that this description is reproduced by the chiral ring of the gauge theories!
Let me sketch the relation for the $Y^{2,1}$ model:

recall the fan / cone $\quad v_1 = (1,0,0) \quad v_2 = (1,1,0) \quad v_3 = (1,2,2)$

- the dual cone $C^\vee$ has 9 generators: $\quad v_4 = (1,0,1)$

	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3	c_4
$m_i =$	$(0,0,1)$	$(0,1,0)$	$(1,-1,1)$	$(1,0,0)$	$(1,1,-1)$	$(2,-2,1)$	$(2,-1,0)$	$(2,0,-1)$	$(2,1,-2,$
$w^m =$	w_3	w_2	$w_1 w_2^{-1} w_3$	w_1	$w_1 w_2 w_3^{-1}$	$w_1^2 w_2^{-2} w_3$	$w_1^2 w_2^{-1}$	$w_1^2 w_3^{-1}$	$w_1^2 w_2 w$

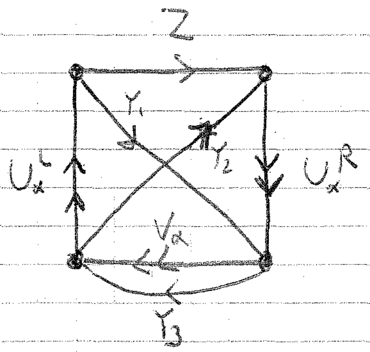
These 9 m generate $C^\vee$, however are not independent over $\mathbb{Z}$ $\to$ there are 20 relations e.g. $(0,0,1) + (2,0,-1) = 2(1,0,0)$

$\Rightarrow \quad a_1 c_3 = b_2^2$, and so on.

Where we regard a_i, b_i, c_i as complex variables in $\mathbb{C}^9$. These 20 relations "monomial=monomial" form a toric ideal $I = \langle \underbrace{a_1c_3 - b_2^2}_{1}, a_2c_2 - b_2^2, \ldots, \underbrace{c_1c_3 - c_2^2}_{20} \rangle$

$\Rightarrow$ polynomial ring of $C(Y^{2,1})$ is $\dfrac{\mathbb{C}[a_1, \ldots, c_4]}{I}$

In the gauge theory:



F-term equations: $\dfrac{\partial W}{\partial X_i} = 0$

$$\left.\begin{aligned} U_L^1 V^2 &= U_L^2 V^1 \\ V^1 U_R^2 &= V^2 U_R^1 \\ &\vdots \\ Y_1 U_L^\alpha &= U_R^\alpha Y_2 \end{aligned}\right\} \text{10 - equations}$$

One can show that any single trace chiral primary operator

$\mathcal{O} = tr(X_{i_1} \ldots X_{i_n}) \; / \;$ F-terms $\leftrightarrow$ loop in the quiver

can be written in terms of precisely 9 building blocks, corresponding to loops that wind just once. These are

$a_1 = U_R^1 Y_2 Y_3 \quad a_2 = U_R^2 Y_2 Y_3 \quad b_1 = U_R^1 Z U_L^1 Y_3 \quad b_2 = U_R^1 Z U_L^2 Y_3$

$b_3 = U_R^2 Z U_L^2 Y_3 \quad c_1 = U_R^1 Z U_L^1 V^1 \quad c_2 = U_1^R Z U_1^L V^2 \quad c_3 = U_R^1 Z U_L^2 V^2$

$c_4 = U_R^2 Z U_L^2 V^2$

These are the generators of the chiral ring of the gauge theory, and we have called them with the same names as for the generators of the ring of holomorphic functions of the $Y^{2,1}$ toric singularity, because we can identify them.
They satisfy the same relation

e.g. $$a_1 c_3 \sim U_R^1 Y_2 Y_3 U_R^1 Z U_L^2 V^2 \sim [U_R^1 Z U_L^2 Y_3]^2 \sim b_2^2$$

use F-term: $Y_2 V^2 = Z U_L^2 Y_3$

Of course, each variable a_i, b_i, c_i can also be expressed in terms of the Z_a. For instance

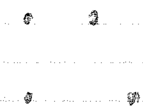
$$a_1 \sim U_R^1 Y_2 Y_3 \sim Z_4 Z_3 Z_3 = Z_4 Z_3^2$$

this is consistent with:

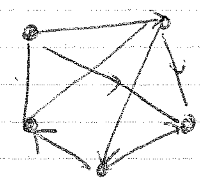
$$a_1 \sim w_1^0 w_2^0 w_3^1 \qquad w_i = Z_1^{v_1^i} Z_2^{v_2^i} Z_3^{v_3^i} Z_4^{v_4^i}$$

$$\Rightarrow w_1 = Z_1 Z_2 Z_3 Z_4$$
$$w_2 = Z_2 Z_3^2 Z_4$$
$$w_3 = Z_3^2 Z_4$$

Consider the class of <u>toric quiver gauge theories</u>:

start with a <u>toric</u> <u>singularity</u>:  $\{v_a\}$

$\downarrow$

quiver gauge theory: $\Big[$  $+ \; W \Big] \; (v_a)$

$\downarrow$

construct a_{trial} : $a_{trial}(s^I; v_a)$

$\downarrow$ a-maximisation

fixes $a = a_{max}(v_a)$, $R = R_{max}(v_a)$

$\downarrow$ AdS / CFT

L : toric SE

$$a = \frac{1}{vol(L)} \qquad R_a = \frac{vol(\Sigma_a)}{vol(L)}$$

Σ_a : susy submanifolds

$$\Rightarrow \quad vol(L) = vol_L(v_a)$$
$$vol(\Sigma_a) = vol_{\Sigma_a}(v_a)$$

It must be possible to determine the volumes of toric Sasaki-Einstein manifolds only in terms of the toric data!

