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SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

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Quantum Information via Non-Additive Entropy

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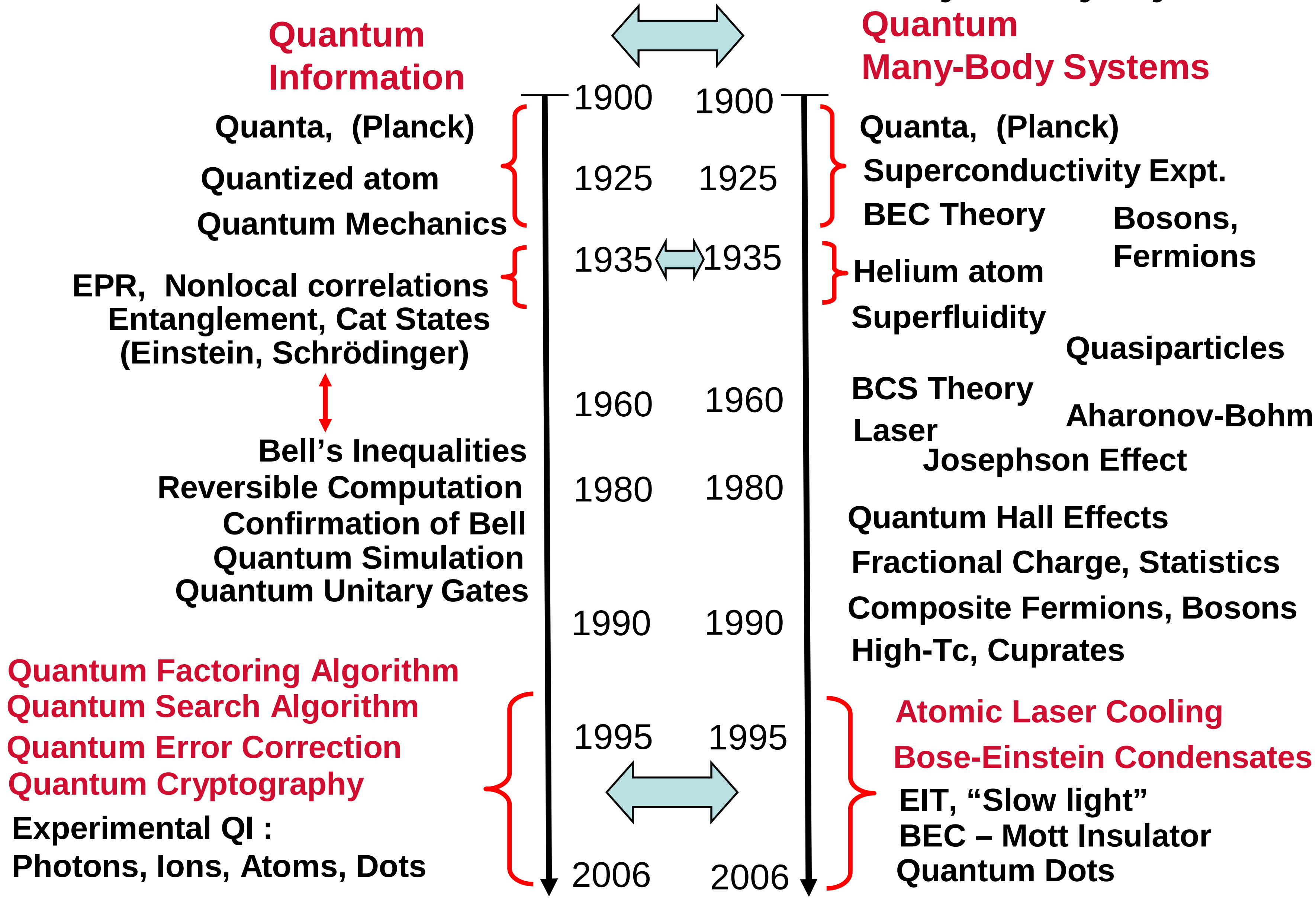
ABSTRACT

This lecture will address some issues of quantum information based on the framework of quantum non-additive entropy. This is a four part presentation:
(a) General introduction to the issues by discussing known many-body systems;
(b) Exposition of basic concepts of quantum theory – uncertainty and superposition principles - in relation to quantum information illustrated in simple discrete systems (spins) leading to notions of separability and entanglement;
(c) Quantification of quantum information based on concepts of entropy illustrated with the simple systems discussed in (b) and exhibit implications of non-additive entropy in this connection; and finally
(d) Concluding remarks will include topics not covered in this introductory presentation such as continuous systems (light), some open problems and issues as well as current ongoing attempts at practical implementation of these ideas.

Collaborators: Sumiyoshi Abe and Ronald W. Rendell

Interplay of Concepts: Quantum Information and Quantum Many-Body Systems



BRIEF REVIEW OF QUANTUM MECHANICS

THIS REVIEW IS NOT MEANT TO BE COMPLETE
(For a full account, see references 1 and 2)

CLOSED SYSTEM	OPEN SYSTEM
Pure quantum state $\vert\Psi(A)\rangle$ e.g., Isolated harmonic oscillator Solution to Schrodinger equation $i\frac{\partial}{\partial t}\vert\Psi\rangle = \hat{H}\vert\Psi\rangle$ (Unitary evolution)	Mixed quantum state Density matrix $\hat{\rho}(A)$ e.g. oscillator at finite T (heat bath) Solution to von Neumann equation $i\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$ (Unitary for closed; non-unitary for subsystem evolution)

Density matrix $\hat{\rho}$: positive, semi-definite, Hermitian operator with unit trace.

$$\hat{\rho} = \sum_i p_i |i\rangle\langle i|, \quad \sum_i p_i = 1, \ \ 0 \le p_i \le 1$$

If $p_i = 1/N$, $i = 1, 2, \cdots, N$, chaotic (random) or noisy state density matrix: $\hat{\rho}_{ch}$

The most general transformation that maintains these properties is POVM

$$\hat{\rho}' = \sum_i \hat{V}_i \hat{\rho} \hat{V}_i^+, \quad \sum_i \hat{V}_i^+ \hat{V}_i = 1$$

Unitary transformation: special case when i=1:

$$\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^+$$

For a pure state, $\hat{\rho}^2 = \hat{\rho} \Rightarrow \hat{\rho} = |\Psi\rangle\langle\Psi|$

For a composite closed system (A, B), $\hat{\rho}(A, B) = |\Psi(A, B)\rangle\langle\Psi(A, B)|$

Marginal density matrix gives description of system A irrespective of B:

$$\hat{\rho}(A) = Tr_B \rho(A,B) \quad \text{(in general a mixed state)}$$

Introduction of wave functions as “probability amplitudes”.

