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Basic Physics of Magnetoplasmas-I: Single Particle Drift Montions

Vladimir CADEZ
Astronomical Observatory
Volgina 7
11160 Belgrade
SERBIA AND MONTENEGRO

LECTURE 1

SINGLE PARTICLE DRIFT MOTIONS

Vladimir M. Čadež

Astronomical Observatory Belgrade
Volgina 7, 11160 Belgrade, Serbia&Montenegro
Email: vladimir.cadez@phy.bg.ac.yu

Gaseous plasma is a mixture of moving particles of different species α having mass m_α and charge q_α. Usually but not necessarily always, such a plasma is globally electro-neutral, i.e. $\sum_\alpha q_\alpha = 0$.

In astrophysical plasmas, we often have mixtures of two species $\alpha = e, p$ or electron-proton plasmas as protons are the ionized atoms of Hydrogen, the most abundant element in the universe. Another plasma constituent of astrophysical significance are dust particles of various sizes and charges. For example, the electron-dust and electron-proton-dust plasmas ($\alpha = e, d$ and $\alpha = e, p, d$ resp.) are now frequently studied in scientific literature. Some astrophysical configurations allow for more exotic mixtures like electron-positron plasmas ($\alpha = e_-, e_+$) which are steadily gaining interest among theoretical astrophysicists.

In what follows, we shall primarily deal with the electron-proton, electro-neutral plasmas in magnetic field configurations typical of many solar-terrestrial phenomena.

To understand the physics of plasma processes in detail it is necessary to apply complex mathematical treatments of kinetic theory of ionized gases. For practical reasons, numerous approximations are introduced to the full kinetic approach which results in simplified and more applicable plasma theories. This lecture will show what can be learned about plasma dynamics by looking at motions of individual particles in external or predefined magnetic and electric fields assuming that the induced electromagnetic fields, produced by such moving charged plasma particles, are negligible in comparison with fields externally prescribed. Such an approximation of unaffected external fields significantly simplifies the analysis of particle dynamics and can be applied in many low plasma density configurations including those existing in the solar corona and in planetary magnetospheres and ionospheres.

Let us now consider a series of examples of different external magnetic and electric field configurations.

1.1 Case with $\vec{\mathbf{E}}$=const and $\vec{\mathbf{B}} = 0$

The equation of motion for a particle with mass m and charge q is

$$m\frac{d\vec{v}}{dt} = q\vec{E} \tag{1}$$

whose solution for the particle velocity

$$\vec{v} = \frac{q}{m}(t - t_0)\vec{E} + \vec{v}_0$$

represents a uniformly accelerated particle motion along a constant $\vec{E}$-field.

In this example, particles with charges of different sign move in opposite directions which results in electric currents:

$$\vec{j} = q_e\vec{v}_e + q_p\vec{v}_p$$

.

1.2 Case with $\vec{\mathbf{B}}$=const and $\vec{\mathbf{E}} = 0$

Motion of a charged particle in a constant magnetic field is described by

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B} \tag{2}$$

which immediately yields

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = 0 \qquad \Rightarrow \qquad |\vec{v}| = v = \text{const} \tag{3}$$

i.e. constancy of the velocity vector intensity. This further tells us that a charged particle does not gain any kinetic energy form the considered magnetic field.

Decomposing the velocity vector into two components

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$$

in directions parallel and normal to $\vec{B}$, the equation of motion (2) reduces to two equations for each of the velocity components:

$$m\frac{d\vec{v}_{\perp}}{dt} = q\vec{v}_{\perp} \times \vec{B} \qquad \text{and} \qquad \frac{d\vec{v}_{||}}{dt} = 0. \tag{4}$$

We see that the velocity component $\vec{v}_{||}$ along $\vec{B}$ remains constant in this case, i.e. unaffected by the presence of magnetic field and its magnitude $v_{||}$ is simply prescribed as the initial condition. Now, if $v_{||}$=const the same must also be true for the magnitude of the normal component $v_{\perp}$ since $v^2 \equiv v_{||}^2 + v_{\perp}^2 = \text{const}$ as already obtained in Eq (3). Thus:

$$v_{\perp},\, v_{||},\, v \equiv \sqrt{v_{\perp}^2, v_{||}^2} = \text{const} \tag{5}$$

