The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR.1763- 5

SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

31 July - 8 August 2006

Generalized statistical mechanics

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Collaboration with

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Motivations

- **Experiments**
  - **Power-law distributions**
    - Physical systems: plasmas, high-energy processes, phase transitions, turbulence, …
    - Other fields: temporal series, economics, geology,…
- **Different approaches**
  - Deformations of distributions functions (ad hoc)
  - Deformations of distributions functions (axiomatic)
  - Modified algebra:
    - Sum, derivative, differential equation, solutions
  - Kinetical approach:
    - *microscopic,/mesoscopic physics, modified coefficients (Fokker-Plank, …)*

    Quarati, Kaniadakis:

    Physica A 192 (1993) 677 (A set of non-Maxwellian distributions);

    PRE 49 (1994) 11529; PRE 49 (1994) 1529;
  - **New entropic form**

Motivations

Success of the Boltzmann-Gibbs (BG) theory suggests that new formulations of statistical mechanics should preserve most of the mathematical and epistemological structure of the classical theory.

New entropic form:
physical interpretation or convenient functional that generate a consistent framework?

Supply a different perspective:

- **q-statistical mechanics unifies different existing pieces (deformed exponential, information entropy, …)**
- **Other coherent frameworks?**

Motivations

- **More specific aim**

  Derive the largest class of extended entropies (coherent statistical mechanics, distributions) with "reasonable" properties:
  - **Trace form**
  - **"exponential" distributions**
- **Three-parameter class of entropies**
- **Existing 1- 2-parameter cases**
  - discuss in an unified way
  - comparison
  - asymptotic behavior

Motivations

- **Physically motivated models**
  - q-entropy (Tsallis): *fractal space, long range, …*
  - q-entropy (Abe): *quantum groups*
  - κ-entropy (Kaniadakis): *relativistic transformations*

- **Not only theory**
  - **Chaotic systems**

New entropic forms generalize the one introduced by Boltzmann, Gibbs and Shannon (BGS entropy)

$$S_{\mathrm{BGS}}(n) = -k_{\mathrm{B}} \sum_i n_i \log(n_i) = -k_{\mathrm{B}} \langle \log(n) \rangle$$

First request: trace form generalization

$$S(n) = -k_{\mathrm{B}} \sum_i n_i \Lambda(n_i) = -k_{\mathrm{B}} \langle \Lambda(n) \rangle$$

$\Lambda(n)$ **is an analytic function:**
we have in mind a generalized (deformed) logarithm

Canonical formalism

Entropic functional

$$\mathcal{F} = S(n) + \beta\left(\sum_i E_i n_i - U\right) + \beta'\left(\sum_i n_i - N\right)$$

Stationary for variation of Lagrange multipliers: constraints

$\delta\mathcal{F}/\delta\beta = 0$ $\quad \sum_i E_i n_i = U$ $\quad \delta\mathcal{F}/\delta\beta' = 0$ $\quad \sum_i n_i = N$

Stationary for variation of probabilities

$$\frac{\delta}{\delta n_j}\sum_i\left[-k_{\mathrm{B}}\, n_i \log(n_i) - \frac{1}{T}(E_i n_i - U) + \frac{\mu}{T}(n_i - N)\right] = 0$$

$$\frac{\delta}{\delta n_i} n_i \log(n_i) = \log(n_i) + 1 = -\frac{E_i - \mu}{k_{\mathrm{B}} T}$$

$$n_i \propto \exp\left\{-\frac{E_i - \mu}{k_{\mathrm{B}} T}\right\}$$

Canonical formalism

Trace form generalization

$$\frac{\delta}{\delta n_j} \sum_i \left[-k_{\rm B}\, n_i\, \Lambda(n_i) - \frac{1}{T}(E_i\, n_i - U) + \frac{\mu}{T}(n_i - N) \right] = 0$$

$$\frac{\delta}{\delta n_i} n_i\, \Lambda(n_i) = -\frac{E_i - \mu}{k_{\rm B} T}$$

Without loss of generality

- **"*Exponential*" ansatz**

$$\Rightarrow \quad n_j = \alpha\, \mathcal{E}\left[-\frac{1}{\lambda} \left(\frac{E_j - \mu}{k_{\rm B}\, T} + \eta \right) \right]$$

Three parameters

$\mathcal{E}(x)$: unspecified invertible function

Canonical formalism

$$\frac{\delta}{\delta n_i} n_i \Lambda(n_i) = \lambda \mathcal{E}^{-1}\left(\frac{n_i}{\alpha}\right) + \eta$$

Second request (analogy with relation between log and exp):

$\mathcal{E}(x)$ **be inverse of** $\Lambda(x)$

$\Lambda(x)$ is invertible, $\mathcal{E}(x)$ generalized exponential, distribution has modified exponential form:

$$n_j = \alpha\, \mathcal{E}\left[-\frac{1}{\lambda}\left(\frac{E_j - \mu}{k_B T} + \eta\right)\right]$$

$\Lambda(x)$ **must verify a differential-functional equation**

$$\frac{\delta}{\delta n_i} n_i \Lambda(n_i) = \lambda \Lambda\left(\frac{n_i}{\alpha}\right) + \eta$$

Two boundary conditions: $\Lambda(1) = 0$ $\quad d\Lambda(x)/dx|_{x=1} = 1$

From 3 to 2 parameters

η can be easily eliminated

$$[x\Lambda(x)]' = \lambda\Lambda(x/\alpha)$$

Define: $\bar{\Lambda}(x) \equiv a\Lambda(\sigma x) + b$

$$\bar{\Lambda}(1) = 0$$
$$d\bar{\Lambda}(x)/dx\big|_{x=1} = 1$$

$$\Rightarrow \quad a = \frac{1}{\sigma\Lambda'(\sigma)} \qquad b = \frac{-\Lambda(\sigma)}{\sigma\Lambda'(\sigma)}$$

$$[x\bar{\Lambda}(x)]' = \lambda\Lambda(x/\alpha) + \eta$$

$$\eta = (1-\lambda)b = (\lambda - 1)\frac{\Lambda(\sigma)}{\sigma\Lambda'(\sigma)}$$

From 3 to 2 parameters

$$[x\bar{\Lambda}(x)]' = \bar{\Lambda}(x) + x\,[a\Lambda(\sigma x) + b]' = \bar{\Lambda}(x) + xa[\Lambda(\sigma x)]'$$

$$= \bar{\Lambda}(x) + a(\sigma x)\,[\Lambda'(y)]_{y=\sigma x} = \bar{\Lambda}(x) + a\,[y\Lambda'(y)]_{y=\sigma x}$$

$$= \bar{\Lambda}(x) + a\,[\lambda\Lambda(y/\alpha) - \Lambda(y)]_{y=\sigma x} = a\Lambda(\sigma x) + b + a\lambda\Lambda(\sigma x/\alpha) - a\Lambda(\sigma x)$$

$$= b + a\lambda\Lambda(\sigma x/\alpha) = b + \lambda(\bar{\Lambda}(x/\alpha) - b) = \lambda\bar{\Lambda}(x/\alpha) + b(1-\lambda)$$

If $\Lambda(x)$ verifies $[x\Lambda(x)]' = \lambda\Lambda(x/\alpha)$ then

$$\bar{\Lambda}(x) \equiv \frac{\Lambda(\sigma x) - \Lambda(\sigma)}{\sigma\Lambda'(\sigma)}$$ verifies $$[x\bar{\Lambda}(x)]' = \lambda\Lambda(x/\alpha) + \eta$$

with $$\eta = (\lambda - 1)\frac{\Lambda(\sigma)}{\sigma\Lambda'(\sigma)}$$

Two-parameter solution

Solve

$$\frac{d}{dx} x\,\Lambda(x) = \lambda\,\Lambda\left(\frac{x}{\alpha}\right)$$

Can be transformed into

Delay equation

$$\frac{d\,f(t)}{dt} = f(t - t_0)$$

$$\begin{aligned} f(t) &= x\,\Lambda(x) \\ x &= \exp\left(\frac{t}{\lambda\alpha}\right) \\ t_0 &= \lambda\alpha\ln\alpha \end{aligned}$$

Laplace transform, exponential basis

Original variables: powers

$$\Lambda(x) = \sum_i^m A_i x^{\kappa_i}$$

$$1 + \kappa = \lambda\,\alpha^{-\kappa}$$

Two-parameter solution

simple root

$$\boxed{\Lambda(x) = x^{\kappa}}$$

$$\frac{d}{dx}[x\Lambda(x)] - \lambda\Lambda(\frac{x}{\alpha}) = (\kappa + 1)x^{\kappa} - \lambda\frac{x^{\kappa}}{\alpha^{\kappa}}$$

$$= x^{\kappa}\left[\kappa + 1 - \lambda\alpha^{-\kappa}\right] = 0$$

$$\boxed{1 + \kappa - \lambda\,\alpha^{-\kappa} = 0}$$

Two-parameter solution

multiple root

$$\boxed{\Lambda(x) = x^\kappa \log^n x} = (d/d\kappa)^n \, x^\kappa$$

$$\frac{d}{dx}\left[x\left(\frac{d}{d\kappa}\right)^n x^\kappa\right] - \lambda\left(\frac{d}{d\kappa}\right)^n \left(\frac{x}{\alpha}\right)^\kappa = 0$$

$$\left(\frac{d}{d\kappa}\right)^n \left[\frac{d}{dx}x^{\kappa+1} - \lambda x^\kappa \alpha^{-\kappa}\right] = \left(\frac{d}{d\kappa}\right)^n \left[\left(1+\kappa-\lambda\,\alpha^{-\kappa}\right) x^\kappa\right] = 0$$

$$\boxed{\left(\frac{d}{d\kappa}\right)^j \left(1+\kappa-\lambda\,\alpha^{-\kappa}\right) = 0} \quad j = 0, \ldots, n$$

