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Basic Physics of Magnetoplasmas-II: Fluid Description and MHD

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LECTURE 2

FLUID DESCRIPTION AND MHD

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In many magnetized plasmas in nature and laboratory, we may introduce certain realistic assumptions that simplify mathematical procedures and help making a better insight into physical processes involved. Thus, a common way to study a system with a large number of charged particles is to introduce the fluid approximation in which we deal with a number of macroscopic quantities of the system that are averages of microscopic physical properties of individual particles. This approach assumes a sufficiently large number of particles in an elementary plasma volume dV = dxdydz so that their mean free path $\bar{\ell}$ obeys dx, dy, dz $\gg \bar{\ell}$. In other words, the plasma is considered as a continuous fluid characterized by its macroscopic properties: the density $\rho \equiv$dm/dV, temperature T, pressure p, fluid velocity $\vec{v}$ etc.

When motions of a single charged particle were considered, we took the external electric and magnetic fields unaffected which, however, may easily not be true in presence of a large number of particles. Namely, each moving charged particle creates its own electric and magnetic field that add up and may significantly modify the initial magnetic and electric field configurations. This has a further feedback effect on particle motions and, consequently, the plasma dynamics becomes more complex and requires a full self-consistent description of plasma dynamics with macroscopic electromagnetic field and macroscopic plasma parameters mutually coupled. From the mathematical point of view, this means that we have to use the full system of fluid dynamic equations together with Maxwells equations.

Plasma can often be considered globally electro-neutral meaning the equality of the total positive and negative electric charge. On a local scale, however, a charge separation may occur which results into local electric fields according to the Gauss law.

The fluid approach can be utilized also for a plasma composed of several different species when each of them is treated as a separate fluid with some interaction force acting among them like, for example, electric force, viscous friction etc. A multi-component plasma fluid approach is commonly used in plasma studies with typical components composed of electrons, ions, electro neutral particles, and dust particles of practically arbitrary size and charge.

In many situations, one can treat plasma as a single fluid when electrons and their motions are not recognizable as a separate plasma component, instead they are treated as electric currents in a moving electro-conducting fluid satisfying Maxwells equations and fluid equations of motion. Such a single fluid treatment of a plasma in a magnetic field is the essence of the magneto-hydrodynamical or MHD approximation. Although simplified, the MHD treatment reveals and describes many real plasma features existing and observed in nature.

2.1 Basic equations of fluid dynamics

All physical variables of plasma dynamics are field variables in the hydrodynamic approach meaning they are functions of the position $\vec{r}$ and time t:

$$\vec{v} = \vec{v}(\vec{r}, t);\ p = p(\vec{r}, t);\ \rho = \rho(\vec{r}, t);\ \vec{B} = \vec{B}(\vec{r}, t);\ \text{ etc.} \tag{1}$$

Their space-time behavior is governed by standard fluid equations which are derived either directly from the conservation laws of mass, momentum and energy, for an arbitrary fluid volume, or by averaging statistic equations of the kinetic theory for individual particles. These equations are supplemented by macroscopic Maxwells equations obtained from microscopic equations for a system of individual particles by the same averaging procedure.

The fluid equations govern the dynamics of a fluid parcel with mass dm occupying the elementary volume dV at location $\vec{r} \equiv (x, y, z)$ at time t. After the division by dV, the resulting equations involve physical quantities related to the unit volume i.e. their densities: the mass density $\rho \equiv \mathrm{dm/dV}$, force density $\vec{f} \equiv d\vec{F}/dV$, charge density $\rho_q \equiv \mathrm{dq/dV}$, electric current density $\vec{j}$ etc.

