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q-Generalization of Symmetric α-Stable Distributions

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q-GENERALIZATION OF SYMMETRIC α-STABLE DISTRIBUTIONS. PART I

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Abstract

The classic and the Lévy-Gnedenko central limit theorems play a key role in theory of probabilities, and also in Boltzmann-Gibbs (BG) statistical mechanics. They both concern the paradigmatic case of probabilistic independence of the random variables that are being summed. A generalization of the BG theory, usually referred to as nonextensive statistical mechanics and characterized by the index q ($q = 1$ recovers the BG theory), introduces global correlations between the random variables, and recovers independence for $q = 1$. The classic central limit theorem was recently q-generalized by some of us. In the present paper we q-generalize the Lévy-Gnedenko central limit theorem.

1 Introduction

In the recent paper by some of us [1], a generalization of the classic central limit theorem applicable to nonextensive statistical mechanics [2, 3] (which recovers the usual, Boltzmann-Gibbs statistical mechanics as the $q = 1$ particular instance), was presented. We follow here along the lines of that paper. One of the important aspects of this generalization is that it concerns the case of globally correlated random variables. On the basis of the q-Fourier transform F_q introduced there (F_1 being the standard Fourier transform), and the function

$$z(s) = \frac{1+s}{3-s},$$

we described attractors of conveniently scaled limits of sums of q-correlated random variables [1] with a finite $(2q-1)$-variance [2]. This description was essentially based on the mapping

$$F_q : \mathcal{G}_q[2] \to \mathcal{G}_{z(q)}[2], \tag{1}$$

[1] q-correlation corresponds to standard probabilistic independence if $q = 1$, and to specific global correlations if $q \neq 1$.

[2] We required there $q < 2$. Denoting $Q = 2q - 1$, it is easy to see that this condition is equivalent to the finiteness of the Q-variance with $Q < 3$.

where $\mathcal{G}_q[2]$ is the set of q-Gaussians (the number 2 in the notation will soon become transparent).

In the current paper, which consists of two parts, we will introduce and study a q-analog of the α-stable Lévy distributions, and establish a q-generalization of the Lévy-Gnedenko central limit theorem. In this sense, the present paper is a conceptual continuation of paper [1]. The classic theory of the α-stable distributions was originated and developed by Lévy, Gnedenko, Feller and others (see, for instance, [4, 5, 6, 7, 8] and references therein for details and history). The α-stable distributions found a huge number of applications in various practical studies [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], confirming the frequent nature of these distributions.

For simplicity we will analise only symmetric densities in the one-dimensional case. Stable distributions with skewness and multivariate stable distributions can be studied in the same way applying the known classic techniques.

In Part 1 we study a q-generalization of the α-stable Lévy distributions. Namely, we consider the symmetric densities $f(x)$ with asymptotics $f \sim C|x|^{-\frac{1+\alpha}{1+\alpha(q-1)}}$, $|x| \to \infty$, where C is a positive constant [3]. We classify these distributions in terms of their densities depending on the parameters $q < 2$ (or equivalently $Q < 3$, $Q = 2q - 1$) and $0 < \alpha \le 2$. We establish the mapping

$$F_q : \mathcal{G}_{q^L}[2] \to \mathcal{G}_q[\alpha], \tag{2}$$

where $\mathcal{G}_q[\alpha]$ is the set of all densities $\{be_q^{-\beta|\xi|^\alpha},\ b > 0,\ \beta > 0\}$, and

$$q^L = \frac{3 + Q\alpha}{1+\alpha},\ Q = 2q - 1\,,$$

i.e.,

$$\frac{2}{q^L - 1} = \frac{1+\alpha}{1+\alpha(q-1)}\,.$$

The particular case $q = Q = 1$ recovers $q^L = \frac{3+\alpha}{1+\alpha}$, already known in the literature [3]. We consider the values of parameters Q and α ranging in the set

$$\mathcal{Q}_0 = \{(Q,\alpha) : -1 < Q < 3,\ 0 < \alpha < 2,\ \alpha < \frac{2}{1-Q}\}.$$

The values of Q and α in

$$\mathcal{Q}_2 = \{(Q,\alpha) : -1 < Q < 3,\ \alpha = 2\}$$

were studied in [1]. Note that for Q and α in

$$\mathcal{Q}_1 = \{(Q,\alpha) : \frac{2}{1-Q} \le \alpha < 2,\ -1 < Q < 0\},$$

the densities have finite $(2q-1)$-variance. Consequently, the theorem obtained in [1] is again applicable. For $(Q,\alpha) \in \mathcal{Q}_0$ the Q-variance is infinite. We will focus our analysis namely on this case. Note that the case $\alpha = 2$, in the framework of the present description like in that of the classic α-stable distributions, becomes peculiar.

In Part 2 we study the attractors of scaled sums, and expand the results of the paper [1] to the region

$$\mathcal{Q} = \{(Q,\alpha) : -1 < Q < 3, 0 < \alpha \le 2\}\,,$$

generalizing the mapping (1) into the form

$$F_{\zeta_\alpha(q)} : \mathcal{G}_q[\alpha] \to \mathcal{G}_{z_\alpha(q)}[\alpha],\ q < 2,\ 0 < \alpha \le 2, \tag{3}$$

[3] Hereafter $g(x) \sim h(x), x \to a$, means that $\lim_{x\to a} \frac{g(x)}{h(x)} = 1$.

where

$$\zeta_\alpha(s) = \frac{\alpha - 2(1-q)}{\alpha} \text{ and } z_\alpha(s) = \frac{\alpha q + 1 - q}{\alpha + 1 - q}.$$

Note that, if $\alpha = 2$, then $\zeta_2(q) = q$ and $z_2(q) = (1+q)/(3-q)$, thus recovering the mapping (1), and consequently, the result of the paper [1].

These two types of q-generalized descriptions of the standard symmetric α-stable distributions, based on mappings (2) and (3) respectively, allow us to draw a full picture of the q-generalization of the Lévy-Gnedenko central limit theorem that we have obtained.

2 Basic operations of q-mathematics

We recall briefly basics of q-mathematics. Indeed, the analysis we conduct is entirely based on the q-structure (for more details see [20, 21, 22, 23, 24] and references therein). Let x and y be two given real numbers. By definition, the *q-sum* of these numbers is defined as $x \oplus_q y = x + y + (1-q)xy$. The q-sum is commutative, associative, recovers the usual summing operation if $q = 1$ (i.e. $x \oplus_1 y = x + y$), and preserves 0 as the neutral element (i.e. $x \oplus_q 0 = x$). By inversion, we can define the *q-subtraction* as $x \ominus_q y = \frac{x-y}{1+(1-q)y}$. The *$q$-product* for x, y is defined by the binary relation $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$. This operation also commutative, associative, recovers the usual product when $q = 1$, and preserves 1 as the unity. The q-product is defined if $x^{1-q} + y^{1-q} \geq 1$. Again by inversion, it can be defined the *q-division*: $x \oslash_q y = (x^{1-q} - y^{1-q} + 1)^{\frac{1}{1-q}}$. Note that, for $q \neq 1$, $x \otimes_q 0 \neq 0$, and division by zero is allowed.

3 q-generalisation of the exponential and cyclic functions

Now we introduce the q-exponential and q-logarithm [20], which play an important role in the nonextensive theory. These functions are denoted by e_q^x and $ln_q x$ and respectively defined as $e_q^x = [1 + (1-q)x]_+^{\frac{1}{1-q}}$ and $\ln_q x = \frac{x^{1-q}-1}{1-q}$, $(x > 0)$. Here the symbol $[x]_+$ means that $[x]_+ = x$ if $x \geq 0$, and $[x]_+ = 0$ if $x < 0$. We mention the main properties of these functions, which we will use essentially in this paper. For the q-exponential the relations $e_q^{x \oplus_q y} = e_q^x e_q^y$ and $e_q^{x+y} = e_q^x \otimes_q e_q^y$ hold true. These relations can be written equivalently as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ [4], and $\ln_q(xy) = (\ln_q x) \oplus_q (\ln_q y)$. The q-exponential and q-logarithm have the asymptotics

$$e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2),\ x \to 0, \tag{4}$$

and

$$\ln_q(1+x) = x - \frac{q}{2}x^2 + o(x^2),\ x \to 0, \tag{5}$$

respectively. If $q < 1$, then, for real x, $|e_q^{ix}| \geq 1$ and $|e_q^{ix}| \sim (1+x^2)^{\frac{1}{2(1-q)}}$, $x \to \infty$. Similarly, if $q > 1$, then $0 < |e_q^{ix}| \leq 1$ and $|e_q^{ix}| \to 0$ if $|x| \to \infty$.

Lemma 3.1 *Let $A_n(q) = \prod_{k=0}^n a_k(q)$ where $a_k(q) = q - k(1-q)$. Then there holds the series expansion*

$$e_q^x = 1 + x + x^2 \sum_{n=0}^{\infty} \frac{A_n(q)}{(n+2)!} x^n,\ \forall x \in R.$$

[4]This property reflects the possible extensivity of S_q in the presence of special correlations [25, 26, 27, 28].

Corollary 3.2 *For arbitrary real number x the equation*

$$e_q^{ix} = \{1 - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n}\} + i\{x - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}\}$$

holds.

Define q-cos and q-sin by formulas

$$\cos_q(x) = 1 - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n}, \tag{6}$$

and

$$\sin_q(x) = x - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}. \tag{7}$$

Properties of q-sin, q-cos, and corresponding q-hyperbolic functions, were studied in [22]. Here we note that the q-analogs of the well known Euler's formulas read

Corollary 3.3 *(i)* $e_q^{ix} = \cos_q(x) + i\ \sin_q(x)$;

(ii) $\cos_q(x) = \frac{e_q^{ix} + e_q^{-ix}}{2}$;

(iii) $\sin_q(x) = \frac{e_q^{ix} - e_q^{-ix}}{2i}$.

Lemma 3.4 *The following equality holds:*

$$\cos_q(2x) = e_{2q-1}^{2(1-q)x^2} - 2\ \sin^2_{2q-1}(x). \tag{8}$$

Proof. The proof follows from the definitions of $\cos_q(x)$ and $\sin_q(x)$, and from the fact that $(e_q^x)^2 = e_{(1+q)/2}^{2x}$ (see Lemma 2.1 in [1]).

