Generalized Complex Geometry in String Theory

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Introduction

What?

String theory beyond Calabi-Yau manifolds

Why?

- Emerging language might be of fundamental significance
- Are CY vacua special?
- Mirror symmetry, topological models...

Generalized complex geometry is the emerging language

reformulates metric in terms of differential forms

g determined by pair (Φ_+, Φ_-) $\qquad \Phi = \sum_k \mathrm{form}_k$

compatible Cl(6,6) pure spinors

Example: $\Phi_+ = e^{iJ}$, $\Phi_- = \Omega$

Ω defines Gl(3,C) structure $\qquad J \wedge \Omega = 0$

J defines Sp(6,R) structure $\qquad J^3/3! = i\Omega \wedge \bar{\Omega}$

- makes natural to relate metric and fluxes (as susy should require)

vacua | black hole attractor

IIB:

$$d\Phi_- = 0 \qquad \partial_r \Phi_- = f + i * f + |Z|\bar{\Phi}_-$$

$$d\Phi_+ = F + i * F + dA \wedge \bar{\Phi}_+ \qquad \partial_r \Phi_+ = 0$$

- puts order in arrays of possibilities
  - example: (2,2) models ($\sim \mathcal{N}=2$ supergravity vacua, RR=0)
    - know and love: Kähler
    - [GatesHullRocek'84]: J_+ for right-movers
      $\neq J_-$ for left-movers
      ... neither is Kähler

  all these cases are generalized Kähler $\left[\Leftarrow (d + H\wedge)\Phi_\pm = 0 \right]$
  [Gualtieri'04]

  - similarly for supergravity:
    - SU(3), SU(2)...
    - different fluxes

Plan

- Worldsheet models; an application (K3s)
    - Supergravity vacua
        - (Black holes and topological models)

Pure spinors

Internal spinors define geometries:

Ex.:

$$\epsilon_1^{10} = \epsilon_+^4 \otimes \eta_1^+ + \text{c. c.}$$
$$\epsilon_2^{10} = \epsilon_\pm^4 \otimes \eta_2^+ + \text{c. c.}$$

6d spinors

$$\eta_+^1 \otimes \eta_+^{1\,t} = \frac{1}{6}\Omega_{mnp}\gamma^{mnp} \equiv \not{\Omega}$$

[Fierz]

via $v^{1,0} \wedge \Omega = 0$:

almost complex structure $(c_1 = 0)$
(dec.) $\Omega \mid \Omega \wedge \bar{\Omega}$ nowhere zero

$+ \; d\Omega = 0 \;\Rightarrow$ complex $(K = 0)$

$$\eta_+^1 \otimes \eta_+^{1\,\dagger} = \not{e}^{iJ} \equiv 1 + i\not{J} - \frac{1}{2}\not{J}^2 - \frac{i}{6}\not{J}^3$$

almost symplectic structure
$J \mid J^3$ nowhere zero

$+ \; de^{iJ} = 0 \;\Rightarrow$ symplectic

(CY: complex and symplectic)

Ω and e^{iJ} share a property

pure Cl(6,6) spinors

bispinors:	differential forms:
$\not{C}$ rep. for Clifford(6)xClifford(6)	rep. forClifford(6,6) C

"gamma matrices":

contractions $v\llcorner$

wedges $\omega\wedge$

For tensor products:

$$\gamma^m \not{C} = [(dx^m \wedge + g^{mn}\iota_n)C]$$

$$\not{C}_\pm \gamma^m = \pm\, [(dx^m \wedge - g^{mn}\iota_n)C_\pm]$$

since

$$\gamma^{\bar{i}}\eta_+ = 0$$

$$\Downarrow$$

$$Ann(\eta^1_+ \otimes \eta^{2\,\dagger}_+) = 3 + 3$$

Φ is called pure if

$$Ann(\Phi) = 6$$

Generalized Calabi-Yau: $d\Phi = 0$ [Hitchin,Gualtieri...]

