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Basic Physics of Magnetoplasmas-III: Statics and MHD Waves in Gravitational Field

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LECTURE 3

MAGNETOHYDROSTATICS AND MHD WAVES IN GRAVITATIONAL FIELD

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The presence of gravitational field introduces the additional non-magnetic gravity force $\vec{f}_g = \rho\vec{g}$ into MHD equations. We shall assume a uniform, initially prescribed, gravitational field with constant acceleration $\vec{g}$. This means that the considered plasma does not affect the gravitational field as typical of stellar atmospheres and planetary plasmaspheres for example. Some other media, however, like stellar interiors and large interstellar plasma clouds would require a self-consistent approach with the gravitational field resulting from the density distribution $\rho(\vec{r}, t)$ and gravitational potential $\phi_g(\vec{r}, t)$ computed by the Poisson equation $\nabla^2\phi_g = 4\pi G\rho$. Such self-gravitating plasmas with $\vec{g} = -\nabla\phi_g$ fall off the scope of this lecture.

In what follows, we shall deal with two aspects of gravitational effects on magnetized plasmas: static configurations of plasma and magnetic field, and propagation of linear MHD waves.

3.1 Magnetohydrostatics

An ideal dissipationless plasma in the state of magnetohydrostatic equilibrium is described by the balance of all acting forces: the pressure gradient force, magnetic force and gravitational force with $\vec{g} = g\hat{e}_z$, g=const:

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} - \nabla p - \rho\vec{g} = 0. \tag{1}$$

As Eq (1) contains three unknown quantities p, $\vec{B}$ and ρ, only one of them can be computed while the other two have to be either specified initially or mutually related by some additional equations. We consider our plasma to be an ideal and fully ionized gas obeying the perfect gas law:

$$p = \mathcal{R}_{\mathcal{M}}\rho T \quad \text{where} \quad \mathcal{R}_{\mathcal{M}} \equiv \frac{\mathcal{R}}{\mathcal{M}} \tag{2}$$

which introduces the temperature profile T as a new unknown physical quantity. In further treatment, we take $T(x, y, z)$ to be initially prescribed.

Let us now examine some typical examples of magnetohydrostatic equilibria.

3.1.1 Horizontal $\vec{B}$-field and T=const

Taking $\vec{B} = B(z)\hat{e_z}$ and T=const, Eqs (1)-(2) yield:

$$\frac{d}{dr} \ln \rho \;=\; -\frac{\gamma}{v_s^2} \left(g + v_A^2 \frac{d}{dz} \ln B \right) \tag{3}$$

which is the equation for the density distribution with the height z in presence of a magnetic field with given z-dependence $B(z)$.

Eq (3) shows that a horizontal magnetic field influences the plasma density distribution only through the z-derivative of $B(z)$ i.e. if the $\vec{B}$-field is nonuniform. Moreover, we may conclude that the effect of a magnetic field whose strength grows with z, i.e. $dB/dz > 0$, is equivalent to a nonmagnetic case with enlarged gravitational acceleration g. In other words, a horizontal magnetic with $dB/dz > 0$ compresses the plasma which makes its density to fall off with z more steeply. The opposite effect occurs if $B(z)$ decreases with z when $dB/dz < 0$ reduces the action of gravity in Eq (3), and the inhomogeneity of the plasma density becomes less pronounced. The density is thus more evenly distributed with z meaning that such a magnetic field configuration supports mass against the gravity and increases the potential energy of the system. This extra potential energy can be released through instability leading to a rearranged distributions of both the density and magnetic field.

The described effect of magnetic field on plasma density is nicely seen in a simple plasma configuration with two distinct regions: Region 1 with a uniform horizontal magnetic field and Region 2 with no magnetic field. At the horizontal interface separating these two plasma domains, the boundary condition requires the continuity of the total pressure $p_{\text{tot}} \equiv p + p_m$:

$$p_{\text{tot}}(1) \;=\; p_{\text{tot}}(2) \quad \Rightarrow \quad p(1) + p_m(1) \;=\; p(2)$$

which can be written as follows:

$$\rho(2) - \rho(1) \;=\; \frac{B^2(1)}{2\mu_0 \mathcal{R}_{\mathcal{M}} T} \tag{4}$$

where $T \equiv T(1) = T(2)$=const. The plasma occupying the domain with the magnetic field is therefore less dense than plasma in the domain with $B = 0$: $\rho(2) > \rho(1)$. The magnetic field therefore tends to reduce the plasma density.