Denote $\Psi_q(x) = \cos_q 2x - 1$. It follows from Equation (8) that

$$\Psi_q(x) = (e_{2q-1}^{2(1-q)x^2} - 1) - 2\ \sin^2_{2q-1}(x). \tag{9}$$

Lemma 3.5 *Let $q \geq 1$. Then we have*

1. $-2 \leq \Psi_q(x) \leq 0$;
2. $\Psi_q(x) = -2\,q\,x^2 + o(x^3),\ x \to \infty$.

Proof. It follows from (9) that $\Psi_q(x) \leq 0$. Further, $\sin_q(x)$ can be written in the form (see [22]) $\sin_q(x) = \rho_q(x)\sin[\varphi_q(x)]$, where $\rho_q(x) = (e_q^{(1-q)x^2})^{1/2}$ and $\varphi_q(x) = \frac{\arctan(1-q)x}{1-q}$. This yields $\Psi_q(x) \geq -2$ if $q \geq 1$. Using the asymptotic relation (4), we get

$$e_{2q-1}^{2(1-q)x^2} - 1 = 2(1-q)x^2 + o(x^3),\ x \to 0. \tag{10}$$

It follows from (7) that

$$-\ 2\ \sin^2_{2q-1}(x) = -2\,x^2 + o(x^3), x \to 0. \tag{11}$$

The relations (9), (10) and (11) imply the second part of the statement.

4 q-Fourier transform for symmetric functions

The *q-Fourier transform*, based on the q-product, was introduced in [1] and played a central role in establishing the q-analog of the standard central limit theorem. Formally the q-Fourier transform for a given function $f(x)$ is defined by the formula

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx \,. \tag{12}$$

For discrete functions $f_k, k = 0, \pm 1, ...$, this definition takes the form

$$F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e_q^{ik\xi} \otimes_q f(k) \,. \tag{13}$$

In the future we use the same notation in both cases. We also call (12) or (13) the *q-characteristic function* of a given random variable X with an associated density $f(x)$, using the notations $F_q(X)$ or $F_q(f)$ equivalently.

It should be noted that, if in the formal definition (12), f is compactly supported, then integration has to be taken over this support, although, in contrast with the usual analysis, the function $e_q^{ix\xi} \otimes_q f(x)$ under the integral does not vanish outside the support of f. This is an effect of the q-product.

The following lemma establishes the relation of the q-Fourier transform without using the q-product.

Lemma 4.1 *The q-Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{(f(x))^{1-q}}} dx. \tag{14}$$

Remark 4.2 *Note that, if the q-Fourier transform of a given function $f(x)$ defined by the formal definition in (12) exists, then it coincides with the expression in (14). The q-Fourier transform determined by the formula (14) has an advantage when compared to the formal definition: it does not use the q-product, which is, as we noticed above, restrictive in use. From now on we refer to (14) when we speak about the q-Fourier transform.*

Further to the properties of the q-Fourier transform established in [1], we note that, for symmetric densities, the assertion analogous to Lemma 4.1 is true with the q-cos.

Lemma 4.3 *Let $f(x)$ be a symmetric density. Then its q-Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \cos_q(x\xi[f(x)]^{q-1}) dx. \tag{15}$$

Proof. Notice that, because of the symmetry of f,

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx = \int_{-\infty}^{\infty} e_q^{-ix\xi} \otimes_q f(x) dx \,.$$

Taking this into account, we have

$$F_q[f](\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \left(e_q^{ix\xi} \otimes_q f(x) + e_q^{-ix\xi} \otimes_q f(x) \right) dx \,.$$

Applying Lemma 4.1 we obtain

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \frac{e_q^{ix\xi[f(x)]^{q-1}} + e_q^{-ix\xi[f(x)]^{q-1}}}{2} dx \,,$$

which coincides with (15).

Let us now refer to the three sets:

$$\mathcal{Q}_0 = \{(Q,\alpha) : -1 < Q < 3,\, 0 < \alpha < 2,\, \alpha < \frac{2}{1-Q}\},$$

$$\mathcal{Q}_1 = \{(Q,\alpha) : \frac{2}{1-Q} \leq \alpha < 2,\, -1 < Q < 0\},$$

$$\mathcal{Q}_2 = \{(Q,\alpha) : -1 < Q < 3,\, \alpha = 2\},$$

where $Q = 2q-1$. Obviously $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ gives the semi-strip

$$\mathcal{Q} = \{-1 < Q < 3,\ 0 < \alpha \leq 2\}$$

which contains the top boundary. A q-generalization of the central limit theorem for Q and α in the set $\mathcal{Q}_2$ was studied in [1]. It is not hard to verify that any density corresponding to $(Q,\alpha) \in \mathcal{Q}_1$ has a *finite* Q-variance. Hence, for $\mathcal{Q}_1$ also, the theorem obtained in [1] is applicable. For $(Q,\alpha) \in \mathcal{Q}_0$, the Q-variance of densities considered in the following lemma is *infinite*. From now on, we focus our studies on this case.

Lemma 4.4 *Let $f(x)$, $x \in R$, be a symmetric probability density function of a given random vector. Further, let either*

(i) the $(2q-1)$-variance $\sigma^2_{2q-1}(f) < \infty$, (associated with $\alpha = 2$), or

(ii) $f(x) \sim C(|x|^{-\frac{\alpha+1}{1+\alpha(q-1)}})$, $|x| \to \infty$, with $C > 0$ and $(2q-1,\alpha) \in \mathcal{Q}_0$.

Then, for the q-Fourier transform of $f(x)$, the following asymptotic relation holds true:

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^{\alpha} + o(|\xi|^{\alpha}), \xi \to 0, \tag{16}$$

where

$$\mu_{q,\alpha} = \begin{cases} \frac{q}{2}\sigma^2_{2q-1}\nu_{2q-1}, & \text{if } \alpha = 2 \ ; \\ \\ \frac{2^{2-\alpha}(1+\alpha(q-1))}{2-q} \int_0^{\infty} \frac{-\Psi_q(y)}{y^{\alpha+1}} dy, & \text{if } 0 < \alpha < 2 \ . \end{cases} \tag{17}$$

with $\nu_{2q-1}(f) = \int_{-\infty}^{\infty} [f(x)]^{2q-1}\, dx$.

Remark 4.5 *Stable distributions require $\mu_{q,\alpha}$ to be positive. We have seen (Lemma 3.5) that if $q \geq 1$, then $\Psi_q(x) \leq 0$ (not being identically zero), which yields $\mu_{q,\alpha} > 0$. If $q = 0$, we can check by strightforward calculations that $\mu_{q,\alpha} = 0$. We denote by $\mathcal{Q}_0^+$ the subset of $\mathcal{Q}_0$, where $\mu_{q,\alpha} > 0$.*

Proof. First, assume $\alpha = 2$. Evaluate $F_q[f](\xi)$. Using Lemma 4.1 we have

$$F_q[f](\xi) = \int_{-\infty}^{\infty} (e_q^{ix\xi}) \otimes_q f(x) dx = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} dx \,. \tag{18}$$

Making use of the asymptotic expansion (4) we can rewrite the right hand side of (18) in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \left(1 + \frac{ix\xi}{[f(x)]^{1-q}} - q/2 \frac{x^2\xi^2}{[f(x)]^{2(1-q)}} + o(\frac{x^2\xi^2}{[f(x)]^{2(1-q)}})\right) dx =$$

$$1 - (q/2)\xi^2 \sigma_{2q-1}^2 \nu_{2q-1} + o(\xi^2), \; \xi \to 0,$$

from which the first part of Lemma follows.

Now, assume $(2q-1, \alpha) \in \mathcal{Q}_0^+$. Apply Lemma 4.3 to obtain

$$F_q[f](\xi) - 1 = \int_{-\infty}^{\infty} f(x)[\cos_q(x\xi[f(x)]^{q-1}) - 1] dx =$$

$$2\int_0^N f(x)\Psi_q(\frac{x\xi[f(x)]^{q-1}}{2})dx + 2\int_N^{\infty} f(x)\Psi_q(\frac{x\xi[f(x)]^{q-1}}{2})dx \,,$$

where N is a sufficiently large finite number. In the first integral we use the asymptotic relation $\Psi(\frac{x}{2}) = -\frac{q}{2}x^2 + o(x^3)$, which follows from Lemma 3.5, and get

$2\int_0^N f(x)\Psi_q(\frac{x\xi[f(x)]^{q-1}}{2})dx =$

$$-q\xi^2 \int_0^N x^2 f^{2q-1}(x) dx + o(\xi^3), \; \xi \to 0. \tag{19}$$

In the second integral taking into account the hypothesis of the lemma with respect to $f(x)$, we have

$$2\int_N^{\infty} f(x)\Psi_q(\frac{x\xi[f(x)]^{q-1}}{2})dx = 2\int_N^{\infty} \frac{1}{x^{\frac{\alpha+1}{1+\alpha(1-q)}}} \Psi_q(\frac{x^{1-\frac{(\alpha+1)(q-1)}{1+\alpha(q-1)}}\xi}{2})dx \,.$$

We use the substitution

$$x^{\frac{2-q}{1+\alpha(q-1)}} = \frac{2y}{\xi}$$

in the last integral, and obtain

$$2\int_N^{\infty} f(x)\Psi_q(\frac{x\xi[f(x)]^{q-1}}{2})dx =$$

$$-\frac{2^{2-\alpha}(1+\alpha(q-1))}{2-q}|\xi|^{\alpha} \int_0^{\infty} \frac{-\Psi_q(y)}{y^{\alpha+1}} dy + o(\xi^{\alpha}), \; \xi \to 0. \tag{20}$$

Hence, the obtained asymptotic relations (19) and (20) complete the proof.

5 (q,α)-stable distributions

Two random variables X and Y are called to be q-correlated if

$$F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi). \tag{21}$$

In terms of densities, relation (21) can be rewritten as follows. Let f_X and f_Y be densities of X and Y respectively, and let f_{X+Y} be the density of $X+Y$. Then

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f_{X+Y}(x)dx = F_q[f_X](\xi) \otimes_q F_q[f_Y](\xi). \tag{22}$$

Definition 5.1 *A random variable X is said to have a (q,α)-stable distribution if its q-Fourier transform is represented in the form $b\, e_q^{-\beta|\xi|^\alpha}$, with some real constants $b>0$ and $\beta>0$. We denote by $\mathcal{L}_q(\alpha)$ the set of all (q,α)-stable distributions.*

Denote $\mathcal{G}_q(\alpha) = \{b\, e_q^{-\beta|\xi|^\alpha},\ b>0,\ \beta>0\}$. In other words $X \in \mathcal{L}_q(\alpha)$ if $F_q[f] \in \mathcal{G}_q(\alpha)$. We will study limits of sums

$$Z_N = \frac{1}{D_N(q)}(X_1 + \dots + X_N), N = 1,2,\dots$$

where $D_N(q)$, $N = 1,2,\dots$, are some reals (scaling parameter), that belong to $\mathcal{L}_q(\alpha)$, when $N \to \infty$.