To find the shape of the trajectory the particle moves along, we take the Cartesian geometry with $\vec{B} = (0,0,B)$, $\vec{v}_\perp = (v_x, v_y, 0)$ and $\vec{v}_\| = (0,0,v_z)$ and write Eq (4) as:

$$m\frac{dv_x}{dt} \;=\; qBv_y, \quad m\frac{dv_y}{dt} \;=\; -qBv_x, \quad m\frac{dv_z}{dt} \;=\; 0.$$

After some elementary rearrangements, we obtain the following set of equations:

$$\frac{d^2v_x}{dt^2} \;=\; -\omega_L^2 v_x, \quad v_x^2 + v_y^2 = v_\perp^2, \quad v_z \;=\; v_\|. \tag{6}$$

where $v_\|$ and $v_\perp$ are constants given as initial conditions.

Three velocity components follow from Eq (6) as:

$$\begin{aligned} v_x \;&\equiv\; \frac{dx}{dt} \;=\; v_\perp \cos(\omega_L t) \\ v_y \;&\equiv\; \frac{dy}{dt} \;=\; -v_\perp \sin(\omega_L t) \\ v_z \;&\equiv\; \frac{dz}{dt} \;=\; v_\| \end{aligned} \tag{7}$$

where:

$$\omega_L \;\equiv\; \frac{qB}{m} \tag{8}$$

is known as Larmor frequency (also the gyro or cyclotron frequency).

Eqs (9) finally yield the particle trajectory equation after one time-integration:

$$\begin{aligned} x - x_0 \;&=\; r_L \sin(\omega_L t) \\ y - y_0 \;&=\; r_L \cos(\omega_L t) \\ z - z_0 \;&=\; v_\| t, \end{aligned} \tag{9}$$

with

$$r_L \;\equiv\; \frac{v_\perp}{\omega_L} \tag{10}$$

known as the Larmor radius(also gyro-radius).

The trajectory (9) is a helix along the z-axis (i.e. in the direction of $\vec{B}$) with the pitch angle α given through the relation $v_\perp = v_\| \tan\alpha$.

The vector of the gyration angular velocity $\vec{\omega}_L$ follow also directly from the first equation Eqd4 integrated over time:

$$m\int \frac{d\vec{v}_\perp}{dt}dt \;=\; q\int \vec{v}_\perp dt \times \vec{B} \quad \Rightarrow \quad \vec{v}_\perp \;=\; \vec{\omega}_L \times \vec{r}_\perp \tag{11}$$

where

$$\vec{\omega}_L \;\equiv\; -\frac{q\vec{B}}{m}.$$

The considered charged particle motion is therefore a superposition of gyration (with the gyration radius and gyration frequency r_L and ω_L respectively) in a plane normal to magnetic filed lines, and a uniform motion of the center of gyration, the so called guiding center, along the field lines. In the guiding center description, such charged particle motion is identified as the motion of its guiding center.

According to Eqd5c, the positively charged particles gyrate about the magnetic field line in the clockwise direction while particles with a negative charge move in the opposite direction.

1.3 Case with constant $\vec{\mathbf{B}}$ and $\vec{\mathbf{E}}$

If a constant electric field is added to the previous configuration, the dynamics of a charged particle is governed by the equation

$$m\frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v} \times \vec{B} \tag{12}$$

which can be analyzed and solved in a similar way as done in Case 1.2. Thus, we decompose $\vec{v}$ and $\vec{E}$

$$\vec{v} = \vec{v}_\perp + \vec{v}_\parallel, \qquad \vec{E} = \vec{E}_\perp + \vec{E}_\parallel,$$

substitute these into Eq (12) and obtain two equations for propagations parallel and normal to the magnetic field:

$$m\frac{d\vec{v}_\parallel}{dt} = q\vec{E}_\parallel, \quad \text{and} \quad m\frac{d\vec{v}_\perp}{dt} = q\vec{E}_\perp + q\vec{v}_\perp \times \vec{B} \tag{13}$$