The set of equations fully describing the dynamics of an electro-conducting plasma is as follows:

The continuity equation resulting from the conservation of mass law:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \; = \; 0. \tag{2}$$

The momentum equation resulting from the momentum conservation law:

$$\rho \frac{d\vec{v}}{dt} \; = \; -\nabla p + \vec{f}_g + \vec{f}_L + \vec{f}. \tag{3}$$

The acceleration term in Eq (3) can be expressed as:

$$\frac{d\vec{v}}{dt} \; \equiv \; \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \tag{4}$$

The derivative $\partial \vec{v}/\partial t$ is related to the local time variation of the velocity field, and $\vec{v} \cdot \nabla \vec{v}$ is the convection term resulting from velocity variation as the fluid moves with velocity $\vec{v}$ in a spatially dependent velocity field $\vec{v}(\vec{r})$. The convective term can further be written as:

$$\vec{v} \cdot \nabla \vec{v} = \nabla \frac{v^2}{2} + (\nabla \times \vec{v}) \times \vec{v}. \tag{5}$$

On the right-hand side of Eq (3) we have acting forces per unit volume. Typically they are the pressure gradient force $-\nabla p$, the gravitational force $\vec{f}_g$:

$$\vec{f}_g \;=\; -\nabla\phi_g \quad \text{where} \quad \nabla^2\phi_g = 4\pi G\rho \tag{6}$$

(G=6.672× 10^{-11} N $\mathrm{m^2kg^{-2}}$ - the gravitational constant), the Lorentz force $\vec{f}_L$:

$$\vec{f}_L \;=\; \rho_q\vec{E} + \vec{j}\times\vec{B}, \tag{7}$$

with ρ_q and $\vec{j}$ being the charge density and electric current density respectively. In addition, there is $\vec{f}$ that stands for any other possible force acting per unit volume that may be present in Eq (3).

The energy equation depends on types of energy involved and on dissipative properties of plasma in the considered problem. In this lecture, we make a frequent assumption of plasma behaving as an ideal gas obeying the perfect gas law:

$$p = \frac{\mathcal{R}}{\mathcal{M}}\rho T,$$

($\mathcal{R}$=8.314 J $\mathrm{mol^{-1}K^{-1}}$ - the universal gas constant; $\mathcal{M}$ - molar mass) with negligible effects of thermal, viscous and Joule heating energy dissipations. In absence of other energy sources (possible chemical reactions, latent heat in phase transitions, etc), the energy conservation can be expressed as the adiabatic law:

$$\delta Q \;\equiv\; c_v dT + p d(1/\rho) = 0$$

which together with the perfect gas law yields the equation for adiabatic processes:

$$\frac{d}{dt}\ln p \;=\; \gamma\frac{d}{dt}\ln\rho \tag{8}$$

(γ=c_p/c_v; c_v, c_p=c_v+$\mathcal{R}$ - specific heats at constant volume and pressure respectively).

Macroscopic Maxwell's equations are represented by:

$$\begin{aligned} \nabla\cdot\vec{E} \;&=\; \frac{1}{\epsilon}\rho_q, \quad \nabla\times\vec{E} \;=\; -\frac{\partial\vec{B}}{\partial t}, \\ \nabla\cdot\vec{B} \;&=\; 0, \quad \nabla\times\vec{B} \;=\; \mu_0\vec{j} + \mu_0\epsilon\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{9}$$

and they have to be supplemented by Ohm's law for a moving electro-conducting plasma:

$$\vec{j} \;=\; \sigma\vec{E}' \equiv \sigma(\vec{E} + \vec{v}\times\vec{B}) \tag{10}$$

If the plasma electrical conductivity σ is finite, the Joule heating is not negligible and, strictly speaking, the adiabatic law of no heat transfer in the considered process is not applicable. Yet, in many cases, one may keep σ finite and assume the Joule heating ineffective to influence the energy balance appreciably.

2.2 MHD approximation

In the non-relativistic MHD approximation, we shall now be dealing with, the plasma is considered an electro-conducting fluid moving with typical speed v^* that is much smaller than the speed of light c: $v^* \ll c$.

The plasma is fully ionized, globally electro-neutral, permeated by a macroscopic external magnetic field but no macroscopic external electric field is assumed to be present. All electric fields are therefore induced by motions of a conducting plasma in a magnetic field.

The dynamics of plasma motions and changes in electric and magnetic fields are characterized by the same typical time and length scales τ^* and ℓ^* respectively.