Definition 5.2 *A sequence of random variables Z_N is said to be q-convergent to a (q,α)-stable distribution, if $\lim_{N\to\infty} F_q[Z_N](\xi) \in \mathcal{G}_q(\alpha)$ locally uniformly by ξ.*

Theorem 1. *Assume $(2q-1,\alpha) \in \mathcal{Q}_0^+$. Let $X_1, X_2, \dots, X_N, \dots$ be symmetric random variables mutually q-correlated and all having the same probability density function $f(x)$ satisfying the conditions of Lemma 4.4.*

Then Z_N, with $D_N(q) = (\mu_{q,\alpha}N)^{\frac{1}{\alpha(2-q)}}$, is q-convergent to a (q,α)-stable distribution, as $N \to \infty$.

Remark 5.3 *By definition $\mathcal{Q}_0$ excludes the value $\alpha = 2$. The case $\alpha = 2$, in accordance with the first part of Lemma 4.4, coincides with Theorem 2 of [1]. Note in this case $\mathcal{L}_q(2) = \mathcal{G}_{q^*}(2)$, where $q^* = \frac{3q-1}{q+1}$.*

Proof. Assume $(Q,\alpha) \in \mathcal{Q}_0^+$. Let f be the density associated with X_1. First we evaluate $F_q(X_1) = F_q(f(x))$. Using Lemma 4.4 we have

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^\alpha + o(|\xi|^\alpha), \xi \to 0. \tag{23}$$

Denote $Y_j = N^{-\frac{1}{\alpha}} X_j, j = 1,2,\dots$. Then $Z_N = Y_1 + \dots + Y_N$. Further, it is readily seen that, for a given random variable X and real $a > 0$, there holds $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$. It follows from this relation that $F_q(Y_j) = F_q[f](\frac{\xi}{(\mu_{q,\alpha}N)^{1/\alpha}})$, $j = 1,2,\dots$ Moreover, it follows from the q-correlation of $Y_1, Y_2, \dots$ (which is an obvious consequence of the q-correlation of $X_1, X_2, \dots$) and the associativity of the q-product that

$$F_q[Z_N](\xi) = F_q[f]((\mu_{q,\alpha}N)^{-\frac{1}{\alpha}}\xi) \otimes_q \dots \otimes_q F_q[f]((\mu_{q,\alpha}N)^{-\frac{1}{\alpha}}\xi) \text{ (N factors)}. \tag{24}$$

Hence, making use of the expansion (5) for the q-logarithm, Eq. (24) implies

$$\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f]((\mu_{q,\alpha}N)^{-\frac{1}{\alpha}}\xi) = N \ln_q(1 - \frac{|\xi|^\alpha}{N} + o(\frac{|\xi|^\alpha}{N})) =$$

$$- |\xi|^\alpha + o(1),\ N \to \infty, \tag{25}$$

locally uniformly by ξ.

Hence, locally uniformly by ξ,

$$\lim_{N\to\infty} F_q(Z_N) = e_q^{-|\xi|^\alpha} \in \mathcal{G}_q(\alpha). \tag{26}$$

Thus, Z_N is q-convergent to a (q, α)-stable distribution, as $N \to \infty$. Q.E.D.

This theorem links the classic Lévy distributions with their q_α^L-Gaussian counterparts. Indeed, in accordance with this theorem, a function f, for which

$$f \sim C/x^{(\alpha+1)/(1+\alpha(q-1))},\ |x| \to \infty,$$

is in $\mathcal{L}_q(\alpha)$, i.e. $F_q[f](\xi) \in \mathcal{G}_q(\alpha)$. It is not hard to verify that there exists a q_α^L-Gaussian, which is asymptotically equivalent to f. Let us now find q_α^L. Any q_α^L-Gaussian behaves asymptotically like $C/|x|^\eta = C/|x|^{2/(q_\alpha^L-1)}$, $C = const$, i.e. $\eta = 2/(q_\alpha^L - 1)$. Hence, we reobtain the relation

$$\frac{\alpha+1}{1+\alpha(q-1)} = \frac{2}{q_\alpha^L - 1}. \tag{27}$$

Solving this equation with respect to q_α^L, we have

$$q_\alpha^L = \frac{3+Q\alpha}{\alpha+1},\ Q = 2q - 1\,, \tag{28}$$

linking three parameters: α, the parameter of the α-stable Lévy distributions, q, the parameter of correlation, and q_α^L, the parameter of attractors in terms of q_α^L-Gaussians (see Fig. 2). Equation (28) identifies all (Q, α)-stable distributions with the same index of attractor $G_{q_\alpha^L}$ (See Fig. 1).

In the particular case $Q = 1$, we recover the known link between the classical Lévy distributions ($q = Q = 1$) and corresponding q_α^L-Gaussians. Put $Q = 1$ in Eq. (28) to obtain

$$q_\alpha^L = \frac{3+\alpha}{1+\alpha},\ 0 < \alpha < 2. \tag{29}$$

When α increases between 0 and 2 (i.e. $0 < \alpha < 2$), q_α^L decreases between 3 and 5/3 (i.e. $5/3 < q_\alpha^L < 3$): See Figs. 2 and 4(a).

It is useful to find the relationship between $\eta = \frac{2}{q_\alpha^L-1}$, which corresponds to the asymptotic behaviour of the attractor, and (α, Q). Using formula (28), we obtain (Fig. 3)

$$\eta = \frac{2(\alpha+1)}{2+\alpha(Q-1)}. \tag{30}$$

If $Q = 1$ (classic Lévy distributions), then $\eta = \alpha + 1$, as well known.

Analogous relationships can be obtained for other values of Q. We call, for convenience, a (Q, α)-stable distribution to be a Q-Cauchy distribution, if its parameter $\alpha = 1$. We obtain the classic Cauchy-Poisson distribution if $Q = 1$. The corresponding line can be obtained cutting the surface in Fig. 3 along the line $\alpha = 1$. For Q-Cauchy distributions we have

$$q_1^L(Q) = \frac{3+Q}{2} \text{ and } \eta = \frac{4}{Q+1}, \tag{31}$$

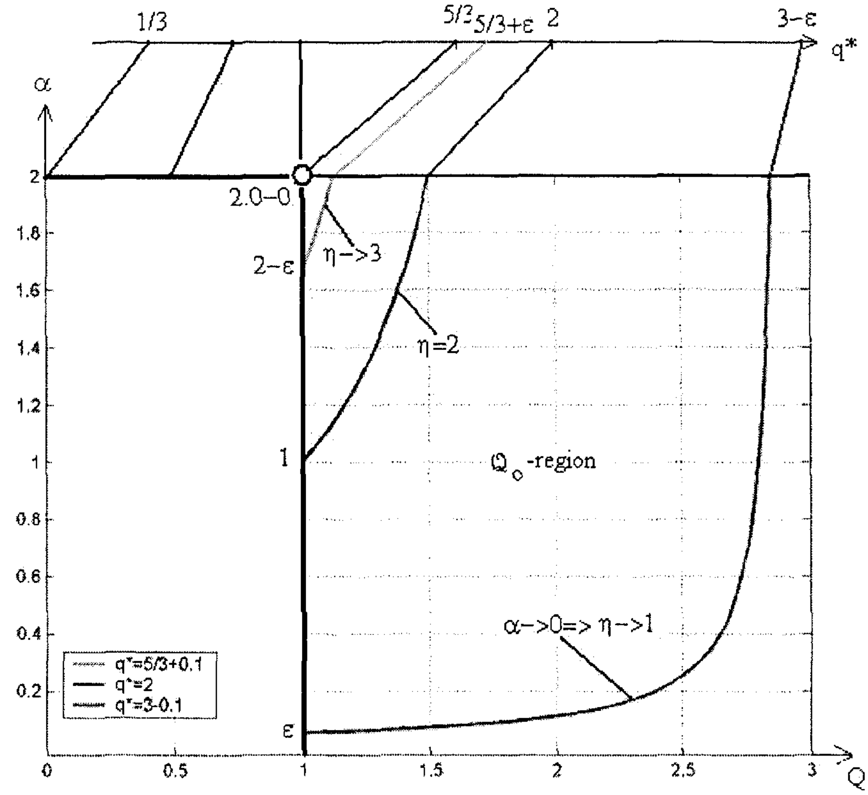

Figure 1: All pairs of (Q, α) on the indicated curves are associated with the same q_α^L-Gaussian. Two curves corresponding to two different values of q_α^L do not intersect. In this sense these curves represent the constant levels of q_α^L or $\eta = 2/(q_\alpha^L - 1)$. The line $\eta = 1$ joins the points $(Q, \alpha) = (1, 0.0 - 0)$ and $(3 - 0, 2)$; the line $\eta = 2$ joins the Cauchy distribution (noted C) with itself at $(Q, \alpha) = (1, 1)$ and at $(2, 2)$; the $\eta = 3$ line joins the points $(Q, \alpha) = (1, 2.0 - 0)$ and $(5/3, 2)$ (by ϵ we simply mean to give an indication, and not that both infinitesimals coincide). The entire line at $Q = 1$ and $0 < \alpha < 2$ is mapped into the line at $\alpha = 2$ and $5/3 \leq q^* < 3$.

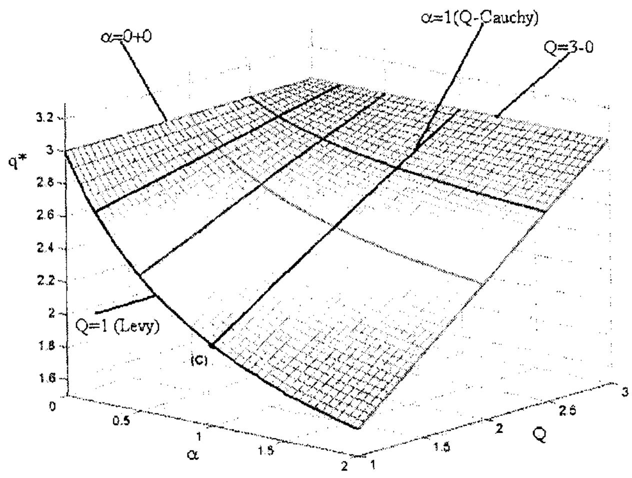

Figure 2: $q_\alpha^L \equiv q^*$ as function of (Q, α).

