Random Matrix Ensembles for String Inflation

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hep-th/0512102

Plan of the Talk

I. String Inflation Status Report
   i. The Moduli Problem
   ii. The Eta Problem
   iii. Current Models
   iv. The Challenge of Large-Field Models

II. A Large-Field Model in String Theory
   i. Tool 1: Assisted Inflation
   ii. Tool 2: Random Matrix Theory
   iii. Results and Predictions

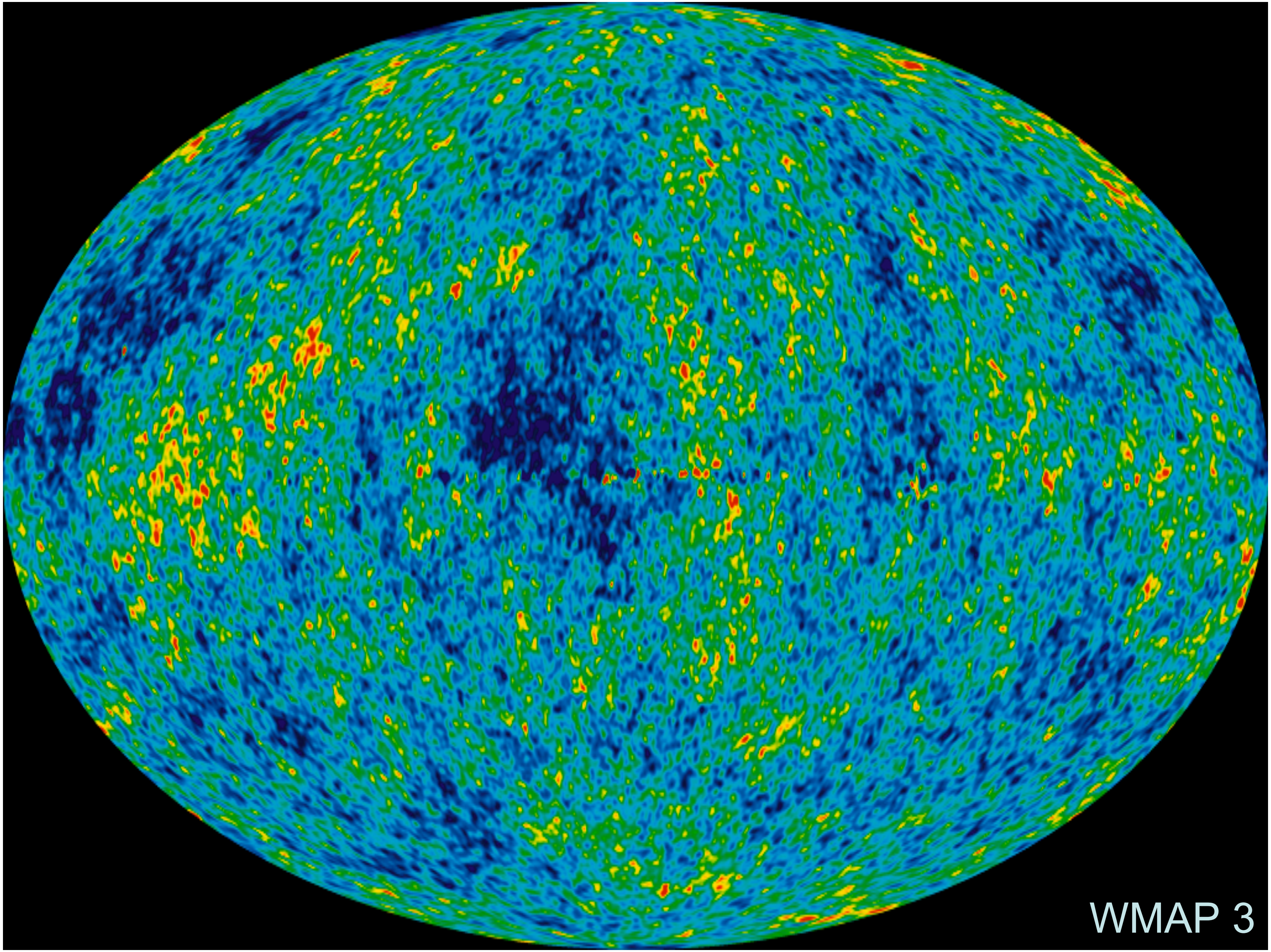
Key Questions:

What can string theory teach us about inflation?

Can we use inflation to constrain string theory?

Why String Inflation?

- Inflationary paradigm is highly successful!
- After 25 years, still no compelling microscopic theory. Maybe this is just too hard in QFT+GR?
- Sensible to try for a realization (or toy model) in full QG.
- String theory can provide:
  - fundamental scalar fields
  - some amount of UV control
  - fresh ideas on which systems are natural
- Now is a good time:
  - spectacular, ongoing observational progress
  - advances in string theory (moduli stabilization, D-branes, *etc.*) may have put solution within reach


WMAP 3

Achieving Inflation

- Typically requires a scalar field ϕ with a rather flat potential $V(\phi)$.

$$\eta \equiv M_{pl}^2 \frac{V''}{V} \ll 1 \quad \text{and} \quad \varepsilon \equiv \frac{1}{2} M_{pl}^2 \left(\frac{V'}{V} \right)^2 \ll 1$$

- Key goal of inflationary model-building: find such a field and such a potential in a controllable, well-motivated, natural setting.

Can one find _____ examples of _____ inflation in string theory?

any	slow-roll
satisfactory	large-field
non-fine-tuned	small-field
controllable	hybrid
consistent	k-
natural	my favorite
predictive	natural
	stringy

Problems from Moduli

- Store energy during inflation:
  - spoil BBN (100 MeV < m < 30 TeV)
  - overclose universe (m < 100 MeV)
- Fluctuate now:
  - cause ‘constants’ to vary
  - create fifth-force effects
- Cause runaway decompactification!
- Solution: stabilize the moduli.

The Eta Problem

In supergravity and string theory,

$$m_\phi^2 \approx H^2 \quad \Leftrightarrow \quad \eta \approx 1$$

is generic.