As individual particles are undistinguishable in a fluid approach, we take $r_L^* \ll \ell^*$ for a typical Larmor radius r_L^*.

Introducing typical values X^* for each physical quantity X in Eqs (2)-(10) we can compare contributions of individual terms in these equations and show that some of them are negligible and can be ignored and, as a result, we obtain what is called a set of MHD equations. In this sense, let us first look at Ohm's law (10) and take both terms to be of the same order of magnitude:

$$|\vec{E}| \sim |\vec{v} \times \vec{B}| \quad \Rightarrow \quad E^* \sim v^* B^* \quad \text{where} \quad v^* \sim \frac{\ell^*}{\tau^*} \tag{11}$$

which gives an estimate of the magnitude of induced electric field in terms of typical fluid speed and magnetic field.

Looking at Maxwell's equations (9) and taking for the derivatives:

$$|\nabla| \sim \frac{1}{\ell^*} \quad \text{and} \quad \left|\frac{\partial}{\partial t}\right| \sim \frac{1}{\tau^*},$$

we establish additional relations among typical values.

Thus:

$$|\nabla \cdot \vec{E}| = \left|\frac{1}{\epsilon}\rho_q\right| \quad \Rightarrow \quad \rho_q^* \sim \frac{E^*\epsilon}{\ell^*} = \frac{B^*}{\tau^*}\epsilon \tag{12}$$

offers an estimate for the net charge separation induced by fluid motions in a $\vec{B}$-field.

The ratio of two terms in the fourth Maxwell's equation i.e. in Ampère's law:

$$\frac{\left|\mu_0\epsilon\dfrac{\partial\vec{E}}{\partial t}\right|}{\left|\nabla \times \vec{B}\right|} \sim \frac{\dfrac{E^*}{\tau^*}}{c^2\dfrac{B^*}{\ell^*}} = \left(\frac{v^*}{c}\right)^2 \ll 1 \tag{13}$$

shows a negligible contribution of the polarization current and Ampère's law in the MHD approximation reads as:

$$\nabla \times \vec{B} = \mu_0\vec{j}. \tag{14}$$

A comparison of two terms in the Lorentz force (7):

$$\frac{\left|\rho_q \vec{E}\right|}{\left|\vec{j} \times \vec{B}\right|} \sim \frac{\rho_q^* E^*}{j^* B^*} \sim \frac{\epsilon \frac{E^{*2}}{\ell^*}}{\frac{B^{*2}}{\mu_0 \ell^*}} = \left(\frac{v^*}{c}\right)^2 \ll 1 \tag{15}$$

indicates the major contribution to $\vec{f}_L$ coming from the magnetic field only and we may write in the MHD approximation:

$$\vec{f}_L = \vec{j} \times \vec{B}. \tag{16}$$

As the net charge density ρ_q appears solely in the Lorenz force through the term $\rho_q \vec{E}$ which turned out to be negligible in Eq (16), we conclude that ρ_q, determined from the Gauss law for the $\vec{E}$-field (the second Maxwell equation) plays no role in the MHD dynamics.

The third Maxwell's equation, the Lenz rule, combined with Ohm's law (10) and Ampère's law (14) becomes

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B} \quad \text{assuming} \quad \eta \equiv \frac{1}{\mu_0 \sigma} = \text{const}, \tag{17}$$

known as the magnetic induction equation for a plasma with constant electric conductivity: σ=const. This is an important MHD equation telling us that the magnetic field changes for two reasons: the plasma motions $\vec{v}$, and the resistive dissipations given by the diffusion term with magnetic diffusivity η in Eq (17).