It is now evident that this system of two plasma domains can be both stable and unstable depending on the vertical arrangement of the domains: The system is stable if a less dense magnetized plasma with the magnetic field is laid above a denser plasma without magnetic field . In the opposite case with a denser plasma located above a lower density plasma with the magnetic field, the system is buoyantly unstable and the less dense plasma together with the embedded magnetic field tends to move in the upward direction.

This magnetic buoyancy effect of expelling magnetic field is also typical of isolated magnetic flux tubes like those related to sunspot formation during the solar cycle for example.

3.1.2 Statics of magnetic arcades

Eq (1) can also be used to model static magnetic field topologies under some realistic assumptions. For example, let the considered magnetic field $\vec{B}$ be predominantly in the (x,z)-plane and $y-$invariant like in the case of magnetic arcades in the solar corona. Thus:

$$\vec{B} \equiv (B_x(x,z), 0, B_z(x,z)) = \nabla \times \vec{A} = \nabla A \times \hat{e}_y \tag{5}$$

where $\vec{A} = A(x,z)\hat{e}_y$ is the vector potential of the $\vec{B}$-filed.

Eq (5) implies

$$\vec{B} \cdot \nabla A = 0 \tag{6}$$

meaning that A=const along the magnetic field lines whose analytical expressions are then $A(x,z)$=const curves.

Substituting the expression (5) for $\vec{B}$ into Eq (1) and taking $\hat{e}_z = \nabla z$, we obtain:

$$\nabla p = -\rho g \nabla z - \frac{1}{\mu_0}\left(\nabla^2 A\right)\nabla A \tag{7}$$

indicating $p = p(A,z)$ and:

$$\frac{\partial p}{\partial A} = -\frac{1}{\mu_0}\nabla^2 A, \tag{8}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{9}$$

Let us further assume an isothermal plasma with T=const and integrate Eq (9) which yields:

$$p \equiv p(A,z) = p_0(A)\mathrm{e}^{-z/H} \tag{10}$$

where $p_0(A)$ is the boundary value for the pressure distribution on magnetic field lines $A(x,z)$ =const at $z = 0$, and $H \equiv \mathcal{R}_{\mathcal{M}} T/g$. Inserting Eq (10) for $p(A,z)$ into Eq (8) we obtain the final equation for the potential $A = A(x,z)$:

$$\nabla^2 A + \mu_0 \frac{dp_0(A)}{dA}\mathrm{e}^{-z/H} = 0 \tag{11}$$

and the magnetic field components:

$$\begin{aligned} B_x(x,z) &= -\frac{\partial A}{\partial z}, \\ B_y(x,z) &= 0, \\ B_z(x,z) &= \frac{\partial A}{\partial x}. \end{aligned} \tag{12}$$

Knowing the pressure $p(A,z)$, we obtain the density ρ from the perfect gas law $\rho(A,z) = p(A,z)/\mathcal{R}_{\mathcal{M}} T$ with T=const given initially.

3.2 MHD waves in stratified plasmas

Let us now briefly look at MHD waves in an isothermal magnetized ideal plasma in a gravitational field with $\vec{g} = g\hat{e_z}$, g=const. The magnetic field is taken horizontal $\vec{B} = B(z)\hat{e_x}$ and we choose the functional dependence $B(z)$ so that v_A =const:

$$v_A^2 \equiv \frac{B^2(z)}{\mu_0 \rho(z)} = \text{const} \quad \Rightarrow \quad B(z) = v_A \sqrt{\mu_0 \rho(z)}. \tag{13}$$