Reason: corrections from gravity.

e.g. $R\phi^2$ coupling

$$\Delta V \qquad \frac{V}{M_p^2}\phi^2 \qquad H^2\phi^2$$

Extremely hard to forbid all such terms!

Copeland,Liddle,Lyth,Stewart,Wands

Moduli stabilization often generates such terms,
even if naively absent.

Kachru,Kallosh,Linde,Maldacena,L.M.,Trivedi

Approaches to the Eta Problem

Fine-tuning:

Try to find special parameter values where η happens to be small.

- Hard to achieve explicitly.
- Considerable loss of predictivity.

Symmetry ⟹ We'll try this.

Impose a symmetry that forbids undesired mass terms.

- Surprisingly hard to achieve in concrete models.

String Inflation Status Summary

- Much recent progress due to moduli stabilization, *i.e.* solution of moduli problem.
- But most models still suffer from eta problem.
- Fundamentally superior models would be nice.
- Lacking that, much detailed work still needed to make known models fully explicit: more precise potentials, reheating, perturbations, *etc.*

Some Models

- Brane-Antibrane Dvali&Tye; Alexander; Dvali,Shafi,Solganik; Burgess,Majumdar,Nolte,Rajesh,Zhang; Sarangi&Tye.
- Branes at Angles. Garcia-Bellido, Rabadan, Zamora.
- D3-D7. Dasgupta,Herdeiro,Hirano, Kallosh; Hsu,Kallosh, Prokushkin; Hsu&Kallosh.
- warped brane-antibrane Kachru,Kallosh,Linde,Maldacena,L.M.,Trivedi; Firouzjahi&Tye; Burgess,Cline,Stoica,Quevedo; Iizuka&Trivedi; Berg,Haack, Körs; Cline&Stoica; Kofman&Yi; Frey, Mazumdar, Myers; Chialva, Shiu, Underwood; Shandera&Tye.
- DBI. Silverstein&Tong; Alishahiha,Silverstein,Tong; Chen; Kecskemeti, Maiden, Shiu, Underwood.
- Giant Inflaton. DeWolfe,Kachru,Verlinde.
- Racetrack. Blanco-Pillado,Burgess,Cline,Escoda,Gomez-Reino,Kallosh,Linde,Quevedo; Greene&Weltman.
- M5-brane. Buchbinder; Becker, Becker, Krause.
- Warped tachyonic. Cremades, Quevedo, Sinha.
- N-flation. Dimopoulos, Kachru, McGreevy, Wacker; Easther&L.M.
- Kahler. Conlon&Quevedo; Westphal; Simon, Jimenez, Verde, Berglund, Balasubramanian.

The Problem of Planckian Vevs

The simplest model of inflation: a scalar field with a mass.

$$V = \frac{1}{2} m^2 \phi^2$$

Linde 83

The conditions for slow-roll inflation:

$$\eta \equiv M_{pl}^2 \frac{V''}{V} \quad 1 \quad \text{and} \quad \varepsilon \equiv \frac{1}{2} M_{pl}^2 \left(\frac{V'}{V} \right)^2 \quad 1$$

are satisfied only when ϕ ÿ M_{pl}

Super-Planckian vevs characteristic of large-field inflation.

But, in an effective field theory with cutoff M, the quadratic potential is shorthand for

$$V = \frac{1}{2}m^2\phi^2 + \phi^4 \sum_{p=0}^{\infty} \lambda_p \left(\frac{\phi}{M}\right)^p$$

Flatness over distance $\Delta\phi > M$ requires tuning **all** the λ's: "functional fine-tuning"!

Clearly we can't take the cutoff $M \quad M_{pl}$

Do parametrically super-Planckian vevs even make sense? Can we get them in string theory to check this?

SUSY theory:

$$K = \Phi\Phi^{\dagger} + \Phi\Phi^{\dagger}\sum_{Q=1}^{\infty}\alpha_Q\left(\frac{\Psi_i\Psi_i^{\dagger}}{M_p^2}\right)^Q$$

Hard to forbid these!

Espcially hard to avoid coupling to curvature, $R\phi^2$

So far, supergravity and string theory haven’t helped at all. Super-Planckian distances are surprisingly rare in string theory.

EFT vs. Detectable Tensors

- 'Lyth Bound:'

$$\Delta\phi \geq M_p \sqrt{\frac{r}{r_{\min}}}\left(\frac{N_e}{10}\right)$$

$$r \equiv \frac{P_g}{P_R}$$

$$r_{\min} \equiv .07$$

- *If* primordial tensors are seen:
  - It will be difficult to explain them in a satisfactory field theory.
  - Even harder in string theory! (Expected it to be easier there, because of UV data.)

Large-Field Models are Worth the Trouble

- Super-Planckian vevs are very troublesome in an EFT with a Planckian cutoff.
- But forthcoming observations might force us to address this problem.
- This is a true QG problem whose solution(s) may well be testable.
- Large field models are falsifiable: they predict a red spectrum + substantial gravitational waves.
- Large-field models can have chaotic initial conditions, and can easily be eternal.
- As I'll show in Part II, certain large-field models turn out to be surprisingly common and computable in string theory.

End of Status Report

Goal

- Construct a reliable string theory realization of $m^2\phi^2$ inflation.
- To achieve this: develop methods for computing the axion mass spectrum in a stabilized compactification.

Motivation

- $m^2\phi^2$ inflation is an important paradigm (e.g. the key example of chaotic, large-field inflation) but is very hard to realize in field theory or string theory.
- The key obstacle is a need for trans-Planckian vevs.
- This problem is common in inflation, but especially stark here. Solving it here could be instructive.
- I'll describe a solution that is natural in stabilized vacua, and, thanks to random matrix theory, is surprisingly predictive.