<u>Hints</u> : 1) an extremal problem! very geometrical

2) R-symmetry $\leftrightarrow$ Reeb vector of a SE

The dual of determining R using an extremal problem, has to be an extremal problem in toric geometry, that determines the Reeb vector.

In the following, we will show how this expectation is fulfilled.

To formulate a variational problem, we enlarge the space of metrics

toric Sasaki-Einstein Y_5 $\longrightarrow$ toric Sasaki Y_5

$\Updownarrow$

toric Calabi-Yau cone X_6 toric Kähler cone X_6

$$ds^2(X) = dr^2 + r^2 ds^2(L)$$

Kähler form $\omega = d(r^2\theta)$

toric $\Rightarrow$ $T^n = \{\frac{\partial}{\partial\phi_i}\}$ preserves metric & ω

moment map $\mu : X \to \mathbb{R}^n$

$$\omega = \sum_{i=1}^{n} dy_i \wedge d\phi_i \qquad y_i = r^2\langle \theta, \frac{\partial}{\partial\phi_i}\rangle$$

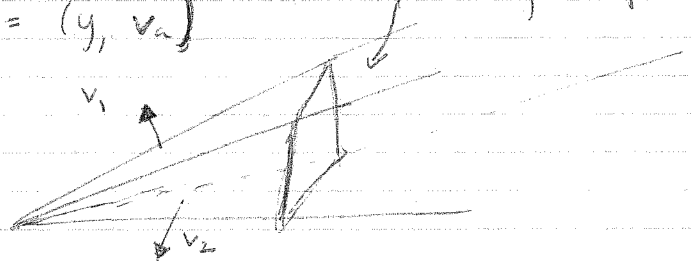
"symplectic coordinates"

The image of the moment map in $\mathbb{R}^n$ is

$$\mu(X) = C = \{ y \in \mathbb{R}^n \mid l_a(y) \geq 0 \ , \ a=1,\dots,d \}$$

"facet": $F = \{ l_a(y) = 0 \}$

$l_a(y) = (y, v_a)$



roughly : $X \sim \{v_a\}$

more precisely: $T^n \to X \to C$

Metric in symplectic coordinates:

any $\mathbb{T}^n$-invariant Kähler metric on X is

$$ds^2 = G_{ij}\, dy_i\, dy_j + G^{ij}\, d\phi_i\, d\phi_j$$

$\mathbb{C}$-structure $J_i{}^j = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix}$

integrability of $J_i{}^j \Rightarrow \frac{\partial}{\partial y_k} G_{ij} = \frac{\partial}{\partial y_j} G_{ik}$

$$\Rightarrow \quad G_{ij} = \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} G(y)$$

$\hookrightarrow$ SYMPLECTIC POTENTIAL

$ds^2(X)$ is cone $\Leftrightarrow$ $G_{ij}(y)$ is homogeneous deg. -1 in y

This $G(y)$ is related to the more familiar Kähler potential in the complex point of view

$$\omega = 2i\, \partial\bar{\partial} F$$

$$ds^2 = F_{ij}\, dx^i dx^j + F_{ij}\, d\phi_i d\phi_j \qquad F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

$$z_i = x_i + i\phi_i$$

But we won't need this formalism.

On X there is a <u>canonical</u> Kähler metric, which is constructed in terms of the toric data, which is inherited from the flat metric on $\mathbb{C}^d$ via the reduction

$$X = \mathbb{C}^d // U(1)^{d-n}$$

This is completely specified by the symp. potential

$$G^{can}(y) = \frac{1}{2} \sum_{a=1}^{d} l_a(y) \log l_a(y)$$

$$\Rightarrow \quad G^{can}_{ij} = \frac{1}{2} \sum_a v_i^a v_j^a \frac{1}{l_a(y)}$$