The first equation in Eq (13) says that the particle is accelerated along the magnetic field by the parallel component of the electric field in the same way as it happens in Case 1.1:

$$\vec{v}_\parallel = \frac{q}{m}(t - t_0)\vec{E}_\parallel + \vec{v}_0. \tag{14}$$

To solve the second equation, we switch to a new frame of reference moving with some constant speed $\vec{V}_E$ in a direction normal to $\vec{B}$ so that

$$\vec{v}_\perp = \vec{V}_E + \vec{u}_\perp \tag{15}$$

where $\vec{u}_\perp$ is the normal velocity component relative to the moving frame. The second Eq (13) now becomes:

$$m\left(\frac{d\vec{u}_\perp}{dt} + \frac{d\vec{V}_E}{dt}\right) = q\vec{E}_\perp + q\vec{V}_E \times \vec{B} + q\vec{u}_\perp \times \vec{B} \tag{16}$$

where $d\vec{V}_E/dt = 0$ as assumed in this case.

Since $\vec{V}_E$ has not been specified so far, we shall so choose it now that the first two terms on the right hand side of Eq (16) mutually cancel out:

$$q\vec{E}_\perp + q\vec{V}_E \times \vec{B} = 0 \quad \Rightarrow \quad \vec{V}_E = \frac{\vec{E}_\perp \times \vec{B}}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}. \tag{17}$$

The remaining part of Eq (16) is then:

$$m\frac{d\vec{u}_\perp}{dt} = q\vec{u}_\perp \times \vec{B} \tag{18}$$

which is the same type of equation as Eq (4) in Case 1.2. The velocity component $\vec{u}_\perp$ therefore describes an orbiting motion with Larmor frequency ω_L and Larmor radius r_L around magnetic field lines as viewed in the frame of reference moving with a constant velocity $\vec{V}_E$. Finally, the particle velocity components in the rest frame are given by Eqs (14)-(15) indicating an accelerated guiding center motion along magnetic field lines with $\vec{v}_\parallel$ and a superimposed perpendicular drift motion with

$$\vec{V}_E = \frac{\vec{E} \times \vec{B}}{B^2}. \tag{19}$$

This drift, called the 'E×B' drift, is charge independent and therefore induces no electric currents as both the positive and negative charges move in the same direction as seen in Eq (19) for $\vec{V}_E$.

1.4 Case with constant $\vec{\mathbf{B}}$ and $\vec{\mathbf{F}}$

If some additional constant external force $\vec{F}$ acts on a charged particle moving in a constant magnetic field we start from the equation of motion:

$$m\frac{d\vec{v}}{dt} = \vec{F} + q\vec{v} \times \vec{B} \tag{20}$$

Comparing this equation with the equation of motion (12) in Case 1.3 we see that the only difference between them is the replacement of $q\vec{E}$ by $\vec{F}$. This means that all derivations performed in Case 1.3 can now be repeated here by taking $\vec{F}/q$ instead of $\vec{E}$. Thus, we conclude that a charged particle spirals around magnetic field lines and its guiding center velocity has two components describing an accelerated motion along the magnetic field lines due to $\vec{F}_\parallel$, and a drift motion across the filed lines with velocity $\vec{V}_F$:

$$\vec{V}_F = \frac{\vec{F}_\perp \times \vec{B}}{qB^2} = \frac{\vec{F} \times \vec{B}}{qB^2} \tag{21}$$

whose orientation is charge dependent. As a result, this drift, also called the force drift, produces electric currents as charges with opposite signs drift in opposite directions.

One of interesting examples of a force drift is the gravitational drift occurring in presence of a uniform gravitational field $\vec{g}$ when $\vec{F} = m\vec{g}$. In this case, the gravitational drift velocity $\vec{V}_g$ follows from Eq (21) as:

$$\vec{V}_g \;=\; \frac{m\vec{g} \times \vec{B}}{qB^2} \tag{22}$$

In astrophysical plasmas, this drift contributes to formation of ring currents among other things.