Solution of the equation for κ depending on the parameters λ and α

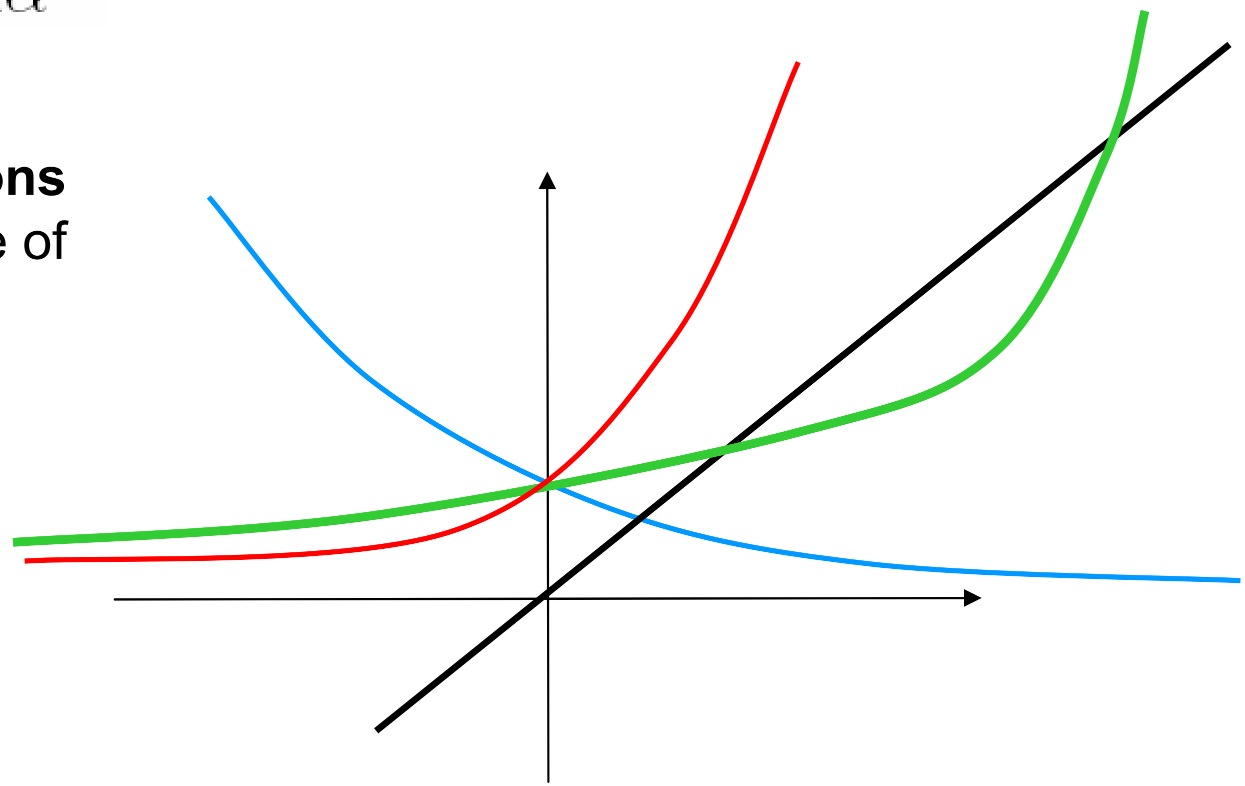
$$1 + \kappa = \lambda\, \alpha^{-\kappa}$$

$$\frac{1+\kappa}{\lambda\alpha} = \alpha^{-(\kappa+1)} = e^{-(\kappa+1)\log\alpha} = e^{-\frac{1+\kappa}{\lambda\alpha}\lambda\alpha\log\alpha}$$

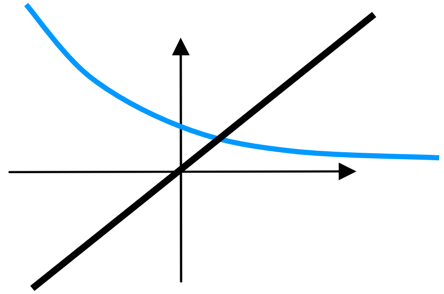
defining $s = \dfrac{1+\kappa}{\lambda\alpha}$ $\quad t = \lambda\alpha\log\alpha$ becomes $s = e^{-st}$

Zero, one, two solutions depending on the value of

$$t = \lambda\alpha\log\alpha$$



$t = \lambda\alpha \log \alpha \geq 0$ one solution



Single power, non interesting trivial solution $\Lambda(x) = a x^{\kappa}$

When boundary imposed $\Lambda(x) = 0$

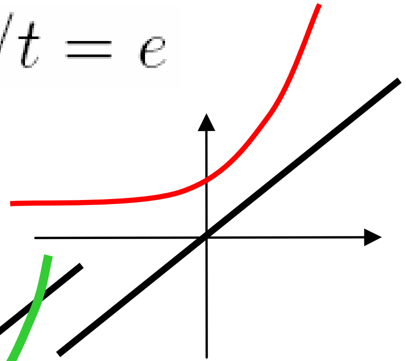
$t = \lambda\alpha \log \alpha < 0$: no solution or two solutions

border case: double root

$$s = e^{-st}$$
$$1 = -t e^{-st}$$

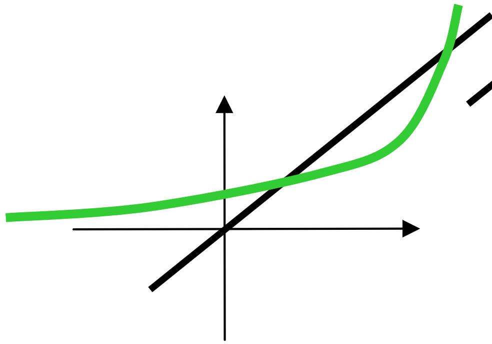
$\Rightarrow$

$$st = -1$$
$$s = -1/t = e$$



$t = \lambda\alpha \log \alpha < -1/e$ no solution

$-1/e < \lambda\alpha \log \alpha < 0$ two solutions



$$\Lambda(x) = A_1(\kappa_1, \kappa_2)\, x^{\kappa_1} + A_2(\kappa_1, \kappa_2)\, x^{\kappa_2}$$

Boundary conditions (log as limiting value)

$$\Lambda(1) = 0$$

$$\frac{d\,\Lambda(x)}{d\,x}\Big|_{x=1} = 1$$

$$\forall\, \kappa_1,\ \kappa_2$$

⇒

$$\Lambda(x) = \frac{x^{\kappa_1} - x^{\kappa_2}}{\kappa_1 - \kappa_2}$$

Symmetric choice of parameters

$$\kappa = (\kappa_1 - \kappa_2)/2$$

$$r = (\kappa_1 + \kappa_2)/2$$

$$\ln_{\{\kappa,r\}}(x) = \Lambda(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\,\kappa}\, x^{r}$$

Since $\ln_{\{\kappa,r\}}(x) = \ln_{\{-\kappa,r\}}(x)$ **consider** $\kappa \geq 0$

Appropriate range of parameters

Λ **bijective** ⇒ **invertible**

$$\mathcal{E}(x) \equiv \Lambda^{-1}(x) = \exp_{\{\kappa,r\}}(x)$$

Properties (required) of the whole class of logarithms

Analyticity

$$\ln_{\{\kappa,\, r\}}(x) \in C^{\infty}(\mathbf{R}^{+})$$

Monotonic increasing function (invertible)

$$\frac{d}{d\,x} \ln_{\{\kappa,\, r\}}(x) > 0 \quad \Rightarrow \quad \boxed{-\kappa < r < \kappa}$$

Stability

$$\frac{d^2}{d\,x^2} \ln_{\{\kappa,\, r\}}(x) < 0 \quad \Rightarrow \quad \boxed{-\kappa < r < 1 - \kappa}$$

Asymptotic property

$$\ln_{\{\kappa,r\}}(x) \underset{x \to 0^+}{\sim} -\frac{1}{2\,|\kappa|}\,\frac{1}{x^{|\kappa|-r}} \to -\infty$$

(exception: q-log)

Normalizability

$$\int_0^1 \ln_{\{\kappa,\, r\}}(x)\, dx = -\frac{1}{(1+r)^2 - \kappa^2} \quad \Rightarrow \quad \boxed{r > \kappa - 1}$$