Pairs of pure spinors determine a metric

Just as $(J, j) \mapsto g_{mn} = j_m{}^p J_{pn}$

almost symplectic structure

almost complex structure

g symmetric:

$(J,\ j)$ "compatible"

$$(\Phi_+, \Phi_-) \mapsto g^{mn} = \mathcal{J}_+^{mp}\mathcal{J}_{-p}{}^n + \mathcal{J}_+^m{}_p \mathcal{J}_-^{pn}$$

SU(3)xSU(3) structure

$$\Phi \mapsto \mathcal{J} \equiv \qquad (f = \mathrm{Re}\Phi)$$

$$\left[\begin{array}{c|c} \mathcal{J}^{mn} = (f \wedge dx^m \wedge dx^n \wedge f)_{\rm top} & \mathcal{J}_m{}^n = (f \wedge \partial_m \llcorner dx^n \wedge f)_{\rm top} \\ \hline \mathcal{J}^m{}_n = (f \wedge dx^m \wedge \partial_n \llcorner f)_{\rm top} & \mathcal{J}_m{}^n = (f \wedge \partial_m \llcorner \partial_n \llcorner f)_{\rm top} \end{array}\right]$$

[Hitchin;Gualtieri] [Kapustin,Orlov'00]

$$\mathcal{J}^2 = -1_{6+6}$$

its eigenspaces are closed under Courant

"Lie bracket for $T \oplus T^*$"

really: generalized trivial can. bundle

$\mathcal{J}$ integrable: generalized complex $\subset$ $(d + H\wedge)\Phi = 0$: generalized Calabi-Yau

$\cap$ $\qquad\qquad$ $\cap$

$(\mathcal{J}_+, \mathcal{J}_-)$ integrable: generalized Kahler $\subset$ $(d + H\wedge)\Phi_\pm = 0$: generalized Calabi-Yau metric

Worldsheet

(1,1) models: no conditions on geometry $\delta X^m = \epsilon Q X^m$

(2,1): impose extra susy $\delta_+ X^m = \epsilon^+ j^m{}_n(X) D_+ X$

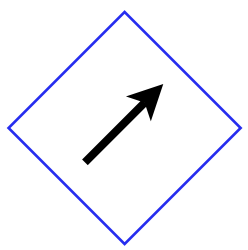
susy algebra $\Rightarrow$ j is complex structure

[Hull]

invariance of action $\Rightarrow$ $[J = j \cdot g]$ $\partial J = i H^{2,1}$ $(\Leftrightarrow dJ = j \cdot H)$

$\mathrm{Ann}(\Phi_+) \cap \mathrm{Ann}(\Phi_-)$
closed under Courant

[Grana,Minasian,Petrini,AT]



"one $\mathcal{J}$ and a half
are integrable"

less directly related to GCG

$\exists$ (3,0) form Ω for j $\Rightarrow$ $\mathcal{N} = 1$ supergravity vacuum

[Strominger]

and $d\Omega = W\Omega = 0$

("$K = 0$")

[toy model: $\int d\theta^+ S_m D_+ X^m$ (2,0) $\Leftrightarrow$ generalized complex]

[Lindstrom,Minasian,AT,Zabzine]

(2,2): $\delta_\pm X^m = \epsilon^\pm j_{(\pm)}{}^m{}_n D_\pm X$

chiral $\mathcal{N}=2$ superfields

- usual case: $j_+ = j_- \Rightarrow$ Kähler

also: $\int d^2\theta d^2\bar\theta\, K(\mathbb{X},\bar{\mathbb{X}})$

- in general: $j_\pm$ complex, $dJ_\pm = \pm j_\pm \cdot H$ $\Longrightarrow$ generalized Kähler

[GatesHullRocek] [Gualtieri]

- $[j_+, j_-] = 0 \Rightarrow \int d^2\theta d^2\bar\theta\, K(\mathbb{X},\bar{\mathbb{X}},\mathbb{Y},\bar{\mathbb{Y}})$ "twisted chirals"

- $[j_+, j_-] \neq 0$ algebra only works on-shell

add extra fields $\Rightarrow$ off-shell formulation exists! $\int d^2\theta d^2\bar\theta\, K($chirals, twisted chirals, "semichirals"$)$

generator for canonical transformation from z^i_+ to z^i_-.

