SCHOOL and CONFERENCE  
on  
COMPLEX SYSTEMS  
and  
NONEXTENSIVE STATISTICAL MECHANICS  

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Nonextensive Statistical Mechanics:  
Theoretical, Experimental, Observational and  
Computational Aspects

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NONEXTENSIVE STATISTICAL MECHANICS: THEORETICAL, EXPERIMENTAL, OBSERVATIONAL AND COMPUTATIONAL ASPECTS

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S. Umarov, C.T., M. Gell-Mann and S. Steinberg, cond-mat/0603593, 0606038, 0606040

Trieste, July 2006
NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS

Nonextensive Statistical Mechanics and Thermodynamics
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Brazilian Journal of Physics 29, Number 1 (1999)

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P Grigolini, C Tsallis and BJ West, eds

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M. Sugiyama, ed

Anomalous Distributions, Nonlinear Dynamics, and Nonextensivity
HL Swinney and C Tsallis, eds

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G Kaniadakis and M Lissia, eds

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H Herrmann, MBarbosa and E Curado, eds

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C Beck, GBenedek, ARapisarda and C Tsallis, eds
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G Kaniadakis, A Carbone and M. Lissia, eds
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(Europhysics News, Nov-Dec 2005, European Physical Society)

http://www.europhysicsnews.com

Full bibliography
(29 July 2006: 1935 manuscripts)

http://tsallis.cat.cbpf.br/biblio.htm
UBIQUITOUS LAWS IN COMPLEX SYSTEMS

ORDINARY DIFFERENTIAL EQUATIONS

FURTHER APPLICATIONS
(Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)

ENTROPY $S_q$
(Nonextensive statistical mechanics)

PARTIAL DIFFERENTIAL EQUATIONS
(Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)

STOCHASTIC DIFFERENTIAL EQUATIONS
(Langevin, multiplicative noise)

CENTRAL LIMIT THEOREMS
(Gauss, Levy-Gnedenko)

STOCHASTIC DIFFERENTIAL EQUATIONS
(Langevin, multiplicative noise)

NONLINEAR DYNAMICS
(Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)

q-TRIPLET

q-ALGEBRA

CORRELATIONS IN PHASE SPACE

GEOMETRY
(Scale-free networks)

SUPERSTATISTICS
(Other generalizations)

AGING (metastability)

LONG-RANGE INTERACTIONS
(Hamiltonians, coupled maps)

SIGNAL PROCESSING
(ARCH, GARCH)

IMAGE PROCESSING

GLOBAL OPTIMIZATION
(Simulated annealing)

THERMODYNAMICS

GLOBAL OPTIMIZATION
(Simulated annealing)

SIGNAL PROCESSING
(ARCH, GARCH)
It is the natural (or artificial or social) system itself which, through its geometrical-dynamical properties, determines the specific informational tool --- entropy --- to be meaningfully used for the study of its (thermo) statistical properties.

\[ d_f = \frac{\ln 2}{\ln 3} = 0.6309... \]

Hence the interesting measure is given by

\[ (10 \, cm)^{0.6309...} \approx 4.275 \, cm^{0.6309} \]
### Entropic Forms

<table>
<thead>
<tr>
<th>BG Entropy ( (q = 1) )</th>
<th>Nonextensive Entropy ( (q = \mathcal{R}) ) ( (q \neq 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i = \frac{1}{W} ) ( (\forall i) ) equi-probability</td>
<td>( k \ln W )</td>
</tr>
<tr>
<td>( \forall p_i \ (0 \leq p_i \leq 1) ) ( \left( \sum_{i=1}^{w} p_i = 1 \right) )</td>
<td>( -k \sum_{i=1}^{w} p_i \ln p_i )</td>
</tr>
</tbody>
</table>

Possible generalization of Boltzmann-Gibbs statistical mechanics

Ludwig BOLTZMANN

Vorlesungen uber Gastheorie (Leipzig, 1896)
Lectures on Gas Theory, transl. S. Brush

The forces that two molecules impose one onto the other during an interaction can be completely arbitrary, only assuming that their sphere of action is very small compared to their mean free path.
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite value, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.
The situation is different for the additivity postulate $Pa2$, the validity of which cannot be inferred from general principles. We have to require that the interaction energy between thermodynamic systems be negligible. This assumption is closely related to the homogeneity postulate $Pd1$. From the molecular point of view, additivity and homogeneity can be expected to be reasonable approximations for systems containing many particles, provided that the intramolecular forces have a short range character.
The presence of long-range forces causes important amendments to thermodynamics, some of which are not fully investigated as yet.

Is equilibrium always an entropy maximum?

[...] in the case of systems with long-range forces and which are therefore nonextensive (in some sense) some thermodynamic results do not hold. [...] The failure of some thermodynamic results, normally taken to be standard for black hole and other nonextensive systems has recently been discussed. [...] If two identical black holes are merged, the presence of long-range forces in the form of gravity leads to a more complicated situation, and the entropy is nonextensive.
This means that the total energy of any finite collection of self-gravitating mass points does not have a finite, extensive (e.g., proportional to the number of particles) lower bound. Without such a property there can be no rigorous basis for the statistical mechanics of such a system (Fisher and Ruelle 1966). Basically it is that simple. One can ignore the fact that one knows that there is no rigorous basis for one's computer manipulations; one can try to improve the situation, or one can look for another job.
The formalism of equilibrium statistical mechanics -- which we shall call thermodynamic formalism -- has been developed since G.W. Gibbs to describe the properties of certain physical systems. [...] While the physical justification of the thermodynamic formalism remains quite insufficient, this formalism has proved remarkably successful at explaining facts. The mathematical investigation of the thermodynamic formalism is in fact not completed: the theory is a young one, with emphasis still more on imagination than on technical difficulties. This situation is reminiscent of pre-classic art forms, where inspiration has not been castrated by the necessity to conform to standard technical patterns.