Thus, we consider the unperturbed basic state with v_s,v_A=const in a magnetohydrostatic equilibrium described by Eq (3) with $B_0(z)$ given by Eq (13):

$$\frac{d}{dz} \ln \rho_0 \; = \; -\frac{g}{\mathcal{R}_{\mathcal{M}} T_0} \frac{\beta}{1+\beta} \quad \text{where} \quad \beta \equiv \frac{p_0}{p_{0m}} = \frac{2}{\gamma} \frac{v_s^2}{v_A^2} \tag{14}$$

The solution for $\rho_0(z)$ is then:

$$\rho_0 \; = \; \rho_0(0)\mathrm{e}^{-z/H} \quad \text{where} \quad H \equiv (1+\beta)\frac{\mathcal{R}_{\mathcal{M}} T_0}{g\beta}. \tag{15}$$

while $B_0(z)$ follows from Eq (13) as:

$$B_0 \; = \; B_0(0)\mathrm{e}^{-z/2H} \quad \text{where} \quad B_0(0) \equiv v_A \sqrt{\mu_0 \rho_0(0)}. \tag{16}$$

This magnetic field decreases with z which, as mentioned before, may result into a magnetic buoyancy instability of linear perturbations.

Let us now disturb a z-dependent basic state by small amplitude perturbations whose dynamics is governed by the linearized set of ideal MHD equations with the gravitational force $\vec{f_g} = -g\hat{e}_z$ included:

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v_1} + \vec{v_1} \cdot \nabla \rho_0 \; = \; 0, \\
&\rho_0 \frac{\partial \vec{v_1}}{\partial t} \; = \; -\nabla p_1 + \rho_1 \vec{g} + \frac{1}{\mu_0}\left(\nabla \times \vec{B_0}\right) \times \vec{B_1} + \frac{1}{\mu_0}\left(\nabla \times \vec{B_1}\right) \times \vec{B_0}, \\
&\frac{\partial \vec{B_1}}{\partial t} \; = \; \nabla \times \left(\vec{v_1} \times \vec{B_0}\right), \\
&\frac{\partial \rho_1}{\partial t} \; = \; \frac{1}{v_s^2}\frac{\partial p_1}{\partial t} + \left(\frac{1}{\gamma} - 1\right)\frac{d\rho_0}{dz} v_{1z}.
\end{aligned} \tag{17}$$

Since the basic state quantities Ψ_0 in Eq (17) are z-dependent we can take all perturbations Ψ_1 as harmonic functions of time t and two spacial coordinates x and y while their amplitudes are some functions of z:

$$\Psi_1(x, y, z, t) \; = \; \hat{\Psi}_1(k_x, k_y, \omega; z)\mathrm{e}^{-i\omega t + i(k_x x + k_y y)} \tag{18}$$

The full set of linearized MHD equations Eq (17), reduce to the following system of two equations for the total pressure perturbation $\hat{P}_1 \equiv \hat{p}_1 + B_0 \hat{B}_1/\mu_0$ and the Lagrangian displacement $\hat{\xi}_z \equiv i\hat{v}_{1z}/\omega$:

$$D\frac{d\hat{\xi}_{1z}}{dz} = C_1\hat{\xi}_{1z} - C_2\hat{P}_1,$$
$$D\frac{d\hat{P}_1}{dz} = C_3\hat{\xi}_{1z} - C_1\hat{P}_1. \tag{19}$$

where:

$$\begin{aligned}
D &= \rho_0(z)(v_s^2+v_A^2)(\omega^2-\omega_A^2)(\omega^2-\omega_c^2),\\
C_1 &= \rho_0(z)g\omega^2(\omega^2-\omega_A^2),\\
C_2 &= (\omega^2-\omega_A^2)(\omega^2-\omega_s^2) - k_y^2(v_s^2+v_A^2)(\omega^2-\omega_c^2),\\
C_3 &= \rho_0^2(\omega^2-\omega_A^2)[(v_s^2+v_A^2)(\omega^2-\omega_A^2)(\omega^2-\omega_c^2)+g^2(\omega^2-\omega_A^2)\\
&\quad +g(v_s^2+v_A^2)(\omega^2-\omega_c^2)\frac{d}{dz}\ln\rho_0]
\end{aligned} \tag{20}$$

with:

$$\omega_A^2 \equiv v_A^2k_x^2, \quad \omega_c^2 \equiv \frac{v_A^2v_s^2}{v_A^2+v_s^2}k_x^2 \quad \text{and} \quad \omega_s^2 \equiv v_s^2k_x^2.$$