Properties (required) of the whole class of logarithms

$$\ln_{\{\kappa,r\}}(x) = \frac{x^{r+\kappa} - x^{r-\kappa}}{2\,\kappa}$$

Monotonic increasing function (invertible)

$$(r+\kappa)x^{r+\kappa-1} - (r-\kappa)x^{r-\kappa-1} > 0$$ dividing positive factor

$$(r+\kappa)x^{2\kappa} > r-\kappa$$ ⟹ $r+\kappa > 0$ and $r-\kappa < 0$

also $r = \pm\kappa$

eliminates possibility of degenerate root $\kappa_1 = \kappa_2$ ⟹ $\kappa = 0$

$$\Lambda(x) = x^r \log(x)$$ $\kappa = 0$ ⟹ $r = 0$

Stability

$$(r+\kappa)(r+\kappa-1)x^{r+\kappa-2} - (r-\kappa)(r-\kappa-1)x^{r-\kappa-2} < 0$$

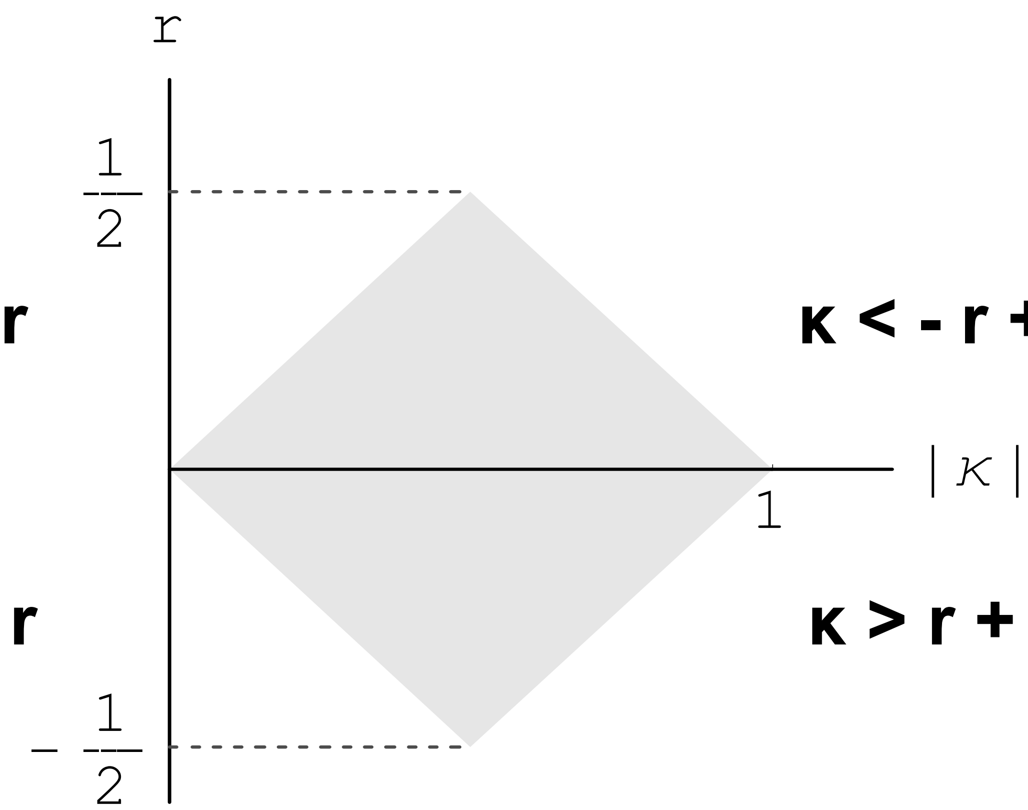
$$(r+\kappa)(r+\kappa-1)x^{2\kappa} < (r-\kappa)(r-\kappa-1) > 0$$

$$r+\kappa-1 < 0$$

range of parameters

κ < r

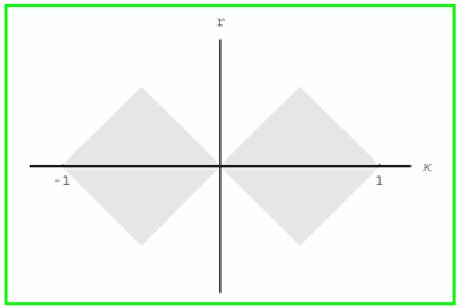
κ > - r



κ < - r + 1

κ > r + 1

If –κ is also shown



Corresponding properties of the deformed exponential

$$\exp_{\{\kappa,\,r\}}(x) \in C^{\infty}(\mathbf{R})$$

$$\exp_{\{\kappa,\,r\}}(0) = 1$$

$$\frac{d}{d\,x} \exp_{\{\kappa,\,r\}}(x) > 0$$

$$\frac{d^2}{d\,x^2} \exp_{\{\kappa,\,r\}}(x) > 0$$

Asymptotic behavior and normalizability

$$\exp_{\{\kappa,r\}}(x) \underset{x\to\pm\infty}{\sim} |2\,\kappa\,x|^{1/(r\pm|\kappa|)}$$

$$\exp_{\{\kappa,\,r\}}(-\infty) = 0^{+}$$

$$\int_{-\infty}^{0} \exp_{\{\kappa,r\}}(x)\,dx = \frac{1}{(1+r)^2-\kappa^2}$$

Two-parameter solution

$$\ln_{\{\kappa,r\}}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\,\kappa}\,x^{r}$$

verifies

$$d[x\ln_{\{\kappa,r\}}(x)]/dx = \lambda\ln_{\{\kappa,r\}}(x/\alpha)$$

$$(r+\kappa+1)x^{r+\kappa} - (r-\kappa+1)x^{r-\kappa} = \lambda\left[\frac{x^{r+\kappa}}{\alpha^{r+\kappa}} - \frac{x^{r-\kappa}}{\alpha^{r-\kappa}}\right]$$

$$(r+\kappa+1) = \lambda\alpha^{-r-\kappa} \qquad (r-\kappa+1) = \lambda\alpha^{-r+\kappa}$$

$$\alpha = \left(\frac{1+r-\kappa}{1+r+\kappa}\right)^{1/2\,\kappa}$$

$$\lambda = \frac{(1+r-\kappa)^{(r+\kappa)/2\,\kappa}}{(1+r+\kappa)^{(r-\kappa)/2\,\kappa}}$$

Three-parameter class

$$\ln_{\{\kappa,r\}}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\,\kappa}\, x^{r}$$

verifies

$$d[x \ln_{\{\kappa,r\}}(x)]/dx = \lambda \ln_{\{\kappa,r\}}(x/\alpha)$$

$$\ln_{\{\kappa,r,\sigma\}}(x) = \frac{\ln_{\{\kappa,r\}}(\sigma x) - \ln_{\{\kappa,r\}}(x)}{\sigma \ln'_{\{\kappa,r\}}(\sigma)} = \frac{x^{r}[(\sigma x)^{\kappa} - (\sigma x)^{-\kappa}] - \sigma^{\kappa} + \sigma^{-\kappa}}{(\kappa + r)\sigma^{\kappa} + (\kappa - r)\sigma^{-\kappa}}$$

verifies

$$d[x \ln_{\{\kappa,r,\sigma\}}(x)]/dx = \lambda \ln_{\{\kappa,r,\sigma\}}(x/\alpha) + \eta$$

$$\alpha = \left(\frac{1 + r - \kappa}{1 + r + \kappa}\right)^{1/2\,\kappa}$$

$$\eta = (\lambda - 1)\frac{\sigma^{\kappa} - \sigma^{-\kappa}}{(\kappa + r)\sigma^{\kappa} + (\kappa - r)\sigma^{-\kappa}}$$

$$\lambda = \frac{(1 + r - \kappa)^{(r+\kappa)/2\,\kappa}}{(1 + r + \kappa)^{(r-\kappa)/2\,\kappa}}$$

In general this class of logarithms does not verify

$$\ln_{\{\kappa,r\}}(1/x) \neq -\ln_{\{\kappa,r\}}(x)$$

$$\ln_{\{\kappa,r\}}(x^{-1}) = \frac{(x^{-1})^{\kappa} - (x^{-1})^{-\kappa}}{2\,\kappa}\,(x^{-1})^{r} = \frac{x^{-\kappa} - x^{\kappa}}{2\,\kappa}\,x^{-r} = -\ln_{\{\kappa,-r\}}(x)$$

and the corresponding exponential does not verify

$$\exp_{\{\kappa,r\}}(-x) \neq 1/\exp_{\{\kappa,r\}}(x)$$

Two parameter class of entropies

$$S(n) = -k_{\rm B} \sum_i n_i \ln_{\{\kappa,r\}}(n_i) = -k_{\rm B} \sum_i n_i^{1+r} \frac{n_i^{\kappa} - n_i^{-\kappa}}{2\kappa}$$

- **Information theory (1975}**
  Sharma, Mittal, Taneja

- **Statistical mechanics (1998)**
  Borges, Roditi

Two-parameter extended sum

$$x \overset{\kappa,r}{\oplus} y = \ln_{\{\kappa,r\}} \left[\exp_{\{\kappa,r\}}(x)\, \exp_{\{\kappa,r\}}(y)\right]$$

$$x \longrightarrow \ln_{\{\kappa,r\}}(x) \qquad y \longrightarrow \ln_{\{\kappa,r\}}(y)$$

$$\ln_{\{\kappa,r\}}(x\,y) = \ln_{\{\kappa,r\}}(x) \overset{\kappa,r}{\oplus} \ln_{\{\kappa,r\}}(y)$$

generalizes $\log(x) + \log(y) = \log(xy)$

Different approach to deformed distributions

Define deformed sum (*subset of properties of usual sum*)