The problem of why the Gibbs ensemble describes thermal equilibrium (at least for “large systems”) when the above physical identifications have been made is deep and incompletely clarified.

[The first equation is dedicated to define the BG entropy form. It is introduced after the words “we define its entropy” without any kind of justification or physical motivation.]
The values of $p_i$ are determined by the following dogma:

if the energy of the system in the $i$-th state is $E_i$ and if the temperature of the system is $T$ then:

$$p_i = \frac{e^{-E_i/kT}}{Z(T)}$$

where $Z(T) = \sum_i e^{-E_i/kT}$

(this last constant is taken so that $\sum_i p_i = 1$).

This choice of $p_i$ is called the Gibbs distribution. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as "deep and incompletely clarified".
ABOUT ORDINARY DIFFERENTIAL EQUATIONS:

(i) \( \frac{dy}{dx} = 0 \) with \( y(0) = 1 \)
\[ \Rightarrow y = 1 \]
Inverse function: \( x = 1 \)

(ii) \( \frac{dy}{dx} = 1 \) with \( y(0) = 1 \)
\[ \Rightarrow y = 1 + x \]
Inverse function: \( x = y - 1 \)

(iii) \( \frac{dy}{dx} = y \) with \( y(0) = 1 \)
\[ \Rightarrow y = e^x \text{ (EXPONENTIAL)} \]
Inverse function: \( x = \ln y \)
Property: \( \ln(y_A y_B) = \ln y_A + \ln y_B \)
ABOUT ORDINARY DIFFERENTIAL EQUATIONS:

(i) \[ \frac{dy}{dx} = 0 \quad \text{with} \quad y(0) = 1 \]
\[ \Rightarrow y = 1 \]
Inverse function: \( x = 1 \)

(in the sense of \( x = y \) symmetry)

(ii) \[ \frac{dy}{dx} = 1 \quad \text{with} \quad y(0) = 1 \]
\[ \Rightarrow y = 1 + x \]
Inverse function: \( x = y - 1 \)

(iii) \[ \frac{dy}{dx} = y \quad \text{with} \quad y(0) = 1 \]
\[ \Rightarrow y = e^x \quad \text{(EXPONENTIAL)} \]
Inverse function: \( x = \ln y \)
Property: \( \ln(y_A y_B) = \ln y_A + \ln y_B \)

(iv) Unification:
\[ \frac{dy}{dx} = y^q \quad (q \in \mathbb{R}) \quad \text{with} \quad y(0) = 1 \]
\[ \Rightarrow y = [1 + (1-q)x]^{1-q} = e_q^x \quad \text{(POWER-LAW)} \]
Inverse function: \( x = \frac{y^{1-q-1}}{1-q} \equiv \ln_q y \)
Property: \( \ln_q(y_A y_B) = \ln_q y_A + \ln_q y_B + (1-q)(\ln_q y_A)(\ln_q y_B) \)
\[ [q = 1, \quad q = 0 \quad \text{and} \quad q \to \infty \quad \text{recover the three previous cases}] \]
ABOUT MEAN VALUES:

(i) **Equiprobability**: \( p_i = \frac{1}{W} \) \((\forall i)\)

\[ S_{BG} = k \ln W \quad (k = \text{positive constant}) \]

Naturally generalized into:

\[ S_q = k \ln_q W \]

(ii) **General**: (not necessarily equiprobability)

\[ S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i} \equiv \langle k \ln \frac{1}{p_i} \rangle \]

"surprise" or "unexpectedness"

[We verify that \( p_i = \frac{1}{W} \) \((\forall i)\) recovers \( S_{BG} = k \ln W \) ]

Naturally generalized into:

\[ S_q(\{p_i\}) \equiv \langle k \ln_q \frac{1}{p_i} \rangle = k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} \]

"q-surprise" or "q-unexpectedness"

hence:

\[ S_q(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \]

[We verify that \( p_i = \frac{1}{W} \) \((\forall i)\) recovers \( S_q = k \ln_q W \) ]

Property:

\[ p_{ij}^{A+B} = p_i^A p_j^B \Rightarrow \]

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \]
ABOUT BIAS:

\[ 0 < p_i < 1 \quad (\forall i) \]

\[ p_i^q > p_i \quad if \quad q < 1 \]

\[ < p_i \quad if \quad q > 1 \]

\[ = p_i \quad if \quad q = 1 \quad (BG) \]

(i) \( S_q = \{p_i\} \) should be invariant under permutation.
The simplest manner is to be
\[ S_q(\{p_i\}) = f(\sum_{i=1}^{W} p_i^q) \]