1.5 Case with $\vec{\mathbf{B}}$ = const and $\vec{\mathbf{E}} = \vec{E}(t)$

Take now that the uniform electric field from Case 1.3 varies in time, $\vec{E} = \vec{E}(t)$, and let us examine how this effects plasma particle motions.

The time variation is assumed small on the time scale of one gyration period $\tau_L = 2\pi/\omega_L$ (taking $q > 0$), and only the first time derivative is retained in series expansions meaning that $d\vec{E}/dt \approx$const. Now, we can go back to Case 1.3 and repeat the whole analytical procedure up to Eq (16):

$$m\left(\frac{d\vec{u}_\perp}{dt} + \frac{d\vec{V}_E}{dt}\right) \;=\; q\vec{E}_\perp + q\vec{V}_E \times \vec{B} + q\vec{u}_\perp \times \vec{B}. \tag{23}$$

The term $d\vec{V}_E/dt$ now remains in the equation and its presence represents effects of a slowly time-varying electric field. Same as in Case 1.3, we go to a new coordinate system moving with drift velocity $\vec{V}_E$ given by Eq (17)

$$\vec{V}_E \;=\; \frac{\vec{E} \times \vec{B}}{B^2}$$

which reduces Eq (23) to:

$$m\frac{d\vec{u}_\perp}{dt} \;=\; -m\frac{d\vec{V}_E}{dt} + q\vec{u}_\perp \times \vec{B} \tag{24}$$

This equations is equivalent to Eq (20) with the external force $\vec{F}$ given by:

$$\vec{F} \;=\; -m\frac{d\vec{V}_E}{dt}$$

which promotes an additional drift motion of the guiding center with velocity $\vec{V}_F \equiv \vec{V}_P$ called the polarization drift:

$$\vec{V}_P \;=\; -\frac{m}{qB^2}\frac{d\vec{V}_E}{dt} \times \vec{B} \;=\; -\frac{m}{qB^2}\frac{d}{dt}\left(\frac{\vec{E}_\perp \times \vec{B}}{B^2}\right) \times \vec{B}$$

or

$$\vec{V}_P \;=\; \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}. \tag{25}$$

The polarization drift $\vec{V}_P$ is charge sign dependent and therefore induces electric currents in plasmas.

In this example, the total drift velocity $\vec{V}_D$ is then:

$$\vec{V}_D = \vec{V}_P + \vec{V}_E \tag{26}$$

with the following ordering:

$$\frac{V_P}{V_E} \equiv \frac{\left|\dfrac{m}{qB^2}\dfrac{d\vec{E}_\perp}{dt}\right|}{\left|\dfrac{\vec{E}_\perp \times \vec{B}}{B^2}\right|} \sim \frac{\tau_L}{E_\perp}\frac{dE_\perp}{dt} \ll 1.$$

1.6 Case with nonuniform magnetic field: $\vec{\mathbf{B}}$=(0,0,B(x))

Examine now an example of a charged particle with $q > 0$ moving in a magnetic field with straight lines parallel to the z-axis whose density varies in the x-direction: $\vec{B} = B(x)\hat{e}_z$. Let this x-dependence be sufficiently weak as to allow the first order series expansion as an acceptable approximation

$$B(x) \;=\; B(0) + x\,\left.\frac{dB}{dx}\right|_{x=0} \tag{27}$$

and let the change of $B(x)$ over the distance of the gyration radius r_L be small relative to $B(x)$ itself: $r_L dB/dx \ll B$.