$G^{can}_{ij}(y)$ is homogenous deg -1 in y $\Rightarrow$ $ds^2(X)$ is a <u>cone</u>

it's Kähler by construction, however <u>it is not Ricci-flat.</u>

The key object to consider is the <u>Reeb vector</u>:

$$\xi \stackrel{def}{=} J \cdot \left(r \frac{\partial}{\partial r} \right)$$

$\mathbb{C}$-structure Euler vector

indeed this can be defined for <u>any Kähler cone</u>

it is Killing, and unit-norm

toric $\Rightarrow$ $\xi = b_i \frac{\partial}{\partial \phi_i}$ $\qquad b^i \in \mathbb{R}^n$

we can show that b^i is a <u>constant vector</u> :

$$r\frac{\partial}{\partial r} = 2y_i \frac{\partial}{\partial y_i} \Rightarrow b_i = 2 G_{ij} y_j$$

$$\frac{\partial}{\partial y_k} b_i = 2y_j G_{ij,k} + 2G_{ik} = 2 y_j \frac{\partial}{\partial y_j} G_{ik} + 2 G_{ik} = 0$$

use integrability of G_{ij}

use Euler theorem and G_{ij} hom deg -1

$$b^{can} = 2 G^{can}_{ij} y_j = \sum_a v^a \rightarrow \text{def: } l_\infty(y) = (b^{can}, y)$$

- One can show (exercise) that

$$g = G' - G \in \mathcal{H}(1) \Leftrightarrow b = b'$$

$\mathcal{H}(1) = \{$ functions homogeneous in y_i of degree 1 $\}$

<u>Result</u>: any symplectic potential is written as

$$G(y) = G^{can} + G_b + g$$

$$G_b = \frac{1}{2} l_b(y) \log l_b(y) - \frac{1}{2} l_\infty \log l_\infty(y)$$

↳ smooth function

by construction, Reeb $= 2 G_{ij} y_j = b$ (check)

$\Rightarrow$ moduli space $\boxed{S = C^* \times \mathcal{H}(1)}$

$b \in C^* = \{$ dual cone $\}$, $g \in \mathcal{H}(1)$

The problem of finding toric Sasakian metrics with prescribed properties (e.g. Einstein) has split in two parts:

1) Reeb : finite dimensional (n)

2) g : infinite dimensional

In the remaining part of the lecture, we will see that for our purposes <u>g is irrelevant</u>.

We now require X to be Ricci-flat, which is equivalent to Y being Sasaki-Einstein

Ricci form $\rho = -i\partial\bar{\partial}\log\det(F_{ij})$

Ricci-flat $\Leftrightarrow$ $\rho = 0$ $\Leftrightarrow$ $\log\det(F_{ij}) = -2\gamma_i x_i + c$

passing to symplectic coordinates:

$$\det(G_{ij}) = \exp\left[2\gamma_i \frac{\partial G}{\partial y_i} - c\right] \quad \text{"Monge-Ampère"}$$

We can show that regularity of this equation alone already fixes one component of the Reeb vector:

$$b_1 = n$$

recall that the CY condition on the polyhedral cone $\{v^a\}$ implies that there is always a basis such that

$$v_a = (1, w_a) \qquad w_a \in \mathbb{Z}^{n-1}$$

MA $\Rightarrow$ $(b, \gamma) = -n$ (because $\det G_{ij}$ is hom deg $-n$ in y_i)

expanding MA: $$\det(G_{ij}) = \prod_a \left[\frac{l_a(y)}{l_\infty(y)}\right]^{(v^a, \gamma)} l_b(y)^{-n} \exp\left(2\gamma_i \frac{\partial g}{\partial y_i}\right)$$

$$\parallel$$

$$\prod_a [l_a(y)]^{-1}$$

$\Rightarrow$ $(v_a, \gamma) = -1 \qquad \forall a$

$v_a = (1, w_a) \quad \Rightarrow \quad \gamma = (-1, 0, \dots, 0) \quad \Rightarrow \quad b_1 = n$

$\blacksquare$

To summarize, we have shown that the Reeb vector of any toric Sasaki-Einstein 5-manifold is

$$b = (3, y, t)$$

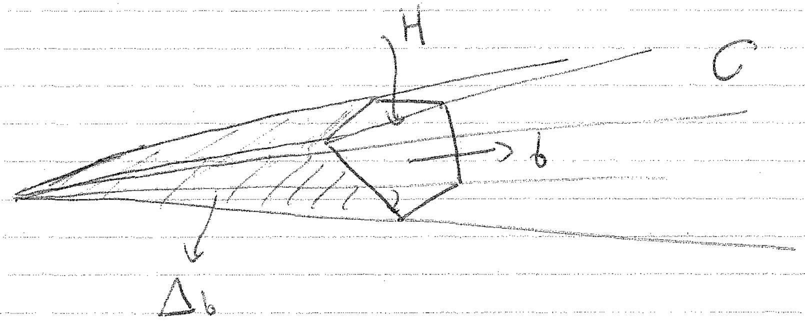
i.e. depends only on 2 variables.