(ii) The simplest function \( f(x) \) is
\[ S_q(\{p_i\}) = a + b\sum_{i=1}^{W} p_i^q \]

(iii) Certainty must correspond to \( S_q = 0 \)
In other words \( p_i = 1 \quad for \quad i = i_0 \)
\[ = 0 \quad otherwise \]

hence \( a + b = 0 \)

hence \( S_q(\{p_i\}) = a(1 - \sum_{i=1}^{W} p_i^q) \)

(iv) For \( q \to 1 \) we must recover \( S_{BG}(\{p_i\}) \).
Using \( p_i^{q-1} \sim 1 + (1 - q) \ln p_i \) we obtain
\[ S_q(\{p_i\}) \sim -a(q - 1) \sum_{i=1}^{W} p_i \ln p_i \]

consequently, by identifying \( a(q - 1) = k \) we obtain
\[ S_q(\{p_i\}) = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \]
ABOUT REACTION UNDER BIAS:

[See Abe, Phys. Lett. A 224, 326 (1997)]

(i) Testing the function under **translation** of the bias $x$:

$$S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = -k \left[ \frac{d}{dx} \sum_{i=1}^{W} p_i^x \right]_{x=1}$$

(ii) Testing the function under **dilatation** of the bias $x$:

We replace $\frac{d}{dx}$ by Jackson’s 1909 generalized derivative

$$D_q h(x) \equiv \frac{h(qx) - h(x)}{qx-x} \quad [D_1 h(x) = \frac{dh(x)}{dx}]$$

and obtain

$$S_q(\{p_i\}) = -k \left[ D_q \sum_{i=1}^{W} p_i^x \right]_{x=1}$$

hence

$$S_q(\{p_i\}) = k \left( 1 - \sum_{i=1}^{W} p_i^q \right)$$
\[
\frac{dy}{dx} = -a_q y^q \quad \text{with} \quad y(0) = 1
\]

\[
\Rightarrow y = \frac{1}{\left[1 + (q-1)a_q x\right]^{q-1}} = e_{a_q}^{-a_q x}
\]
CONCAVITY:

Let us assume two arbitrary and different probability sets, namely \(\{p_i\}\) and \(\{p'_i\}\), associated with a single system having \(W\) states. We define an intermediate probability set as follows:

\[
p''_i = \lambda p_i + (1 - \lambda)p'_i \quad (\forall i; 0 < \lambda < 1)
\]

then

\[
S_q(\{p''_i\}) > \lambda S_q(\{p_i\}) + (1 - \lambda) S_q(\{p'_i\}) \quad (q > 0)
\]
STABILITY
(or CONTINUITY or EXPERIMENTAL ROBUSTNESS)


The entropy $S$ is said **stable** iff, for any given $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that, independently from $W$,

$$\sum_{i=1}^{W} |p_i - p_i'| \leq \delta_\varepsilon \quad \Rightarrow \quad \left| \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} \right| < \varepsilon$$

Hence

$$\lim_{\delta \to 0} \lim_{W \to \infty} \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} = 0$$

$S_{BG}$ and $S_q \ (\forall q > 0)$ are **stable**


$$S_q^R(\{p_i\}) \equiv \frac{\ln \sum_{i=1}^{W} p_i^q}{q-1} \quad \text{(Renyi entropy)}$$

$$S_q^N(\{p_i\}) \equiv \frac{S_q(\{p_i\})}{\sum_{i=1}^{W} p_i^q} \quad \text{(Normalized entropy)}$$

$$S_q^E(\{p_i\}) \equiv \frac{1 - \left(\sum_{i=1}^{W} p_i^{1/q}\right)^{-q}}{q-1} \quad \text{(Escort entropy)}$$

are **unstable**

B. Lesche (1982); S. Abe (2002); C.T. and E. Brigatti (2003)
\( q = 0.5 \) (QC)

\( q = 2 \) (QEP)

\( q = 0.5 \) (QEP)

\( q = 2 \) (QC)

QEP = quasi equal probabilities
QC  = quasi certainty

**SANTOS THEOREM:** RJV Santos, J Math Phys 38, 4104 (1997)

(q - generalization of Shannon 1948 theorem)

**IF**  
$S(\{p_i\})$ continuous function of $\{p_i\}$

**AND**  
$S(p_i = 1/W, \forall i)$ monotonically increases with $W$

**AND**  
$$\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B)}{k} + (1-q) \frac{S(A) S(B)}{k} \quad \text{(with } p_{ij}^{A+B} = p_i^A p_j^B \text{)}$$

**AND**  
$$S(\{p_i\}) = S(p_L, p_M) + p_L^q S(\{p_i / p_L\}) + p_M^q S(\{p_m / p_M\}) \quad \text{(with } p_L + p_M = 1 \text{)}$$

**THEN AND ONLY THEN**

$$S(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad \left( q = 1 \Rightarrow S(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i \right)$$

CE SHANNON (The Mathematical Theory of Communication):

"This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications."
**ABE THEOREM:**  S Abe, Phys Lett A 271, 74 (2000)

(q - generalization of Khinchin 1953 theorem)

*IF*  \( S(\{p_i\}) \) *continuous function of* \( \{p_i\} \)

*AND*  \( S(p_i = 1/W, \forall i) \) *monotonically increases with* \( W \)