To summarize, the full set of MHD equations for a plasma with a constant electrical conductivity σ consists of standard equations of fluid dynamics supplemented by expressions containing the magnetic field:

$$\begin{aligned} &\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}, \quad \nabla \times \vec{B} = \mu_0 \vec{j}, \\ &\vec{f}_L = \vec{j} \times \vec{B}, \quad \vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}). \end{aligned} \tag{18}$$

2.3 Some properties of Lorentz force

Let us look at the Lorentz force and see how it acts in an MHD fluid. For this purpose, we substitute Ampère's law (14) into Eq (16) which gives:

$$\vec{f}_L \equiv \vec{j} \times \vec{B} = \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} = -\nabla \frac{B^2}{2\mu_0} + \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B}. \tag{19}$$

The Lorentz force Eq (19) is thus a sum of two terms each related to a particular type of magnetic force:

The first term, $-\nabla p_m$, with $p_m \equiv B^2/2\mu_0$ known as the magnetic pressure, represents a magnetic pressure gradient force acting in the same way as the thermal pressure gradient force in the momentum equation Eq (3) and we can introduce the total pressure by $p_{\text{tot}} \equiv p + p_m$. The relative significance of these two pressures to a given MHD process is defined by a dimensionless number $\beta \equiv p/p_m$ called the plasma beta parameter. For example, in a case of $\beta \ll 1$, the magnetic pressure dominates and the dynamics of the considered process involves primarily magnetic forces.

The second term in Eq (19), $(\vec{B}\cdot\nabla)\vec{B}/\mu_0$, is a force arising from the curvature of magnetic field lines as it depends on the change of $\vec{B}$ in the direction of the $\vec{B}$-field itself. This type of force appears also in mechanics as a result of curvature stress in elastic strings and rods. Magnetic field lines thus behave as elastic material strings whose mass and inertia come from the attached surrounding plasma.

In MHD, the Lorentz force acting as a combination of magnetic pressure gradient force and elastic curvature stress plays an important role in formation of small amplitude disturbances with properties of both the longitudinal (from the magnetic pressure) and transverse (from the curvature stress) waves.

2.4 Ideal MHD

In a spacial case of a perfectly electro-conducting fluid with infinite electrical conductivity $\sigma \to \infty$, i.e. $\eta = 0$, we are talking about the ideal MHD. The induction equation (17) is then:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \tag{20}$$

while Ohm's law (10) with $\sigma \to \infty$ reduces to:

$$\vec{E}' = 0 \quad \Rightarrow \quad \vec{E} = -\vec{v} \times \vec{B} \tag{21}$$

as the electric current $\vec{j}$ has to stay finite according to Ampére's law even if $\sigma$is very large. In the ideal MHD, the electric current $\vec{j}$ cannot be determined by Ohm's law (10) and only Ampère's law (14) has to be used.

The ideal MHD approach can also be applied to real plasmas with some finite σ provided the scaling parameters, relevant to the considered plasma process, yield negligible effects of a finite conductivity. In other words, this means that in the induction equation Eq (17), the diffusive term has to be comparatively small:

$$\frac{|\eta \nabla^2 \vec{B}|}{|\nabla \times (\vec{v} \times \vec{B})|} \ll 1 \quad \Rightarrow \quad \frac{1}{\mu_0 \sigma \ell^* v^*} \equiv \frac{1}{R_m} \ll 1 \tag{22}$$

where $R_m = \mu_0 \sigma \ell^* v^*$ is a dimensionless quantity known as the magnetic Reynolds number. The ideal MHD thus requires the condition $R_m \gg 1$ which is easily satisfied for many large scale processes in the Solar and planetary plasmas.

A large Reynolds number also yields a negligible electric field $\vec{E}'$ observed in a reference frame moving with velocity $\vec{v}$. Namely, according the expression Eq (10) for Ohm's law and taking Ampère's law (14) into account, we can write:

$$\frac{|\vec{E}'|}{|\vec{E}|} = \frac{j^*}{\sigma v^* B^*} = \frac{1}{\mu_0 \sigma \ell^* v^*} = \frac{1}{R_m} \ll 1, \tag{23}$$

which agrees with Eq (21).

