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Bose-Einstein Condensation with
Entangled Order Parameter

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Bose-Einstein Condensation with Entangled Order Parameter

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Two-component BEC

A mixture of A-atoms and B-atoms:

\[ \psi \approx \phi(\mathbf{r}_{a1}) \cdots \phi(\mathbf{r}_{aN_a}) \otimes \phi(\mathbf{r}_{b1}) \cdots \phi(\mathbf{r}_{bN_b}) \]

\[ |\psi\rangle = \frac{1}{\sqrt{N_a N_b}} (a^\dagger)^{N_a} (b^\dagger)^{N_b} |0\rangle = |\psi\rangle_a \otimes |\psi\rangle_b \]

A-atoms and B-atoms separately condense, with separate order parameters (classically coupled). Mean field theory.

Similar is a mixture of one species of atoms with two spin states, the numbers of which are conserved respectively.
Spin-1 condensate

\[ |\psi\rangle \sim \left[ (a_0^\dagger)^2 - 2a_{-1}^\dagger a_1^\dagger \right]^{N/2} |0\rangle \]

- Similar is the spin-1/2 BEC (Kuklov-Svistunov)
- Non mean-field state.
- But the particles are all identical! Each particle can flip spins.
- Practically very difficult to realize, as the energy difference with the symmetry breaking mean-field state vanishes as \( N \to \infty \).
2 species \( \times \) 2 spin states

- Each atom can flip the spin, but cannot transit between the atom species.

- \( N_{i\uparrow} \) and \( N_{i\downarrow} \) (i=a,b) are not conserved.
Only consider single particle orbital ground state; ignore depletion.
Scattering channels

Intra-species:

Similar is the b-species.
Scattering channels (continued)

Inter-species, without spin-exchange
Scattering channels (continued)

Inter-species, with spin-exchange
Requirements

- Energy conservation in each scattering
- Conservation of total z-component spin in each scattering
Experimental feasibility

For given $I$ and $J$, Hyperfine-Zeeman energy levels depend only on $F, m_F$, not on atom species.
Merits

- Interesting spinful BEC can thus also be realized in magnetic traps.
- Call for experiments on multichannel scattering between different species of atoms.
- But what is the goodness?
  It realizes, in the ground state, entanglement between BECs.
Entanglement between BECs

Like a pure two-particle entangled state, where each particle is not in any pure spin state, there is no simple BEC of either species; there is only a global simple BEC.

BEC occurs in an entangled inter-species pair state.

Hamiltonian

\[ H = \sum_{\sigma} f_{i\sigma} N_{i\sigma} + \frac{1}{2} \sum_{\sigma\sigma'} |K^{(ii)}_{\sigma\sigma'}| N_{i\sigma} N_{i\sigma'} + \sum_{\sigma\sigma'} K^{(ab)}_{\sigma\sigma'} N_{a\sigma} N_{b\sigma'} + \frac{K_e}{2} (a_{\uparrow}\downarrow b_{\uparrow\downarrow} + a_{\downarrow\uparrow} b_{\downarrow\uparrow}) \]

\[ K^{(ij)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \equiv (2\pi \hbar^2 \xi^{(ij)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} / \mu_{ij}) \int \phi^*_{i\sigma_1} (r) \phi^*_{j\sigma_2} (r) \phi_{j\sigma_3} (r) \phi_{i\sigma_4} (r) d^3 r \]

\[ |K^{(ii)}_{\sigma\sigma} = |K^{(ii)}_{\sigma\sigma} \]

\[ K^{(ii)}_{\sigma\bar{\sigma}} \equiv 2K^{(ii)}_{\sigma\bar{\sigma}\sigma\bar{\sigma}} = 2K^{(ii)}_{\sigma\sigma\bar{\sigma}\bar{\sigma}} \]

\[ K^{(ab)}_{\sigma\sigma'} \equiv K^{(ab)}_{\sigma\sigma'\sigma'} \]

\[ K_e \equiv 2K^{(ab)}_{\uparrow\downarrow\uparrow\downarrow} = 2K^{(ab)}_{\downarrow\uparrow\downarrow\uparrow} \]

\[ f_{i\sigma} \equiv \epsilon_{i\sigma} - |K^{(ii)}_{\sigma\sigma}| / 2 \]

\[ \epsilon_{a\uparrow} - \epsilon_{a\downarrow} = \epsilon_{b\downarrow} - \epsilon_{b\uparrow} \]
Spin representation

\[ S_a = \sum_{\sigma, \sigma'} a_{\sigma}^\dagger s_{\sigma \sigma'} a_{\sigma'}, \quad S_b = \sum_{\sigma, \sigma'} b_{\sigma}^\dagger s_{\sigma \sigma'} b_{\sigma'} \]