*AND*  \( S(p_1, p_2, ..., p_W, 0) = S(p_1, p_2, ..., p_W) \)

*AND*  \( \frac{S(A + B)}{k} = \frac{S(A)}{k} + \frac{S(B \mid A)}{k} + (1-q) \frac{S(A)}{k} \frac{S(B \mid A)}{k} \)

*THEN AND ONLY THEN*

\[
S(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad \left( q = 1 \Rightarrow S(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i \right)
\]

The possibility of such theorem was conjectured by AR Plastino and A Plastino (1996, 1999).
$S_q(N,t)$ versus $t$
**LOGISTIC MAP:**

\[ x_{t+1} = 1 - a \, x_t^2 \quad (0 \leq a \leq 2; \quad -1 \leq x_t \leq 1; \quad t = 0, 1, 2, \ldots) \]

(strong chaos, i.e., **positive Lyapunov exponent**)

We verify

\[ K_1 = \lambda_1 \quad (\text{Pesin–like identity}) \]

where

\[ K_1 \equiv \lim_{t \to \infty} \frac{S_1(t)}{t} \]

and

\[ \xi(t) \equiv \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1 \ t} \]
\[ x_{t+1} = 1 - a \, x_t^2 \]
\[ a = 1.4011552 \]
\[ N = W = 2.5 \times 10^6 \]
\[ \# \text{ realizations} = 15115 \]

(weak chaos, i.e., zero Lyapunov exponent)

E. Mayoral and A. Robledo, Phys. Rev. E 72, 026209 (2005), and references therein
We verify

\[ K_q = \lambda_q \quad (q - \text{generalized Pesin-like identity}) \]

where

\[ K_q \equiv \lim_{t \to \infty} \sup \left\{ \frac{S_q(t)}{t} \right\} \]

and

\[ \xi(t) \equiv \sup \left\{ \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} \right\} = e^{\lambda_q t} \]

with

\[ \frac{1}{1 - q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}} = \frac{\ln \alpha_F}{\ln 2} \quad \text{and} \quad \lambda_q = \frac{1}{1 - q} \]

\[ x_{t+1} = 1 - a |x_t|^z \quad \Rightarrow \quad \frac{1}{1 - q(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = (z - 1) \frac{\ln \alpha_F(z)}{\ln 2} \]
THE CASATI-PROSEN TRIANGLE MAP:
G. Casati and T. Prosen,

“While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary.”

\[ y_{t+1} = y_t + \alpha \text{ sgn}(x_t) + \beta \pmod{2} \]
\[ x_{t+1} = x_t + y_{t+1} \pmod{2} \]
(\(\alpha\) and \(\beta\) independent irrationals)

\text{e.g., } (\alpha, \beta) = \left( \frac{1}{2}(\sqrt{5} - 1) - (1/e), \frac{1}{2}(\sqrt{5} - 1) + (1/e) \right)

This map is conservative, mixing, ergodic and nevertheless with zero Lyapunov exponent!

Furthermore \( \xi \equiv \lim_{\Delta X(0) \to 0} \frac{\Delta X(t)}{\Delta X(0)} \propto t \)
CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett 83, 4729 (1999) and 85, 4261 (2000)]
(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)

NONEXTENSIVITY OF THE CASATI-PROSEN MAP:

Answer to the above equation:  


It is not necessary: a meaningful statistical description is possible with zero Lyapunov exponent!

[Essentially because an integrable system has zero Lyapunov exponent but the opposite is not true]

In general, \( \xi = [1 + (1-q)\lambda q t]^{1/(1-q)} \)

hence, \( \xi \propto t \Rightarrow q = 0 \)

Consistently, we expect

\[
1 - \sum_{i=1}^{W} [p_i(t)]^q \propto t \quad \text{only for } q = 0
\]

\((i)\) \( S_q(t) \equiv \frac{1 - \sum_{i=1}^{W} [p_i(t)]^q}{q - 1} \propto t \quad \text{only for } q = 0
\]

\((ii)\) \( K_q \equiv \lim_{t \to \infty} \frac{S_q(t)}{t} = \lambda q \quad \text{for } q = 0
\]
CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett 83, 4729 (1999) and 85, 4261 (2000)]
(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)

\[ W = 4000 \times 4000 \text{ cells} \]
\[ N = 1000 \text{ initial conditions randomly chosen in one cell} \]
\[ \text{Average done over 100 initial cells} \]

\[ [q = 0 \rightarrow \text{linear correlation} = 0.99993] \]

Also \[ \xi = e^{\lambda_0 t} \]
with \[ \lambda_0 = \lim_{n \to \infty} \frac{S_0(n)}{n} = 1 \]

\[ q - \text{generalization of Pesin (- like) theorem} \]

$S_q(N, t) \ versus \ N$
HYBRID PASCAL - LEIBNITZ TRIANGLE

(N = 0)
1 × \frac{1}{1}

(N = 1)
1 × \frac{1}{2} \quad 1 × \frac{1}{2}

(N = 2)
1 × \frac{1}{3} \quad 2 × \frac{1}{6} \quad 1 × \frac{1}{3}

(N = 3)
1 × \frac{1}{4} \quad 3 × \frac{1}{12} \quad 3 × \frac{1}{12} \quad 1 × \frac{1}{4}