Fix a toric singularity $\{v^a\} \to C \in \mathbb{R}^n$

$$ds^2(X) = dr^2 + r^2\, ds^2(L) \qquad \{r^2 = 1\} = L$$

$$\tfrac{1}{2} r^2 = (b, y) \Rightarrow L = \{ y \in \mathbb{R}^n \mid (b,y) = \tfrac{1}{2} \}$$

define $H = \{ y \in \mathbb{R}^n \mid (b,y) = \frac{1}{2} \} \cap C$ "characteristic hyperplane"



to any toric singularity, with Reeb vector, we can associate a finite polytope Δ_b

$$X_1 \equiv X_{r<1} \qquad \mathrm{vol}(X_1) = \int_0^1 dr\, r^{2n-1}\, \mathrm{vol}(L) = \frac{1}{2n}\mathrm{vol}(L)$$

on the other hand,

$$\mathrm{vol}(X_1) = \int \frac{1}{n!}\omega^n = \int dy_1 \cdots dy_n\, d\phi_1 \cdots d\phi_n = (2\pi)^n\, \mathrm{vol}(\Delta_b)$$

$$\Rightarrow \mathrm{vol}(L) = 2n\,(2\pi)^n\, \mathrm{vol}(\Delta_b) \qquad \blacksquare$$

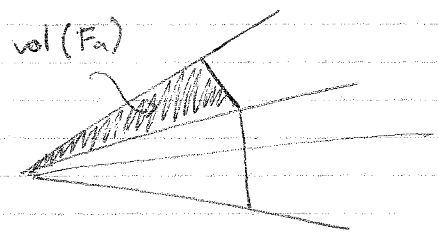
<u>Note</u>: to compute the volume of L we don't need the metric, but only b!

Similarly, one can compute the volume of supersymmetric submanifolds Σ of codimension 2, e.g. 3-submanifolds inside Sasaki-Einstein 5-manifolds

complex divisors D_a correspond to $\{Z_a = 0\}$ in the GLSM description

calibrated by $\frac{1}{(n-1)!}\omega^{n-1}$

$$\Rightarrow \quad \mathrm{vol}(\Sigma_a) = 2(n-1)(2\pi)^{n-1} \frac{1}{|v_a|} \mathrm{vol}(F_a)$$



- Integrated Ricci Scalar

In the same spirit, we can compute the integrated Ricci scalar of a toric kähler metric

$$\int_{X_1} R_X = (2\pi)^n \int_{\Delta_b} R_X \, dy^1 \dots dy^n$$

the crucial thing is that the Ricci scalar is a <u>total derivative</u>

$$R_X = - \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} G^{ij}$$

we can then apply Stoke's theorem

$$= \sum_a \int_{\Sigma_a} \frac{\partial}{\partial y_i} G^{ij} \frac{v_j^a}{|v|} \, d\sigma - \int_H \frac{\partial}{\partial y_i} G^{ij} \frac{b_j}{|b|} \, d\sigma$$

the second term is straightforward to deal with:

$$\frac{\partial}{\partial y_i}(G^{ij}) b_j = 2 \frac{\partial}{\partial y_i}\left(G^{ij} b_j \right) = 2 \frac{\partial}{\partial y_i}\left(G^{ij} G_{jk} y_k \right) =$$

$$= 2 \frac{\partial}{\partial y_i}\left(\delta_{ik} y_k \right) = 2n$$

$$\Rightarrow \text{2nd term} = -\frac{2n}{|b|} \int_H d\sigma = -2n \underbrace{\frac{\mathrm{vol}(H)}{|b|}}$$

generalizes the formula for the triangle area $\longrightarrow$ $\frac{1}{|b|} \mathrm{vol}(H) = 2n \, \mathrm{vol}(\Delta)$

eventually, we obtain

$$\int_{X_i} R_X \, dy^1 \cdots dy^n = \frac{2\pi}{(n-1)} \sum_a \mathrm{vol}(\Sigma_a) - 2n \, \mathrm{vol}(\Delta)$$

■

Using similar tricks, we can obtain a nice formula involving all the geometrical objects around:

$$\pi \sum_a \mathrm{vol}(\Sigma_a) \vec{v}_a = (n-1) \, \mathrm{vol}(\Delta) \, \vec{b}$$

<u>Exercise</u>: prove this formula. Hint: use Stokes' theorem applied to Δ_b.

We are finally ready to obtain the <u>extremal problem</u>.

We are interested in Sasakian metrics, that are also Einstein metrics:

$$R_{\mu\nu} = 2(n-1)\, h_{\mu\nu}$$

This equation is obtained from a metric functional, that is the Einstein-Hilbert action

$$S[h] = \int_{L} \left(R_L + 2(n-1)(3-2n) \right) d\mu_L$$

We want to reduce the integral of $\int_L R_L$ to some integral on the polytope.

For this, all we have to do is obtain an expression for R_L

<u>Exercise</u>: $ds^2(X) = dr^2 + r^2 ds^2(L)$

$$\Rightarrow \quad R_X = \frac{1}{r^2}\left[R_L + 2(2n-1)(1-n) \right]$$

Inserting this into $\int_{X_1} R_X = \int_{r\leq 1} dr \int_L R_X$

we obtain an expression for the Einstein-Hilbert action, which now depends only on $\underline{\underline{b}}$

$$S[b] = 4\pi \sum_a \mathrm{vol}(\Sigma_a) - 4(n-1)^2 \mathrm{vol}(L)$$

Now $\quad \frac{\delta S}{\delta h_{\mu\nu}} = 0 \quad \longrightarrow \quad \frac{\partial S}{\partial b_i} = 0$

there are 3 equations that determine a Reeb vector b, and the infinite dimensional part of the problem, depending on the function $g \in H(1)$ has been <u>integrated out</u>.