SUPERPOSITION PRINCIPLE:

$$|\Psi\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle, \quad \langle\Psi_i|\Psi_i\rangle = 1, \quad \langle\Psi_1|\Psi_2\rangle = |\langle\Psi_1|\Psi_2\rangle| \exp i\phi(12)$$

(INTERFERENCE)

Expressed in terms of density matrix, this is:

$$\hat{\rho} = |a|^2 \hat{\rho}_1 + |b|^2 \hat{\rho}_2 + \frac{|a||b|}{\sqrt{Tr(\hat{\rho}_1 \hat{\rho}_2)}} \{\exp i(\alpha - \phi(12)) \hat{\rho}_1 \hat{\rho}_2 + h.c.\}$$

where $a\,b^* = |ab| \exp i\alpha$ and $\hat{\rho}_i \equiv |\Psi_i\rangle\langle\Psi_i| \, (i = 1,2)$.

Trace condition gives

$$1 = |a|^2 + |b|^2 + 2|a||b|\sqrt{Tr(\hat{\rho}_1 \hat{\rho}_2)} \cos(\alpha - \phi(12))$$

A more complete discussion of composition law of two pure state density matrices which are orthogonal to each other, see V. I. Manko et al, PLA **273**, 31 (2000) and quant-ph/0207033 (2002).

Physical quantities are represented by Hermitian operators (have REAL eigenvalues) (compare classical description using real numbers).

Measurement of two or more physical properties

"commuting" and "non-commuting" operators

"simultaneous measurement of two quantities" & "Uncertainty Principle".

Outcome of measurement $\langle \hat{O} \rangle \equiv Tr\left(\hat{O}\hat{\rho}\right)$; define $\Delta\hat{O} \equiv \hat{O} - \langle \hat{O} \rangle$

Correlation matrix: $V_{ij} = \frac{1}{2}\left\langle \Delta\hat{O}_i\,\Delta\hat{O}_j + \Delta\hat{O}_j\,\Delta\hat{O}_i \right\rangle$

Uncertainty Relations: $V_{ii}\,V_{jj} - V_{ij}^2 \geq \frac{1}{4}\left|\left\langle \Delta\hat{O}_i\,\Delta\hat{O}_j - \Delta\hat{O}_j\,\Delta\hat{O}_i \right\rangle\right|^2$

This completes our brief QM review! **ANY QUESTIONS???**

Classical Information content in a stochastic system

$$P(a,b), \quad \sum_{a,b} P(a,b) = 1$$

Marginal probability distribution is given by $p(a) = \sum_b P(a,b)$, etc.

Classical certainty is $P(a,b) = 1$; Uncorrelated systems $P(a,b) = p(a)q(b)$

A standard measure to examine statistical correlation between A and B:

Shannon entropy $H(A,B) = -\sum_{a,b} P(a,b) \ln P(a,b) \geq 0$,

And $H(A) = -\sum_a p(a) \ln p(a)$

Comparison of two probability densities: **Kullback-Leibler relative entropy**

$$K\big(p(A) \| q(B)\big) = \sum_{a,b} p(a) \log \big(p(a)/q(b)\big) \geq 0$$

Uncorrelated systems, $H(A,B) = H(A) + H(B)$ (ADDITIVE ENTROPY)

Important CLASSICAL RESULT: $H(A,B) \geq H(A) \; or \; H(B)$

follows from $\sum_b \big(P(a,b)/p(a)\big) = 1 \Rightarrow 0 \leq \big(P(a,b)/p(a)\big) \leq 1$

Intuitively, Info content in the composite system is always greater than in its parts!

AN IMPORTANT ASPECT OF CLASSICAL PROBABILISTIC CORRELATIONS BETWEEN A and B!

In **Quantum version**,

Von Neumann entropy $S(A,B) = -Tr\,\rho(A,B)\ln\rho(A,B) \geq 0$,

and $$S(A) = -Tr_A\,\rho(A)\ln\rho(A) \geq 0$$

$$S\left(\hat{\rho}_{Pure}\right) = 0 \qquad S\left(\hat{\rho}_{ch}\right) = \ln N$$

Uncorrelated (Non-interacting) quantum systems: $\hat{\rho}(A,B) = \hat{\rho}(A)\otimes\hat{\sigma}(B)$

$$S(A,B) = S(A) + S(B) \quad \text{(Additive property)}$$

S does not change under Unitary transformation but does under POVM!

QUANTUM RESULT IS COUNTER INTUITIVE

$$S(A,B)\ \text{can be} \quad \geq \quad \text{or} \quad \leq \quad \mathrm{S(A)}\ \text{or}\ \mathrm{S(B)}$$

AN IMPORTANT ASPECT OF QUANTUM CORRELATIONS BETWEEN A and B BEYOND CLASSICAL CORRELATION !

TWO LESSONS:

(a) CORRELATIONS BETWEEN SUBSYSTEMS

(b) IMPORTANCE OF A MEASURE TO QUANTIFY THEM

Motivation for “entropy” in quantum information theory in composite systems.

ANY QUESTIONS?

Quantum Theory

Comparison of two density matrices,
Kullback-Leibler Relative Entropy

$$K\big(\rho(A)\big|\,\sigma(B)\big) = Tr\,\rho(A)\{\log\rho(A) - \log\sigma(B)\} \geq 0$$

if the two density matrices have the same domain.

In the respective eigenexpansions of the two density matrices

$$K\big(\rho(A)\big|\,\sigma(B)\big) = \sum_{a,b}^{N} \left|\langle a|b\rangle\right|^2 p(a)\log\big(p(a)/q(b)\big) \geq 0$$

CLASSICAL VERSION does not have the overlap matrix element.