Plan of Part II

A Large-Field Model in String Theory

i. Tool 1: Assisted Inflation
ii. The N-flation proposal
iii. Tool 2: Random Matrix Theory
iv. N-flation via RMT
v. Predictions
vi. Conclusions

Assisted Inflation

Liddle, Mazumdar, and Schunck, astro-ph/9804177

Given N fields with identical potentials:

$$\ddot{\phi}_i + 3H\dot{\phi}_i = -V_{,\phi_i}$$

$$H \propto N^{1/2}$$

The Hubble friction is enhanced!

Challenge: how to get many **identical** potentials?

N-flation.

Dimopoulos, Kachru, McGreevy, and Wacker, hep-th/0507205.

I would call it "assisted quadratic inflation with string axions"

$$V = \sum_{i=1}^{N} \tfrac{1}{2} m^2 \phi_i^{\,2}$$

$$\Phi^2 \equiv \sum \phi_i^2 = N\bar{\phi}^2 \qquad\qquad \phi_i = \bar{\phi}$$

Effective single-field model: $$V = \frac{1}{2} m^2 \Phi^2$$

Result: $\phi_i \qquad M_{pl} \qquad \Phi$ is possible.

Having N fields does not suffice

$$V = \tfrac{1}{2} m^2 \phi^T \phi$$

$$\Delta V \qquad \frac{(m^2 \phi^T \phi)}{M_p^2} \phi^T \phi \qquad H^2 \phi^T \phi$$

Eta-problem terms are consistent with O(N) symmetry.

We need a protective symmetry.

Why N-flation Works

Axions have shift symmetries, only broken nonperturbatively.

$$V = V_0 + \sum_i \Lambda_i^4 \cos(\phi_i / f_i) + \sum_i \phi_i^2 \frac{V_{other}}{M_{pl}^2} + \ldots$$

Eta-problem terms $H^2 \phi_i^2$ are forbidden.

$$\Rightarrow \quad V = \sum_{i=1}^{N} \tfrac{1}{2} m^2 \phi_i^2$$

+ negligible corrections for small displacements

Axion Shift Symmetries

Symmetry comes from 10D gauge invariance.

Broken only nonperturbatively.

$\phi \leftrightarrow \int B_{ij}$ heterotic

$\phi \leftrightarrow \int C_{ijkl}$ IIB (partners of Kähler moduli)

Crucial Questions

- Realize in a stable compactification?
- Masses can't be identical, so:
  1. What **spectrum of masses** can we expect?
  2. Does the model still work with these masses?
  3. How do the predictions change?

$$V = \sum_{i=1}^{N} \frac{1}{2} m_i^2 \phi_i^2$$

- Radiative stability? (not today's focus)

Realizing N-flation

String compactifications have many axions.
Nonperturbatively-stabilized vacua have axion potentials.

IIB:
$$W_{KKLT} = \underbrace{\int G \wedge \Omega}_{W_0} + \sum_{i=1}^{h^{1,1}} \underbrace{A_i(\chi_a) e^{-2\pi\rho_i}}_{C_i} e^{2\pi i \phi_i}$$

$$\Rightarrow \quad V = \underbrace{\left(e^K K^{AB} D_A C_i D_B C_j \right)}_{M_{ij}} \phi_i \phi_j + O(\phi^3)$$

for small displacements from the F-flat vacuum.

$i, j : 1 \ldots h^{1,1}$

$a, b : 0 \ldots h^{2,1}$

$A, B \supset \{i, a\}$

Redefine and simplify:

$$\mathsf{L} = \tfrac{1}{2} M_{pl}^2 K_{ij} \partial_\mu \phi_i \partial^\mu \phi_j - \tfrac{1}{2} \mathbf{M_{ij}} \phi_i \phi_j$$

rotate to canonical fields then diagonalize

$$\mathsf{L} = \tfrac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_j - \tfrac{1}{2} m_i^2 \varphi_i^2$$

Complete data of a model: the masses

Can We Compute the Masses?

$$\mathbf{M_{ij}} = \left(e^{K} K^{AB} D_A C_{\mathbf{i}} D_B C_{\mathbf{j}} \right)$$

In a KKLT vacuum, C_i depends on threshold corrections to gaugino condensates or fluctuation determinants for Euclidean D3-branes, both hard to compute. Even K can be complicated.

Moreover, this matrix is at least 300x300!

With present techniques,
computing all the entries of M is out of reach.

Can we take advantage of the large size of M?

Random Matrix Theory (RMT)

- Specifically, we will use the spectral theory of large random matrices.
- This is a rich and beautiful subject, but we only need a few easy results.
- We just need to know
  - how to compute the eigenvalue spectrum of a large matrix whose entries are drawn from a given distribution.
  - which matrix ensemble describes N string axions.
  - how to ensure that this gives a faithful description of the system

The Beginning of RMT in Physics

Classic problem: to find the energy spectrum of a highly-excited nucleus.
Full quantum mechanics problem? Impossible.

However, if we only need the density of levels,

$$\rho(E_0) \equiv \frac{d}{dE} N(E < E_0)$$

AND using the bold guess (Wigner 1958) that this is determined just by the symmetries of the Hamiltonian (without microscopic details) then the associated toy model is highly tractable.

The problem is therefore to *find the spectrum (density ρ) of eigenvalues of a large matrix whose symmetries are given.*

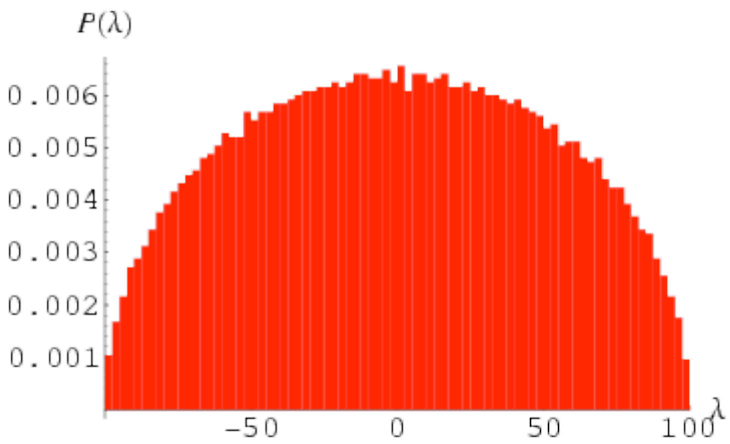
Wigner's Semicircle Law

- The simplest sort of toy Hamiltonian is a symmetric real matrix.