(N = 4)
1 × \frac{1}{5} \quad 4 × \frac{1}{20} \quad 6 × \frac{1}{30} \quad 4 × \frac{1}{20} \quad 1 × \frac{1}{5}

(N = 5)
1 × \frac{1}{6} \quad 5 × \frac{1}{30} \quad 10 × \frac{1}{60} \quad 10 × \frac{1}{60} \quad 5 × \frac{1}{30} \quad 1 × \frac{1}{6}

\sum = 1 \quad (\forall \ N)

Blaise Pascal (1623-1662)
Gottfried Wilhelm Leibnitz (1646-1716)
Daniel Bernoulli (1700-1782)
EQUIVALENTLY:

$(N=0)$

$(N=1)$

$(N=2)$
$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N$ ($N \to \infty$)

Leibnitz triangle

\[
\left( p_{N,0} = \frac{1}{N+1} \right)
\]

$N$ independent coins

\[
\begin{align*}
    p_{N,0} &= p^N \\
    \text{with } p &= 1/2
\end{align*}
\]

Stretched exponential

\[
\begin{align*}
    p_{N,0} &= p^{N\alpha} \\
    \text{with } p &= \alpha = 1/2
\end{align*}
\]

(All three examples strictly satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA 102, 15377 (2005)
Asymptotically scale-invariant (d=2)

\[
\begin{array}{cccc}
(N = 0) & 1 \\
(N = 1) & 1/2 & 1/2 \\
(N = 2) & 1/3 & 1/6 & 1/3 \\
(N = 3) & 3/8 & 5/48 & 5/48 & 0 \\
(N = 4) & 2/5 & 3/40 & 1/20 & 0 & 0 \\
\end{array}
\]

(\text{It asymptotically satisfies the Leibnitz rule})

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA 102, 15377 (2005)
\( q \neq 1 \) SYSTEMS

i.e., such that \( S_q(N) \propto N \ (N \to \infty) \)

\[ q = 1 - \frac{1}{d} \]

(All three examples asymptotically satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA 102, 15377 (2005)
C.T., M. Gell-Mann and Y. Sato
Europhysics News 36 (6), 186 (2005) [European Physical Society]
A conjecture for $S_q(N,t)$:

For $q = q_{sen}$, $N \to \infty$ and $t \to \infty$ play essentially the same role.

In particular,

i) Under conditions of infinitely fine graining in phase space,

$$S_{q_{sen}}(N,t) \sim K_{q_{sen}}(N) \; t \propto N \; t$$

($q_{sen} = 1 \Rightarrow K_1 = \sum_{j/\lambda_1^{(j)}>0} \lambda_1^{(j)}$, i.e., Pesin–like identity for finite $N$)

ii) Under conditions of finite graining in phase space,

$$\lim_{t \to \infty} S_{q_{sen}}(N,t) \propto N$$

(Clausius)

C.T., M. Gell-Mann and Y. Sato
Europhysics News 36, 186 (2005)
C.T., M. Gell-Mann and Y. Sato
Europhysics News Special Issue *Nonextensive Statistical Mechanics – New Trends, New Perspectives*
(European Physical Society, Nov-Dec 2005), in press
If $A$ and $B$ are independent,
i.e., if $p_{ij}^{A+B} = p_i^A \cdot p_j^B$,
then
$$S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B)$$
whereas
$$S_q(A + B) = S_q(A) + S_q(B) + \frac{1-q}{k_B} S_q(A) \cdot S_q(B)$$
$$\neq S_q(A) + S_q(B) \quad (if \ q \neq 1)$$

But if $A$ and $B$ are especially (globally) correlated,
then
$$S_q(A + B) = S_q(A) + S_q(B)$$
whereas
$$S_{BG}(A + B) \neq S_{BG}(A) + S_{BG}(B)$$
<table>
<thead>
<tr>
<th>PROPERTY</th>
<th>[a]</th>
<th>[b]</th>
<th>[c]</th>
<th>[d]</th>
<th>[e]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extensive ($\forall q$) [ $\rho^{qs} = \rho_q \rho_s$ ]</td>
<td>YES</td>
<td>NO</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Extensive ($q \neq 1$) [special global correlations]</td>
<td>NO</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Concave ($\forall q &gt; 0$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Lesche-stable ($\forall q &gt; 0$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
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<tr>
<td>Jackson $q$-derivative application on $-\sum \rho_i$ ($\forall q$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Finite entropy production per unit time ($\forall q = q_{max} \leq 1$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>$\exists \hat{S} / \hat{S}$ and $\hat{S} = \hat{S}$ obey same composition [ $\rho^{qs} = \rho_q \rho_s$ ] ($\forall q$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
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<tr>
<td>$\exists \hat{S} / \hat{S}(\hat{\rho}^{-1})$ has same functional form as $S(\rho_i - 1/P)$ ($\forall q$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Same functional form for $Z_q(\beta F_q)$ and $Z_q(\beta E_q)$ ($\forall q$)</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Energy eigenvalues scaling temperature coincides with inverse Lagrange parameter</td>
<td>YES</td>
<td>NO</td>
<td>YES</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Optimizing distribution ($\forall q$)</td>
<td>Exponential</td>
<td>Power-law</td>
<td>Power-law</td>
<td>Power-law</td>
<td>Power-law</td>
</tr>
</tbody>
</table>
q - CENTRAL LIMIT THEOREM: (conjecture)

\[ \frac{\partial p(x,t)}{\partial t} = D \frac{\partial^\gamma \left[ p(x,t) \right]^{2-q}}{\partial \left| x \right|^\gamma} \quad (0 < \gamma \leq 2; \; q < 3) \]