After pulling out some numerical normalization, and using an earlier formula, we *finally* introduce the Z-function

$$Z[b] = (b_1 - n + 1)\, 2n\, \mathrm{vol}(\Delta_b)$$

It is interesting to note that the condition imposed by the MA equation is easyly reproduced:

$$0 = b^i \frac{\partial}{\partial b_i} Z = -2n(n-1)(b_1 - n)\, \mathrm{vol}(\Delta)$$

$$\Rightarrow \quad b_1 = n$$

It is not difficult to show that:

- an ~~...~~ extremal point of Z always exists and is unique (it is in fact a local minimum of Z).

[To show this the only non-trivial formula needed is

$$\frac{\partial^2}{\partial b_i \partial b_j} \mathrm{vol}(\Delta_b) = \frac{2(n+1)}{|b|} \int_H y_i y_j \, d\sigma > 0$$

that proves convexity of Δ_b. The rest are standandard arguments in analysis.]

In dimension $n=3$ - $\dim(Y) = 5$ - we can write down elementary expressions for the volumes and Z.

For instance

$$\mathrm{vol}(\Delta_b) = \frac{1}{6b_1} \cdot \frac{1}{8} \sum_a \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}$$

This can be proved by elementary vectorial manipulations

$$(a, b, c) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Z-minimisation in general and characters

Problem: a-maximisation is valid for any SCFT, not only for "toric quivers". In particular, it predicts via AdS/CFT, that "Z-minimisation" must exist for arbitrary Sasaki-Einstein manifolds!

General set-up:

Kähler cone metric $ds^2(M) = dr^2 + r^2 ds^2(L)$

link L is Sasakian

From previous lecture: $ds^2(L) = \underset{\substack{\uparrow \\ \text{"transverse" Kähler}}}{ds_T^2} + \eta \otimes \eta$

$\omega_T = \frac{1}{2} d\eta$

Reeb killing vector:

$$\xi = J\left(r\frac{\partial}{\partial r}\right) = \eta^{\#} = \sum_{i=1}^{r} b_i \frac{\partial}{\partial \phi_i} \qquad r \leq n$$

relevant case is $n=3$:

$r=3 \rightarrow$ toric

$r=2 \rightarrow$ generic

$r=1 \rightarrow$ must be quasi-regular

[We also require the topological condition of the existence of $\Omega \in \Lambda^{n,0}(M)$ which is necessary for Ricci-flat metrics to exist

$\longrightarrow$ the singularities are then called "Gorenstein"]

Start from Einstein-Hilbert action on L

eq. of motion: $R^L_{\mu\nu} = 2(n-1) g^L_{\mu\nu}$ $\leftarrow$ $S[g_L] = \int_L \left[R(g_L) + 2(n-1)(3-2n) \right] d\mu$

$$\Downarrow$$

$S[g_L] = 4(n-1)\,\mathrm{vol}[L]$ [only used: 1) g_M is kähler 2) $\mathcal{L}_{r\frac{\partial}{\partial r}} \Omega = n\Omega$]

$\Updownarrow$ $b_1 = n$ in toric ca

- Moduli space of Sasakian metrics = $\{\text{Reeb}\} \times \{\text{transverse kähler deformations}\}$

$\{\text{Reeb}\}$: finite

$\{\text{transverse kähler deformations}\}$: infinite (in toric case was encoded in g)

$\mathrm{vol}[L]$ is independent of transverse kähler deformations

$$\left[\underbrace{r^2(t)}_{\text{new kähler potential}} = r^2 \exp(t\phi) \quad \longrightarrow \quad \frac{d\,\mathrm{vol}[L]}{dt}(t=0) = 0 \right]$$

$b \in C_0$

$$\mathrm{vol}[L] : C_0 \longrightarrow \mathbb{R}$$

C_0 = interior of $C \subset t_r \cong \mathbb{R}^n$

C dual to C^*

Recall that there are still moment maps (even if $r < n$)

$$d\mu_i = - \frac{\partial}{\partial \phi_i} \lrcorner\, \omega \quad \Rightarrow \quad \mu_i = \frac{1}{2} r^2 \eta\left(\frac{\partial}{\partial \phi_i}\right)$$

$$\mu_i : M \longrightarrow t_r^* \cong \mathbb{R}^r \supset C^*$$

now, write $\xi(t) = \xi + tX$

$X \in \mathfrak{t}_r$ is holomorphic and killing

along X

$\Rightarrow$ 1) $d\,\mathrm{vol}[L] = -n \int_L \eta(X)\, d\mu$

2) $d^2 \mathrm{vol}[L] = n(n+1) \int_L \eta(X)\,\eta(Y)\, d\mu \rightarrow$ $\mathrm{vol}[L]$ is convex $\Rightarrow \exists$ unique critical Reeb

suppose L is toric :

$\Rightarrow \quad \eta(X) = \eta\left(\frac{\partial}{\partial \phi_i}\right) = y_i \qquad i = 2, 3$

$\Rightarrow \quad d\mathrm{vol} = -n \int_L y_i \, d\mu \quad , \quad d^2\mathrm{vol} = n(n+1) \int_L y_i y_j \, d\mu$

as we found before, by more explicit computations.