  Suppose **A** is an NxN matrix whose real entries A_{ij} are drawn from a Gaussian distribution with mean zero and variance σ^2/N.

Let **M** = **A** + $\mathbf{A}^T$.
Then the spectrum of M is

$$\rho(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}$$



(mathworld)

Marčenko-Pastur Law

Suppose **R** is an N x (N+P) matrix whose real entries R_{ij} are drawn from a distribution Ω with mean zero and variance σ^2/N.

And let $\mathbf{M} = \mathbf{R}\mathbf{R}^T$ (cf. **M** = **A** + $\mathbf{A}^T$ for Wigner Law)

Spectrum of **M** (Marčenko-Pastur 1967):

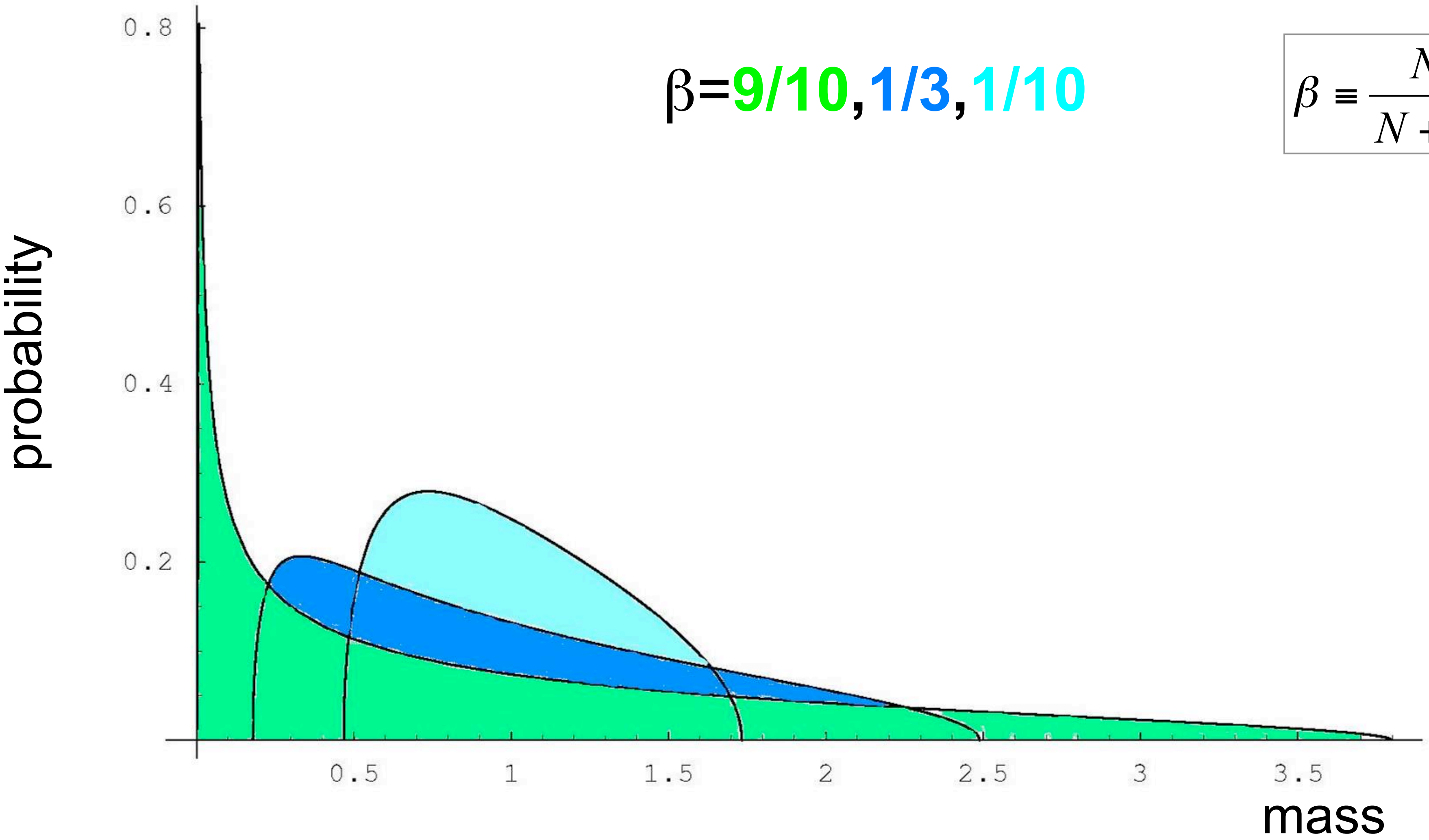
$$\mathrm{MP}(\beta,\sigma): \quad \rho(\lambda) = \frac{1}{2\pi\lambda\beta\sigma^2}\sqrt{(b-\lambda)(\lambda-a)}$$

$$a \equiv \sigma^2\left(1-\sqrt{\beta}\right)^2$$

$$b \equiv \sigma^2\left(1+\sqrt{\beta}\right)^2$$

$$\beta \equiv \frac{N}{N+P}$$

Marčenko-Pastur Spectrum



$$\beta \equiv \frac{N}{N+P}$$

Marčenko-Pastur Governs N Axions

$$\mathbf{M}_{\mathbf{ij}} = \left(e^{K} K^{AB} D_A C_{\mathbf{i}} D_B C_{\mathbf{j}}\right) \approx \left(\mathbf{R}\mathbf{R}^{\mathrm{T}}\right)_{\mathbf{ij}}$$

Can incorporate rotation
diagonalizing kinetic terms (not shown).

$$\rho(m^2) = \mathrm{MP}(\beta, \sigma)$$

$$\beta \equiv \frac{N}{N+P} = \frac{h^{1,1}}{h^{1,1} + h^{2,1} + 1}$$

An excellent approximation when N is large.