- globally correlated variables; finite q-variance; q-Gaussian attractor
- independent variables; finite variance; Gaussian attractor
- independent variables; divergent variance; Levy attractor

C.T., Milan J. Math. 73, 145 (2005)
$q$-GAUSSIANS:
q - CENTRAL LIMIT THEOREM (q-product and de Moivre-Laplace theorem):

The q-product is defined as follows:

\[ x \otimes_q y \equiv \left[ x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}} \]

**Properties:**

i) \( x \otimes_1 y = x y \)

ii) \( \ln_q (x \otimes_q y) = \ln_q x + \ln_q y \)

[whereas \( \ln_q (x y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y) \)]

E.P. Borges, Physica A 340, 95 (2004)]

The de Moivre-Laplace theorem can be constructed with

\[ p_{N,0} = p^N \quad \text{with} \quad p = 1/2 \]

and

*Leibnitz rule*
q - CENTRAL LIMIT THEOREM: (numerical indications)

We q - generalize the de Moivre–Laplace theorem with

\[ \frac{1}{p_{N,0}} = \left( \frac{1}{p} \right) \otimes_q \left( \frac{1}{p} \right) \otimes_q \cdots \left( \frac{1}{p} \right) \quad (N \text{ terms}) \]

i.e.,

\[ p_{N,0} = \left[ N \, p^{q-1} - (N-1) \right]^{1-q} \quad (\text{with } p = 1/2) \]

(q = 3/10)

[Hence q \to 2 - q (additive duality) and q \to 1/q (multiplicative duality) are involved]

**q - GENERALIZED CENTRAL LIMIT THEOREM:**  (mathematical proof)

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

\[
F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) \, dx = \int_{-\infty}^{\infty} e_q^{ix\xi} [f(x)]^{-q} f(x) \, dx \quad \text{(nonlinear!)}
\]

q-correlation:

*Two random variables \( X \) [with density \( f_X(x) \)] and \( Y \) [with density \( f_Y(y) \)] are said *\( q \)-correlated if

\[
F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi),
\]

i.e., if

\[
\int_{-\infty}^{\infty} dz \, e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[ \int_{-\infty}^{\infty} dx \, e_q^{ix\xi} \otimes_q f_X(x) \right] \otimes_q \left[ \int_{-\infty}^{\infty} dy \, e_q^{iy\xi} \otimes_q f_Y(y) \right],
\]

with \( f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, h(x, y) \, \delta(x + y - z) = \int_{-\infty}^{\infty} dx \, h(x, z - x) = \int_{-\infty}^{\infty} dy \, h(z - y, y) \)

where \( h(x, y) \) is the joint density.

\[
\begin{cases}
\text{q-correlation means independence if } q = 1, \ i.e., \ h(x, y) = f_X(x) f_Y(y) \\
\text{global correlation if } q \neq 1, \ hence \ h(x, y) \neq f_X(x) f_Y(y)
\end{cases}
\]
Closure:

The q-Fourier transform of a q-Gaussian is a z(q)-Gaussian with

\[ z(q) = \frac{1+q}{3-q} \in (-\infty,3) \]

Iteration:

\[ q_n \equiv z_n(q) \equiv z(z_{n-1}(q)) = \frac{2q + n(1-q)}{2 + n(1-q)} \quad (n = 0, \pm 1, \pm 2, \ldots; \quad q_0 = q) \]

(the same as in R.S. Mendes and C.T. [Phys Lett A 285, 273 (2005)] when calculating marginal probabilities!)

hence

(i) \( q_n(1) = 1 \) (\( \forall n \)), \quad \( q_{\pm n}(q) = 1 \) (\( \forall q \)),

(ii) \( q_{n+1} = 2 - \frac{1}{q_{n+1}} \),

(the same as in L.G. Moyano, C.T. and M. Gell-Mann (2005)!) (the same as in A. Robledo [Physica D 193, 153 (2004)] for pitchfork and tangent bifurcations!)

(iii) \( n = 2m = 0, \pm 2, \pm 4, \ldots \) yields \( q_{(m)} \equiv q_{2m} = \frac{q + m(1-q)}{1 + m(1-q)} \)

(the same obtained in C.T., M. Gell-Mann and Y. Sato [Proc Natl Acad Sci (USA) 102, 15377 (2005)], by combining only additive and multiplicative dualities, and which was conjectured to be a possible explanation for the NASA-detected q-triangle for \( m = 0, \pm 1 \))
\[
\frac{\alpha}{1-q_{\alpha,n}} = \frac{\alpha}{1-q} + n
\]