Deformed derivative

Differential equation → **Exponential and logarithm**

κ-entropy (Kaniadakis)   *q-entropy (Borges, Wang)*

PHYSICAL REVIEW E **66**, 056125 (2002)

Statistical mechanics in the context of special relativity

G. Kaniadakis*

One-parameter special examples

Shannon entropy

Usual log and exp independent of direction or value of σ

$$\kappa\,,r \to 0$$

$$\ln_{\{\kappa,r\}}(x) \to \ln(x)$$

$$\exp_{\{\kappa,r\}}(x) \to \exp(x)$$

$$\ln(1/x) = -\ln(x)$$

Usual sum $x \oplus y = x + y$

One-parameter special examples

q-entropy (Tsallis)

$$r = \pm\kappa$$

$$q = 1 \pm 2\kappa$$

$$\log_q(x) = \pm\frac{1 - x^{\mp 2\kappa}}{2\kappa} = \frac{1 - x^{q-1}}{1-q}$$

Solve linear equation

$$\exp_q(x) = [1 + (q-1)\,x]^{-1/(1-q)}$$

$$r = \kappa \quad q = 1 + 2\kappa$$

$$\log_q(0+) = -1/(q-1)$$

$$\log_q(1/x) = -\log_{2-q}(x)$$

Non-extensive sum

$$x \overset{q}{\oplus} y = x + y + (q-1)\,x\,y$$

One-parameter special examples

q-entropy (Abe)

$$r = \sqrt{1+\kappa^2} - 1$$

$$q = \sqrt{1+\kappa^2} - \kappa$$

$$\log_q(x) = \frac{x^{1/q-1} - x^{q-1}}{1/q - q}$$

Invariant under $q \leftrightarrow 1/q$

Inverse exists, but not in terms of elementary functions

No explicit form for the q-exponential

No explicit form for the q-sum

One-parameter special examples

κ-entropy (Kaniadakis)

$$r = 0$$

Invariant under

$$\kappa \to -\kappa$$

$$\ln_{\{\kappa\}}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\,\kappa}$$

Solve quadratic equation

$$\exp_{\{\kappa\}}(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa\, x\right)^{1/\kappa}$$

$$\ln_{\{\kappa\}}(1/x) = -\ln_{\{\kappa\}}(x)$$

$$\exp_{\{\kappa\}}(-x) = 1/\exp_{\{\kappa\}}(x)$$

$$x \overset{\kappa}{\oplus} y = x\sqrt{1 + \kappa^2 y^2} + y\sqrt{1 + \kappa^2 x^2}$$

Relativistic sum of momenta

κ-entropy: special properties

$$\ln_{\{\kappa\}}(x^{-1}) = \frac{x^{-\kappa} - x^{\kappa}}{2\,\kappa} = -\frac{x^{\kappa} - x^{-\kappa}}{2\,\kappa} = -\ln_{\{\kappa,\}}(x)$$

$$\begin{aligned}
\exp_{\{\kappa\}}(-x) &= \left(\sqrt{1+\kappa^2\,x^2} - \kappa\,x\right)^{1/\kappa} \\
&= \left(\frac{\left(\sqrt{1+\kappa^2\,x^2} - \kappa\,x\right)\left(\sqrt{1+\kappa^2\,x^2} + \kappa\,x\right)}{\sqrt{1+\kappa^2\,x^2} + \kappa\,x}\right)^{1/\kappa} \\
&= \left(\frac{1}{\sqrt{1+\kappa^2\,x^2} + \kappa\,x}\right)^{1/\kappa} = \frac{1}{\exp_{\{\kappa\}}(x)}
\end{aligned}$$

κ-entropy: deformed sum

$$x \overset{\kappa}{\oplus} y = \ln_{\{\kappa\}} \left[\exp_{\{\kappa\}}(x)\ \exp_{\{\kappa\}}(y)\right]$$

$$= \frac{1}{2\kappa}\left\{\left[\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(y)\right]^{\kappa} - \left[\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(y)\right]^{-\kappa}\right\}$$

$$= \frac{1}{2\kappa}\left\{(\sqrt{1+\kappa^2 x^2} + \kappa\, x)(\sqrt{1+\kappa^2 y^2} + \kappa\, y)\right.$$

$$\left. -(\sqrt{1+\kappa^2 x^2} - \kappa\, x)(\sqrt{1+\kappa^2 y^2} - \kappa\, y)\right\}$$

$$= \frac{1}{2\kappa}\left\{2\kappa\, y\sqrt{1+\kappa^2 x^2} + 2\kappa\, x\sqrt{1+\kappa^2 y^2}\right\}$$

$$= x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2}$$

NOTE: $x \overset{\kappa}{\oplus} (-x) = 0$ $-\,x$ inverse of x

RELATIVISTIC SUM OF MOMENTA

Particle of mass m and momentum p in a given frame

has energy $E = \sqrt{(mc^2)^2 + (pc)^2}$ and velocity $v = \dfrac{pc^2}{E}$

In a new frame with velocity V relative to the previous one has momentum

$$p' = \gamma[p - VE/c^2] \qquad \text{where} \qquad \frac{1}{\gamma^2} = 1 - \frac{V^2}{c^2}$$

If we consider two particles p_1 p_2

New frame where $p_1' = 0$ p_2'

$$V = v_1 = \frac{p_1 c^2}{E_1}$$

$$\frac{1}{\gamma_1^2} = 1 - \frac{v_1^2}{c^2} = 1 - \frac{p_1^2 c^2}{E_1^2} = \frac{E_1^2 - p_1^2 c^2}{E_1^2} = \frac{m_1^2 c^4}{E_1^2}$$

$$\gamma_1 = E_1/(m_1 c^2)$$

RELATIVISTIC SUM OF MOMENTA

$$p_2' = \gamma_1[p_2 - \frac{v_1 E_2}{c^2}] = \frac{E_1}{m_1 c^2}\left[p_2 - \frac{p_1}{E_1}E_2\right] = p_2\frac{E_1}{m_1 c^2} - p_1\frac{E_2}{m_1 c^2}$$

Scale particle momentum with mass

$$\frac{p_2'}{m_2 c} = \frac{p_2}{m_2 c} \times \frac{E_1}{m_1 c^2} - \frac{p_1}{m_1 c} \times \frac{E_2}{m_2 c^2}$$

Use definition of energy as function of momentum $E = \sqrt{(mc^2)^2 + (pc)^2}$

$$\frac{p_2'}{m_1 c} = \frac{p_2}{m_2 c}\sqrt{1 + \left(\frac{p_1}{m_1 c}\right)^2} - \frac{p_1}{m_1 c}\sqrt{1 + \left(\frac{p_2}{m_2 c}\right)^2}$$

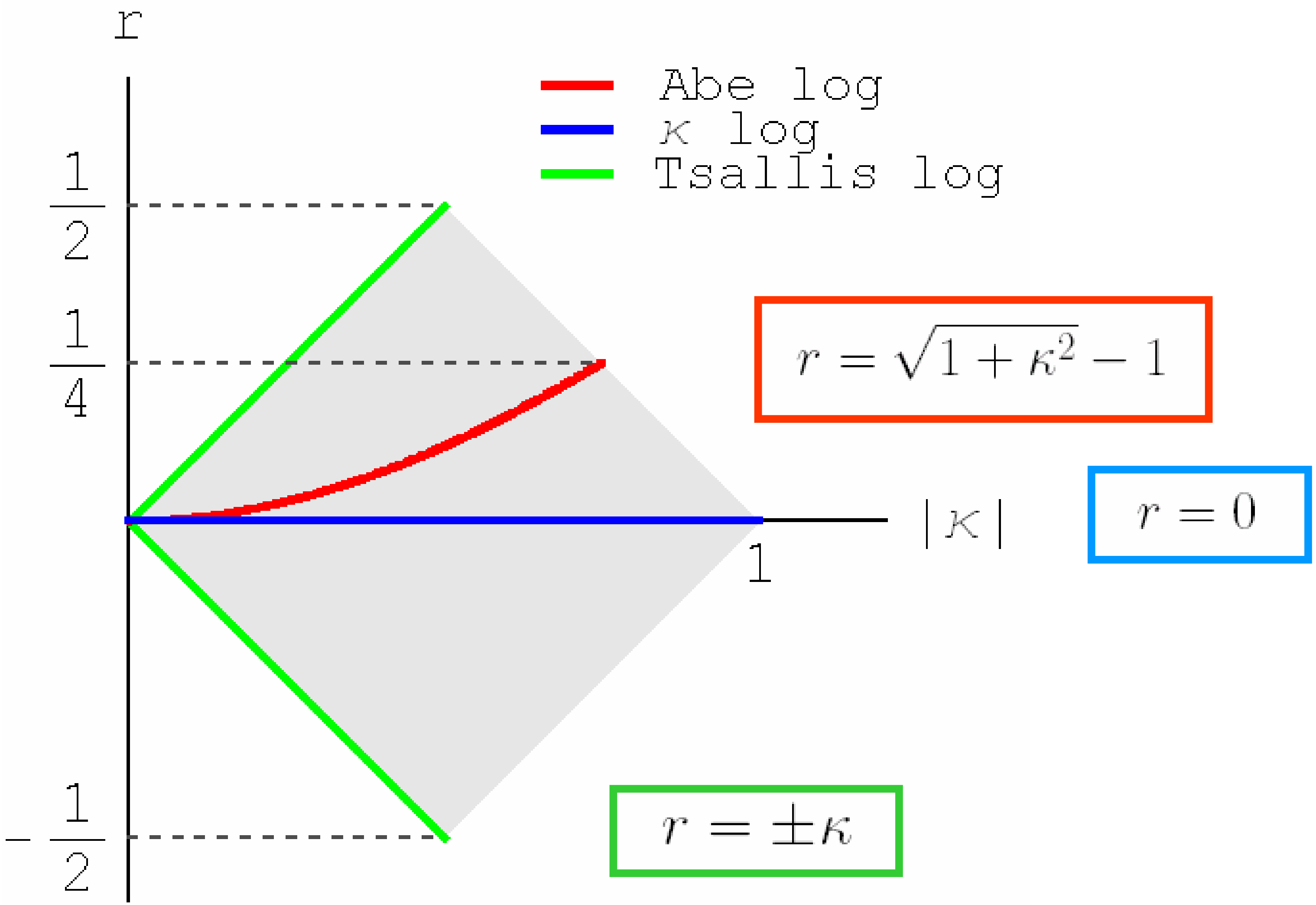
This addition law can be view as a case of deformed sum