Scope of the RMT Approach

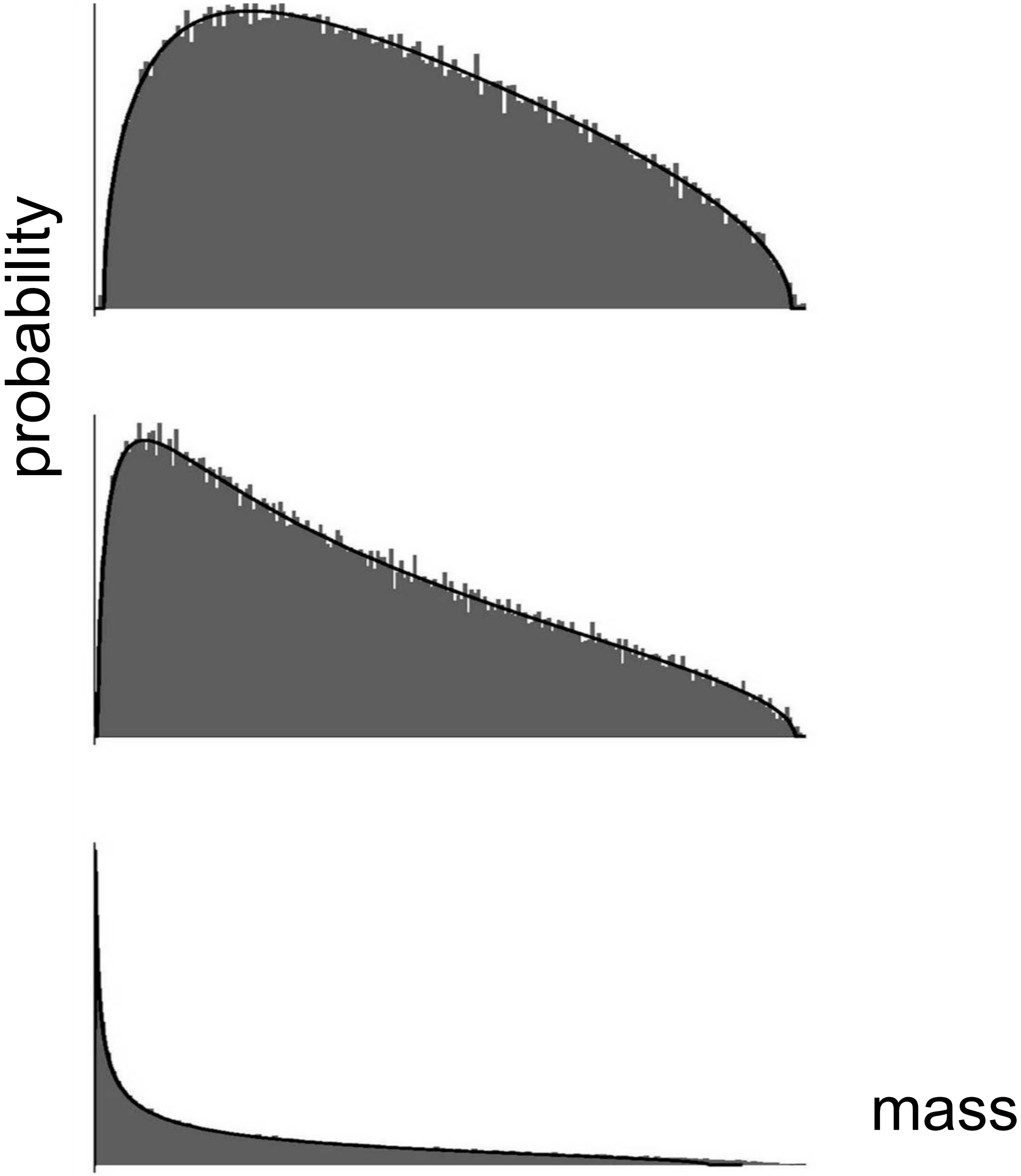
The Marčenko-Pastur law is **robust**:
• the distribution Ω need not be Gaussian
• the mean of Ω need not be small
• the entries can have some correlations (more soon)

General result from Z.D.Bai's review:

If M is a matrix whose constituent entries are drawn from ***any*** distribution with appropriately bounded moments, then **in the large N limit** the spectrum of M approaches that of a matrix whose constituent entries have a Gaussian distribution with mean zero!

We won't need to invoke this universality explicitly, but it is a strong motivation.

Monte Carlo Check



$\beta = 1/10$

$\beta = 1/3$

$\beta = 9/10$

$N = 300, \# = 100$

What is the ensemble?

- RMT gives an eigenvalue distribution that is a good guide even for a *single*, large matrix.
- It is not necessary to average over an ensemble of matrices.
- Roughly, the size of the matrix already gives an ‘ensemble’.

Central Limit Analogy

- recall CLT: the sum of many (appropriately bounded) i.i.d. distributions Ω_i is a Gaussian distribution.
- Spectrum of one matrix is analogous to this CLT gaussian!
- Individual matrix entries ~ Ω_i
- Sum in row-column contraction ~ sum over Ω_i
- But, particular structure of matrix contraction breaks analogy and gives a spectrum richer than a Gaussian (here, Marčenko-Pastur)

Many Fields, not Many Vacua

- We've used *large dimension of scalar field space* in a given vacuum
- Large mass matrix $\Longrightarrow$ RMT useful
- Don't need huge space of vacua, except to tune **Λ**. (Bousso-Polchinski).
- This is not the statistical reasoning most often used in the landscape.
- But, *cf*. Denef & Douglas.

Did Compactification Details Matter?

Required data:

W_0	→	average mass
$h^{1,1}$	→	N
$h^{2,1}$	→	spread of masses

Cf. D-brane models in KKLT vacua of IIB, where inflaton mass depends in detail on:

$$W_{KKLT} = \int G \wedge \Omega + \sum_{i=1}^{h^{1,1}} A_i(\varphi_b) e^{-2\pi\rho_i} e^{2\pi i\phi}$$ and on the Kähler potential.