The Hamiltonian becomes that of two big spins

\[ S_a = \frac{N_a}{2} \quad \text{and} \quad S_b = \frac{N_b}{2} \]

\[ \frac{\mathcal{H}}{J} = \frac{K}{J_z} (s_{ax} s_{bx} + s_{ay} s_{by}) + s_{az} s_{bz} + B_a s_{az} + B_b s_{bz} + C_a s_{az}^2 + C_b s_{bz}^2 + \frac{E_0}{J_z} \]

Coefficients are functions of K’s.
Conserved Quantities

\[ N_i = N_i^\uparrow + N_i^\downarrow \]

Total

\[ S_z = \left( N_a^\uparrow - N_a^\downarrow + N_b^\uparrow - N_b^\downarrow \right) / 2 \]
Isotropic point

\[ \mathcal{H} = J_z S_a \cdot S_b \]

Ground states:

\[ |G_{S_z}\rangle = |S_a - S_b, S_z\rangle = A (a_\uparrow^{\dagger} n_\uparrow (a_\downarrow^{\dagger} n_\downarrow (a_\uparrow^{\dagger} b_\downarrow^{\dagger} - a_\downarrow^{\dagger} b_\uparrow^{\dagger})^{N_b} |0\rangle \]

\[ n_\uparrow = N_a/2 - N_b/2 + S_z, \quad n_\downarrow = N_a/2 - N_b/2 - S_z \]

Degenerate but unique for a given \( S_z \).

For \( N_a = N_b = N \):

\[ |G_0\rangle = (\sqrt{N} + 1N!)^{-1} (a_\uparrow^{\dagger} b_\downarrow^{\dagger} - a_\downarrow^{\dagger} b_\uparrow^{\dagger})^N |0\rangle \]
Concept of quantum entanglement

\[ |\Psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B \]

E.g. \( \frac{1}{2} (|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle) \)

The most important concept distinguishing quantum mechanics from classical theory.

Can be quantified as

\[ S = \log \rho_A, \quad \rho_A = Tr_B |\Psi\rangle \langle \Psi|, \]
	hanks{thanks to quantum information theory.}
Using entanglement to characterize
the non-mean field nature

\[ |G_0\rangle = (\sqrt{N+1})^{-1} \sum_{m=0}^{N} (-1)^{N-m} |m\rangle_{a\uparrow} |N-m\rangle_{a\downarrow} |N-m\rangle_{b\uparrow} |m\rangle_{b\downarrow} \]

- Consider its occupation entanglement
  Method: YS, Phys.Rev.A 67, 024301 (03);
- The subsystems are the single particle basis states envolved.
- Entanglement entropy: von Neumann entropy of the reduced density matrix of a subsystem, which measures the entanglement with the rest of the system.
Using entanglement to characterize the non-mean field nature (continued)

For each single particle basis state, the occupation number is \( N + 1 \)-valued, so the base of the entanglement entropy is set to be \( N + 1 \).

\[
|G_0\rangle = \frac{1}{\sqrt{(N+1)}} \sum_{m=0}^{N} (-1)^{N-m} |m\rangle_a\uparrow |N-m\rangle_a\downarrow |N-m\rangle_b\uparrow |m\rangle_b\downarrow
\]

is an equal superposition of quart-orthogonal states, consequently the entanglement entropy for each single particle basis state is 1.
Entanglement between the two species

The basis of A species is chosen to be

\[(a \uparrow, a \downarrow)\]

The occupation [always \((m, N-m)\)] is still \(N+1\)-valued.