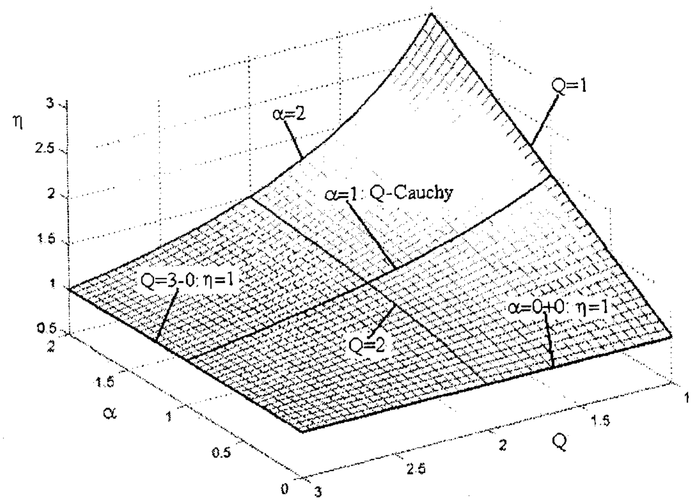

Figure 3: η as the function of (Q, α).

respectively (see Figs. 2 and 3).

The relationship between α and q_α^L for typical fixed values of Q are given in Fig. 4 (a). In this figure we can also see, that $\alpha = 1$ (Cauchy) corresponds to $q_1^L = 2$ (in the $Q = 1$ curve). In Fig. 4 (b) the relationships between Q ($Q = 2q - 1$) and q_α^L are represented for typical fixed values of α.

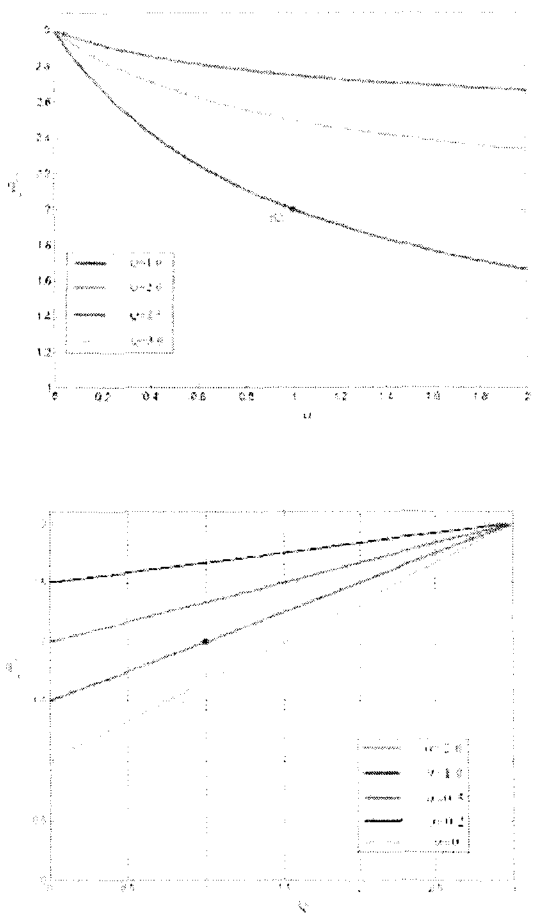

Figure 4: Constant Q and constant α sections of Fig. 2.

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q-GENERALIZATION OF SYMMETRIC α-STABLE DISTRIBUTIONS. PART II

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Abstract

The classic and the Lévy-Gnedenko central limit theorems play a key role in theory of probabilities, and also in Boltzmann-Gibbs (BG) statistical mechanics. They both concern the paradigmatic case of probabilistic independence of the random variables that are being summed. A generalization of the BG theory, usually referred to as nonextensive statistical mechanics and characterized by the index q ($q = 1$ recovers the BG theory), introduces global correlations between the random variables, and recovers independence for $q = 1$. The classic central limit theorem was recently q-generalized by some of us. In the present paper we q-generalize the Lévy-Gnedenko central limit theorem. In Part I we described the q-version of the α-stable Lévy distributions. In Part II we study the $(q^*, q, q_*)-$triplet, for which the mapping $F_{q^*} : \mathcal{G}_q \to \mathcal{G}_{q_*}$ holds. This fact allows to study the corresponding attractors and to obtain a complete generalization of the q-central limit theorem for random variables with infinite $(2q-1)$-variance.

1 Introduction

In the recent paper [1], and in Part I [2] of the current paper, we discussed a q-generalization of the classic central limit theorem applicable to the nonextensive statistical mechanics (see [3, 4, 5] and references therein), as well as a q-generalization of the Lévy-Gnedenko central limit theorem. One of the important aspects of the obtained generalization is that it works for globally correlated random variables. In [1] we introduced a new Fourier transform, i.e. the q-Fourier transform [1] and the function

$$z(s) = \frac{1+s}{3-s}$$

to describe attractors of scaling limits of sums of q-correlated random variables with a finite $(2q-1)$-variance [2]. This description was essentially based on the mapping

$$F_q : \mathcal{G}_q \to \mathcal{G}_{z(q)}, \tag{1}$$

[1]The q-Fourier transform formally is defined as $F_q[f](\xi) = \int_R f \otimes_q e_q^{ix\xi} dx$ and is a nonlinear operator if $q \neq 1$.

[2]We required there $q < 2$. Denoting $Q = 2q - 1$, it is easy to see that this condition is equivalent to the finitness of the Q-variance with $Q < 3$.

where $\mathcal{G}_q$ is the set of q-Gaussians.

In the current paper (consisting of two parts) we study a q-analog of the α-stable Lévy distributions. In this sense the present paper is a conceptual continuation of [1]. For simplicity we are dealing only with the standard symmetric one-dimensional case. Stable distributions with skewness and multivariate stable distributions can be studied in the same way applying the known classic techniques. Earlier studies devoted to the foundations of nonextensivity can be found in [5, 6, 7, 8, 9, 10, 11, 12], limit theorems in nonextensive statistical mechanics in [4, 13, 14, 15, 16], and various applications in [17, 18, 19, 20, 21, 22].

We study limits of sums of random variables with densities having the asymptotic behavior $f \sim C|x|^{-\frac{\alpha+1}{1+\alpha(q-1)}}$, $|x| \to \infty$, where $0 < \alpha < 2$, $0 < q < 2$, and $C > 0$. These random variables have infinite Q-variance ($Q = 2q-1$) for all $0 < \alpha < 2$ if $0 < Q < 3$ and for all $\alpha < 2/(1-Q)$, if $-1 < Q < 0$. The obtained limits represent a q-generalization of the α-stable Lévy distributions. In Part I we have given one of the possible classifications of these distributions in terms of their densities depending on the parameters Q and α, where $0 < \alpha < 2$. The classification was based on the mapping

$$F_q : \mathcal{G}_{q^L}[2] \to \mathcal{G}_q[\alpha], \tag{2}$$

where $\mathcal{G}_q[\alpha]$ is the set of all densities $\{be_q^{-\beta|\xi|^\alpha},\ b > 0,\ \beta > 0\}$ and

$$q^L = \frac{3+Q\alpha}{\alpha+1},\ Q = 2q-1.$$

Note that the values of parameters Q and α range in the set

$$\mathcal{Q}_0 = \{-1 < Q < 0,\ \alpha < \frac{2}{1-Q}\} \cup \{0 < Q < 3,\ 0 < \alpha < 2\},$$

i.e. the case $\alpha = 2$, in the framework of this description, is peculiar.

We note also that, for the q-generalization of the Lévy-Gnedenko central limit theorem, the following lemma proved in Part I plays an essential role.

Lemma 1.1 *Let $f(x)$, $x \in R$, be a symmetric probability density function of a given random vector. Further, let either*

(i) the $(2q-1)$-variance $\sigma_{2q-1}^2 < \infty$, (associated with $\alpha = 2$), or

(ii) $f(x) \sim C(|x|^{-\frac{\alpha+1}{1+\alpha(q-1)}})$, with $C > 0$ and $0 < \alpha < 2$, as $|x| \to \infty$.

Then for the q-Fourier transform of $f(x)$ the following asymptotic relation holds true:

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^\alpha + o(|\xi|^\alpha), \xi \to 0, \tag{3}$$

where

$$\mu_{q,\alpha} = \begin{cases} \frac{q}{2}\sigma_{2q-1}^2 \nu_{2q-1}, & \text{if } \alpha = 2; \\ \\ \frac{2^{2-\alpha}(1+\alpha(q-1))}{2-q} \int_0^\infty \frac{-\Psi_q(y)}{y^{\alpha+1}} dy, & \text{if } 0 < \alpha < 2, \end{cases} \tag{4}$$

where $\nu_{2q-1}(f) = \int_{-\infty}^{\infty} [f(x)]^{2q-1}\, dx$, and $\Psi_q(y) = \cos_q(2x) - 1$. [3]

[3] For the definition of q-cos see [1, 8].

In Part 2 we study a description of (q,α)-stable distributions differing from the description used in Part I. In the frame of the new description the value $\alpha = 2$ is no longer peculiar, but in this case we require $q \neq 1$.[4] More precisely, we expand the result of the paper [1] to the region

$$\mathcal{Q} = \{(Q,\alpha) : -1 < Q < 3, 0 < \alpha \leq 2\},$$

generalizing the mapping (1) in the form

$$F_{\zeta_\alpha(q)} : \mathcal{G}_q[\alpha] \to \mathcal{G}_{z_\alpha(q)}[\alpha],\ 0 < q < 2,\ 0 < \alpha \leq 2, \tag{5}$$

where

$$\zeta_\alpha(s) = \frac{\alpha - 2(1-q)}{\alpha} \text{ and } z_\alpha(s) = \frac{\alpha q + 1 - q}{\alpha + 1 - q}.$$

Note that if $\alpha = 2$, then $\zeta_2(q) = q$ and $z_2(q) = (1+q)/(3-q)$, recovering the mapping (1).