RMT N-flation: 3 numbers

IIB brane inflation: at least N functions

Large N is actually simpler!

What were the assumptions?

- Correlations between entries in $D_A C_i$ are not extremely strong.

Wigner Law survives substantial correlations: Schenker&Schulz-Baldes,math-ph/0505003, show that N^2 correlations per row are allowed!

- $h^{1,1} \geq 240$ (300 is better)

- Standard requirements of moduli stabilization.

- W_0 has been fine-tuned to set overall scale,

$$\sigma \approx 10^{-5} M_{pl}$$

Qualitative Results

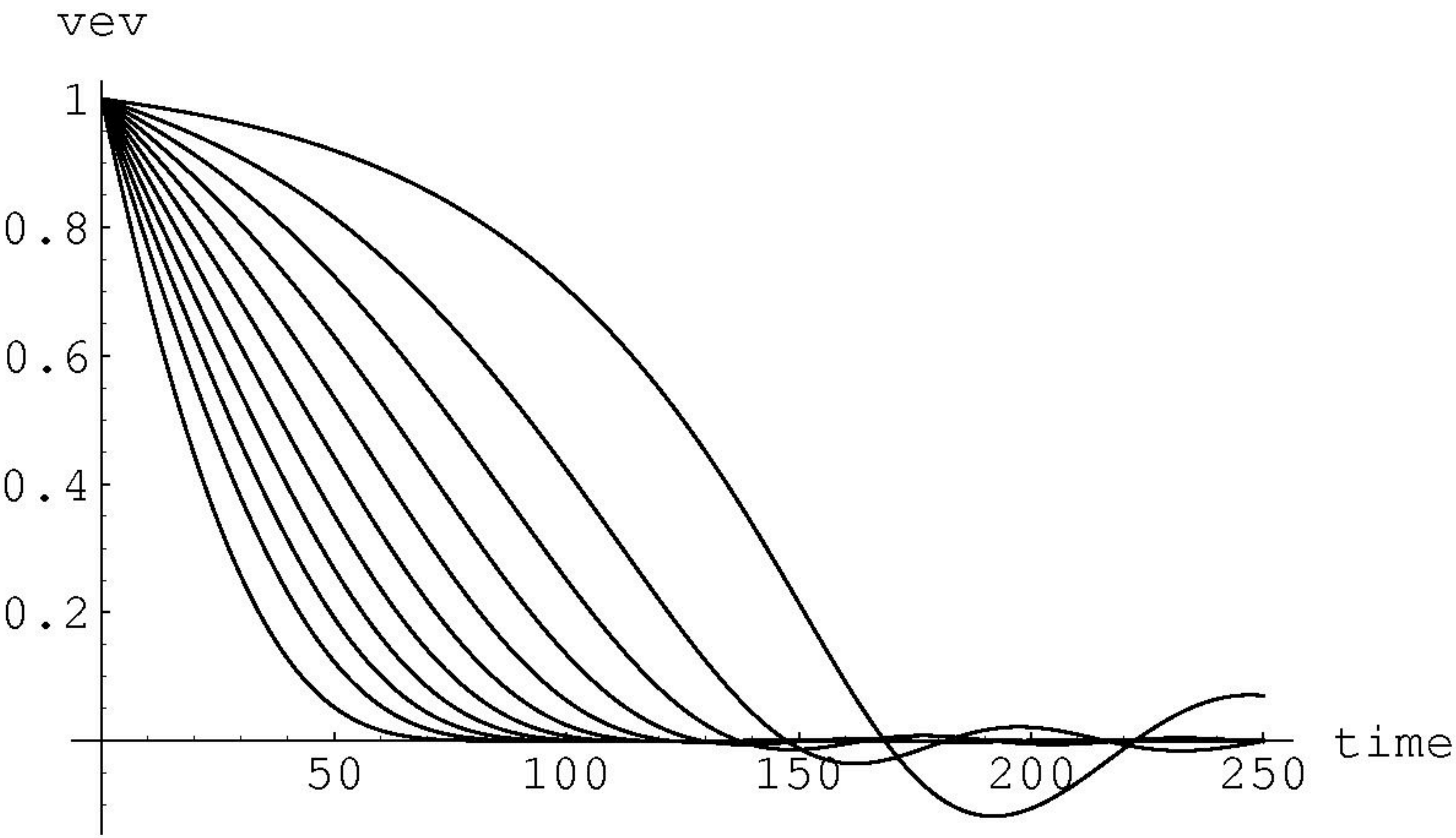
$\beta \quad 1 \quad (h^{1,1} \quad h^{2,1})$ → small spread in masses

$\beta \approx 1 \quad (h^{1,1} \quad h^{2,1})$ → large spread in masses

Mass ratio: $$\frac{m_{\max}}{m_{\min}} = \frac{1+\sqrt{\beta}}{1-\sqrt{\beta}}$$