$$\boxed{\frac{p_2'}{m_1 c} = \frac{p_2}{m_2 c} \overset{\kappa}{\oplus} \left(-\frac{p_1}{m_1 c}\right)} \qquad \kappa = 1$$

Relativity and κ-statistics

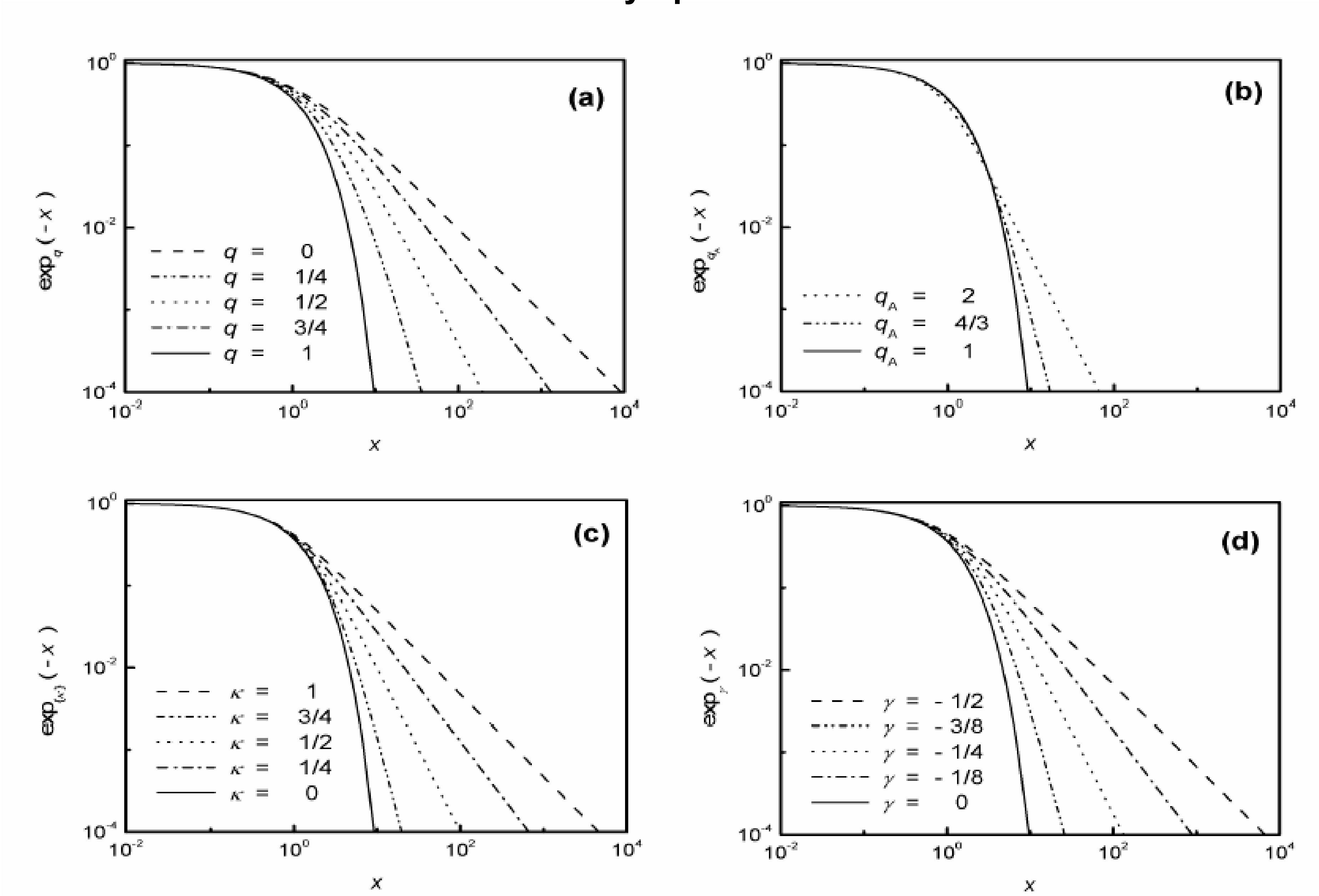
- Only a formal analogy coming from the sharing the same algebraic properties of the sum?
- More analogies: integration, …
- Deeper relation between relativity and the κ-statistics?

One-parameter special cases



Comparison of distributions

with same asymptotic behavior



Comparison of distributions
asymptotic behavior

No need explicit form of exponential

$$y = \ln_{\{\kappa,r\}}(x) = \frac{x^{r+\kappa} - x^{r-\kappa}}{2\kappa} \rightarrow -\infty$$ when $x \rightarrow 0$

$$y \sim -\frac{x^{r-\kappa}}{2\kappa}$$

$$x = \exp_{\{\kappa,r\}}(-y) \rightarrow (-2\kappa y)^{1/(r-\kappa)}$$ for $y \rightarrow -\infty$

Same asymptotic behavior for same $r - \kappa$

Different definitions differ only in the central range

One-parameter special cases

Only in a few case the exponential can be obtained explicitly (invert the logarithm):

- q-exp : solve linear equation
- κ-exp : solve quadratic equation
- cubic and quartic equations

γ-exp

$r = \pm\kappa/3$

$\gamma = \pm 2\kappa/3$

$$\log_\gamma(x) = \frac{x^{2\gamma} - x^{-\gamma}}{3\gamma}$$

$$-1/2 < \gamma < 1/2$$

$$\exp_\gamma(x) = \left[\left(\frac{1+\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3} + \left(\frac{1-\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3}\right]^{1/\gamma}$$

Another 2-parameter class

from the 3-parameter case with r=0

Scaled κ-log

$$\log_{\{\kappa,\sigma\}}(x) = \frac{\ln_{\{\kappa\}}(\sigma x) - \ln_{\{\kappa\}}(\sigma)}{(\sigma^\kappa + \sigma^{-\kappa})/2} = \frac{(\sigma x)^\kappa + (\sigma x)^{-\kappa} - (\sigma^\kappa - \sigma^{-\kappa})}{\kappa(\sigma^\kappa + \sigma^{-\kappa})}$$

$$\exp_{\{\kappa,\sigma\}}(y) = \frac{1}{\sigma}\exp_{\{\kappa\}}\left[\frac{\sigma^\kappa + \sigma^{-\kappa}}{2}y + \frac{\sigma^\kappa - \sigma^{-\kappa}}{2\kappa}\right]$$

$$\log_{\{\kappa,\sigma\}}(1/x) = -\log_{\{\kappa,1/\sigma\}}(x)$$

Interpolates continuously from q-log to κ-log

$\sigma = 1$ $\quad\log_{\{\kappa,\sigma\}}(x) = \ln_{\{\kappa\}}(x)$

$\sigma \to \infty$ $\quad\log_{\{\kappa,\sigma\}}(x) \to \frac{x^\kappa - 1}{\kappa}$

Conclusions (1)

- Coherent general framework for generalized distributions

- Three-parameter class of entropies
  - trace form
  - "exponential" distribution

$$S(n) = -k_B \sum_i n_i \Lambda(n_i)$$

$$n_j = \alpha\, \mathcal{E}\left[-\frac{1}{\lambda}\left(\frac{E_j-\mu}{k_B T} + \eta\right)\right]$$

$$\Lambda(x) = \ln_{\{\kappa,r,\sigma\}}(x) = \frac{x^r\left[(\sigma x)^\kappa - (\sigma x)^{-\kappa}\right] - \sigma^\kappa + \sigma^{-\kappa}}{(\kappa+r)\sigma^\kappa + (\kappa-r)\sigma^{-\kappa}}$$

- Unified discussion of properties of different generalizations
  - **Tsallis**
  - **Abe**
  - **Kaniadakis**

Conclusions (1)

- Comparison: similarities and differences
  - Asymptotic behavior: same power exponent
  - Simplicity of equations
  - Cut-offs
- Entropy interpretation

- κ-statistics
  - More symmetric algebra
  - Intriguing analogy with the formalism of relativity

PHYSICAL REVIEW E **72**, 036108 (2005)

Statistical mechanics in the context of special relativity. II.