If the structure is quasi-regular, we can also relate $d\mathrm{vol}[L]$ to the Futaki invariant of the $\underbrace{\text{Fano}}$ orbifold V :

anticanonical bundle K^{-1} is ~~very~~ ample $\rightarrow c_1(V) > 0$ i.e. del Pezzo's

K^{-n} is very ample: i.e. it can be embedded in $\mathbb{CP}^K$

$$d\mathrm{vol}[L](X) = -\frac{\ell}{2} F[J_V X_V]$$

ℓ = lenght of the $U(1)$ fiber $= \frac{2\pi\beta}{n}$

J_V = complex structure of V

X_V = X restricted to V

$\rho_V - \lambda \omega_V = i\partial\bar\partial f$

$$F[Y] = \int_V (\mathcal{L}_Y f)\, \frac{\omega_V^{n-1}}{(n-1)!} \qquad F_{\mathbb{C}} : \mathrm{aut}(V) \rightarrow \mathbb{C}$$

- independent of metric (topological invariant)
- Lie algebra homomorphism
- F vanishes if V admits Kähler-Einstein metrics (i.e. f=const

So, to recap, we have shown:

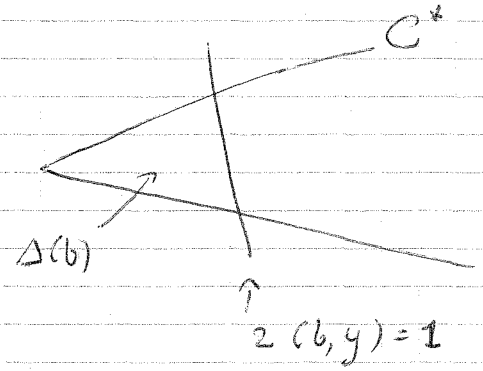
1) $S_{EH}[L] \to \mathrm{vol}[L]$

2) $\mathrm{vol}[L](b)$
  ↳ only

3) $\mathrm{vol}[L](b)$ is convex $\to$ $\exists$ unique minimum

Great. But in the toric setting we could actually compute $\mathrm{vol}[L](b)$ in terms of the toric data. To compute the volume in general, we have to use more sophisticated techniques.

Note: there is still a cone image of the moment map:



here $r=2$

$b \in C_0 \subset C$

$$\mathrm{vol}[L] \neq (2\pi)^{\#}\, \mathrm{vol}\, \Delta(b) \quad !!$$

$\frac{\omega^n}{n!} = r^{2n-1} dr \wedge d\mu$

So far, we used $\mathrm{vol}[L] = \int_L d\mu \int_{r\leq 1} \frac{\omega^n}{n!}$

but is also true that $\mathrm{vol}[L] = \frac{1}{2^{n-1}(n-1)!} \int_M e^{-r^2/2} \frac{\omega^n}{n!}$

This simple observation has important consequences.

- the integral is over the full cone M
- $e^{-r^2/2}$ acts as a convergence factor
- recall the Hamiltonian function associated to a vector field X is $y_X = \frac{1}{2} r^2 \eta(X)$ $\Rightarrow$
  $H = y_\xi = \frac{1}{2} r^2 \eta(\xi) = \frac{1}{2} r^2 \to$ generates flow along Reeb

We can then write suggestively :

$$\mathrm{vol}[L] = \frac{1}{2^{n-1}(n-1)!} \underbrace{\int_M e^{-H+\omega}}$$

this can now be evaluated in terms of the fixed points of the Reeb vector field (generated by H)

This is the Duistermaat - Heckman theorem.
It is a special case of a more general localisation theorem which uses the fact that the integrand is equivariantly closed
$\left[(d + i_\xi)\, e^{-H+\omega} = 0 \right]$

- on the Kähler cone M the only fixed point is $r=0$ (the norm of ξ is precisely r^2)
- in order to apply the theorem we must resolve the singularity at $r=0$
- we compute the integral on a resolved M. Then we take the limit $r \to 0$.
- The result is independent of the choice of resolution.

The fixed points don't have to be isolated, although when they are the general formula simplifies considerably.

Let me write down the DH theorem and explain the various ingredients

$$\text{DH theorem:} \qquad \int e^{-H}\frac{\omega^n}{n!} = \sum_{\{F\}} \int_F \frac{\overbrace{e^{-H} e^{\omega}}^{\text{restricted to } F}}{\det\left(\frac{L\xi - \Omega}{2\pi i}\right)}$$

- $\sum_{\{F\}}$ is sum over connected components of fixed point sets F

- normal bundle to F : $\underset{\text{rank } k}{\mathcal{E}} = \mathcal{E}_1 \oplus \cdots \oplus \underset{\text{rank } n_R}{\mathcal{E}_R}$ $\qquad \sum_{\lambda=1}^{R} n_\lambda = k$

  ξ acts as $L\xi = i \operatorname{diag}\left(\mathbb{1}_{n_1}(b,u_1), \ldots, \mathbb{1}_{n_R}(b,u_R)\right)$