In what follows, we shall consider only the normal component of the particle velocity vector $\vec{v}$ as the parallel component remains unaffected by the $\vec{B}$-field and we take $\vec{v}_\parallel = 0$ by the initial condition so that $\vec{v} \cdot \vec{B} = 0$. The particle velocity has thus only two components:

$$v_x = \frac{dx}{dt} \qquad \text{and} \qquad v_y = \frac{dy}{dt},$$

and the vector equation of motion Eq (2) can be expressed as a system of two scalar equations in the following way:

$$\begin{aligned} m\frac{dv_x}{dt} &= q\left(B(0) + x\,\left.\frac{dB}{dx}\right|_0\right)\frac{dy}{dt} \\ m\frac{dv_y}{dt} &= -q\left(B(0) + x\,\left.\frac{dB}{dx}\right|_0\right)\frac{dx}{dt}. \end{aligned} \tag{28}$$

The trajectory of particle motion described by Eq (28) would be circular with the radius r_L and the gyration period τ_L if the magnetic field were uniform: $dB/dx = 0$. The presence of a weak magnetic field non uniformity dB/dx slightly modifies the trajectory in the sense that it is not a closed circle any more and the particle position shifts by some Δy along the y-axis after each gyration time τ_L. This results into a drift motion along the y-axis known as the magnetic gradient drift with speed $V_G = \Delta y/\tau_L$. To obtain the displacement Δy we integrate the first equation in Eq (28) over one time period τ_L. As the particle motion remains periodic in the x-direction we have:

$$m \int_0^{\tau_L} \frac{dv_x}{dt} dt \;=\; mv_x(\tau_L) - mv_x(0) \;\approx\; 0.$$

which further yields:

$$qB(0) \int_0^{\tau_L} \frac{dy}{dt} dt + q \left.\frac{dB}{dx}\right|_0 \int_0^{\tau_L} x \frac{dy}{dt} dt \;=\; 0$$

or

$$B(0)\Delta y - \pi r_L^2 \left.\frac{dB}{dx}\right|_0 \;=\; 0 \quad \Rightarrow \quad \Delta y \;=\; \frac{\pi r_L^2}{B} \frac{dB}{dx} \tag{29}$$

with expressions (9) for $x(t)$ and $y(t)$ taken into account.

The gradient drift velocity is now:

$$\vec{V}_G \;\equiv\; \frac{\Delta y}{\tau_L} \hat{e}_y \;=\; \frac{\omega_L r_L^2}{2B} \frac{dB}{dx} \hat{e}_y = \frac{mv^2}{2qB^2} \frac{dB}{dx} \hat{e}_y$$

or in a full vector form:

$$\vec{V}_G \;=\; \frac{mv^2}{2qB^3} \vec{B} \times \nabla B. \tag{30}$$

As the magnetic field gradient drift (30) depends on the sign of charge q it induces electric currents in plasmas which occurs in planetary magnetospheres for example.

1.7 Case of stationary $\vec{\mathbf{B}}$-field with curved and parallel field lines

A uniform and stationary magnetic field configuration studied in Case 1.2 is now assumed to be modified by adding a small curvature to its field lines that causes corrections to particle motion of the first order of smallness. In this example, the curved magnetic field lines are parallel and uniformly distributed through any perpendicular plane. In other words, the magnetic field intensity B does not change in the direction along the magnetic field vector $\vec{B} = B\hat{e}_s$.

The basic type of a plasma particle motion in this case is a gyration with its guiding center moving along slightly curved magnetic field lines with velocity $\vec{v}_\parallel = v_\parallel \hat{e}_s$ which introduces a centrifugal force $\vec{F}_c$. This results into a force drift described in Case 1.4 with the drift velocity:

$$\vec{V}_c \;=\; \frac{\vec{F}_c \times \vec{B}}{qB^2} \tag{31}$$

known as the centrifugal drift.