TWO ALTERNATIVE ENTROPY MEASURES

RÉNYI ENTROPY:

$$S_\alpha^R(\hat{\rho}) = \frac{\ln Tr(\hat{\rho}^\alpha)}{1-\alpha}, \quad \alpha : \text{real}\,\#$$

TSALLIS ENTROPY:

$$S_q^T(\hat{\rho}) = \frac{Tr\,\hat{\rho}^q - 1}{1-q}, \quad q : \text{real}\,\#$$

Both reduce to von Neumann entropy when $\alpha = 1, \quad q = 1$

For uncorrelated bipartite system, $\hat{\rho}(A,B) = \hat{\rho}(A) \otimes \hat{\sigma}(B)$**:**

$$S_\alpha^R(\hat{\rho}(A,B)) = S_\alpha^R(\hat{\rho}(A)) + S_\alpha^R(\hat{\sigma}(B)) \quad \textbf{(Additive)}$$

$$S_q^T(\hat{\rho}(A,B)) = S_q^T(\hat{\rho}(A)) + S_q^T(\hat{\sigma}(B)) + (1-q)S_q^T(\hat{\rho}(A))S_q^T(\hat{\sigma}(B))$$

(Non-additive)

Non-additive Formulation - Tsallis

Comparison of two density matrices, q-KL entropy
(Abe, AKR) for $0 \le q \le 1$

$$K_q(\rho|\sigma) = Tr\,\rho^q\left(\ln_q \rho - \ln_q \sigma\right)$$

$$= \frac{1}{1-q}\left[1 - Tr\,\rho^q\,\sigma^{1-q}\right]$$

$$= \frac{1}{1-q}\sum_{a,b}\left|\langle a|b\rangle\right|^2 p(a)\left[1-\left(\frac{q(b)}{p(a)}\right)^{1-q}\right] \ge 0$$

After using $\left(1-x^p\right)/p \ge (1-x),\ \left(x \ge 0,\ 0 < p < 1\right)$

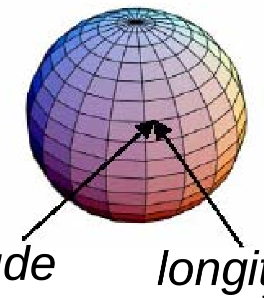
Bloch-Sphere Representation of a Quantum Bit

Start with one two-level state

Physically implement states using properties of atoms, photons, electrons, Cooper pairs, nuclei, etc.

Classical state (bit) $|\uparrow\rangle$ *or* $|\downarrow\rangle$

Quantum state (qubit)



$|\uparrow\rangle$ *North Pole*

$|\downarrow\rangle$ *South Pole*

Many possible two-level states

$$|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\varphi}\sin(\theta/2)|\downarrow\rangle$$

Superposition Principle

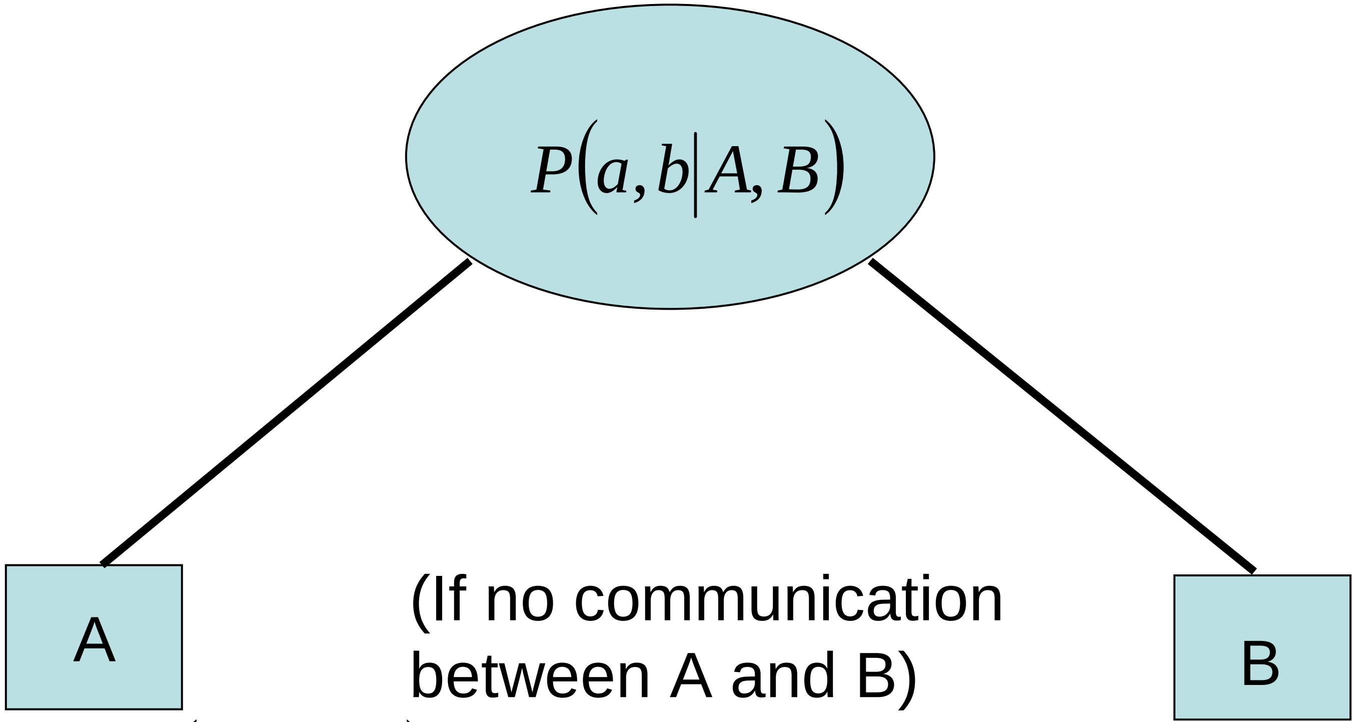
Unitary Evolution
(processing quantum information)