\( (n = 0, \pm 1, \pm 2, \ldots) \)
\[ q\text{-Fourier Transform} \left[ \frac{\sqrt{\beta}}{C_q} e^{-\beta t^2} \right] = e^{-\beta_1 \omega^2} \]

where
\[ q_1 = \frac{1+q}{3-q} \]

and
\[ \beta_1 = \frac{3-q}{8 \beta^{2-q} C_q^{2(1-q)}} \]

\[ = \begin{cases} 
2\sqrt{\pi} \Gamma \left( \frac{1}{q-1} \right) \\ (3-q)\sqrt{(1-q)} \Gamma \left( \frac{3-q}{2(1-q)} \right) \end{cases} \text{ if } q < 1 \]

with
\[ C_q = \begin{cases} 
\sqrt{\pi} \text{ if } q = 1 \\
\sqrt{\pi} \Gamma \left( \frac{3-q}{2(q-1)} \right) \text{ if } 1 < q < 3 \\
\sqrt{q-1} \Gamma \left( \frac{1}{q-1} \right) \text{ if } q > 3 \end{cases} \]
$q = 2$
$\beta = 0.5$

$q_1 = 3$
$\beta_1 = 0.012$
A random variable $X$ is said to have a \((q, \alpha)\)-stable distribution $L_{q, \alpha}(x)$ if its $q$-Fourier transform has the form $a \ e_{q}^{-b \ |\xi|^{\alpha}} \quad (a > 0, \ b > 0, \ 0 < \alpha \leq 2)$

i.e., if

$$F_{q}[L_{q,\alpha}](\xi) \equiv \int_{-\infty}^{\infty} e_{q}^{ix} \otimes_{q} L_{q,\alpha}(x) \ dx = \int_{-\infty}^{\infty} e_{q}^{\left[L_{q,\alpha}(x)\right]^{-q}} L_{q,\alpha}(x) \ dx = a \ e_{q}^{-b \ |\xi|^{\alpha}}$$

$L_{1,2}(x) \equiv G(x) \quad \text{(Gaussian)}$

$L_{1,\alpha}(x) \equiv L_{\alpha}(x) \quad \text{($\alpha$-stable Levy distribution)}$

$L_{q,2}(x) \equiv G_{q}(x) \quad \text{(q-Gaussian)}$


cond-mat/0606038

cond-mat/0606040
### Central Limit Theorems: $N^{1/[\alpha(2-q)]}$ - Scaled Attractor $\mathbb{F}(x)$ When Summing $N \to \infty$

$q$-Correlated Identical Random Variables with Symmetric Distribution $f(x)$

<table>
<thead>
<tr>
<th>$q = 1$ [independent]</th>
<th>$q \neq 1$ (i.e., $Q \equiv 2q - 1 \neq 1$) [globally correlated]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Q &lt; \infty$ ((\alpha = 2))</td>
<td>$\mathbb{F}(x) = G_{3q-1}(x) \equiv \frac{3q-1}{q+1}$ - Gaussian,</td>
</tr>
<tr>
<td>$\sigma_Q \to \infty$ ((0 &lt; \alpha &lt; 2))</td>
<td>with same $\sigma_Q \left[ \equiv \int dx x^2 [f(x)]^Q / \int dx [f(x)]^Q \right]$ of $f(x)$</td>
</tr>
</tbody>
</table>
| $L_\alpha(x)$ | $G_{3q-1}(x) \begin{cases} 
G(x) & \text{if } |x| << x_c(q, 2) 
\sim f(x) \sim C_q / |x|^{(q+1)/(q-1)} & \text{if } |x| >> x_c(q, 2) 
\end{cases}$ |
| $\lim_{\alpha \to 2} x_c(1, \alpha) = \infty$ | with $\lim_{q \to 1} x_c(q, 2) = \infty$ |

S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006) [cond-mat/0606038] and [cond-mat/0606040]
NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS
(CANONICAL ENSEMBLE):

Extremization of the functional

\[ S_q[p_i] \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \]

with the constraints

\[ \sum_{i=1}^{W} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{W} p_i^q E_i = U_q \quad \text{and} \quad \sum_{i=1}^{W} p_i^q = \mathcal{Z}_q \]

yields

\[ p_i = \frac{e^{-\beta_q (E_i - U_q)}}{\mathcal{Z}_q} \]

with \( \beta_q = \frac{\beta}{\sum_{i=1}^{W} p_i^q} \), \( \beta \equiv \text{energy Lagrange parameter} \), and \( \mathcal{Z}_q = \sum_{i=1}^{W} e^{-\beta_q (E_i - U_q)} \).
We can rewrite
\[ p_i = \frac{e^{-\beta'_q E_i}}{Z'_q} \]
with
\[ \beta'_q \equiv \frac{\beta_q}{1 + (1 - q) \beta_q U_q} \]
and
\[ Z'_q \equiv \sum_{i=1}^{W} e^{-\beta'_q E_i} \]

And we can prove

(i) \[ \frac{1}{T} = \frac{\partial S_q}{\partial U_q} \quad \text{with} \quad T \equiv \frac{1}{k \beta} \]

(ii) \[ F_q \equiv U_q - TS_q = -\frac{1}{\beta} \ln_q Z_q \quad \text{where} \quad \ln_q Z_q = \ln_q Z_q - \beta U_q \]

(iii) \[ U_q = -\frac{\partial}{\partial \beta} \ln_q Z_q \]

(iv) \[ C_q \equiv T \frac{\partial S_q}{\partial T} = \frac{\partial U_q}{\partial T} = -T \frac{\partial^2 F_q}{\partial T^2} \]