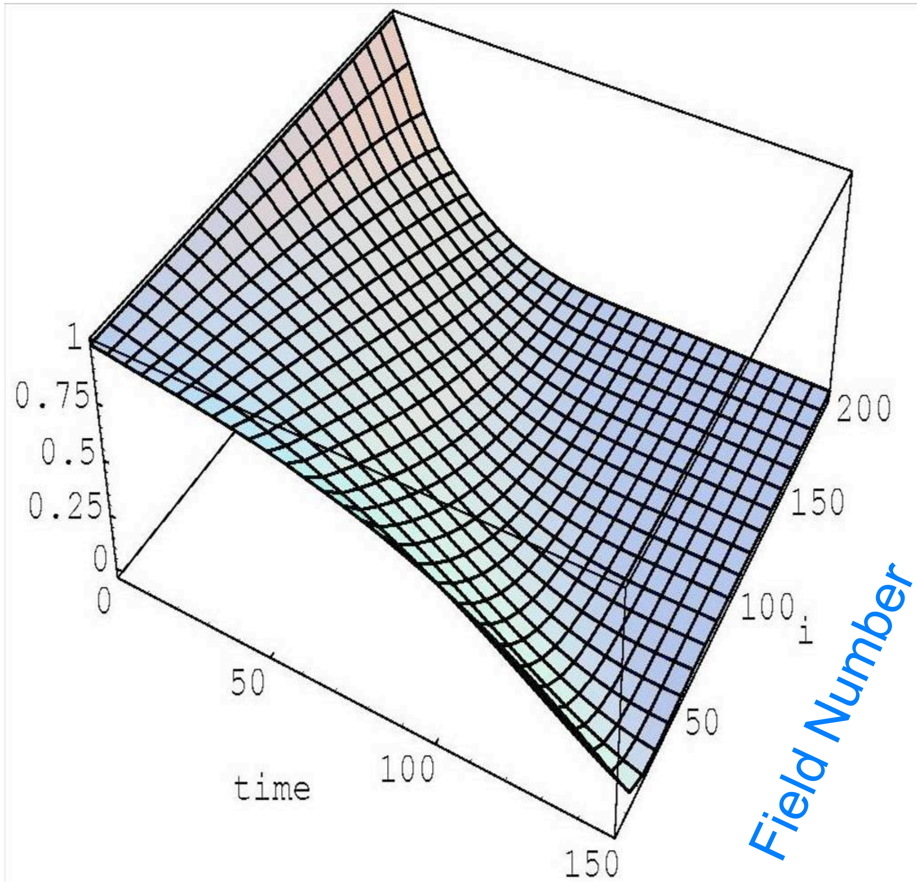
Heavier fields roll faster → Mass spectrum affects the dynamics

Motion of the Fields

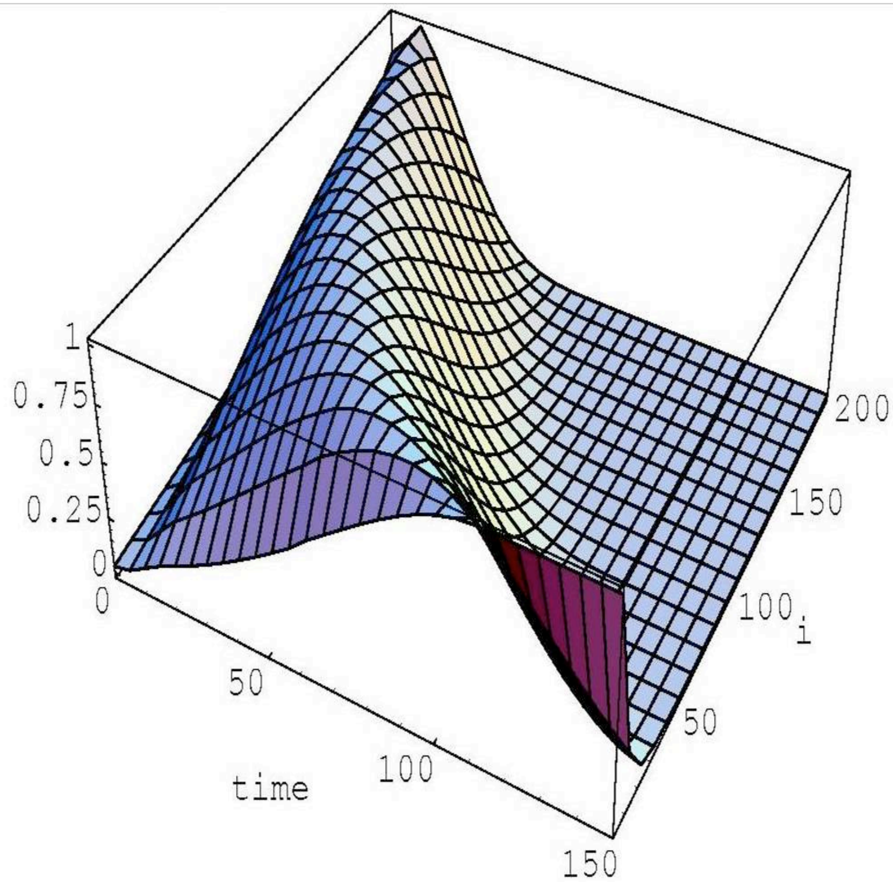


Field Evolution

Displacement



Energy Fraction



Initial Conditions

Quadratic potentials do not give attractor behavior.

So we must specify the initial conditions.

For now:

$$\phi_i \approx \bar{\phi} \frac{\bar{m}}{m_i} \quad \forall i$$

which follows from postulating a uniform distribution in the allowed field space

We're working on a better model for the initial conditions.
(with D. Baumann and R. Easther)

Computing the Predictions

$$P_{\mathrm{R}} \equiv \left(\frac{H}{2\pi}\right)^2 \left(\frac{\partial \mathrm{N}}{\partial \phi_i}\right)^2$$

Sasaki & Stewart '95

use $$\langle m^{2k} \rangle \equiv \frac{1}{N}\sum_{i=1}^{N} m_i^{2k} \approx \int d(m^2)\rho(m^2)m^{2k}$$

analytic result exists

$$P_{\mathrm{R}} = \frac{N^2}{96\pi^2}\frac{\langle m^2\rangle\langle\phi^2\rangle^2}{M_{pl}^6}$$

Predictions (cont'd)

$$n_s - 1 = -\frac{4M_{pl}^2}{N}\left(\frac{1}{\langle\phi^2\rangle} + \frac{\langle m^4\phi^2\rangle}{\langle m^2\phi^2\rangle^2}\right)$$