$$|\psi_{out}\rangle = U(t)|\psi_{in}\rangle$$

Output – Quantum Measurement

Input -State Preparation

Data Collection Experiment



$$P(a|A,B) = \sum_b P(a,b|A,B)$$

$$= P(a|A)$$

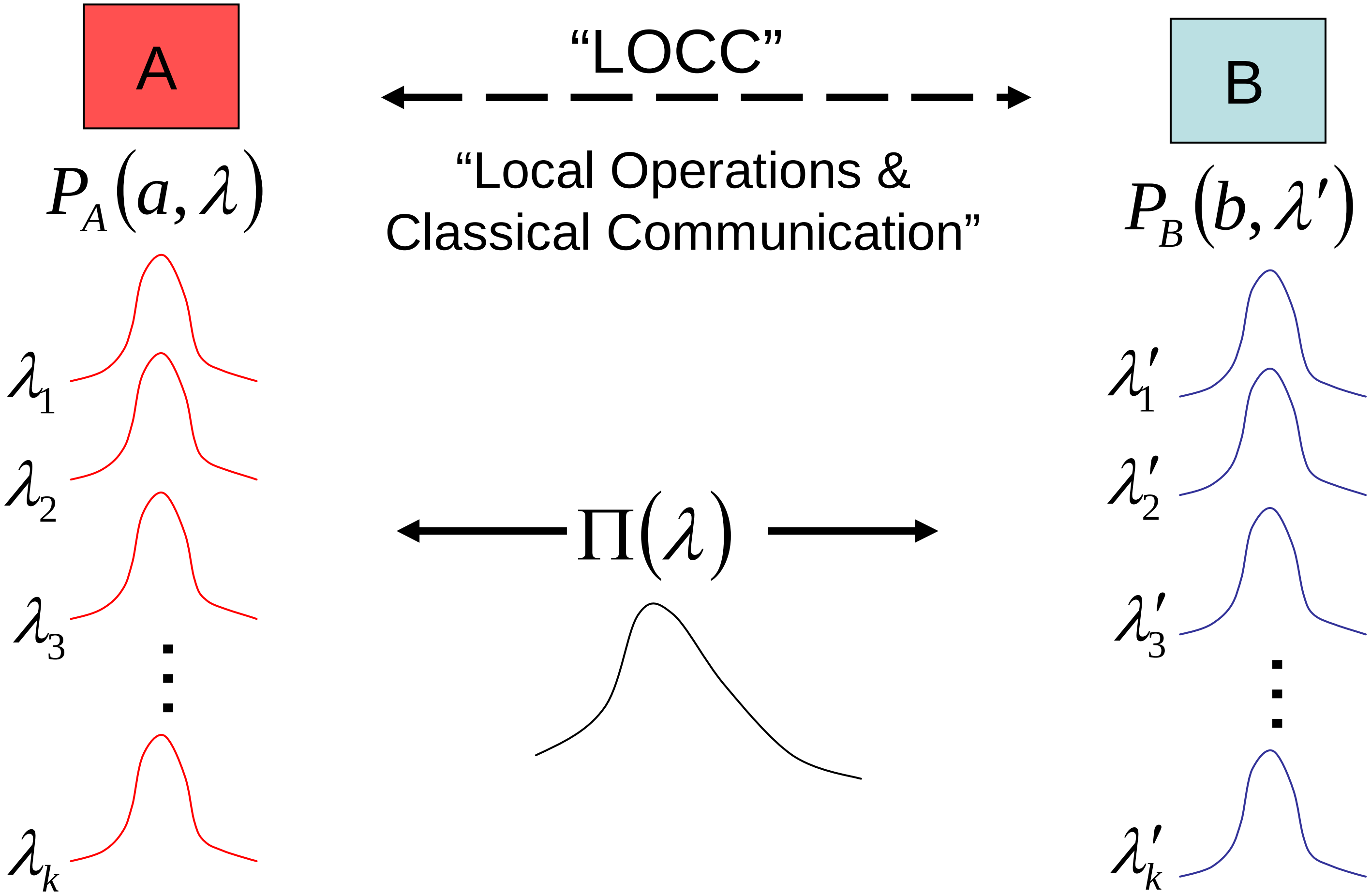
“No Signaling
Condition”
(Einstein)

$$P(b|A,B) = \sum_a P(a,b|A,B)$$

$$= P(b|B)$$

(R.F. Werner & M.M. Wolf: Qu. Inf. & Comp **1**, 1 (2001))

Construction of Joint Probability for the Data



$$\left(\sum_{\lambda} \Pi_A(\lambda) P_A(a,\lambda)\right)\left(\sum_{\lambda'} \Pi_B(\lambda') P_B(b,\lambda')\right)$$

“LOCC” →

$$\sum_{\lambda} \Pi(\lambda) P_A(a,\lambda)\, P_B(b,\lambda) \equiv P(a,b|A,b)$$

Definition of Quantum Entanglement

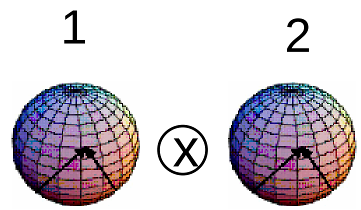
R.F. Werner, Phys. Rev. **A40**, 4273 (1989)

If $\rho(A,B) = \sum_{\lambda} \Pi(\lambda) \rho_{\lambda}(A) \otimes \sigma_{\lambda}(B)$, $\sum_{\lambda} \Pi(\lambda) = 1$

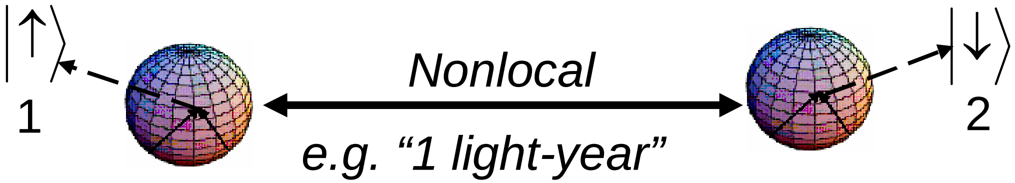
Then $\rho(A,B)$ is "separable"

Otherwise $\rho(A,B)$ is "entangled"