The explicit expression for the centrifugal force and the related drift follows from:

$$\vec{F}_c \equiv -m\frac{d\vec{v}_\parallel}{dt} = -mv_\parallel\frac{d\hat{e}_s}{dt} = -mv_\parallel\frac{d\hat{e}_s}{ds}\frac{ds}{dt} = -\frac{mv_\parallel^2}{B}\frac{d\vec{B}}{ds} \tag{32}$$

where:

$$v_\parallel = \frac{ds}{dt}$$

while the derivative of the magnetic field along the curved filed line s can be written as:

$$\frac{d\vec{B}}{ds} = (\hat{e}_s \cdot \nabla)\vec{B} = \frac{1}{B}(\vec{B}\cdot\nabla)\vec{B}.$$

Eq (32) for the centrifugal force $\vec{F}_c$ then takes the final form:

$$\vec{F}_c = -\frac{mv_\parallel^2}{B^2}(\vec{B}\cdot\nabla)\vec{B} \tag{33}$$

while the centrifugal drift velocity (31) becomes:

$$\vec{V}_c = \frac{mv_\parallel^2}{qB^4}\vec{B}\times\left[(\vec{B}\cdot\nabla)\vec{B}\right]. \tag{34}$$

As can be seen from Eq (34), the centrifugal drift $\vec{V}_c$ is charge dependent and therefore produces electric currents. These effects of curved magnetic fields are commonly present in ring current formation mechanisms in planetary magnetospheres for example.

1.8 Stationary $\vec{\mathbf{B}}$ field with slightly convergent field lines

Let us now assume the magnetic field lines from Case 1.2 slightly convergent and axially symmetric with respect to the z-axis. For this reason, it is more convenient to apply a cylindrical coordinate system $(\hat{e}_r, \hat{e}_\phi, \hat{e}_z)$ oriented along the axis of symmetry so that the considered weakly convergent magnetic field $\vec{B}$ is now given by

$$\vec{B} = (B_r(r,z), 0, B_z(r,z)) \qquad \text{with} \qquad |B_r(r,z)| \ll |B_z(r,z)|.$$

The positively charged particle motion in such a magnetic field configuration retains the two basic properties from Case 1.2 where the magnetic field was uniform: The guiding center motion with velocity $\vec{v}_\parallel$ along a magnetic field line, let it be the line of the axis of symmetry i.e. the z-axis, and a gyration with velocity $\vec{v}_\perp$ about the same field line. The difference now is that the uniform $\vec{B}$-filed considered in Case 2.1 had no effect on $\vec{v}_\parallel$ while the magnetic field with slightly convergent field lines also exerts a parallel force $\vec{F}_\parallel$ along the z-axis which effects the guiding center velocity $\vec{v}_\parallel$:

$$\vec{F}_\parallel \equiv F_z\hat{e}_z = qB_r\vec{v}_\perp \times \hat{e}_r \tag{35}$$

To calculate the radial magnetic field component B_r we start from the Gauss law:

$$\nabla \cdot \vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial}{\partial z}B_z$$

then perform an integration over the coordinate r between $r = 0$ and $r = r_L$ with conditions $\partial B_z/\partial z \approx$ const and $B_r(0, z) = 0$ which yields:

$$r_L B_r(r_L, z) = -\frac{\partial B_z}{\partial z}\int_0^{r_L} r dr$$

or

$$B_r(r_L, z) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z}. \tag{36}$$

Taking Eq(11) for the gyration velocity, i.e.:

$$\vec{v}_\perp = \vec{\omega}_L \times \vec{r}_L = -\omega_L r_L \hat{e}_z \times \hat{e}_r = v_\perp \hat{e}_r \times \hat{e}_z,$$

we finally obtain from Eq (35) the following expression for $\vec{F}_\parallel$:

$$\vec{F}_\parallel = -\frac{1}{2}qr_L v_\perp \frac{\partial B_z}{\partial z}(\hat{e}_r \times \hat{e}_z) \times \hat{e}_r = -\frac{mv_\perp^2}{2B_z}\frac{\partial B_z}{\partial z}\hat{e}_z$$

or:

$$\vec{F}_\parallel \approx -\frac{W_\perp}{B}\frac{dB}{dz}\hat{e}_z \quad \text{with} \quad W_\perp \equiv \frac{1}{2}mv_\perp^2 \tag{37}$$

where $W_\perp$ is the particle kinetic energy of orbital motion, and with the assumption $B_z \gg B_r$ equivalent to $B \approx B_z$ taken into account.