[Lindstrom,Rocek, vonUnge,Zabzine]

Again, $\exists$ closed(3,0) forms $\Omega_\pm$ for $j_\pm \Rightarrow \mathcal{N}=2$ supergravity vacuum

$(d + H\wedge)\Phi_\pm = 0$ [Jeschek,Witt]

Ex.: Enriques surface $c_1 = 0$ but $K \neq 0 \Rightarrow$ no spacetime susy [Sharpe]

Application: K3

[Huybrechts]
[Aspinwall-Morrison]

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- Moduli space of metrics:
  - $H^2(K3)$ has signature (3,19) $\quad (\alpha, \beta) \equiv \int \alpha \wedge \beta$
  - $(\mathrm{Re}(\Omega), \mathrm{Im}(\Omega), J)$ span 3-plane w/ positive signature (and can be rotated in one another)

$$\Rightarrow \quad \mathcal{M}_{\mathrm{hK}} = \{\text{ such planes }\} \equiv Gr(3,19) = \frac{SO(3,19)}{SO(3) \times SO(19)}$$

- Compactify type II on K3: $\quad g : (3 \times 19) \quad B : 22 \quad \phi : 1 \Rightarrow 80$

$$\Downarrow$$

4-planes in $\mathbb{R}^{4,20}$? Why?

$$\mathcal{M}_{\mathrm{string}} = \frac{SO(4,20)}{SO(4) \times SO(20)}$$

- $H^{\mathrm{even}} = H^0 \oplus H^2 \oplus H^4$ has signature (4,20)
- (Φ_+, Φ_-) span 4-plane w/ positive signature

$$\{(\Phi_+, \Phi_-)\} = \mathcal{M}_{2,2}$$

$$S^2 \times S^2 \downarrow$$

$$\{\text{ 4-planes }\} = \mathcal{M}_{4,4}$$

Mirror symmetry: $\quad \Phi_+ \leftrightarrow \Phi_-$

Vacua

[Grana,Minasian,Petrini,AT]

- IIA, IIB: Most general 4d vacua that preserve
  - $\mathcal{N}=1:$ $\delta_\epsilon \psi_m = 0$ $\delta_\epsilon \lambda = 0$
  - RR≠0
  - 4d Poincaré $g_{10} = e^{2A} g_4 + g_6$

$$F^{(10)} = F + \mathrm{vol}_4 \wedge *F \qquad F = \sum_k F_k$$

define differential

$$d_H(\cdot) \equiv e^{-2A+\phi}(d + H\wedge)(e^{2A-\phi}\ \cdot)$$

- then $\exists\ (\Phi_+, \Phi_-)|$

IIA

$$d_H \Phi_+ = 0$$
$$d_H \Phi_- = dA \wedge \bar{\Phi}_- - $$
$$-\frac{1}{16}[c_- e^{A+\phi} F^t - i e^{A-\phi} c_+ *F]$$

IIB

$$d_{-H} \Phi_- = 0$$
$$d_{-H} \Phi_+ = dA \wedge \bar{\Phi}_+ +$$
$$+\frac{1}{16}[c_- e^{A+\phi} F + i c_+ e^{A-\phi} *F^t]$$

$c_\pm$ integration constants

M_6 compact: $\Rightarrow$ orientifold $\Rightarrow$ $c_- = 0$

- all (Minkowski) vacua are generalized Calabi-Yau $(d+H\wedge)\Phi=0$

  [AdS generalization exists; "generalized half-flat"]

- $\mathrm{Re}: \quad c_- F^t = 16(d+H\wedge)(e^{A-\phi}\mathrm{Re}\Phi_-) \quad \Rightarrow (d-H\wedge)F=0$

  $\mathrm{Im}: \quad c_+\, e^{4A} * F = -16(d+H\wedge)(e^{3A-\phi}\mathrm{Im}\Phi_-) \quad \Rightarrow (d+H\wedge)e^{4A}*F=0$

  compact $\Rightarrow c_-=0 \quad \Rightarrow (d-H\wedge)F = \text{sources}$

- brane action DBI $\left(\geq \int e^{3A-\phi}e^B \mathrm{Im}\Phi_-\right)$ +CS $\left(=\int e^{4A}e^B\tilde{C}\right)$