2.5 Concept of frozen-in $\vec{B}$-field

There are two important MHD theorems describing motions of infinitely conducting plasma:

Theorem I: Magnetic flux Φ_B through any closed contour L moving freely with the fluid velocity $\vec{v}$ does not change in time:

$$d\Phi_B \;=\; d\Phi_B^{(1)} + d\Phi_B^{(2)} = 0 \tag{24}$$

where:

$$d\Phi_B^{(1)} \;\equiv\; \frac{\partial \Phi_B}{\partial t} dt, \quad d\Phi_B^{(2)} \;\equiv\; (\vec{v} \cdot \nabla)\Phi_B dt \quad \text{and} \quad \Phi_B = \iint\limits_{S_{(L)}} \vec{B} \cdot d\vec{S},$$

and $S_{(L)}$ is the surface bounded by the contour L.

To prove Eq (24), we note that the $d\Phi_B^{(1)}$ is the local change of the flux Φ_B through the surface $S_{(L)}$ in a time interval dt:

$$d\Phi_B^{(1)} \;=\; \frac{\partial}{\partial t} \iint\limits_{S_{(L)}} \vec{B} \cdot d\vec{S}\, dt = \iint\limits_{S_{(L)}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}\, dt \tag{25}$$

while $d\Phi_B^{(2)}$ is the change of the magnetic flux through $S_{(L)}$ arising from the motion of the contour L when some of magnetic field lines initially permeating the surface $S_{(L)}$, may be lost/gained as they cross the contour L in time interval dt. The quantity $d\Phi_B^{(2)}$ is thus the flux of the $\vec{B}$-field through the cylindrical surface S_V drawn by the closed contour L as it moves with the fluid velocity $\vec{v}(\vec{r}, t)$ in time dt:

$$d\Phi_B^{(2)} \;=\; \iint\limits_{S_V} \vec{B} \cdot d\vec{S}\, dt \quad \text{where} \quad d\vec{S} = \vec{v}\, dt \times d\vec{L}$$

which can further be expressed as follows:

$$\begin{aligned} d\Phi_B^{(2)} \;&=\; \oint_L \vec{B} \cdot (\vec{v} \times d\vec{L})\, dt = -\oint_L (\vec{v} \times \vec{B}) \cdot d\vec{L}\, dt \\ &= -\iint\limits_{S_{(L)}} [\nabla \times (\vec{v} \times \vec{B})] \cdot d\vec{S}\, dt. \end{aligned} \tag{26}$$

Substituting expressions (25)-(26) into Eq (24) and taking the induction equation (20) into account, we obtain:

$$\frac{d\Phi_B}{dt} \;=\; \iint\limits_{S_{(L)}} \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \right] \cdot d\vec{S} = 0. \tag{27}$$

which proves the theorem.

Theorem II: A fluid particle initially on a given magnetic field line remains on it at any later time. This statement follows from Theorem I and we shall give no further proof of it.

These two theorems show that plasma motions in the ideal MHD approximation take place with fluid and magnetic field lines moving together as if they were mutually frozen-in. Magnetic field lines are thus being stretched, twisted and bent by fluid motions $\vec{v}(\vec{r},t)$ and vice-versa, the fluid velocity field is constantly being affected by the magnetic field $\vec{B}(\vec{r},t)$. Such a 'frozen-in' concept of ideal plasma, helps us in making visualizations of plasma dynamics in many natural phenomena: solar wind interaction with planetary magnetic fields, occurrence of MHD waves, dynamo processes of generation and amplification of astrophysical magnetic fields by fluid motions, magnetic field reconnections, etc.

2.6 MHD waves in ideal plasma

In the ideal MHD, a plasma and magnetic field form a unified elastic medium whose dynamics is governed by the full set of the described equations which put together look are:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \;=\; 0, \\
&\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} \;=\; -\nabla \left(p + \frac{\vec{B} \cdot \vec{B}}{2\mu_0} \right) + \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} + \vec{f}_g + \vec{f}, \\
&\frac{d}{dt} \ln p \;=\; \frac{C_p}{C_v} \frac{d}{dt} \ln \rho, \\
&\frac{\partial \vec{B}}{\partial t} \;=\; \nabla \times (\vec{v} \times \vec{B}).
\end{aligned} \tag{28}$$

In what follows, we shall exclude the gravity and all other non-magnetic forces, i.e. $\vec{f}_g$, $\vec{f}$=0.