According to Eq (37), the parallel force $\vec{F}_\parallel$ is oriented in the direction of decreasing of the magnetic field intensity. In other words, this force tends to slow down and eventually to stop the guiding center motion toward the region with stronger magnetic field. The details of this process follow directly from the equation of motion under action of the parallel force Eq (37). Thus, expressing the parallel acceleration term in Eq(37) as:

$$m\frac{d\vec{v}_\parallel}{dt} = m\frac{d\vec{v}_\parallel}{dz}\frac{dz}{dt} = mv_\parallel\frac{dv_\parallel}{dz}\hat{e}_z = \frac{dW_\parallel}{dz}\vec{e}_z,$$

so that Eq (37) reduces to:

$$\frac{dW_\parallel}{dz} = -\frac{W_\perp}{B}\frac{dB}{dz}$$

or to

$$\frac{dW_\perp}{dz} = \frac{W_\perp}{B}\frac{dB}{dz} \tag{38}$$

with the particle energy conservation law $W \equiv W_\perp + W_\parallel$ =const taken into account.

Eq (38) can easily be integrated which yields:

$$M \equiv \frac{W_\perp(z)}{B(z)} = const. \tag{39}$$

The quantity M is the magnetic moment of the particle, it remains constant within the applied approximation of weak divergence of magnetic filed lines, and is known as the first adiabatic invariant. It is easy to show the existence of another adiabatic invariant, the so called third adiabatic invariant, which is the magnetic flux Φ_B through the surface encircled by the quasi-circular particle orbit:

$$\Phi_B \equiv \pi r_L^2(z)B(z) = \pi \frac{m^2 v_\perp^2(z)}{q^2 B(z)} = 2\pi M \frac{m}{q^2} = \text{const.}$$

Constancy of both the particle energy W and magnetic moment M determines the motion of the guiding center along the magnetic field. Namely, if the considered charged particle with total energy $W = W_\perp + W_\parallel$ is moving with the velocity $\vec{v}_\parallel$ toward stronger magnetic field, its parallel kinetic energy $W_\parallel$ is falling off due to the force $\vec{F}_\parallel$ and eventually reaches zero at some location $z = z_r$ called the particle reflection point or the mirror point:

$$W_\parallel(z_r) = W - MB(z_r) = 0 \tag{40}$$

After that, the particle velocity $\vec{v}_\parallel$ changes its orientation and the guiding center of the particle starts moving in the opposite direction toward the region of a weaker magnetic field. The location of the reflection point z_r in a given magnetic field configuration thus depends on two constants of motion specified as initial conditions: the kinetic energy W and magnetic moment M of a moving particle.

From Eq (40) we easily obtain the maximal magnetic field strength $B_{\max} \equiv B(z_r)$ a particle with given W and M prescribed at some initial position $z = z_0$, can reach before being bounced back into the domain of a weaker magnetic field. Namely, from:

$$M = \frac{W_\perp(z)}{B(z)} = \frac{W}{B(z)} \sin^2\theta(z) = \frac{W}{B(z_0)} \sin^2\theta(z_0)$$

we get:

$$B_{\max} = \frac{B(z_0)}{\sin^2\theta(z_0)} \tag{41}$$

where the pitch angle θ is defined as:

$$v_\perp = \tan(\theta) v_\parallel \quad \text{or} \quad v_\perp = \sin(\theta) v.$$

Finally, the same results for drift motions along the magnetic field are valid if the considered slightly divergent/convergent magnetic field topology based on Case 1.2 is replaced by a similarly modified curved field from Case 1.7. The parallel motion of the guiding center then takes place along a new curved coordinate s-line taken to follow a chosen $\vec{B}$-field line rather than along the $z-$axis.

As to the above expressions, they remain as they are only with the coordinate z formally replaced by the coordinate s. Of course, the curved magnetic field now introduces also a transverse centrifugal drift $\vec{V}_c$ given by Eq (34).

The described magnetic field topology and related charged particle motions are typically found in planetary magnetospheres and also in laboratory plasma trapping devices.