2 Description of the (q,α)-stable distributions based on the mapping (5)

In this section we establish the general theorem, representing a family of assertions depending on the type of correlations. Note that the particular case, namely $\alpha = 2$ was proved in [1], where the mapping $F_q : \mathcal{G}_q[2] \to \mathcal{G}_{z(q)}[2]$, $1/2 < q < 2$, was established. An importance of this mapping is its exactness in the sense that it transforms a q-Gaussian into a $z(q)$-Gaussian. For $\alpha < 2$, even in the classic case ($q = 1$), there is no such exact mapping. Asymptotic representation of classic α-stable Lévy distributions were obtained in [23, 24, 25, 26]. Our further arguments are also based on the asymptotic analysis.

Let $1 < q < 2$, or equivalently, $1 < Q < 3$, $Q = 2q - 1$. It follows from the definition of the q-exponential that any density function $g \in \mathcal{G}_q[\alpha]$ has the asymptotic behavior $g \sim b\,|x|^{\frac{-\alpha}{q-1}}$, $b > 0$, for large $|x|$. The set of all functions with this asymptotic we denote by $\mathcal{B}_q[\alpha]$. Obviously, $\mathcal{G}_q[\alpha] \subset \mathcal{B}_q[\alpha]$. At the same time, *for any density $f \in \mathcal{B}_q[\alpha]$, there exists a unique density $g \in \mathcal{G}_q[\alpha]$, such that $f \sim g$, $|x| \to \infty$*. In this sense the two sets $\mathcal{G}_q[\alpha]$ and $\mathcal{B}_q[\alpha]$ are asymptotically equivalent (or *asymptotically equal*). Having this in mind we write (preferably) $\mathcal{G}_q[\alpha]$ instead of $\mathcal{B}_q[\alpha]$.

Lemma 2.1 *Let $0 < \alpha \leq 2$ be fixed. For arbitrary q_1 there exists q_2 and a one-to-one mapping $\mathcal{M}_{q_1,q_2}$ such that*

$$\mathcal{M}_{q_1,q_2} : \mathcal{G}_{q_1}[\alpha] \to \mathcal{G}_{q_2}[2].$$

Obviously, if $\alpha = 2$, then $q_1 = q_2$ and $\mathcal{M}_{q_1,q_2}$ is the identical operator. First we find the relationship between the three indices q, q^* and q_* for which the mapping

$$F_{q^*} : \mathcal{G}_q[\alpha] \overset{(a)}{\to} \mathcal{G}_{q_*}[\alpha], \tag{6}$$

where $\overset{(a)}{\to}$ means that the mapping is in the sense of the asymptotic equivalence explained above, holds with $\alpha \in (0,2]$. The exact meaning of (6) is

$$\mathcal{M}^{-1}_{q_*,z(q_*)} F_{q^*} \mathcal{M}_{q,q^*} : \mathcal{G}_q[\alpha] \to \mathcal{G}_{q_*}[\alpha].$$

In the case $\alpha = 2$, as we mentioned above, $\mathcal{M}_{q_1,q_2} = I$, and the relationships $q^* = q$ and $q_* = \frac{1+q}{3-q}$ were found in [1], giving (1).

[4] $q = Q = 1$ leads to the exponential functions, unlike to $q \neq 1$, which is connected asymptotically with power law functions.

Lemma 2.2 *Assume $0 < \alpha \leq 2$ and that the numbers q^*, q_* and q are connected with the relationships*

$$q^* = \frac{\alpha - 2(1-q)}{\alpha} \text{ and } q_* = \frac{\alpha q + 1 - q}{\alpha + 1 - q.} \tag{7}$$

Then the mapping (6) holds true.

Proof. Let $f \in \mathcal{G}_q(\alpha)$, which means that asymptotically $f(x) \sim b\,|x|^{-\alpha/(q-1)}$, $x \to \infty$ with some $b > 0$. Find the q^*-Gaussian with the same asymptotics at ∞. For $G_{q^*}(\beta; x)$, $(\beta > 0)$, we have the asymptotic equality

$$G_{q^*}(\beta; x) \sim \frac{1}{|x|^{\frac{2}{q^*-1}}} \sim \frac{1}{|x|^{\frac{\alpha}{q-1}}}, \quad |x| \to \infty.$$

Hence

$$q^* = \frac{\alpha - 2(1-q)}{\alpha} = 1 - \frac{2(1-q)}{\alpha}.$$

Further, it follows from Corollary 2.10 of [1], that

$$F_{q^*}: \ \mathcal{G}_{q^*}(2) \to \mathcal{G}_{q_1}(2),$$

where

$$q_1 = \frac{1+q^*}{3-q^*} = \frac{\alpha - (1-q)}{\alpha + (1-q)}.$$

Taking into account the asymptotic equality

$$G_{q_1}(\beta_1; x) \sim \frac{1}{|x|^{\frac{2}{q_1-1}}} \sim \frac{1}{|x|^{\frac{\alpha}{q_*-1}}}, \quad |x| \to \infty,$$

we obtain

$$q_* = \frac{\alpha q + (1-q)}{\alpha + 1 - q} = 1 - \frac{\alpha(1-q)}{\alpha + 1 - q}.$$

Thus, the mapping (6) holds with q^* and q_* in Equation (7).

Let us introduce now two functions that are important for our further analysis:

$$z_\alpha(s) = \frac{\alpha s + (1-s)}{\alpha + 1 - s} = 1 - \frac{\alpha(1-s)}{\alpha + 1 - s}, \ 0 < \alpha \leq 2, \ s < \alpha + 1, \tag{8}$$

and

$$\zeta_\alpha(s) = \frac{\alpha - 2(1-s)}{\alpha} = 1 - \frac{2(1-s)}{\alpha}, \ 0 < \alpha \leq 2. \tag{9}$$

It can be easily verified that $\zeta_\alpha(s) = s$ if $\alpha = 2$.
The inverse, $z_\alpha^{-1}(t)$, $t \in (1-\alpha, \infty)$, of the the first function reads

$$z_\alpha^{-1}(t) = \frac{\alpha t - (1-t)}{\alpha - (1-t)} = 1 - \frac{\alpha(1-t)}{\alpha - 1 + t}. \tag{10}$$

The function $z(s)$ possess the properties: $z_\alpha(\frac{1}{z_\alpha(s)}) = \frac{1}{s}$ and $z_\alpha(\frac{1}{s}) = \frac{1}{z_\alpha^{-1}(s)}$. If we denote $q_{\alpha,1} = z_\alpha(q)$ and $q_{\alpha,-1} = z_\alpha^{-1}(q)$, then

$$z_\alpha(\frac{1}{q_{\alpha,1}}) = \frac{1}{q} \quad \text{and} \quad z_\alpha(\frac{1}{q}) = \frac{1}{q_{\alpha,-1}}. \tag{11}$$

Corollary 2.3 *The following mapping*

$$F_{\zeta_\alpha(q)} : \mathcal{G}_q(\alpha) \overset{(a)}{\to} \mathcal{G}_{z_\alpha(q)}(\alpha), \ q < 1+\alpha,$$

holds.

Corollary 2.4 *There exists the following inverse q-Fourier transform*

$$F^{-1}_{\zeta_\alpha(q)} : \mathcal{G}_{z_\alpha(q)}(\alpha) \overset{(a)}{\to} \mathcal{G}_q(\alpha), \ q < 1+\alpha,$$

Using the above mentioned properties of the function $z_\alpha(s)$ we can derive a number of useful formulas for the q-Fourier transforms. For instance, we get the mappings

$$F_{\zeta_\alpha(\frac{1}{q_{\alpha,1}})} : \ \mathcal{G}_{\frac{1}{q_{\alpha,1}}}(\alpha) \overset{(a)}{\to} \mathcal{G}_{\frac{1}{q}}(\alpha),$$

$$F_{\zeta_\alpha(\frac{1}{q})} : \ \mathcal{G}_{\frac{1}{q}}(\alpha) \overset{(a)}{\to} \mathcal{G}_{\frac{1}{q_{\alpha,-1}}}(\alpha).$$

The analogous formulas hold for the inverse q-Fourier transforms as well.

Introduce the sequence $q_{\alpha,n} = z_{\alpha,n}(q) = z(z_{\alpha,n-1}(q)), n = 1,2,...,$ with a given $q = z_0(q)$, $q < 1+\alpha$. We can extend the sequence $q_{\alpha,n}$ for negative integers $n = -1,-2,...$ as well putting $q_{\alpha,-n} = z_{\alpha,-n}(q) = z_\alpha^{-1}(z_{\alpha,1-n}(q)), n = 1,2,\ldots$. It is not hard to verify that

$$q_{\alpha,n} = 1 - \frac{\alpha(1-q)}{\alpha+n(1-q)} = \frac{\alpha q + n(1-q)}{\alpha + n(1-q)}, \ n = 0,\pm1,\pm2,... \tag{12}$$

$q > 1 + \frac{\alpha}{n}$ for $n < 0$, and $q < 1+\frac{\alpha}{n}$ for $n > 0$. (See Fig. 1). Note that $q_{\alpha,n}$ is a function only of $(q, n/\alpha)$, that $q_{\alpha,n} \equiv 1$ for all $n = 0,\pm1,\pm2,...$, if $q=1$, and that $\lim_{n\to\pm\infty} z_{\alpha,n}(q) = 1$ for all $q \neq 1$. Eq. (12) can be rewritten as follows:

$$\frac{\alpha}{1-q_{\alpha,n}} = \frac{\alpha}{1-q} + n\,, \ \ n = 0,\pm1,\pm2,... \tag{13}$$

This rewriting puts in evidence an interesting property. If we have a q-Gaussian in the variable $|x|^{\alpha/2}$ $(q \geq 1)$, i.e., a q-exponential in the variable $|x|^\alpha$ (whose asymptotic behavior is proportional to $|x|^{\frac{\alpha}{1-q}}$), its successive derivatives and integrations with regard to $|x|^\alpha$ precisely correspond to $q_{\alpha,n}$-exponentials in the same variable $|x|^\alpha$ (whose asymptotic behavior is proportional to $|x|^{\frac{\alpha}{1-q_{\alpha,n}}}$) [5]. Along a similar line, it is also interesting to remark that Eq. (13) coincides with Eq. (13) of [27] (once we identify the present α with the quantity z therein defined), therein obtained through a quite different approach (related to the renormalization of the index q emerging from summing a specific expression over one degree of freedom).