We shall now present a short outline on MHD waves, small amplitude perturbations with harmonic dependence on time t and spatial coordinates x, y, z. Small amplitude here means that each unknown physical quantity $\Psi(x, y, z, t)$ in Eqs (28) can be expressed as a superposition of its initial unperturbed value $\Psi_0(x, y, z, t)$ and a small perturbation $\Psi_1(x, y, z, t)$:

$$\Psi(x, y, z, t) \;=\; \Psi_0(x, y, z, t) + \Psi_1(x, y, z, t) \quad \text{where} \quad |\Psi_1| \ll |\Psi_0|. \tag{29}$$

We shall study small amplitude MHD waves in a uniform, static basic state of a plasma behaving as a perfect gas with a given constant density ρ_0, pressure p_0, temperature T_0, and permeated by a uniform magnetic field $\vec{B}_0 = (B_0, 0, 0)$ oriented along the x-axis. In this case, the MHD equations (28) are automatically satisfied in the zero order while in the first order of approximation, they yield the following set of linear equations for perturbations:

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot (\vec{v}_1) &= 0, \\
\rho_0 \frac{\partial \vec{v}_1}{\partial t} &= -\nabla \left(p_1 + \frac{\vec{B}_0 \cdot \vec{B}_1}{\mu_0} \right) + \frac{1}{\mu_0} (\vec{B}_0 \cdot \nabla) \vec{B}_1, \\
p_1 &= \gamma \frac{p_0}{\rho_0} \rho_1 \qquad \text{where} \qquad \gamma \equiv \frac{C_p}{C_v}, \\
\frac{\partial \vec{B}_1}{\partial t} &= \nabla \times (\vec{v}_1 \times \vec{B}_0).
\end{aligned} \tag{30}$$

We are looking now for harmonic wave solutions of Eqs (30) given by:

$$\Psi_1(x, y, z, t) = \hat{\Psi} \exp i(k_x x + k_y y + k_z z - \omega t) \tag{31}$$

where $\hat{\Psi}$ is the perturbation amplitude depending on the wave parameters k_x, k_y, k_z and ω.

Due to the axial symmetry of the basic state with regard to the uniform $\vec{B}$-field oriented along the x-axis, the wave propagation will be considered in two dimensions only, say in the (x, y)-plane, meaning that we can take $k_z = 0$ without loosing the generality. In this case, Eqs (30) with (31) reduce to a set of homogeneous algebraic equations for perturbation amplitudes that can be grouped in two independent sets:

$$\omega \hat{v}_z + v_A^2 k_x \hat{b}_z = 0 \quad \text{and} \quad k_x \hat{v}_z + \omega \hat{b}_z = 0, \tag{32}$$

and:

$$\begin{aligned}
\omega \hat{r} - k_x \hat{v}_x - k_y \hat{v}_y &= 0, \\
v_s^2 k_x \hat{r} - \omega \hat{v}_x &= 0, \\
v_s^2 k_y \hat{r} - \omega \hat{v}_y + v_A^2 k_y \hat{b}_x - v_A^2 k_x \hat{b}_y &= 0, \\
\omega \hat{r} - k_x \hat{v}_x - \omega \hat{b}_x &= 0, \\
k_x \hat{v}_y + \omega \hat{b}_y &= 0.
\end{aligned} \tag{33}$$

where:

$$\hat{b}_{x,y,z} \equiv \frac{\hat{B}_{x,y,z}}{B_0} \quad \text{and} \quad \hat{r} \equiv \frac{\hat{\rho}}{\rho_0}$$

while:

$$v_A^2 \equiv \frac{B_0^2}{\mu_0 \rho_0} \quad \text{and} \quad v_s^2 \equiv \gamma \frac{p_0}{\rho_0}$$

are squares of the Alfvén speed and adiabatic speed of sound respectively.