G. Kaniadakis*

PHYSICAL REVIEW E **71**, 046128 (2005)

Two-parameter deformations of logarithm, exponential, and entropy: A consistent framework for generalized statistical mechanics

G. Kaniadakis,[1,*] M. Lissia,[2,†] and A. M. Scarfone[1,2,‡]

PHYSICAL REVIEW E **66**, 056125 (2002)

Statistical mechanics in the context of special relativity

G. Kaniadakis*

Chaos and Entropy

Marcello Lissia

Physics Department and INFN Cagliari, Italy

Collaboration with

M. Coraddu and R. Tonelli

R. Tonelli, G. Mezzorani, F. Meloni, M. Lissia, M. Coraddu   Prog.Theor.Phys. 115 (2006) 23

Entropy production and Pesin-like identity at the onset of chaos

Introduction and motivation

Conjecture (1997 Tsallis, Plastino, Zheng)

Statistical mechanics formalism in non linear systems
(sensitivity to initial conditions ⇔ entropy production)
Generalization from chaos to weak chaos

Logistic map as paradigmatic example $$x_{i+1} = 1 - \mu \cdot x_i^2$$

Relevance of chaotic maps for thermodynamics

- **Only formalism?**
- **Perhaps deeper analogy:**
  - **statistics independent of details of dynamics (ergodicity: chaos)**
  - **some properties independent of existence of dynamics?**
- **Numerical cost of calculations at the edge of chaos (critical slowing down)**

Introduction and motivation

Lectures of Robledo and Tirnakli

Chaotic regime

- Asymptotic sensitivity to initial conditions

$$\xi(t) = \lim_{\substack{\Delta x(t)\to\infty \\ \Delta x(0)\to 0}} \frac{\Delta x(t)}{\Delta x(0)} \to \exp(\lambda t)$$

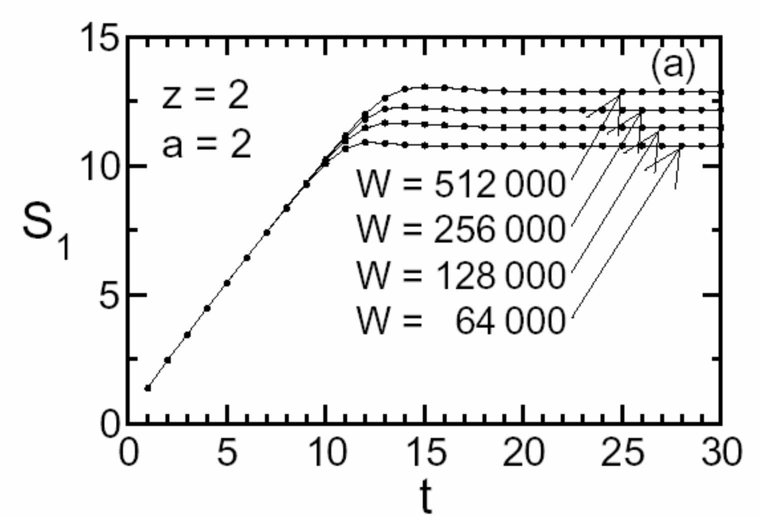
Lyapunov exponent λ

Chaotic regime

Information entropy

W cells, ensemble of *N* systems $p_i(t) = n_i / N \quad i = 1, W$

$$S(t) = -\sum_{i=1}^{W} p_i \log(p_i)$$



Borges et al, Phys Rev Lett 89 (2002) 254103

- Asymptotic linear behavior

$$\frac{S(t)}{t} \to K$$

- Pesin identity $K = \lambda$

Edge of chaos

$\lambda = 0$ power-law divergence of ξ

Generalization

- Sensitivity $\xi \to \exp_q(\lambda_q t) = (1+(1-q)\lambda_q t)^{1/(1-q)}$ ⇒ q λ_q

- Entropy: asymptotic linear behavior for $S_q(t) = \sum_{i=1}^{W} p_i \frac{p_i^{q-1} - 1}{1-q}$

⇒ Same value of *q*

- Pesin identity $S_q(t)/t \to K_q = \lambda_q$

Coherent framework: deformation of usual statistical mechanics

S_q and $\exp_q$ related

Edge of chaos

Numerical evidences and theoretical considerations support this conjecture

Costa et al, Phys Rev E 56 (1997) 245
Lyra & Tsallis, Phys Rev Lett 80 (1998) 53
Anteneodo & Tsallis, Phys.Rev Lett 80 (1998) 5313
Latora et al, Phys Lett A 273 (2000) 97
De Moura et al, Phys Rev E 62 (2000) 6361
Borges et al, Phys Rev Lett 89 (2002) 254103
Tirnakli et al, Phys Lett A 289 (2001) 51
Baldovin & Robledo, Phys Rev E 66 (2002) 045104
Baldovin & Robledo, Europhys Lett E 60 (2002) 518
Baldovin & Robledo, Phys Rev E 69 (2004) 045202
Ananos & Tsallis, arXiv:cond-mat/0401276
Ananos et al, arXiv:cond-mat/0401276

many more …

Question

Only this specific form of S_q and $\exp_q(x)$?

Other deformations with same asymptotic power law?

Test

- Logistic map
- Two-parameter class of deformed statistical mechanics

Two-parameter class of entropies

includes Shannon, Tsallis, Abe, Kaniadakis, …

Trace form

$$S(n) = -k_{\mathrm{B}} \sum_i n_i \Lambda(n_i)$$

generalized logarithm exponential

$$\ln_{\{\kappa,r\}}(x) = \Lambda(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} x^{r}$$