Quantum Entanglement



Two or more qubits : possibilities of nonlocal correlations



Pure states

$$|\Psi(1,2)\rangle = \frac{|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle}{\sqrt{2}}$$

"Entangled state"

$$|\Psi(1,2)\rangle = \frac{|\uparrow_1\downarrow_2\rangle + |\uparrow_1\uparrow_2\rangle}{\sqrt{2}} = |\uparrow\rangle_1 \otimes \left(\frac{|\downarrow\rangle + |\uparrow\rangle}{\sqrt{2}} \right)_2$$

"Separable state" (or unentangled)

$$\rho = |\psi(1,2)\rangle\langle\psi(1,2)|$$

Pure state density matrix

Quantum Theory

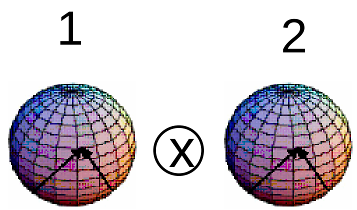
Celebrated example of entangled states (Bell):

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right) \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\right)$$

The pure state density matrices formed from these, all have the same marginals of the Chaotic form
with marginal entropies, log2. "maximally entangled".

Several Implications: No cloning theorem; teleportation; quantum computing algorithms, etc.

Quantum Entanglement



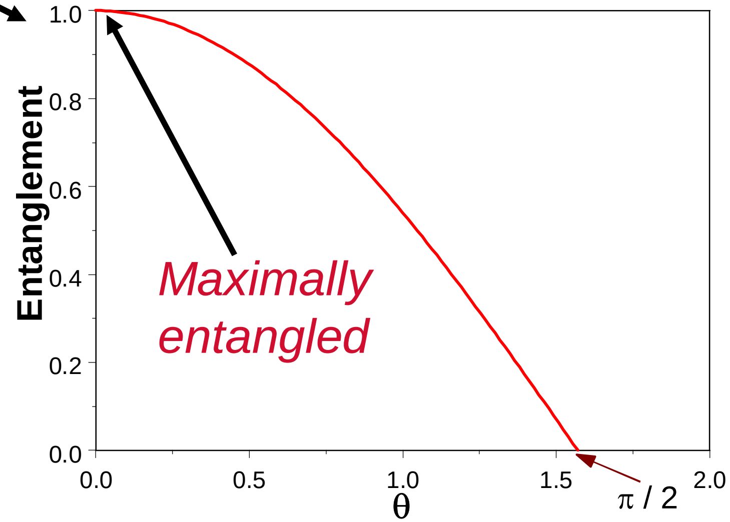
Two or more qubits : possibilities of nonlocal correlations

$$\rho=\frac{1}{2}\left[\cos(\theta)\left(|\uparrow\downarrow\rangle\langle\downarrow\uparrow|+|\downarrow\uparrow\rangle\langle\uparrow\downarrow|\right)+(1-\sin(\theta))|\downarrow\uparrow\rangle\langle\downarrow\uparrow|+(1+\sin(\theta))|\uparrow\downarrow\rangle\langle\uparrow\downarrow|\right]$$

Mixed state density matrix

(θ follows a slice of two-qubit space)

Entanglement as a physical resource



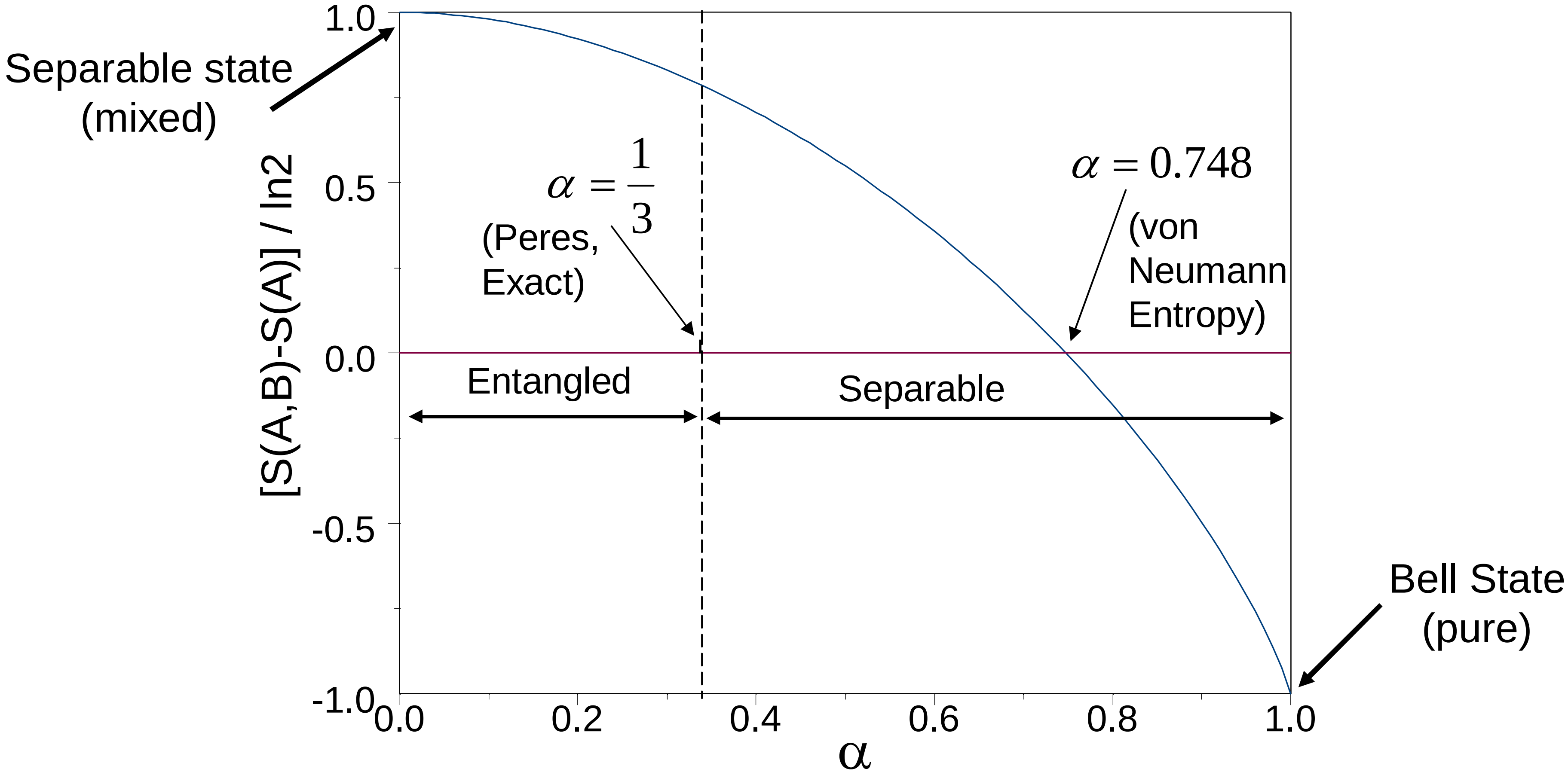
General Transformation: $\rho' = \sum V_i \, \rho V_i^{\dagger}, \quad \sum V_i^{\dagger} V_i = I$