Consequently the entanglement between the two species is 1.

\[
|G_0\rangle = (\sqrt{N+1})^{-1} \sum_{m=0}^{N} (-1)^{N-m} |m\rangle_a \uparrow |N-m\rangle_a \downarrow |N-m\rangle_b \uparrow |m\rangle_b \downarrow
\]
Entanglement as a kind of pairing

**Note**

\[
(a_{\uparrow}^\dagger b_{\downarrow}^\dagger - a_{\downarrow}^\dagger b_{\uparrow}^\dagger)^{N_b} = \left[ \sqrt{2} \int d^3 r_a d^3 r_b \psi_a^\dagger(r_a) \psi_b^\dagger(r_b) \phi(r_a, r_b) \right]^{N_b}
\]

\[
\psi_a(r) = \sum_\sigma a_\sigma \phi_{a\sigma}(r_a) |\sigma\rangle_a, \quad \psi_b(r) = \sum_\sigma b_\sigma \phi_{b\sigma}(r_b) |\sigma\rangle_b
\]

\[
\phi(r_a, r_b) = \frac{1}{\sqrt{2}} \left[ \phi_{a\uparrow}(r_a) |\uparrow\rangle_a \phi_{b\downarrow}(r_b) |\downarrow\rangle_b + \phi_{a\downarrow}(r_a) |\downarrow\rangle_a \phi_{b\uparrow}(r_b) |\uparrow\rangle_b \right]
\]

| \[ G_{S_z} \] \] is thus a condensation of interspecies pairs in the same two-particle entangled state \[ \phi(r_a, r_b) \]

\[ \phi(r_a, r_b) \] is the entangled order parameter.
Entangled pairing lowers the energy

A simple example:

\[
 h(r_a) + h(r_b) + U_1(r_a - r_b) + U_2(r_a - r_b)(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow|)
\]

\[ U_2 > 0 \]

\[
 \phi_a(r_a)\phi_b(r_b)(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}
\]

has lower energy than

\[
 \phi_a(r_a)\phi_b(r_b)|\sigma\rangle|\sigma'\rangle
\]
Detection of the entanglement (1)

(Of course) fluctuations of $N_{i\sigma}$

\[
\sqrt{\langle N_{a\sigma}^2 \rangle - \langle N_{a\sigma} \rangle^2 / \langle N_{a\sigma} \rangle} \approx 1 / \sqrt{3}
\]

Can be obtained from density fluctuation, which is self-averaging, and can be studied in a single image

\[
\rho_{i\sigma}(r_i) = N_{i\sigma} |\phi_{i\sigma}(r_i)|^2
\]

\[
\sqrt{\langle \rho_{i\sigma}(r_i)^2 \rangle - \langle \rho_{i\sigma}(r_i) \rangle^2 / \langle \rho_{i\sigma}(r_i) \rangle} = \sqrt{\langle N_{i\sigma}^2 \rangle - \langle N_{i\sigma} \rangle^2 / \langle N_{i\sigma} \rangle}
\]

Free expansion of the condensate does not affect entanglement
Detection of the entanglement (2)

Nonvanishing of the connected correlations

\[ C_{\sigma,\sigma'} \equiv \langle N_{a\sigma} N_{b\sigma'} \rangle - \langle N_{a\sigma} \rangle \langle N_{b\sigma'} \rangle \]

\[ C_{\sigma,\bar{\sigma}} = -\frac{N(N+2)}{12}, \quad C_{\sigma,\bar{\sigma}} = \frac{N(N+2)}{12} \]

\[ g(r_a, \sigma; r_b, \sigma') \equiv \langle \rho_{a\sigma}(r_a) \rho_{b\sigma'}(r_b) \rangle - \langle \rho_{a\sigma}(r_a) \rangle \langle \rho_{b\sigma'}(r_b) \rangle \]

\[ g(r_a, \sigma; r_b, \sigma') / \langle \rho_{a\sigma}(r_a) \rangle \langle \rho_{b\sigma'}(r_b) \rangle = C_{\sigma,\sigma'} / \langle N_{a\sigma} \rangle \langle N_{b\sigma'} \rangle \]
Detection of entanglement (3)

Measuring spin of an A-atom,

\[ P_{\sigma} = \frac{\langle a_{\sigma}^\dagger a_{\sigma} \rangle}{\sum_{\sigma'} \langle a_{\sigma'}^\dagger a_{\sigma'} \rangle} \]

Joint measurement of the spins of an A-atom and a B-atom which leave the trap

\[ P_{\sigma,\sigma'} = \frac{\langle b_{\sigma'}^\dagger a_{\sigma}^\dagger a_{\sigma} b_{\sigma'} \rangle}{\sum_{\sigma_a,\sigma_b} \langle b_{\sigma_b}^\dagger a_{\sigma_a}^\dagger a_{\sigma_a} b_{\sigma_b} \rangle} \]
Detection of entanglement (3)
(continued)