$$\mathcal{E}(x) \equiv \Lambda^{-1}(x) = \exp_{\{\kappa,r\}}(x)$$

Examples: *q*-statistics $r = -|\kappa| = (q-1)/2$

***k*-statistics** $r = 0$

Weak chaos

asymptotic behavior ⇒ $r + |\kappa| = q - 1$

Three specific cases: *q*- , Abe- and *k*-statistics

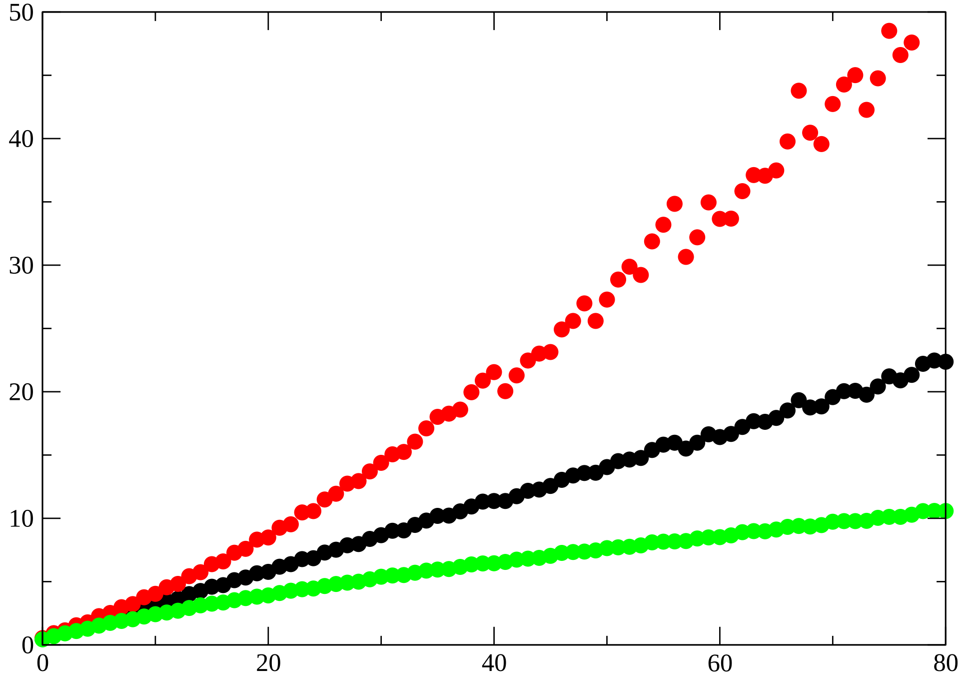
Results: sensitivity

***q*-exponential** (Tsallis)

$$\xi(t) \to [1+(1-q)\lambda_q t]^{1/(1-q)}$$

Plot *q*-log of sensitivity

$$<\log_q(\xi)>=<\frac{\xi^{1-q}-1}{1-q}>=\lambda_q t$$



***q*= 0.24**

***q*= 0.36**

$\lambda_q = 0.27$

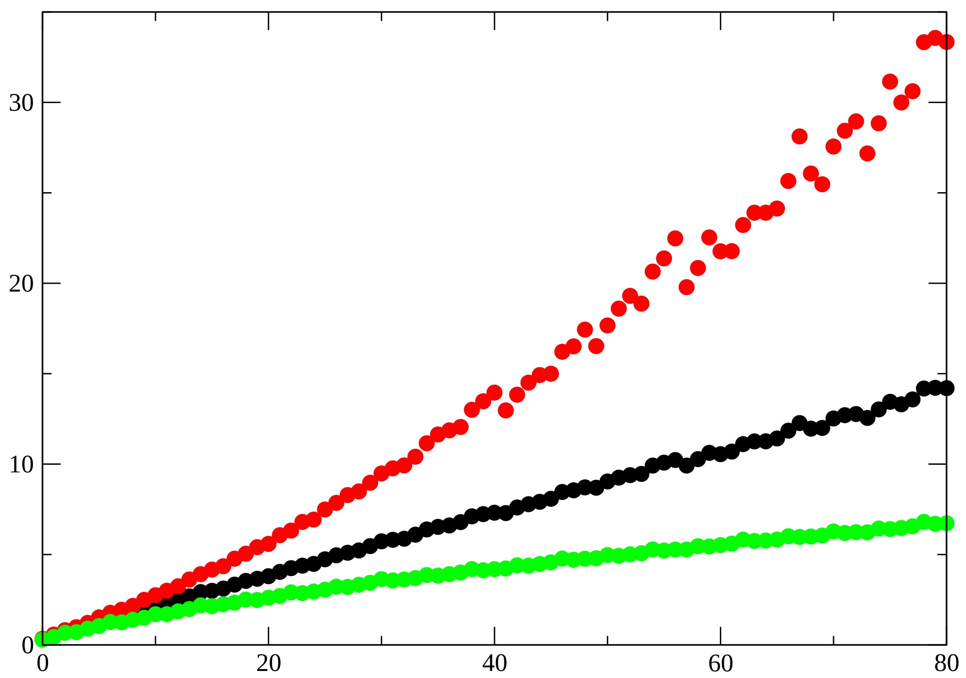
***q*= 0.50**

Results: sensitivity

Plot *q-log* of sensitivity
(Abe)

$$\log_Q(x) = \frac{x^{1/Q-1} - x^{Q-1}}{1/Q - Q}$$

$$Q = 1/(2-q)$$



= 0.24

= 0.36 $\lambda_Q = 0.18$

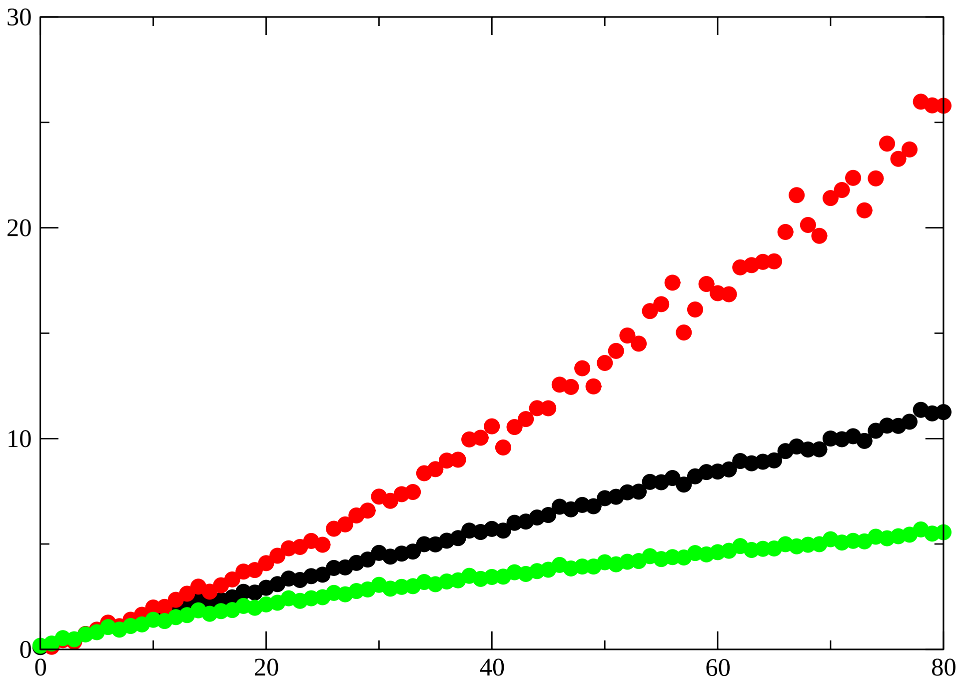
= 0.50

Results: sensitivity

Plot *k*-log of sensitivity
(Kaniadakis)

$$\log_\kappa = \frac{x^\kappa - x^{-\kappa}}{2\kappa}$$

$$\kappa = 1 - q$$



= 0.24

= 0.36

$\lambda_\kappa = 0.15$

= 0.50

Results: sensitivity

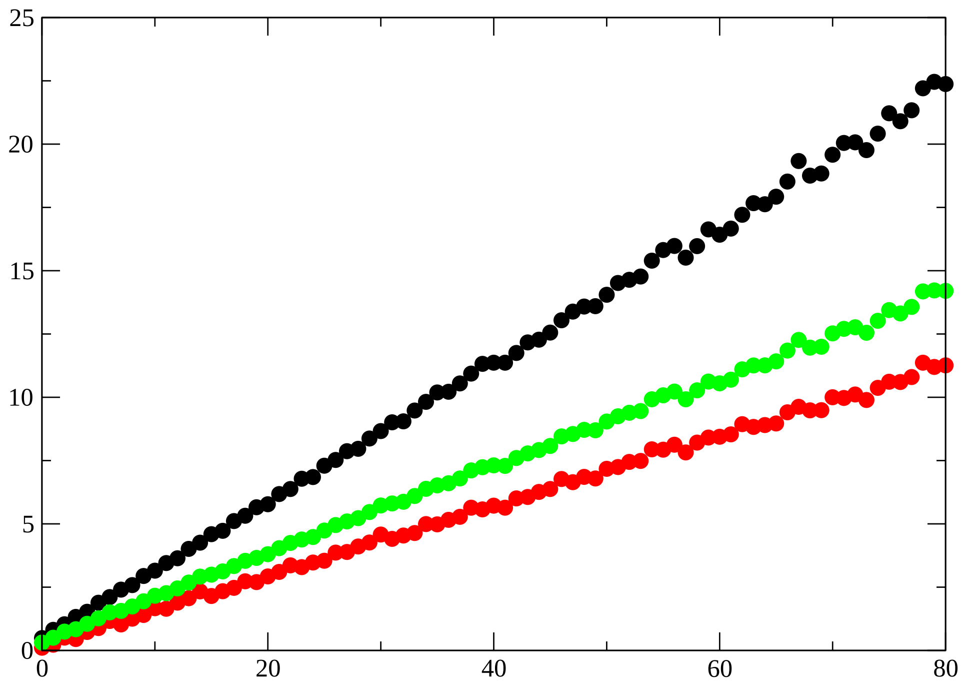
$$\log_q(x) = \frac{x^{1-q} - 1}{1-q}$$

$$\log_Q(x) = \frac{x^{1/Q-1} - x^{Q-1}}{1/Q - Q}$$

$$Q = 1/(2-q)$$

$$\log_\kappa(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa}$$

$$\kappa = 1 - q$$



$$\lambda_q = 0.27$$

$$\lambda_Q = 0.18$$

$$\lambda_\kappa = 0.15$$

same asymptotic power

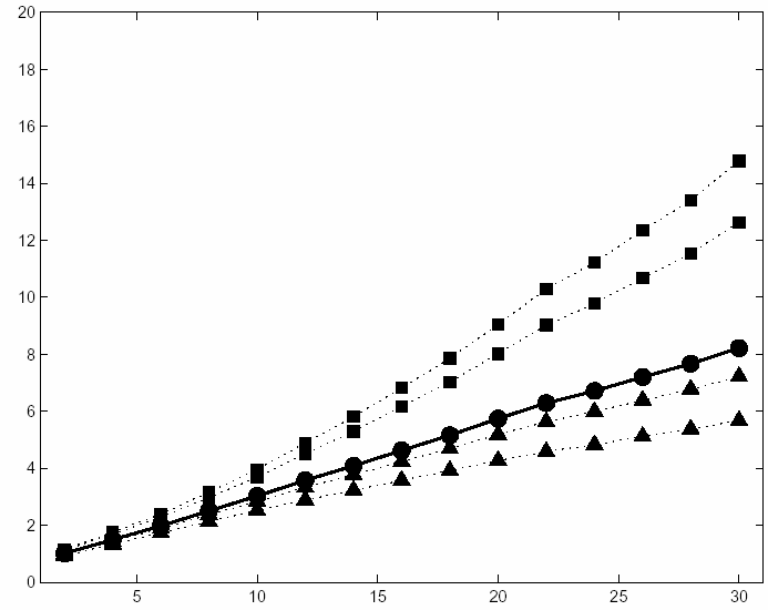
***q*=0.36**

$$\xi \to \exp_\alpha(\lambda_\alpha t)$$

different generalized Lyapunov exponents

Results: entropy (Tsallis)