Entanglement Measure : quantitative degree of entanglement known only for small #'s of qubits or special symmetries.

Entropy Difference for a Werner State

$$\rho_W(A,B) = \left(\frac{1-\alpha}{4}\right) I_2 \otimes I_2 + \alpha \left|\Psi_+\right\rangle\left\langle\Psi_+\right| \qquad \left|\Psi_+\right\rangle = \frac{\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle}{\sqrt{2}}$$

$$[S(A,B) - S(A)]/\ln 2 = \left\{-3\left(\frac{1-\alpha}{4}\right)\ln\left(\frac{1-\alpha}{4}\right) - \left(\frac{1+3\alpha}{4}\right)\ln\left(\frac{1+3\alpha}{4}\right) - \ln 2\right\}/\ln 2$$



Three applications of non-additive formalism to quantum information theory will be given to demonstrate its usefulness.

The problems chosen are of current interest and the procedures employed are novel as will be evident presently.

Mostly we will be concerned with Bell entangled states:

$$\left|\Phi^{\pm}(A,B)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\uparrow\right\rangle \pm \left|\downarrow\downarrow\right\rangle\right) \quad \left|\Psi^{\pm}(A,B)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle \pm \left|\downarrow\uparrow\right\rangle\right)$$

These cannot be written in the form

$$\left(a_1\left|\uparrow\right\rangle + a_2\left|\downarrow\right\rangle\right) \otimes \left(b_1\left|\uparrow\right\rangle + b_2\left|\downarrow\right\rangle\right)$$

Maximum Tsallis Entropy Method

Quantum Entanglement is non-additive:

Consider EPR pair A, B of spin-1/2 with
Bell, Clauser, Horne, Shimony, Holt (Bell-CHSH)
observable:

$$\hat{B} = \sqrt{2}\left\{\hat{\sigma}_{Ax} \otimes \hat{\sigma}_{Bx} + \hat{\sigma}_{Az} \otimes \hat{\sigma}_{Bz}\right\}$$

& in Bell basis this is

$$\hat{B} = 2\sqrt{2}\left\{\left|\Phi^{+}\right\rangle\left\langle\Phi^{+}\right| - \left|\Psi^{-}\right\rangle\left\langle\Psi^{-}\right|\right\}$$

Maximum von Neumann entropy $S_1[\rho]$, given constraint of mean value of this operator

$$b_1 = Tr\hat{\rho}\hat{B}, \quad 0 \le b_1 \le 2\sqrt{2}$$

gave “fake” entanglement (HHH,PRA**59**, 1799 (1999)

Maximum Tsallis Entropy Method

Kullback-Leibler relative entropy is a measure of diff. between two density matrices:

$$K_1(\hat{\rho}_1, \hat{\rho}_2) = Tr\, \hat{\rho}_2 (\ln \hat{\rho}_2 - \ln \hat{\rho}_1) \geq 0$$

Let $\hat{\rho}_2$ maximize S_1 with two constraints, b_1 and another,

then $S_2 \leq S_1$

This shows that a 2nd constraint resolves the puzzle!

We choose $\sigma_1^2 = Tr \hat{\rho} \hat{B}^2$. Uncertainty Principle gives $\sigma_1^2 \geq 2\sqrt{2}\, b_1$.

Equality gives minimum uncertainty.

Entanglement criterion: Eigenvalues of den.matrix < 1/2 leads to

$$\sigma_1^2 > \left(8 - 2\sqrt{2}\, b_1\right)$$

Power of maxent with appropriate constraints!

AKR, PRA**60,** 4338 (1999).

Maximum Tsallis Entropy Method

We now examine the maximum Tsallis entropy

$$S_q[\hat{\rho}] = \frac{1}{1-q}\left\{Tr\,\hat{\rho}^q - 1\right\}$$

subject to normalized q-mean values of the operators above:

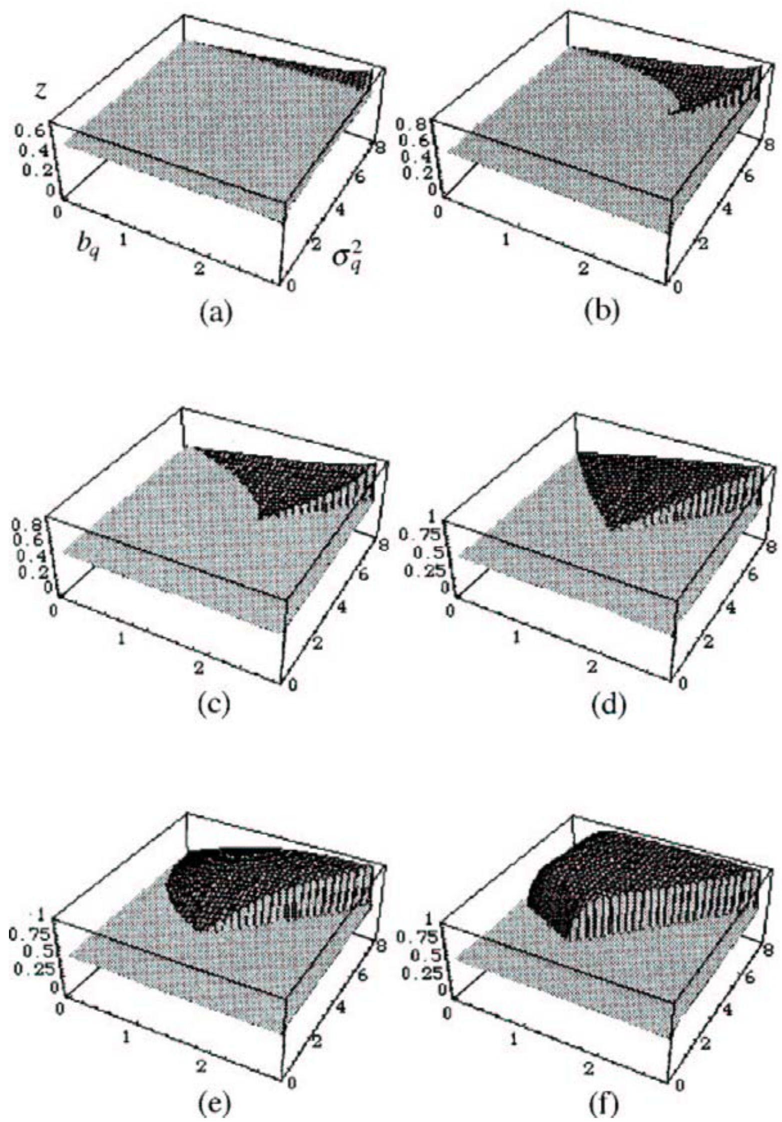
$$\left\langle \hat{Q} \right\rangle_q \equiv \frac{Tr\left(\hat{\rho}^q\,\hat{Q}\right)}{Tr\left(\hat{\rho}^q\right)}$$

$$b_q = \left\langle \hat{B} \right\rangle_q; \quad 0 \le b_q \le 2\sqrt{2}$$

$$\sigma_q^2 = \left\langle \hat{B}^2 \right\rangle_q; \quad \sigma_q^2 \le 8$$

Uncertainty relation still holds:

$$\sigma_q^2 \ge 2\sqrt{2}\,b_q$$



q-Entropy Entanglement Condition :

$$\left[\frac{\sigma_q^2 + 2\sqrt{2}b_q}{16(Z_q)^{q-1}}\right]^{1/q} > \frac{1}{2}$$

Sub-additive :

(a) q=5, (b) q=2, (c) q=1.5

Super-additive :

(d) q=0.9, (e) q=0.5, and (f) q=0.1

S. Abe & A.K. Rajagopal PRA **60**, 3461 (1999)

q - conditional entropy: entanglement criterion of a Werner state

Def.: Separable state has the form

$$\rho(A,B) = \sum_{\lambda} w_{\lambda}\, \rho_{\lambda}(A) \otimes \rho_{\lambda}(B), \quad \sum_{\lambda} w_{\lambda} = 1.$$

The composite density matrix has marginals

$$\hat{\rho}(A) = Tr_B \hat{\rho}(A,B), \quad \hat{\rho}(B) = Tr_A \hat{\rho}(A,B)$$

Quantum entanglement if

$$S_q(B|A),\, S_q(A|B) < 0; \quad S_q(B|A) = \frac{S_q(A,B) - S_q(A)}{1 + (1-q) S_q(A)}.$$

Entanglement criterion for Werner state defined by

$$\hat{\rho}_W(A,B) = \frac{1-x}{4}\, \hat{I}_A \otimes \hat{I}_B + x \left|\Psi^-\right\rangle\left\langle\Psi^-\right|$$

Noise ← $\frac{1-x}{4}$

q - conditional entropy: entanglement criterion of a Werner state

The separability criteria known so far of this state are:

(a) Bell inequality $x < 1/\sqrt{2}$

(b) Renyi entropy $(\alpha = 2)$ $x < 1/\sqrt{3}$

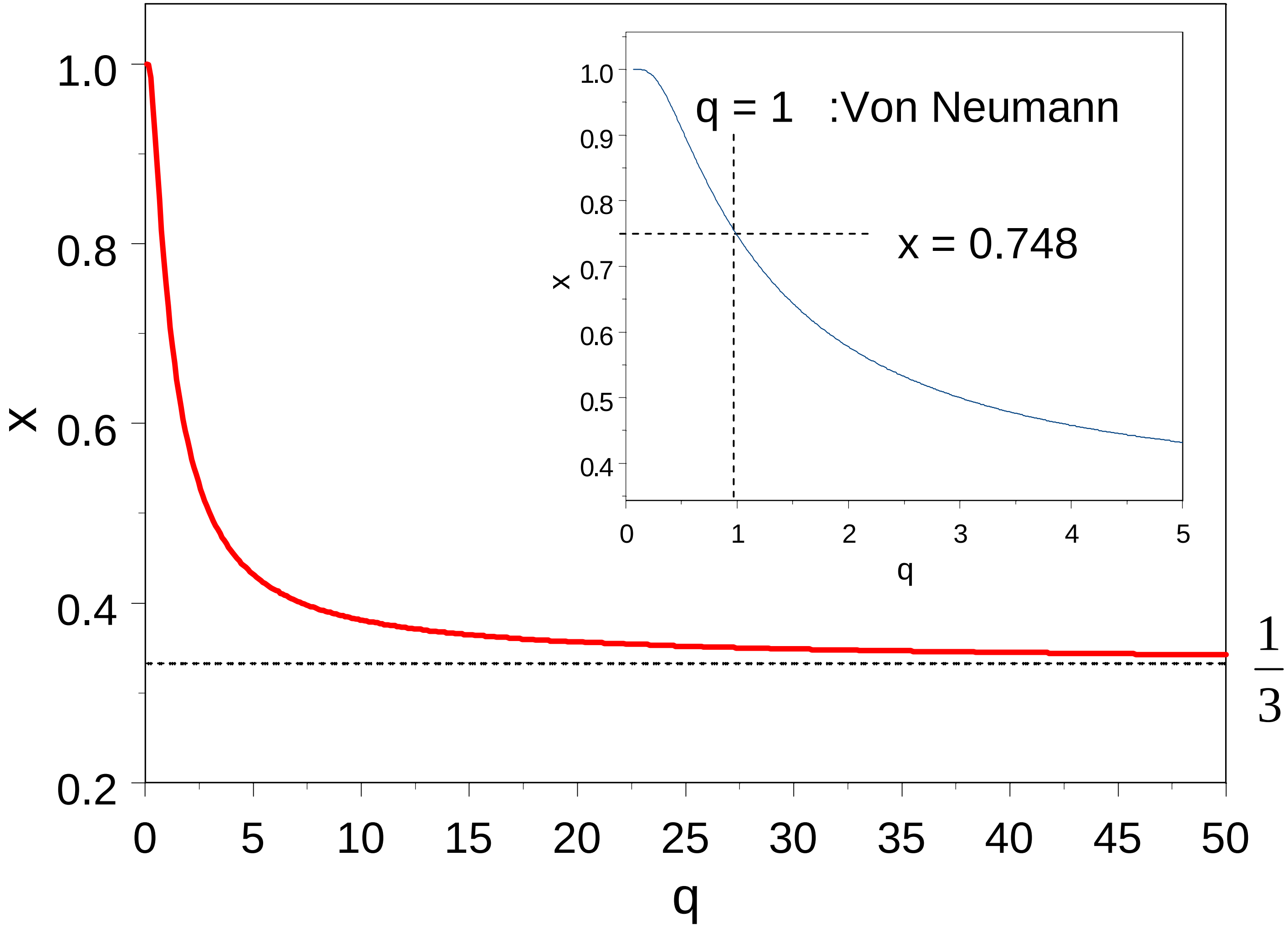
(c) Peres criterion (Exact) $x < 1/3$

We now show that the q - conditional entropy for $q \to \infty$ coincides with the Peres condition (c).

NOTE: Necessary & sufficient for 2x2 and 2x3 systems

Nonadditive q-Conditional Entropy

$$S_q(A|B) = 0$$



S. Abe & A.K. Rajagopal, Physica A **289**, 157 (2001)

(Fig.by Rendell)

q - relative entropy : "fidelity" of a given state

In order to examine how well one has reached a known target state, $\hat{\rho}_1$, after obtaining a state $\hat{\rho}_2$, a measure is introduced called "fidelity":

$$F[\hat{\rho}_1, \hat{\rho}_2] = \left\{ Tr\left(\sqrt{\sqrt{\hat{\rho}_1}\, \hat{\rho}_2 \sqrt{\hat{\rho}_1}}\right) \right\}$$

For a pure state target, $\hat{\rho}_1 = |\psi\rangle\langle\psi|,$ this becomes

$$F[\hat{\rho}_1, \hat{\rho}_2] = \langle\psi|\hat{\rho}_2|\psi\rangle.$$

Fidelity close to 1 is thus desired !

Kullback-Leibler (KL) relative entropy is a convenient measure of diff. between two density matrices:

$$K_1(\hat{\rho}_1, \hat{\rho}_2) = Tr\, \hat{\rho}_2 (\ln \hat{\rho}_2 - \ln \hat{\rho}_1) \geq 0$$

Equality sign is obtained when $\hat{\rho}_1 = \hat{\rho}_2$.

If target state is a pure state, KL is undefined.

q - relative entropy to examine the "fidelity" of a given state

This is where the q-KL for 0 < *q* < 1 becomes useful:

$$K_q(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{1-q} Tr\left[\hat{\rho}_1^q \left(\hat{\rho}_1^{1-q} - \hat{\rho}_2^{1-q}\right)\right] \geq 0$$

In the case when the target state is a pure state as before

$$K_q(\hat{\rho}_1, |\psi\rangle\langle\psi|) = \frac{1}{1-q}\left(1 - \langle\psi|\hat{\rho}^q|\psi\rangle\right) \geq 0$$

Additive limit *q*=1 cannot be taken here.

Example: Degree of purification of a Werner state

$$\hat{\rho}_W(A,B) = \frac{1-x}{4} \hat{I}_A \otimes \hat{I}_B + x|\Psi^-\rangle\langle\Psi^-|$$

Here Fidelity $F = \langle\Psi^-|\hat{\rho}_W|\Psi^-\rangle = (1+3x)/4$

And $K_q(\hat{\rho}_W, |\Psi^-\rangle\langle\Psi^-|) = \frac{1}{1-q}\left(1 - F^q\right) \geq 0$

S. Abe, PRA **68**, 032302 (2003)

Concluding Comments

There has been some discussion about Thermodynamics of Information, in particular Quantum Information. Since there are hints that quantum entanglement may not be additive, and since the concept of entropy has been introduced into the discussion, Abe and AKR (PRA 60, 3461 (1999)) examined the maximum q-entropy principle subject to constraints such as Bell -Clauser-Horne-Shimony-Holt observable, for inferring quantum entanglement.

They also showed that quantum entanglement can be quantified by the q-KL entropy.

Our examples illustrate the usefulness of the q-formalism where q=1 theory is inadequate or unsuitable.

The use of these techniques in quantum computing algorithms have not yet been explored.

State Property : Entanglement

Example of application : Quantum Computation (QC)

1. State preparation

$$|\psi_{in}\rangle = |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$$

2. Unitary evolution (gates)

"computation + error correction"

$$|\psi_{out}\rangle = U(t)|\psi_{in}\rangle$$

3. Quantum Measurement → Output

Role of Entanglement :

- Quantum computation by focusing 'many paths' to desired result (some algorithms: factoring, search, etc.)
- Error correction by delocalizing within larger entangled state.

QC algorithms:

Shor's Factoring (1994), Grover's Search (1997), van Dam (2000), Hallgren (2002),…

Prime factors of large integers → break RSA public key cryptography

Exponential advantage : 250 digits, ~10^7 years classically; ~ hours by QC

Implementation is demanding : ~ 10^5 qubits for 250 digit factorization

Require at least ~ 10^2 qubits to begin useful computations (currently: 7 qubits / NMR)

Error correction / fault tolerance is essential and adds to overhead

More immediate applications of entanglement : quantum cryptography, noise, etc.

SOME REFERENCES:

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(2) M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, U.K.) (2000)

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