- Mean-field (non-entangled) state:

\[
\left(\sqrt{N_1!N_2!N_3!N_4!}\right)^{-1} a_\hat{n}^{N_1} a_{\hat{n}}^{N_2} b_\hat{m}^{N_3} b_{\hat{m}}^{N_4} |0\rangle,
\]

\[
P_{\sigma_a, \sigma_b} = P_{\sigma_a} P_{\sigma_b}
\]
Detection of entanglement (3) (continued)

- Non-mean-field (entangled) BEC:

\[ P_{\sigma_a, \sigma_b} \neq P_{\sigma_a} P_{\sigma_b} \]

- E.g., for \( |G_0\rangle \), \( P_{\sigma_a} = P_{\sigma_b} = 1/2 \),

\[
\begin{align*}
P_{\uparrow\downarrow} &= P_{\downarrow\uparrow} = (2N + 1)/6N, \\
P_{\uparrow\uparrow} &= P_{\downarrow\downarrow} = (N - 1)/6N
\end{align*}
\]
Feedback effect on single-particle orbits

\[ \left\{ -\frac{\hbar^2}{2m}\nabla^2 + U_{a\sigma}(\mathbf{r}) + \left[ N(N-1)/3 \right] g_{\sigma\sigma}^{(aa)} \left| \phi_{a\sigma}(\mathbf{r}) \right|^2 \\
+ \left[ N(N-1)/6 \right] g_{\uparrow\downarrow}^{(aa)} \left| \phi_{a\bar{\sigma}}(\mathbf{r}) \right|^2 + \left[ N(N-1)/6 \right] g_{\sigma\bar{\sigma}}^{(ab)} \left| \phi_{b\sigma}(\mathbf{r}) \right|^2 \\
+ \left[ N(2N+1)/6 \right] g_{\sigma\bar{\sigma}}^{(ab)} \left| \phi_{b\bar{\sigma}}(\mathbf{r}) \right|^2 \right\} \phi_{a\sigma}(\mathbf{r}) \\
- \left[ N(N+2)/12 \right] g_e \phi_{b\bar{\sigma}}^{*}(\mathbf{r}) \phi_{b\sigma}(\mathbf{r}) \phi_{a\bar{\sigma}}(\mathbf{r}) \\
= \mu_{a\sigma} \phi_{a\sigma}(\mathbf{r}) \]

Interference term (proportional to \( g_e \)) emerges.
How the entanglement survives the coupling anisotropy and the nonvanishing of

$$B_a, B_b, C_a, C_b$$

$$\mathcal{H} = \frac{K_e}{J_z} \left( S_{ax} S_{bx} \pm S_{ay} S_{by} \right) + S_{az} S_{bz} + B_a S_{az} + B_b S_{bz} + C_a S_{az}^2 + C_b S_{bz}^2 + \frac{E_0}{J_z}$$
Persistence of entanglement in a wide parameter regime (1)

**Coupling anisotropy** $K_e/J_z$, $B_a = B_b = C_a = C_b = 0$

$S_z = 0$
Persistence of entanglement in a wide parameter regime (2)

- Coupling anisotropy $K_e/J_z$  
  $B_a = B_b = C_a = C_b = 0$

$S_a = 12000$, $S_b = 10000$
Persistence of entanglement in a wide parameter regime (3)

- $C_a$ and $C_b$ nonzero, $B_a = B_b = 0$, under typical values $S_a = 12000, S_b = 10000, S_z = 1000, K_e/J_z = 1.2$

- $J_z, C_a J_z$ and $C_b J_z$ are of the same order of magnitude
Persistence of entanglement in a wide parameter regime (4)

Typically choose $C_a = 0.2$ and $C_b = 0.4$. 
Therefore, in a wide parameter regime, the ground state is of non-mean-field.
Energy difference with the mean-field state

- It is (of course) vanishing at the isotropic limit.
- But it is estimated that when \(|B_a - B_b|\) is of the order of \(N\), or \(C_a + C_b\) is of the order of \(-1\), the energy difference with the lowest mean-field (symmetry breaking) state is of the order of \(N/V\).
- Finite in thermodynamic limit!
- In this regime, as far as \(K_{\text{e}}\) is larger or not much smaller than \(J_2\), the entanglement is significantly nonvanishing.
Summary

- We proposed a non-mean-field ground state of BEC, occurring in an interspecies two-particle entangled state.
- Hence the order parameter is entangled.
- Interspecies entanglement persists in a wide parameter regime.
- In part of the regime, the energy difference with the mean-field state is nonvanishing.
- Call for study of interspecies multichannel scattering.
Thank you for your attention!