  $u_\lambda \in \mathbb{Z}^r$ "weights" of the action on $\mathcal{E}$

- Ω connection on $\mathcal{E}$

  Chern polynomial of $\mathcal{E}_\lambda$

$$\Rightarrow \quad \det\left(\frac{L\xi-\Omega}{2\pi i}\right) = \frac{1}{(2\pi)^k}\det\left(\frac{L\xi}{i}\right)\prod_{\lambda=1}^{R} \underbrace{\det\left(1+\frac{i}{(b,u_\lambda)}\Omega_\lambda\right)}_{\sum_{a\geq 0} c_a(\mathcal{E}_\lambda)\left(\frac{2\pi}{(b,u_\lambda)}\right)^a \,\in\, H^*(F;\mathbb{R})}$$

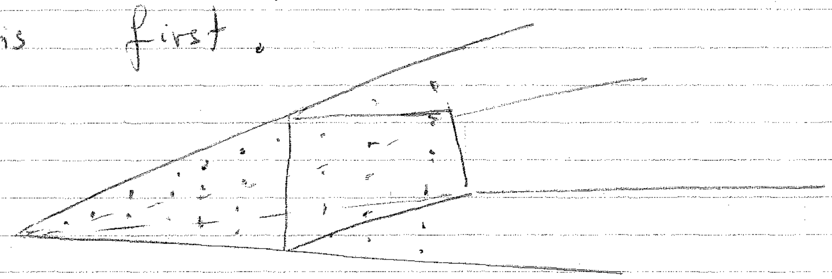
Let's look at the simpler case of isolated fixed points:

$$V(b) = \frac{\mathrm{vol}[L](b)}{\underbrace{\mathrm{vol}[S^{2n-1}]}_{\frac{2\pi^n}{(n-1)!}}} = \sum_{\{\text{points}\}\, p_A} \prod_{i=1}^{n} \frac{1}{(b, u_i^A)}$$

I'll show an example of these formula later.

I will now show that the fixed point formulas for the volume can be extracted from a limit of an equivariant index on the Kähler cone

The toric case is simpler to understand. So I will discuss this first.



$\left\{ \begin{matrix} \text{integral} \\ \text{points inside the cone } C^* \end{matrix} \right\} = C^* \cap \mathbb{Z}^n = S_C$

These points are clearly infinite So I introduce the following "generating function"

$$C(q, M) = \sum_{m \in S_C} q^m \qquad q^m = q_1^{m_1} \cdots q_n^{m_n}$$

claim: $$V(b) = \lim_{t \to 0} t^n C(e^{-tb}, M) \qquad q_i = e^{-tb_i}$$

toy example: $$\underbrace{\lim_{t\to 0} t \sum_{m=0}^{\infty} e^{-tmb}} = \int_0^\infty e^{-yb}\, dy = \frac{1}{b}$$

$$= t \left. \frac{1}{(1-q)} \right|_{q = e^{-tb}}$$

$$\rightarrow \quad \lim_{t \to 0} t^n \sum_{m \in S_C} e^{-t(b,m)} = \underbrace{\int_{C^*} e^{-(b,y)}\, dy_1 \cdots dy_n}_{\frac{(n-1)!}{2\pi^n} \mathrm{vol}[L](b)} \qquad (b,y) = r^2/2 \;!!$$

We would like also to be able to evaluate $C(q, M)$ concretely. To do this we'll have to interpret it

in a more abstract way, i.e. as an <u>index</u>.
In turn, this will allow to obtain a definition which is valid also outside the toric context.

- Recall: points in $S_C = C^{\vee} \cap \mathbb{Z}^n \longleftrightarrow \mathbb{C}[S_C] = \dfrac{\mathbb{C}[z_1, \dots, z_n]}{\langle f_1, \dots, f_s \rangle}$

ring of holomorphic functions on M

→ we want to count holomorphic functions on M, weighted by their charges under the $(\mathbb{C}^*)^r$-action

→ equivariant index of $\bar\partial$ operator on M

$$0 \to \Omega^{0,0}(M) \xrightarrow{\bar\partial} \Omega^{(0,1)}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n}(M) \to 0$$

$\Omega^{0,p} = \{$differential forms of Hodge type $(0,p)\}$

$\mathcal{H}^p(M) = H^{0,p}(M; \mathbb{C})$, $\mathcal{H}^p = 0 \quad p > 0$

comes from Kodaira vanishing theorem

$\mathcal{H}^0(M) = \{$holomorphic functions on $M\}$

$$\Rightarrow \quad C(q, \bar\partial, M) = \sum_{p=1}^{n} (-1)^p \, \mathrm{Tr}\{ q \mid \mathcal{H}^p(M) \}$$

trace over $(\mathbb{C}^*)^r$-action on $\mathcal{H}^p(M)$

$(\mathbb{C}^*)^r : \mathcal{H}^p(M) \to \mathcal{H}^p(M)$

$q = 1 \to$ ordinary index

→ infinite, because M is non-compact

Example : suppose L is regular ; $U(1) \to L$
$\downarrow$
V

focus on the complexified $U(1)$ action : $\mathbb{C}^*$ ↖ Fano (e.g., del Pezzo)