Here, v_A, v_s=const while $\rho_0(z)$ is given by Eq (15). Due to the exponential functional dependence of $\rho_0(z)$, Eqs (20) can be transformed to a system of equations for unknown quantities $[\rho_0(z)\hat{\xi}_{1z}]$ and $\hat{P}_1$ which has constant coefficients:

$$\begin{aligned}
\frac{d}{dz}[\rho_0(z)\hat{\xi}_{1z}] &= \left(a_1-\frac{1}{H}\right)[\rho_0(z)\hat{\xi}_{1z}] - a_2\hat{P}_1,\\
\frac{d\hat{P}}{dz} &= a_3[\rho_0(z)\hat{\xi}_{1z}] - a_1\hat{P}_1,
\end{aligned} \tag{21}$$

where:

$$\begin{aligned}
a_1 &= \frac{g\omega^2}{(v_s^2+v_A^2)(\omega^2-\omega_c^2)},\\
a_2 &= \frac{\omega^2-\omega_s^2}{(v_s^2+v_A^2)(\omega^2-\omega_c^2)} - \frac{k_y^2}{\omega^2-\omega_A^2},\\
a_3 &= \omega^2-\omega_A^2-\frac{g}{H}+\frac{g^2(\omega^2-\omega_A^2)}{(v_s^2+v_A^2)(\omega^2-\omega_c^2)}.
\end{aligned} \tag{22}$$

The solutions of Eqs (21) have now an exponential z-dependence:

$$\rho_0(z)\hat{\xi}_{1z}, \hat{P}_1 \quad \sim \mathrm{e}^{\kappa_z z} \quad \text{with} \quad \kappa_z = -\frac{1}{2H} \pm ik_z$$

where:

$$k_z^2 = a_2a_3 - \left(a_1-\frac{1}{2H}\right)^2 \tag{23}$$

is the dispersion relation for the considered MHD waves.

Finally, the wave amplitudes $\hat{\xi}_{1z}$ and $\hat{P}_1$ have the following z-dependence:

$$\hat{\xi}_{1z} \;=\; \hat{\xi}_{1z}(0)\mathrm{e}^{z/2H}\mathrm{e}^{ik_z} \quad \text{and} \quad \hat{P}_1 \;=\; \hat{P}_1(0)\mathrm{e}^{-z/2H}\mathrm{e}^{ik_z}. \tag{24}$$

The dispersion relation Eq (23) can be written in an explicit form as an algebraic equation that is cubic in ω^2:

$$-\omega^6 + \mathcal{A}_4\omega^4 + \mathcal{A}_2\omega^2 + \mathcal{A}_0 \;=\; 0 \tag{25}$$

with the coefficients:

$$\begin{aligned}
\mathcal{A}_4 \;&=\; v_A^2 k_x^2 + (v_A^2 + v_s^2)\left(k^2 + \frac{1}{4H^2}\right),\\
\mathcal{A}_2 \;&=\; g^2(k_x^2 + k_y^2) - [(v_A^2 + v_s^2)k_y^2 + v_s^2 k_x^2]\frac{g}{H} - (v_A^2 + 2v_s^2)v_A^2\left(k^2 + \frac{1}{4H^2}\right)k_x^2,\\
\mathcal{A}_0 \;&=\; v_A^4 v_s^2\left(k^2 + \frac{1}{4H^2}\right)k_x^4 - g v_A^2\left(g - \frac{v_s^2}{H}\right)(k_x^2 + k_y^2)k_x^2.
\end{aligned} \tag{26}$$

Three solutions of Eq (25) for ω^2 are related to the Alfvén, slow and fast magneto-acoustic MHD modes modified by the gravity.