Let us note also that the definition of the sequence $q_{\alpha,n}$ (Eq. (12)) can be given through the series of mappings

[5] A typical illustration is as follows. Consider $\alpha = 1$, and a normalized q-exponential distribution $f(x)$ which identically vanishes for $x < 0$, and equals $A(q,\beta)\, e_q^{-\beta x}$ $(1 \leq q < 2,\ A(q,\beta) > 0,\ \beta > 0)$ for $x \geq 0$. The *accumulated probability* $F(\geq x) = \int_x^\infty f(x')\,dx'$ decreases from unity to zero when x increases from zero to infinity. This probability, frequently appearing in all kinds of applications, is given by $F(\geq x) = \frac{A}{(2-q)\beta}\,[1+(q-1)\beta x]^{-(2-q)/(q-1)}$, i.e., it is proportional to a $q_{1,1}$-exponential with $\frac{1}{1-q_{1,1}} = \frac{1}{1-q} + 1$.

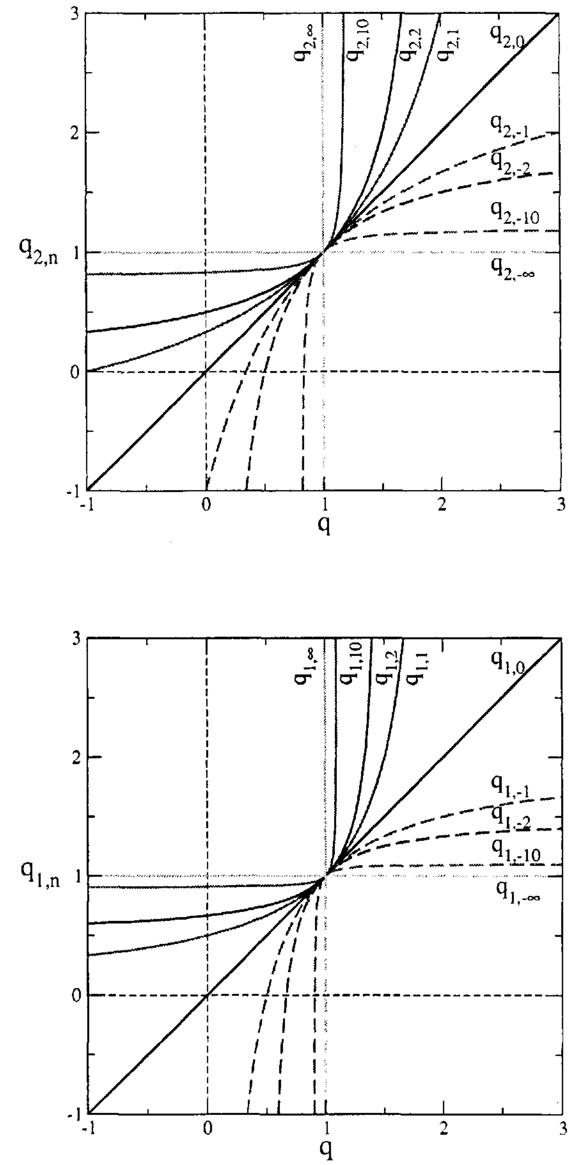

Figure 1: The q-dependences of $q_{2,n}$ (*top*) and $q_{1,n}$ (*bottom*) as given by Eq. (12). We notice the tendency of all the $|n| \to \infty$ curves to collapse onto the $q_{\alpha,n} = 1$ horizontal straight line if $q \neq 1$, and onto the $q = 1$ vertical straight line if $q = 1$. This tendency is gradually intensified, $\forall n$, when α is fixed onto values decreasing from 2 towards zero.

Definition 2.5

$$\dots \overset{z_\alpha}{\to} q_{\alpha,-2} \overset{z_\alpha}{\to} q_{\alpha,-1} \overset{z_\alpha}{\to} q_{\alpha,0} = q \overset{z_\alpha}{\to} q_{\alpha,1} \overset{z_\alpha}{\to} q_{\alpha,2} \overset{z_\alpha}{\to} \dots \tag{14}$$

$$\dots \overset{z_\alpha^{-1}}{\leftarrow} q_{\alpha,-2} \overset{z_\alpha^{-1}}{\leftarrow} q_{\alpha,-1} \overset{z_\alpha^{-1}}{\leftarrow} q_{\alpha,0} = q \overset{z_\alpha^{-1}}{\leftarrow} q_{\alpha,1} \overset{z_\alpha^{-1}}{\leftarrow} q_{\alpha,2} \overset{z_\alpha^{-1}}{\leftarrow} \dots \tag{15}$$

Further, let us introduce the sequence $q^*_{\alpha,n} = \zeta(q_{\alpha,n})$. It is easy to see that

$$q^*_{\alpha,n} = 1 - \frac{2(1-q)}{\alpha + n(1-q)} = \frac{\alpha + (n-2)(1-q)}{\alpha + n(1-q)}, \ n = 0, \pm 1, \dots \tag{16}$$

or, equivalently,

$$\frac{2}{1 - q^*_{\alpha,n}} = \frac{\alpha}{1-q} + n \,, \ n = 0, \pm 1, \dots \tag{17}$$

It follows from Lemma 2.2 and definitions of sequences $q_{\alpha,n}$ and $q^*_{\alpha,n}$ that

$$F_{q^*_{\alpha,n}} : \ \mathcal{G}_{q_{\alpha,n}}(\alpha) \to \mathcal{G}_{q_{\alpha,n+1}}, n = 0, \pm 1, \dots \tag{18}$$

Lemma 2.6 *For all $n = 0, \pm 1, \pm 2, \dots$ there holds the following relations*

$$q^*_{\alpha,n-1} + \frac{1}{q^*_{\alpha,n+1}} = 2, \tag{19}$$

$$q^*_{2,n} = q_{2,n}. \tag{20}$$

Proof. Notice that

$$\frac{1}{q^*_{\alpha,n+1}} = 1 + \frac{2(1-q)}{\alpha + (n-1)(1-q)},$$

which implies (19) immediately. The relation (20) can be checked easily.

Remark 2.7 *The property $q^*_{2,n} = q_{2,n}$ shows that the sequences (12) and (16) coincide if $\alpha = 2$. Hence, the mapping (18) takes the form $F_{q_{2,n}} : \ \mathcal{G}_{q_{2,n}}(2) \to \mathcal{G}_{q_{2,n+1}}(2)$, recovering Lemma 2.16 of [1]. Moreover, in this case the relation (19) holds for the sequence (12) as well. If $\alpha < 2$ [6], then the values of the sequence (16) are splitted from the values of $q_{\alpha,n}$. The shift can be measured as*

$$q_{\alpha,n} - q^*_{\alpha,n} = \frac{(2-\alpha)(1-q)}{\alpha + n(1-q)},$$

vanishing for $\alpha = 2 \,, \forall q$, and for $q = 1 \,, \forall \alpha$.

Define for $n = 0, \pm 1, \dots,\ k = 1, 2, \dots$ the operators

$$F^k_n(f) = F_{q^*_{\alpha,n+k-1}} \circ \dots \circ F_{q^*_{\alpha,n}}[f] = F_{q^*_{\alpha,n+k-1}}[\dots F_{q^*_{\alpha,n+1}}[F_{q^*_{\alpha,n}}[f]]\dots],$$

and

$$F^{-k}_n(f) = F^{-1}_{q^*_{\alpha,n-k}} \circ \dots \circ F^{-1}_{q^*_{\alpha,n-1}}[f] = F^{-1}_{q^*_{\alpha,n-k}}[\dots F^{-1}_{q^*_{\alpha,n-2}}[F^{-1}_{q^*_{\alpha,n-1}}[f]]\dots].$$

In addition, we assume that $F^k_q[f] = f$, if $k = 0$ for any appropriate q.

Summarizing the above mentioned relationships, we obtain the following assertions.

[6] As is known in the classic theory ($q = 1$) this case describes anomalous diffusion processes. If $q = 1$, then $q_{\alpha,n} \equiv q^*_{\alpha,n} \equiv 1$. In the nonextensive systems, as we can see from (18), there exist two separate sequences, which characterize the system under study. A physical confirmation of this theoretical result would be highly interesting.

Lemma 2.8 *The following mappings hold:*

1. $F_{q^*_{\alpha,n}} : \mathcal{G}_{q_{\alpha,n}}(\alpha) \stackrel{(a)}{\to} \mathcal{G}_{q_{\alpha,n+1}}(\alpha), n = 0, \pm 1, ...;$

2. $F^k_n : \mathcal{G}_{q_{\alpha,n}}(\alpha) \stackrel{(a)}{\to} \mathcal{G}_{q_{\alpha,k+n}}(\alpha), \;\; k = 1, 2, ..., \;\; n = 0, \pm 1, ...$

3. $\lim_{k\to\pm\infty} F^k_n \mathcal{G}_q(\alpha) = \mathcal{G}(\alpha), n = 0, \pm 1, ...,$

where $\mathcal{G}(\alpha)$ *is the set of classic* α*-stable Lévy densities.*

Lemma 2.9 *The series of mappings hold:*

$$... \stackrel{F_{q^*_{\alpha,-2}}}{\to} \mathcal{G}_{q_{\alpha,-1}}(\alpha) \stackrel{F_{q^*_{\alpha,-1}}}{\to} \mathcal{G}_q(\alpha) \stackrel{F_{q^*_{\alpha,0}}}{\to} \mathcal{G}_{q_{\alpha,1}}(\alpha) \stackrel{F_{q^*_{\alpha,1}}}{\to} \mathcal{G}_{q_{\alpha,2}}(\alpha) \stackrel{F_{q^*_{\alpha,2}}}{\to} ... \tag{21}$$

$$... \stackrel{F^{-1}_{q^*_{\alpha,-2}}}{\leftarrow} \mathcal{G}_{q_{\alpha,-1}}(\alpha) \stackrel{F^{-1}_{q^*_{\alpha,-1}}}{\leftarrow} \mathcal{G}_q(\alpha) \stackrel{F^{-1}_{q^*_{\alpha,0}}}{\leftarrow} \mathcal{G}_{q_{\alpha,1}}(\alpha) \stackrel{F^{-1}_{q^*_{\alpha,1}}}{\leftarrow} \mathcal{G}_{q_{\alpha,2}}(\alpha) \stackrel{F^{-1}_{q^*_{\alpha,2}}}{\leftarrow} ... \tag{22}$$