use M-P moments

$$= -\frac{8M_{pl}^2}{\Phi^2}\left[\frac{2-\beta}{2-2\beta}\right]$$

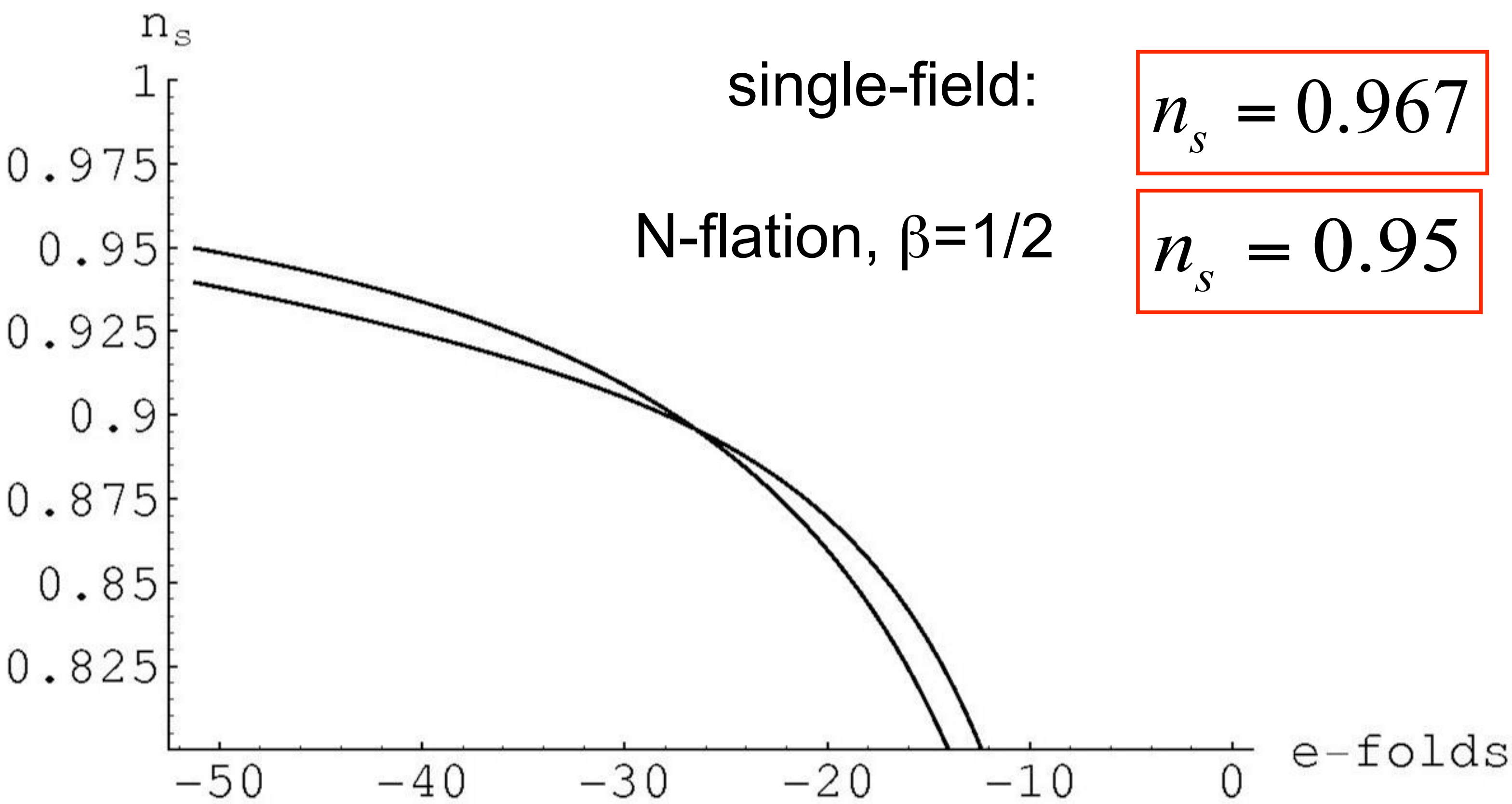
cf. the single-field result

$$n_s - 1 = -\frac{8M_{pl}^2}{\Phi^2}$$

N-field spectrum is more red.

Scalar Spectral Index

Current Bound (WMAP3, via Easther & Peiris): $n_s = 0.982 \pm 0.020$



single-field: $n_s = 0.967$

N-flation, β=1/2 $n_s = 0.95$

Observational Signatures

- Noticeably red spectrum. $n_s < 0.97$
- High tensors. $r \approx 0.13$
- Very modest running.
- Nongaussianity?

Distinguishable from single-field chaotic unless $\beta \approx 0$

Necessary measurements almost certain to happen.

Open Questions

- Understand general initial conditions, and
- Use full cosine potentials, not just masses.

  (in progress with D. Baumann & R. Easther)
- Build explicit examples.
- Verify control of quantum corrections.
- Apply RMT methods to other assisted inflation models.
- Non-Gaussianity? Isocurvature?
- Reheating?

Conclusions

- N-axion systems provide a promising class of string inflation models.
- We characterized general N-axion models using random matrix methods.
- Because **N** is large, compactification details don't matter!
- The scalar power spectrum is more red than in the degenerate case.
- Can be falsified by low tensors or by $n_s > 0.97$
- Can plausibly embed in stabilized vacua of any string theory (unlike brane models).
- Can expect other applications of RMT to fluctuations around string vacua.

Radiative Stability

(From original paper.)

Sigma-model corrections to Einstein-Hilbert action:

$$L_{10} = R_{10} + \zeta(3)\alpha'^3 R_{10}^4 + \ldots$$

$$(2\pi)^3 \int_{CY} R_6 \wedge R_6 \wedge R_6 = \chi$$

$$\Rightarrow \quad \boxed{L_4 = M_{pl}^2 R_4 \left(1 + \frac{\chi \zeta(3) \alpha'^3}{(2\pi)^3 Vol} \right)}$$

N-dependent renormalization of Newton constant!

Controlling the Renormalization

$$\left(\frac{\Delta\Phi}{M_{pl}}\right)^2 \qquad \frac{N}{|\chi|} = \frac{h^{1,1}}{\left|2h^{1,1} - 2h^{2,1}\right|}$$

Models with comparable numbers
of axions and other moduli are preferred.

Totally satisfactory resolution will require
concrete examples with correct order-one factors.

Consequence for us: $\beta \approx \frac{1}{2}$

Large non-Gaussianity?

- Inflationary path curves in field space
- Hence, isocurvature sources super-horizon evolution of curvature (Gordon, Wands, Bassett,Maartens)
- **non-Gaussianity** from curving path
- enhanced by N?