  $\int$ pullback bulk form ($\geq$ DBI) → closed ⇒ calibrated cycles

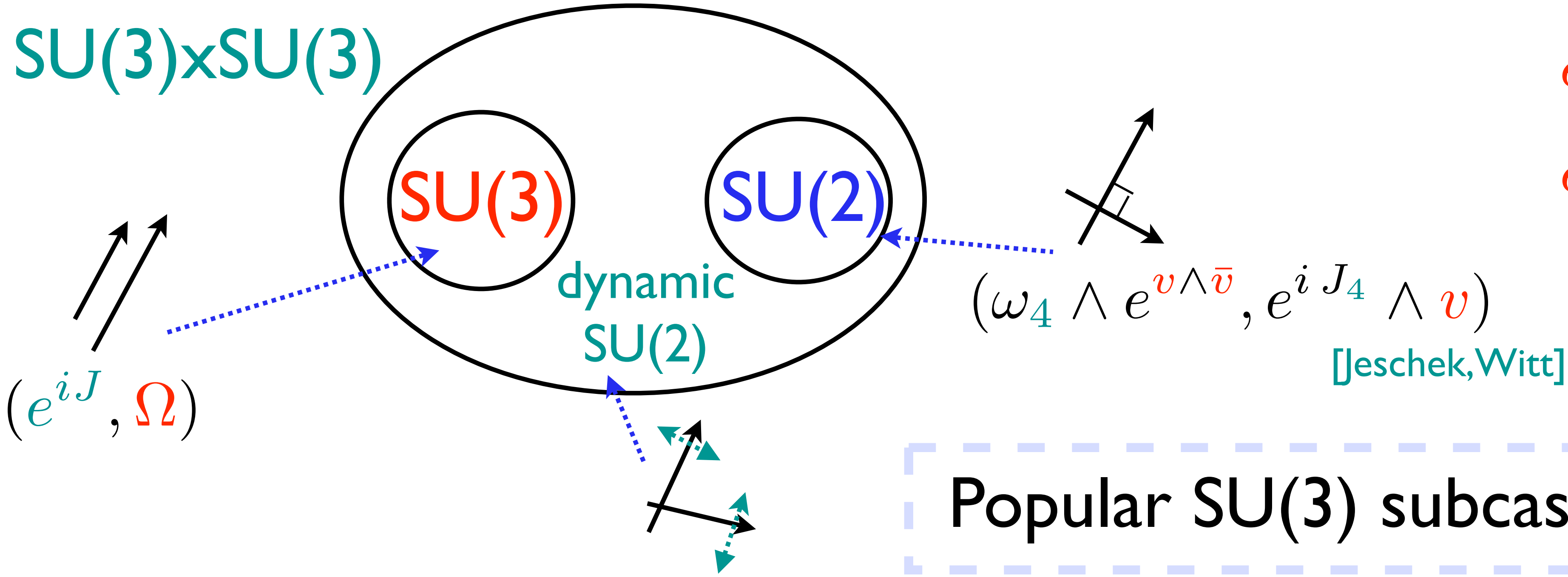
  ⇒ $c_+\, e^{4A} * F = -16(d+H\wedge)(e^{3A-\phi}\mathrm{Im}\Phi_-)$

  [Martucci,Smyth]
  gK:[Koerber]

- should also be derivable from SU(3)xSU(3) prepotentials $\mathcal{P}_\alpha(\Phi_\pm)$

  [Vafa; LawrenceMcGreevy; GMPT]
  [GranaLouisWaldram]

- mirror symmetry? $\Phi_+ \longleftrightarrow \Phi_-$



$$\epsilon_1^{10} = \epsilon_+^4 \otimes \eta_1^+ + \text{c. c.}$$
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6d spinors

Popular SU(3) subcases: (all complex)

- conformal CY; $F_3 + \tau H$ is (2,1) and primitive; $F_3 \wedge H + \text{sources} = dF_5$ $\Rightarrow \not> 0 \Rightarrow$ O3s, D3s

D3 [GranaPolchinski, DasguptaRajeshSethi, GiddingsKachruPolchinski, KlebanovStrassler]

- can combine with F-theory (no longer conf. CY)

D7

but dual to CY fourfold

fake global issue: seemingly $c_1 = 0$, but must allow "S-duality trans. functions"

Less popular (non CY):

- $(dJ - ie^{\phi} F_3)^{1,2}_{\text{prim}} = 0$

D5 [MaldacenaNunez]