Theorem 1. *Assume* $0 < \alpha \le 2$ *and a sequence* $\{..., q_{\alpha,-2}, q_{\alpha,-1}, q_{\alpha,0}, q_{\alpha,1}, q_{\alpha,2}, ...\}$ *as given in (14) with* $q_0 = q \in (1/2, 2)$*. Let* $X_1, X_2, ..., X_N, ...$ *be symmetric random variables mutually* $q_{\alpha,k}$*-correlated for some* k *and all having the same probability density function* $f(x)$ *satisfying the conditions of Lemma 1.1.*

Then Z_N*, with* $D_N(q_{\alpha,k}) = (\mu_{q_{\alpha,k},\alpha} N)^{\frac{1}{\alpha(2-q_{\alpha,k})}}$*, is* $q_{\alpha,k}$*-convergent to a* $(q_{\alpha,k-1}, \alpha)$*-stable distribution, as* $N \to \infty$*.*

Proof. The case $\alpha = 2$ coincides with Theorem 1 of [1]. For $k = 0$, the first part of Theorem (q-convergence) is proved in Part I of the paper. The same method is applicable for $k \ne 1$. For reading convenience we reproduce the proof for arbitrary $k = 0, \pm 1, ...$

Assume $0 < \alpha < 2$. We evaluate $F_{q_{\alpha,k}}(Z_N)$. Denote $Y_j = D_N(q)^{-1}(X_j), j = 1, 2, ...$ Then $Z_N = Y_1 + ... + Y_N$. It is not hard to verify that, for a given random variable X and real $a > 0$, there holds $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$, for arbitrary q. It follows from that $F_{q_{\alpha,k}}(Y_1) = F_{q_{\alpha,k}}[f](\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{-\frac{1}{\alpha}}})$. Moreover, it follows from the $q_{\alpha,k}$-correlation of $Y_1, Y_2, ...$ (which is an obvious consequence of the $q_{\alpha,k}$-correlation of $X_1, X_2, ...$) and the associativity of the q-product that

$$F_{q_{\alpha,k}}[Z_N](\xi) = F_{q_{\alpha,k}}[f](\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{\frac{1}{\alpha}}}) \otimes_{q_{\alpha,k}} ... \otimes_{q_{\alpha,k}} F_{q_{\alpha,k}}[f](\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{\frac{1}{\alpha}}}) \text{ (}N\text{ factors)}. \tag{23}$$

Hence, making use of the properties of the q-logarithm, from (23) we obtain

$$\ln_{q_{\alpha,k}} F_{q_{\alpha,k}}[Z_N](\xi) = N \ln_{q_{\alpha,k}} F_{q_{\alpha,k}}[f](\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{\frac{1}{\alpha}}}) = N \ln_{q_{\alpha,k}}(1 - \frac{|\xi|^\alpha}{N} + o(\frac{|\xi|^\alpha}{N})) =$$
$$- |\xi|^\alpha + o(1), \; N \to \infty, \tag{24}$$

locally uniformly by ξ.

Consequently, locally uniformly by ξ,

$$\lim_{N\to\infty} F_{q_{\alpha,k}}(Z_N) = e_{q_{\alpha,k}}^{-|\xi|^\alpha} \in \mathcal{G}_{q_{\alpha,k}}(\alpha). \tag{25}$$

Thus, Z_N is $q_{\alpha,k}$-convergent.

To show the second part of Theorem we use Lemma 2.9. In accordance with this lemma there exists a density $f(x) \in \mathcal{G}_{q_{\alpha,k-1}}(\alpha)$ that $F_{q^*_{\alpha,k-1}}[f] = e_{q_{\alpha,k}}^{-|\xi|^\alpha}$. Hence, Z_N is $q_{\alpha,k}$-convergent to a $(q_{\alpha,k-1}, \alpha)$-stable distribution, as $N \to \infty$. Q.E.D.

3 Scaling rate analysis

In paper [1] we obtained the formula

$$\beta_k = \Big(\frac{3-q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}}\Big)^{\frac{1}{2-q_{k-1}}}. \tag{26}$$

for the Gaussian coefficient. It follows from this formula that the scaling rate in the case $\alpha = 2$ is

$$\delta = \frac{1}{2-q_{k-1}}, \tag{27}$$

where q_{k-1} is the q-index of the attractor. Moreover, if we insert the 'evolution parameter' t, then the translation of a q-Gaussian to a density in $\mathcal{G}_q(\alpha)$ changes t to $t^{2/\alpha}$. Hence, applying these two facts to the general case, $0 < \alpha \leq 2$, and taking into account that the attractor index in our case is $q^*_{\alpha,k-1}$, we obtain the formula for the scaling rate

$$\delta = \frac{2}{\alpha(2-q^*_{\alpha,k-1})}. \tag{28}$$

In accordance with Lemma 2.6 we have $2 - q^*_{\alpha,k-1} = 1/q^*_{\alpha,k+1}$. Consequently,

$$\delta = \frac{2}{\alpha}q^*_{\alpha,k+1} = \frac{2}{\alpha}\frac{\alpha + (k-1)(1-q)}{\alpha + (k+1)(1-q)}. \tag{29}$$

Finally, in terms of $Q = 2q - 1$ the formula (29) takes the form

$$\delta = \frac{2}{\alpha}\frac{2\alpha + (k-1)(1-Q)}{2\alpha + (k+1)(1-Q)}. \tag{30}$$

In the paper [1] we noticed that the non-linear Fokker-Planck equation corresponds to the case $k = 1$. Taking this fact into account we can conjecture that the fractional generalization of the nonlinear Fokker-Planck equation is linked with the scaling rate $\delta = 2/(\alpha+1-Q)$, which is derived from (30) putting $k = 1$. In the case $\alpha = 2$ we get the known result $\delta = 2/(3-Q)$ obtained in [14].

4 About additive and multiplicative dualities

In the nonextensive statistical mechanical literature, there are two transformations that appear quite frequently in various contexts. They are sometimes referred to as *dualities*. The *multiplicative duality* is defined through

$$\mu(q) = 1/q\,, \tag{31}$$

and the *additive duality* is defined through

$$\nu(q) = 2 - q\,. \tag{32}$$

They satisfy $\mu^2 = \nu^2 = \mathbf{1}$, where $\mathbf{1}$ represents the *identity*, i.e., $\mathbf{1}(q) = q, \forall q$. We also verify that

$$(\mu\nu)^m(\nu\mu)^m = (\nu\mu)^m(\mu\nu)^m = \mathbf{1} \quad (m = 0, 1, 2, ...)\,. \tag{33}$$

Consistently, we define $(\mu\nu)^{-m} \equiv (\nu\mu)^m$, and $(\nu\mu)^{-m} \equiv (\mu\nu)^m$.

Also, for $m = 0, \pm 1, \pm 2, ...,$ and $\forall q$,

$$(\mu\nu)^m(q) = \frac{m - (m-1)\,q}{m+1-m\,q} = \frac{q + m(1-q)}{1 + m(1-q)}\,, \tag{34}$$

$$\nu(\mu\nu)^m(q) = \frac{m + 2 - (m+1)\,q}{m+1-m\,q} = \frac{2 - q + m(1-q)}{1 + m(1-q)}\,, \tag{35}$$

and

$$(\mu\nu)^m\mu(q) = \frac{-m + 1 + m\,q}{-m + (m+1)\,q} = \frac{1 - m(1-q)}{q - m(1-q)}\,. \tag{36}$$

We can easily verify, from Eqs. (12) and (34), that the sequences $q_{2,n}$ $(n = 0, \pm 2, \pm 4, ...)$ and $q_{1,n}$ $(n = 0, \pm 1, \pm 2, ...)$ coincide with the sequence $(\mu\nu)^m(q)$ $(m = 0, \pm 1, \pm, 2, ...)$. This establishes a remarkable connection between sequences which emerge naturally within the context of the q-generalized central limit theorems, and the elementary dualities that we introduced above. However, its physical interpretation is yet to be found. It might be especially interesting if we take into account the fact that such a connection could be a crucial step (see footnote of page 15378 in [10]) for understanding the q-triplet that was observed by NASA using data received from the spacecraft Voyager 1 . Indeed, the existence of a q-triplet, namely $(q_{sen}, q_{rel}, q_{stat})$, (related respectively to sensitivity to the initial conditions, relaxation, and stationary state) was conjectured in [21], and was indeed observed in the solar wind at the distant heliosphere [22].

5 Conclusion

The q-CLT formulated in [1] says that an appropriately scaled limit of sums of q_k-correlated random variables with a finite $(2q_k - 1)$-variance is a q_k-Gaussian, which is the q_k^*-Fourier image of a q_k^*-Gaussian. Here q_k and q_k^* are sequences defined as

$$q_k = \frac{2q + k(1-q)}{2 + k(1-q)},\ k = 0, \pm 1, ...,$$

and

$$q_k^* = q_{k-1},\ k = 0, \pm 1, ...$$

Schematically this theorem can be described as (see Fig. 1 in [1])

$$\{f : \sigma_{2q_k - 1}(f) < \infty\} \xrightarrow{F_{q_k}} \mathcal{G}_{q_k}(2) \xleftarrow{F_{q_k^*}} \mathcal{G}_{q_k^*}(2) \tag{37}$$

where $\mathcal{G}_q(2)$ is the set of all q-Gaussians. We have noted that the processes described by the q-CLT can be effectively described by the triplet $(P_{att}, P_{cor}, P_{scl})$, where P_{att}, P_{cor} and P_{scl} are parameters of *attractor*, *correlation* and *scaling rate*, respectively. We found that

$$(P_{att}, P_{cor}, P_{scl}) \equiv (q_{k-1}, q_k, q_{k+1}). \tag{38}$$

In Part 1 of this paper we have studied a q-generalization of symmetric α-stable Lévy distributions. Schematically the corresponding theorem (Theorem 1 of [2]) is described as

$$\mathcal{L}(q,\alpha) \xrightarrow{F_q} \mathcal{G}_q(\alpha) \xleftarrow{F_q} \mathcal{G}_{q^L}(2),\ 0 < \alpha < 2, \tag{39}$$

where $\mathcal{L}(q,\alpha)$ is the set of (q,α)-stable distributions, $\mathcal{G}_{q^L}(2)$ is the set of q^L-Gaussians asymptotically equivalent to the densities $f \in \mathcal{L}(q,\alpha)$. The index q^L is linked with q as follows

$$q^L = q_\alpha^L(q) = \frac{3+(2q-1)\alpha}{1+\alpha}.$$

Note that the case $\alpha = 2$ is peculiar and we agree to refer to the scheme (37) in this case.