In a very short wavelength limit of $k_x, k_y, k_z \gg 1/H$, the gravitational effect becomes negligible and Eq (25) reduces to the dispersion relation for the MHD waves in a uniform magnetized plasma.

In the opposite case of a large wevelength limit when $k_x, k_y, k_z \ll 1/H$, the dispersion relation Eq (25) has one solution for ω^2:

$$\omega^2 \;=\; \frac{v_A^2 + v_s^2}{4H^2} \tag{27}$$

that describes oscillations of the system as a whole existing only in plasmas in a gravitational field.

Instability occurs if the dispersion relation Eq (25) has non real solutions for ω. This happens if ω^2 is either real and negative, or complex. The latter case cannot occur in ideal non dissipative plasmas while negative solutions for ω^2 are possible in our case.

A simple example of unstable perturbations follows from Eq (25) in the limit of small k_x: $k_x \ll k_y, k_z, 1/H$ when Eq (25) becomes quadratic in ω^2 and easy to analyze.

Thus, taking first $k_x = 0$ i.e. perturbations with straight magnetic field lines, the dispersion relation Eq (25) reduces to:

$$-\omega^6 + \mathcal{A}_4\omega^4 + \mathcal{A}_2\omega^2 + \mathcal{A}_0 \;=\; 0 \tag{28}$$

with the coefficients written as:

$$\begin{aligned}
\mathcal{A}_4 \;&=\; \left(1 + \frac{2}{\gamma\beta}\right)v_s^2\left(k_y^2 + k_z^2 + \frac{1}{4H^2}\right),\\
\mathcal{A}_2 \;&=\; -\frac{1 + (\gamma - 1)\beta}{1 + \beta}g^2 k_y^2,\\
\mathcal{A}_0 \;&=\; 0,
\end{aligned} \tag{29}$$

where $\beta \equiv p_0/p_{m0} = 2v_s^2/(\gamma v_A^2)$.

Among three solutions of Eq (28)-(29) for ω^2 two are non-zero and positive, corresponding to the modified Alfvén and fast magnetoacoustic waves, while the third solution is $\omega^2 = 0$.

Let us next look at perturbations with a small non-zero k_x, i.e. $k_x \ll k_y, k_z, 1/H$. In this case, the magnetic field lines are not straight anymore, they are slightly rippled with comparatively long wavelengths $\lambda_x = 2\pi/k_x$.

Such a small non-zero k_x leaves the coefficients $\mathcal{A}_2$ and $\mathcal{A}_4$ practically unchanged while $\mathcal{A}_0$ is not equal to zero any longer. Consequently, the modified Alfvén and fast magnetoacoustic modes will not be significantly changed by a small finite k_x while the third mode, the modified slow magnetoacoustic mode, will appear too due to the third non vanishing solution for ω^2. This solution, however, can be either positive or negative depending on the sign of the coefficient $\mathcal{A}_0$. Thus, if $\mathcal{A}_0 > 0$ the third solution for ω^2 is positive too and we have a stable gravitationally modified slow magnetoacoustic mode, and v.v. if $\mathcal{A}_0 < 0$, this solution for ω^2 is negative i.e. ω is purely imaginary and perturbations grow exponentially in time indicating the system is unstable. Up to terms of the order $\sim k_x^2$, the coefficient $\mathcal{A}_0$ then reads:

$$\mathcal{A}_0 \; = \; 2\frac{(\gamma-1)\beta-1}{\gamma\beta(1+\beta)} g^2 v_s^2 k_y^2 k_x^2. \tag{30}$$

The instability occurs if $\mathcal{A}_0 < 0$ or:

$$\beta \; < \; \frac{1}{\gamma - 1}.$$

Small β plasmas are thus unstable in this case with the magnetic buoyancy being the main driving mechanism for the instability. On the other side, comparatively weak magnetic fields corresponding to large parameters β cannot provide for a sufficiently intense magnetic buoyancy action and these perturbations are stable oscillations.