[• $(dJ - iH)^{1,2} = 0$ not inside classification (RR≠0)]

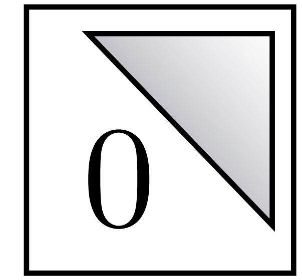
NS5 [Hull; Strominger]

These cases are in correspondence with spacetime-filling branes

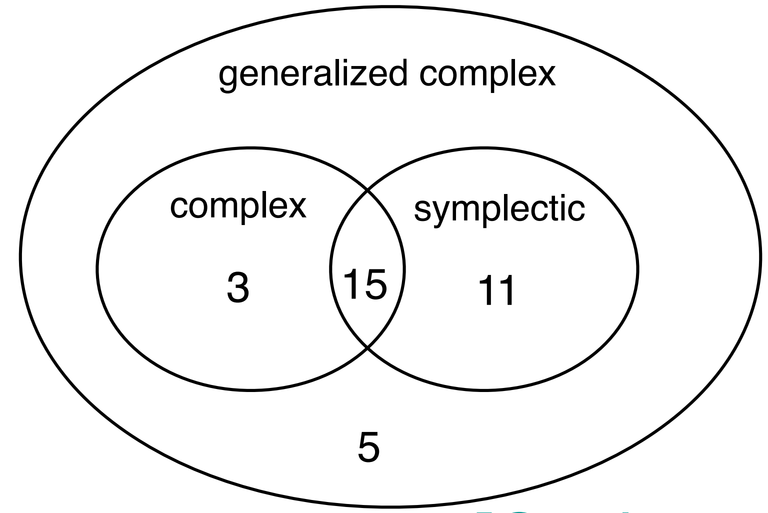
- Look inside a class: nilmanifolds

group manifolds; exponential of Lie algebras

generators=



  - All generalized complex $d\Phi = 0$



  - Action of orientifolds on $\Phi_\pm$ is known

  - All is left is to solve $d\Phi' = *F$

[Cavalcanti,Gualtieri]

  - mod out by T-dualities:

[Kapustin,Orlov'00(!)]

Ex.

$$f^1{}_{ab} \xrightarrow{T_1} H_{1ab}$$

$$e^{i(12+34+56)} \xrightarrow{T_1} (1+i2)e^{i(34+56)}$$

[three T-dualities $\sqrt{|g|}e^B\Omega \longrightarrow \sqrt{|g|}e^{-B}e^{iJ}$]

[Fidanza,Minasian,AT]

  - So far we have found only T-duals of torus

  - Extending to solvmanifolds: compactness issues

[Saito;Auslander]

- take inspiration from generalized Kähler examples?

[Gualtieri]
[Hitchin]
[Lin,Tolman;...]

several (Φ_+, Φ_-) on $\mathbb{CP}^2$

compact, (2,2) models;
but $c_1 \neq 0$

  - from deformations of ordinary Kähler

$$\delta\Omega = \beta \llcorner \Omega$$

bivector

gen. deformations of ordinary complex structure:

$$H^0(\Lambda^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O}) \; (= HH^2)$$

[Gerstenhaber,Schack]

ordinary
complex def.

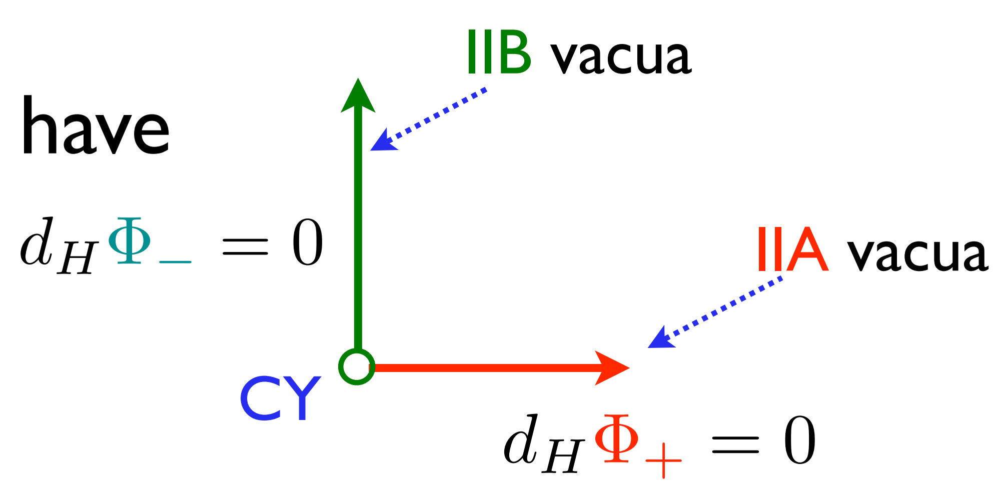
  - from "generalized Kähler quotient"