The first set of equations Eq (32) relates only to the $z-$components of perturbed magnetic field and fluid velocity amplitudes while Eqs (33) involve all the remaining amplitudes not showing up in Eq (32). These two sets of homogeneous algebraic equations then yield two independent conditions for existence of non-trivial solutions, i.e the vanishing determinants Δ_1 of the system Eq (32):

$$\Delta_1 \equiv \begin{vmatrix} \omega & v_A^2 k_x \\ k_x & \omega \end{vmatrix} = 0 \qquad (34)$$

and Δ_2 of the system Eq (33):

$$\Delta_2 = \begin{vmatrix} \omega & -k_x & -k_y & 0 & 0 \\ v_s^2 k_x & -\omega & 0 & 2v_A^2 k_x & 0 \\ v_s^2 k_y & 0 & -\omega & v_A^2 k_y & -v_A^2 k_x \\ \omega & -k_x & 0 & -\omega & 0 \\ 0 & 0 & k_x & 0 & \omega \end{vmatrix} = 0. \qquad (35)$$

Eqs (34) and (35) are the dispersion relations whose three solutions for ω^2 describe three MHD wave modes: the Alfvén wave and two magnetoacoustic waves.

The Alfvén wave is incompressible and therefore induces no density or pressure perturbations in the fluid. Its dispersion relation $\Delta_1 = 0$ is

$$\omega^2 - k^2 v_A^2 \cos^2\theta \;=\; 0 \qquad (36)$$

with the wave vector components expressed in terms of the wave vector intensity k and the angle θ between $\vec{k}$ and $\vec{B}$:

$$k_x \;=\; k\,\cos\theta, \qquad k_y \;=\; k\,\sin\theta.$$

The Alfvén wave has no dispersion as its phase speed $V_A \equiv \omega/k$ does not depend on the wave vector i.e. it is independent of the perturbation wavelength. The fastest phase speed is $V_A = v_A$ occurring for propagation parallel to the $\vec{B}$field when $\theta = 0, \pi$ and $V_A = 0$ for transverse propagation with $\theta = \pi/2$.

The other dispersion relation $\Delta_2 = 0$ is quadratic in ω^2:

$$\omega^4 - (v_s^2 + v_A^2)k^2\omega^2 + v_s^2 v_A^2 k^4 \cos^2\theta \;=\; 0 \qquad (37)$$

with two solutions for ω^2. The bigger ω^2 is related to the fast magnetoacoustic wave:

$$\omega^2 \;=\; \omega_f^2 \equiv 0.5\left(v_s^2 + v_A^2 + \sqrt{\left(v_s^2 + v_A^2\right)^2 - 4v_s^2 v_A^2\cos^2\theta}\right)k^2 \qquad (38)$$

while the lesser solution describes the slow magnetoacoustic wave:

$$\omega^2 \;=\; \omega_s^2 \equiv 0.5\left(v_s^2 + v_A^2 - \sqrt{\left(v_s^2 + v_A^2\right)^2 - 4v_s^2 v_A^2\cos^2\theta}\right)k^2. \qquad (39)$$

These two modes induce also pressure and density perturbations in plasma as can be seen from Eqs (33), and, like the Alfvén wave, they have no dispersion. The fast mode propagates in all directions though with different phase speeds $V_f \equiv \omega_f/k$ depending on θ while the slow mode cannot propagate across the $\vec{B}$-field when $\theta = \pi/2$.

MHD waves are of essential importance to many astrophysical processes, like energy transfer in mechanisms of the non-thermal coronal heating on the Sun, they have a crucial role in transient phenomena in aftermaths of sudden local disturbances, etc.

Many new properties and features occur if linear MHD waves are considered in non-uniform plasmas, if other forces like gravity are present and if the basic state magnetic fields have more complex structures.

Description of small amplitude MHD waves by linearized equations is only approximative as the nonlinear terms retain their second order, yet finite contributions to the dynamics. Typical examples for this are the nonlinear steepening of initially harmonic wave profiles and formation of shock waves, wave-wave interactions, nonlinear saturations of wave amplitude growth during instabilities etc.