In the present paper (Part 2, Theorem 1) we have studied a q-generalization of the CLT to the case when the $(2q-1)-$variance of random variables is infinite. The theorem that we have obtained generalizes the q-CLT, which corresponds to $\alpha = 2$, to the full range $0 < \alpha \le 2$. Schematically this theorem can be described as

$$\mathcal{L}(q_{\alpha,k},\alpha) \xrightarrow{F_{q_{\alpha,k}}} \mathcal{G}_{q_{\alpha,k}}(\alpha) \xleftarrow{F_{q^*_{\alpha,k}}} \mathcal{G}_{q^*_{\alpha,k}}(2),\ 0<\alpha\le 2, \tag{40}$$

having the same meaning as the scheme (37). The sequences $q_{\alpha,k}$ and $q^*_{\alpha,k}$ in this case read

$$q_{\alpha,k} = \frac{\alpha q + k(1-q)}{\alpha + k(1-q)},\ k = 0, \pm 1, ...,$$

and

$$q^*_{\alpha,k} = 1 - \frac{2(1-q)}{\alpha + k(1-q)},\ k = 0, \pm 1, ...$$

Note that the triplet $(P_{att}, P_{cor}, P_{scl})$ mentioned above takes, in this case, the form

$$(P_{att}, P_{cor}, P_{scl}) \equiv (q^*_{\alpha,k-1},\ q_{\alpha,k},\ (2/\alpha)q^*_{\alpha,k+1}),$$

which coincides with (38) if $\alpha = 2$.

Finally, unifying the schemes (39) and (40) we obtain the general picture for the description of stable distributions:

$$\begin{array}{ccc} \mathcal{L}(q_{\alpha,k},\alpha) & \xrightarrow{F_{q_{\alpha,k}}} \mathcal{G}_{q_{\alpha,k}}(\alpha) \xleftrightarrow{F_{q^*_{\alpha,k}}} \mathcal{G}_{q^*_{\alpha,k}}(2) \\ \updownarrow F_q & \\ & \mathcal{G}_{q^L_{\alpha,k}}(2), \end{array} \tag{41}$$

where

$$q^L_{\alpha,k} = q^L_\alpha(q_{\alpha,k}) = \frac{3+(2q_{\alpha,k}-1)\alpha}{1+\alpha}.$$

In Fig. 2 the connection of $(Q,\alpha) \in \mathcal{Q}$ with q^L and q^*, $(k=0)$ is represented. If $Q=1$ and $\alpha = 2$ (the blue box in figure), then the random variables are independent in the usual sense and have *finite* variance. The standard CLT applies, and the attractors are classic Gaussians.

If Q belongs to the interval $(-1,3)$ and $\alpha = 2$ (the blue straight line on the top), the random variables are *not* independent. If the random variables have a *finite* Q-variance, then the paper [1] applies, and the attractors belong to the family of the q^*-Gaussians. The support of attractors is compact if $Q<1$ and infinite if $Q>1$. Note that q^* runs in $(-1,1)$ and in $(1,5/3)$ if $-1<Q<1$ and $1<Q<3$, respectively. Thus, in this case, attractors (q^*-Gaussians) have *finite* classic variance (i.e., 1-variance) in addition to *finite* q^*-variance.

If Q $=1$ and $0<\alpha<2$ (the vertical green line in the figure), we have the classic Lévy distributions), hence the random variables are independent, and have *infinite* variance. The classic

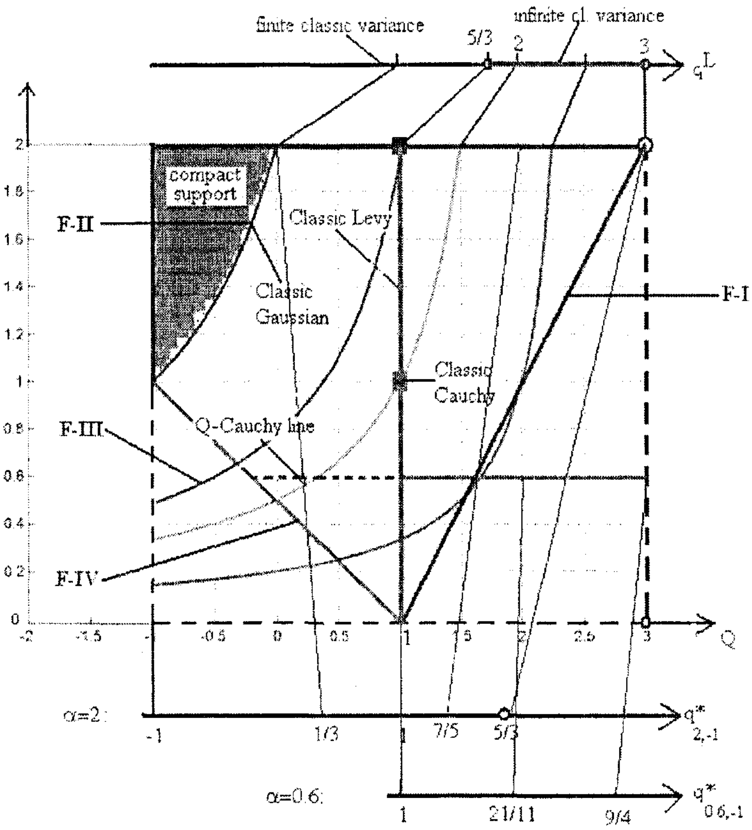

Figure 2: (Q, α)-regions (see the text).

Lévy-Gnedenko CLT applies, and the attractors belong to the family of the α-stable Lévy distributions.

If $0 < \alpha < 2$, and Q belong to the interval $(-1, 3)$ we observe the rich variety of situations that we have described. Let us first consider $1 < Q < 3$. Then the random variables are *not* independent, have *infinite* variance and *infinite* Q*-variance*. The rectangle $\{1 < Q < 3;\ 0 < \alpha < 2\}$, at the right of the classic Lévy line, is covered by non-intersecting curves

$$C_{q^L} \equiv \{(Q,\alpha) : \frac{3+Q\alpha}{\alpha+1} = q^L\},\ 5/3 < q^L < 3\,.$$

Consistently with [1], these families of curves describe all (Q, α)-stable distributions based on the mapping (39) with q-Fourier transform. The constant q^L is the index of the q^L-Gaussian attractor corresponding to the points (Q, α) on the curve C_{q^L}. For example, the green curve corresponding to $q^L = 2$ describes all Q-Cauchy distributions, recovering the classic Cauchy-Poisson distribution if $\alpha = 1$ (the green box in the figure). Every point (Q, α) laying on the brown curve corresponds to $q^L = 2.5$.

At the same time, we have there another description of the (Q, α)-stable distributions, which is based on the mapping (40) with q^*-Fourier transform. Again the rectangle $\{1 < Q < 3;\ 0 < \alpha < 2\}$ is covered by (straight) lines

$$S_{q^*} \equiv \{(Q,\alpha) : \frac{4\alpha}{Q+2\alpha-1} = 3 - q^*\}, \tag{42}$$

which are obtained from (16) replacing $n = -1$ and $2q - 1 = Q$. For instance, every (Q, α) on the line F-I (the blue diagonal of the rectangle in the figure) identifies q^*-Gaussians with $q^* = 5/3$. This line is the frontier of points (Q, α) with finite and infinite classic variances. Namely, all (Q, α) above the line F-I identify attractors with *finite* variance, and points on this line and below identify attractors with *infinite* classic variance. Two bottom lines in Fig. 2 reflect the sets of q^* corresponding to lines $\{(-1 < Q < 3; \alpha = 2)\}$ (the top boundary of the rectangle in the figure) and $\{(1 < Q < 3; \alpha = 0.6)\}$ (the brown horizontal line in the figure).

Finally, we describe the rectangle $\{(Q, \alpha) : -1 < Q < 1; 0 < \alpha < 2)\}$, at the left of the classic Lévy line, analizing the formulas by continuity. In this region we see three frontier lines, F-II, F-III and F-IV. The line F-II splits the regions where the random variables have *finite* and *infinite* Q-variances. More precisely, the random variables corresponding to (Q, α) on and above the line F-II have a *finite* Q-variance, and, consequently, the paper [1] applies. Moreover, as seen in the figure, the q^L-attractors corresponding to the points on the line F-II are the classic Gaussians, because $q^L = 1$ for these (Q, α). It follows from this fact, that q^L-Gaussians corresponding to points above F-II have compact support (the blue region in the figure), and q^L-Gaussians coresponding to points on this line and below have infinite support. The line F-III splits the points (Q, α) whose q^L-attractors have *finite* or *infinite* classic variances. More precisely, the points (Q, α) above this line identify attractors (in terms of q^L-Gaussians) with *finite* classic variance, and the points on this line and below identify atttractors with *infinite* clasic variance. The frontier line F-IV with the equation $Q + 2\alpha - 1 = 0$ and joining the points $(1, 0)$ and $(-1, 1)$ is related to attractors in terms of q^*-Gaussians. It follows from (42) that for (Q, α) laying on the line F-IV, the index $q^* = -\infty$. Thus the horizontal lines corresponding to $\alpha < 1$ can be continued only up to the line F-IV with $q^* \in (-\infty, 3 - \frac{4\alpha}{Q+2\alpha-1})$ (see the dashed horizontal brown line in the figure). If $\alpha \to 0$, the Q-interval becomes narrower, but q^*-interval becomes larger tending to $(-\infty, 3)$.

Let us stress that Fig. 2 corresponds to the case $k = 0$ in the description (41). The cases $k \neq 0$ can be analyzed in the same way. We also note that the method we used for the q-generalization of the classic and the Lévy-Gnedenko CLTs is applicable for $0 < q < 2$ (or equivalently $-1 < Q < 3$). So, the description of (Q, α)-stable distributions corresponding to the region $Q \leq -1$ (the white rectangle and its left side in the figure) remains open at this point.

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