- Proof of concept: $\mathcal{N} = 2$ string theory vacua do exist

[Chuang,Kachru,AT]

  - obtained from effective 4d theory and non-CY transitions
  - $0 \neq g_s \ll 1$; regions of strong curvature $\Longrightarrow$ not in 10d supergravity

- So we have



Black holes

[Hsu,Maloney,AT]

... rather than vacua, again from compactification on M_6

$$ds^2 = e^{2A(y)}\left[\quad \text{4d black hole metric} \quad\right] + g^{\text{internal}}(r, y)$$

$$F^{(10)} = F + \text{vol}_4 \wedge *F + \text{vol}_{S^2} f + *_4 \text{vol}_{S^2} * f$$

$$f = \sum_k f_k$$

I. $\forall\, r$ pure spinors describe $\mathcal{N} = 2$ vacuum (with RR! "twice $\mathcal{N} = 1$")

II. at the horizon: $f = \text{Im}(\bar{C}\Phi_-^{14})$ [attractor equation]

CY: $f = \text{Im}(\bar{C}\Omega_-)$

When can we solve it?

[Hitchin]:

[pointwise:] If and only if $-\text{Tr}[\mathcal{J}(f)]^2 \equiv q(f) > 0$

$\int \sqrt{q(f)}$ is nothing but the horizon area

Clearly it had to be $q>0$.

[but: indirect extra constraint on Φ^{14}]

via [Ooguri,Vafa,Strominger] these black holes should be related to a

Topological model

on a symplectic or complex manifold

- from a generalized Kahler $(\mathcal{J}_+, \mathcal{J}_-)$ $\Rightarrow$ (2,2) model $\xrightarrow{\text{twisting}}$ topological model [Kapustin]

- do we really need both?

suggested picture: [GMPT]

A	$de^{iJ} = 0$
B	$d\Omega = 0$
gen.	$d\Phi = 0$

For SU(3):

- On CY: A independent of complex moduli
- On CY: B independent of symplectic moduli
- in fact: A defined on any symplectic manifold

[Alexandrov,Kontsevich, Schwarz,Zaboronsky] use BV to produce topological models

So far it works well for $[j_+, j_-] \neq 0$ [Zucchini; Pestun]

Summary

	Supergravity			Worldsheet (weaker)		
	conditions ($\sim$)	geometry	compact?		cond.	geometry
$\mathcal{N}=2$ RR=0	$(d+H\wedge)\Phi_{\pm}=0$	"generalized CY metric"	no	(2,2)	$(\mathcal{J}_+,\mathcal{J}_-)$ integrable	generalized Kähler
$\mathcal{N}=1$ RR≠0	$(d+H\wedge)\Phi=0$ $(d+H\wedge)\Phi'=$ $+c_-F+ic_+*F$	generalized CY + "RR=gen.Nij."	$c_-=0$			
$\mathcal{N}=1$ RR=0	$(d+H\wedge)\Omega=0$ $\partial J=iH^{2,1}$	generalized CY +...?	no	(2,1)	"$\mathcal{J}$ +1/2 integrable"	generalized complex +...?

$\mathcal{N}=2$ RR≠0 "=twice $\mathcal{N}=1$ RR≠0 "

Other dimensions:

3+8 [Tsimpis]

3+7 [Jeschek,Witt]

Topological model

classical	anomaly	top. string
$\mathcal{J}$ integrable	$(d+H\wedge)\Phi=0$	?

Conclusions

- GCG nicely classifies vacua
    - What is the use of this geometry?
        - Is this deep?