M is $K^{-1} \to V$
↓
anticanonical bundle

holomorphic functions of charge k : sections of $(K^{-1})^k \to V$

$$\mathrm{Tr}\{ q \mid \mathcal{H}^0(M)_k \} = q^k \underbrace{\dim H^0(V, (K^{-1})^k)}$$

Kodaira $\to$ $\|$

$$\chi(V, (K^{-1})^k) \qquad \text{Euler character}$$

Riemann-Roch $\to$ $\|$

$$\int_V e^{-k c_1(K)} \cdot \mathrm{Todd}(V)$$

$$\Rightarrow \quad C(q, M) = \sum_k q^k \int_V e^{-k c_1(K)} \cdot \mathrm{Todd}(V) =$$

$$= \int_V \frac{\mathrm{Todd}(V)}{1 - q e^{-c_1(K)}}$$

The general form for an arbitrary $(\mathbb{C}^*)^r$ action is (equivariant index theorem) :

$$C(q, M) = \sum_{\{F\}} \int_F \frac{\mathrm{Todd}(F)}{\prod_{\lambda \geq 1}^{R} \prod_j (1 - q^{u_\lambda} e^{-x_j})}$$

- F is the fixed point set of the $(\mathbb{C}^*)^r$ action on any resolution of M
- $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_R$ is the normal bundle to F

$\mathcal{E}_\lambda = \bigoplus_{j\geq 1}^{n_\lambda} \mathcal{L}_j$ $\qquad x_j = c_1(\mathcal{L}_j) \in H^2(F;\mathbb{Z})$

"basic characters" (splitting principle)

[e.g. $\text{Todd} = \prod \frac{x_a}{1-e^{-x_a}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots$

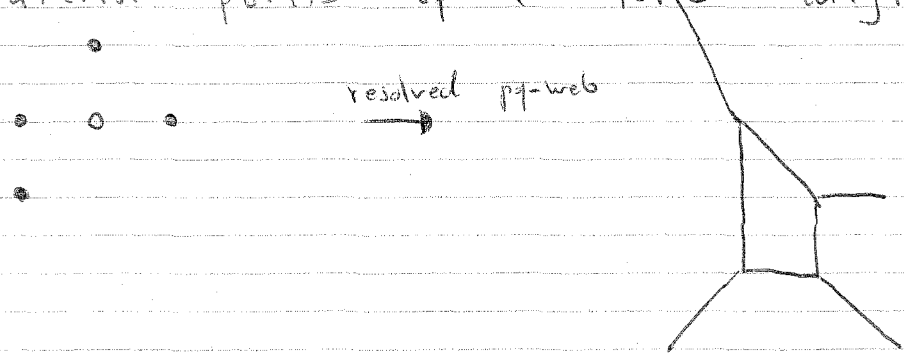
We can show that the general formula for $C(q, M)$ reduces to the general DH fixed point formula for the volume, thus proving in general that

$$V(b) = \lim_{t\to 0} t^n C(e^{-tb}, M)$$

This is very easy to see in the toric case:

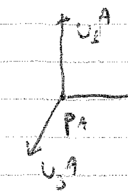
- first we have to resolve our toric singularity: include interior points of a toric diagram

e.g.



(this is a projection in 2d of a 3d picture!

at each point p_A:
(locally $\sim \mathbb{C}^n$)



$u_i^A \in \mathbb{Z}^n$ $(n=3)$
(they actually span $\mathbb{Z}^n \Rightarrow$ smooth,

at p_A $\quad (\mathbb{C}^*)^n : (z_1, \dots, z_n) \to (q^{u_1^A} z_1, \dots, q^{u_n^A} z_n)$

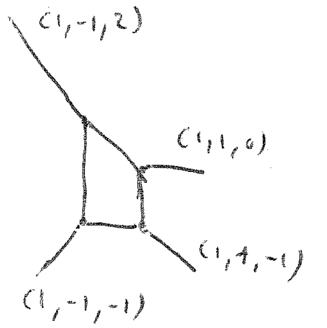
$\Rightarrow$ character $\quad C(q, \mathbb{C}^n_A) = \prod_{i=1}^{n} \frac{1}{(1 - q^{u_i^A})}$

Fixed point theorem then gives:

$$C(q, M) = \sum_{p_A} C(q, \mathbb{C}^n_A)$$

Now it's trivial to proofe that $\lim_{t \to 0} t^n C(e^{-tb}; M) = V(b$
as each single term in the sum over fixed points gives exactly the terms in the DH-type formula for the volume [namely $\prod_{i=1}^{n} \frac{1}{(b, u_i^A)}$].

- Exercise: you can check this very explicitely for the case of $dP_1 = Y^{2,1}$, whose toric resolution I've just shown:
  pick a basis $\quad w_1 = (-1,-1) \quad w_2 = (-1,0) \quad w_3 = (0,1) \quad w_4 = (1,0)$



The rest of the details can be easily filled in.