$$S_q = \sum p_i \frac{p_i^{q-1} - 1}{1-q}$$



t

q

0.20
0.24
0.36
0.40
0.48

same as sensitivity

Sensitivity to initial conditions
Generalized Lyapunov exponent

Rate of entropy production

$$K = \lim_{t \to \infty} \frac{S_q}{t}$$

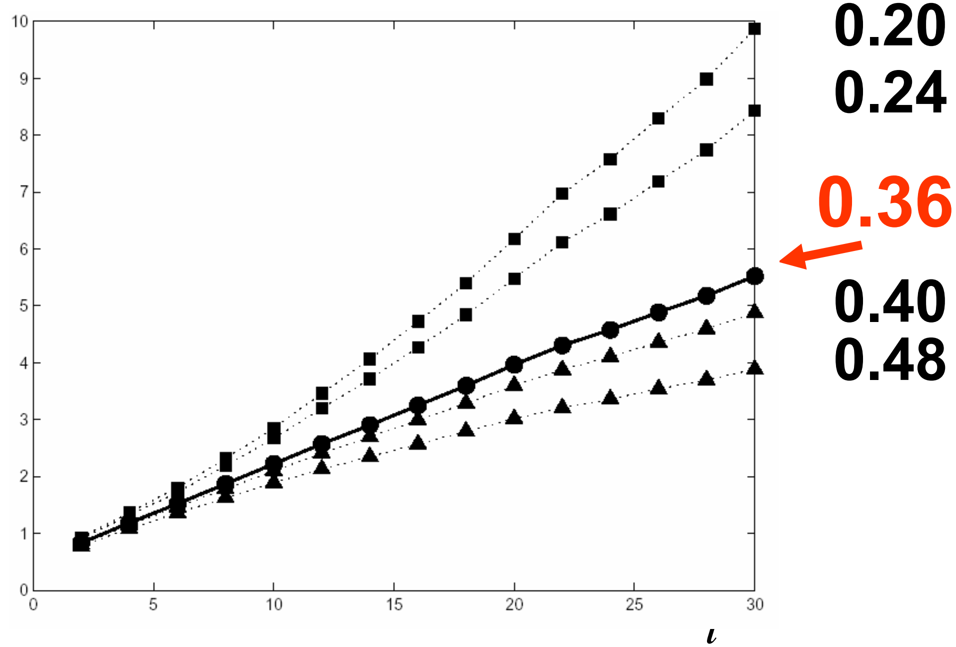
Pesin

$$\lambda_q = 0.27 = K_q$$

Results: entropy (Abe)

$$S_Q = -\sum p_i \frac{p_i^{1/Q-1} - p_i^{Q-1}}{1/Q - Q}$$

$$Q = 1/(2-q)$$



Sensitivity to initial conditions

Generalized Lyapunov exponent

Rate of entropy production

$$K = \lim_{t \to \infty} \frac{S_Q}{t}$$

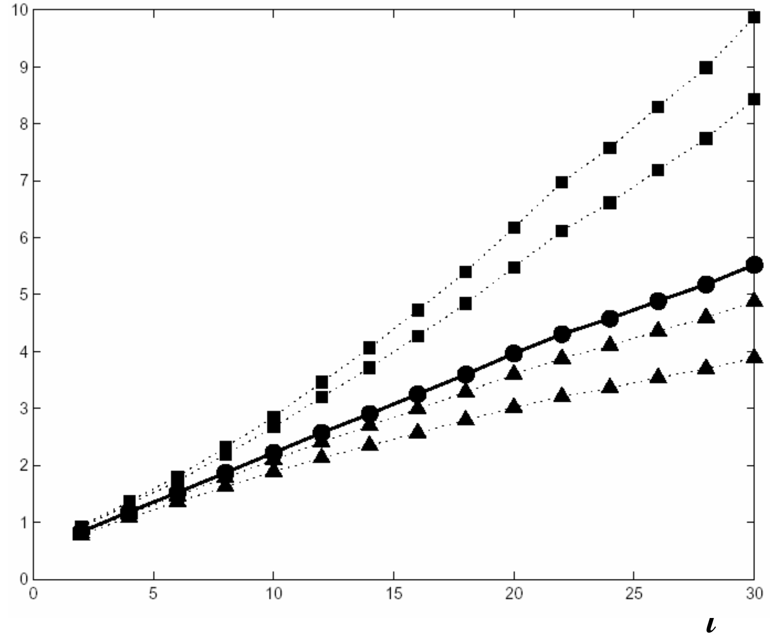
Pesin

$$\lambda_Q = 0.18 = K_Q$$

Results: entropy (Kaniadakis)

$$S_\kappa = -\sum p_i \frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa}$$

$$\kappa = q - 1$$



q

0.20
0.24
0.36
0.40
0.48

Sensitivity to initial conditions
Generalized Lyapunov exponent

Rate of entropy production

$$K = \lim_{t\to\infty} \frac{S_\kappa}{t}$$

Pesin

$$\lambda_\kappa = 0.15 = K_\kappa$$

Intuitive physical explanation

Tonelli et al., Prog.Theor.Phys. 115 (2006) 23

Entropy

$$S(t) \equiv \left\langle \sum_{i=1}^{W} p_i(t) \widetilde{\log}\left(\frac{1}{p_i(t)}\right) \right\rangle = \left\langle \sum_{i=1}^{W} \frac{p_i^{1-\alpha}(t) - p_i^{1+\beta}(t)}{\alpha+\beta} \right\rangle$$

At time t ensemble spread uniformly over ν boxes, while the other $(W-\nu)$ empty

$$S(t) \approx \widetilde{\log}(\nu(t)) \sim [\nu^{\alpha}(t) - \nu^{-\beta}(t)]/(\alpha+\beta)$$

$$\frac{K_\beta}{K_0} = \left(1 + \frac{\beta}{\alpha}\right)^{-1} \times \frac{1-\nu^{-\alpha-\beta}}{1-\nu^{-\alpha}}$$

$W = 10^5$ $\nu \approx 20$

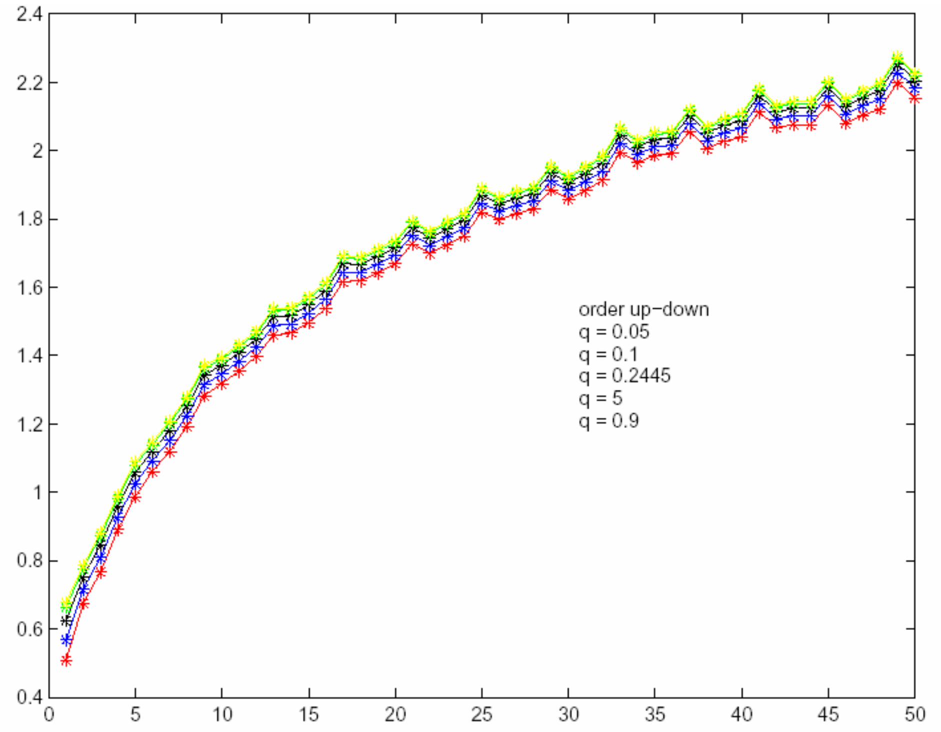
Most of boxes empty!

$\beta = 0$ (Tsallis) $S(t) = \frac{\nu^{\alpha}(t)}{\alpha}$ Abe lecture

Does it work for any entropy? NOT ALWAYS!

Renyi entropy (not trace form)

$$S_\beta = \frac{1}{1-\beta}\log\left(\sum_i p_i^\beta\right)$$



Entropy production not constant

$$0.05 \le q \le 0.9$$

Conclusions (2)

Statistical mechanics formalism for weakly chaotic systems holds for a class of power-law entropies

$$S_{\kappa,r} = -\sum p_i^{1+r} \frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa}$$

Constant asymptotic entropy production rate
Not trivial (ex: Renyi entropy)

$$K_{\kappa,r} = \lim_{t\to\infty} \frac{S_{\kappa,r}}{t}$$

Asymptotic sensitivity goes as corresponding "exponential"

$$\xi(t) \to \exp_{\kappa,r}(\lambda_{\kappa,r} t)$$

same asymptotic ***(q)*** power

Pesin identity verified

$$\lambda_{\kappa,r} = K_{\kappa,r}$$

not same numerical value for all entropies

Numerical evidence for three one-parameter entropies
Tsallis, Abe, Kaniadakis (logistic map)

Physical understanding of entropy production rate in term of occupied boxes: relation between entropy production and asymptotic exponent α

$$K_\beta = \frac{1}{\alpha + \beta}$$

Tsallis: no finite size corrections $\beta = 0$

$$S(t) = \frac{\nu^\alpha(t)}{\alpha}$$