(i.e., the Legendre structure of Thermodynamics is q-invariant!)
SOME FORM-IN Variant RELATIONS (arbitrary q)

CLAUSIUS INEQUALITY AND BOLTZMANN H-THEOREM
(macroscopic time irreversibility)
Abe and Rajagopal, Phys Rev Lett 91 (2003)
\[ \beta \delta Q_q \leq \delta S_q \quad ; \quad q \frac{d S_q}{d t} \geq 0 \]

EHRENFEST THEOREM (correspondence principle)
Plastino and Plastino, Phys Lett A 177 (1993) 177
\[ \frac{d}{d t} \left< \hat{O} \right>_q = \frac{i}{\hbar} \left< \left[ \hat{H} , \hat{O} \right] \right>_q \]

FACTORIZATION OF LIKELIHOOD FUNCTION
(Einstein’s 1910 reversal of Boltzmann’s formula;
thermodynamically independent systems)
Caceres and Tsallis, unpublished (1993); Chame and Mello,
\[ W_q (A + B) = W_q (A) W_q (B) \]

ONSAGER RECIPROCITY THEOREM
(microscopic time reversibility)
Chame and Mello, Phys Lett A 228 (1997) 159
\[ L_{jk} = L_{kj} \]

KRAMERS AND KRONIG RELATION (causality)
Rajagopal, Phys Rev Lett 76 (1996) 3469

PESIN EQUALITY
(mixing; Kolmogorov-Sinai entropy and Lyapunov exponent)
Tsallis, Plastino and Zheng, Chaos, Solitons and Fractals 8 (1997) 885;
Baldovin and Robledo, Phys, Rev. E 69, 045202(R) (2004).
\[ K_q = \begin{cases} \hat{\lambda}_q & \text{if } \hat{\lambda}_q > 0 \\ 0 & \text{otherwise} \end{cases} \]
### Entropic Form and Equilibrium Statistics: Foundations

<table>
<thead>
<tr>
<th>BG (thermal equilibrium)</th>
<th>q ≠ 1 (thermal metaequilibrium, nonequilibrium)</th>
</tr>
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<tbody>
<tr>
<td><strong>Kinetic equation</strong>&lt;br&gt;Molecular chaos hypothesis (Stosszahlansatz)</td>
<td></td>
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<tr>
<td>Boltzmann 1872</td>
<td></td>
</tr>
<tr>
<td><strong>Steepest descent</strong></td>
<td>S. Abe and A.K. Rajagopal&lt;br&gt;J Phys A 33, 8733 (2000)</td>
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<tr>
<td>Darwin-Fowler 1922</td>
<td></td>
</tr>
<tr>
<td><strong>Conditions of uniqueness of S</strong></td>
<td>R. J. V. Santos&lt;br&gt;J Math Phys 38, 4104 (1997)</td>
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<td>Shannon 1948</td>
<td></td>
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<tr>
<td><strong>Law of large numbers</strong></td>
<td>S. Abe and A.K. Rajagopal&lt;br&gt;Europhys Lett 52, 610 (2000)</td>
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<tr>
<td>Khinchin 1949</td>
<td></td>
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<tr>
<td><strong>Compact conditions of uniqueness of S</strong></td>
<td>S. Abe&lt;br&gt;Phys Lett A 271, 74 (2000)</td>
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<td>Khinchin 1953</td>
<td></td>
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<tr>
<td>Balian-Balazs 1987&lt;br&gt;Kubo et al. 1988</td>
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</tr>
</tbody>
</table>
Let us consider $N$ correlated Normal random variables that generate $N$ correlated uniform distributions

$$f(x) = \begin{cases} 
1 & \text{if } -1/2 \leq x \leq 1/2 \\
0 & \text{otherwise}
\end{cases}$$

The $N \times N$ covariance matrix of the $N$ Normal distributions is given by

$$\Sigma_z = \begin{bmatrix}
1 & \rho & \cdots & \cdots & \rho \\
\rho & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \rho \\
\rho & \cdots & \cdots & \rho & 1
\end{bmatrix} \quad (-1 \leq \rho \leq 1)$$

$\rho = 0 \Rightarrow$ independence

$\rho = 1 \Rightarrow$ full correlation
Bivariate normal data with a specified correlation coefficient is generated. The data (red dots) corresponding to the resulting marginal distributions are transformed to uniformly distributed random variables on (-1/2, 1/2). The correlation coefficient of the bivariate uniform marginal data is presented as a function of the normal correlation coefficient. The bisector (continuous black line) is indicated as well as a guide to the eye.
\( (N = 2; \, \rho = 0) \)

\[ \text{Marginal U}_1, \text{ U}_2 \]

\[ \text{Bivariate Uniform Distribution, } \rho = 0 \]

\[ \text{Marginals } U_1, U_2 \]

\[ f_{U_1}(x), f_{U_2}(x) \]

\[ U_1 + U_2, r = 0.011257 \]

\[ U_{\text{sum}} = U_1 + U_2 \]

\[ (\bar{N} ; 0) \]

\[ (\bar{N} ; 0.1) \]

\[ \rho \]

\( (N = 2; \ \rho = 0.5) \)

\( (N = 2; \ \rho = 0.9) \)

Empirical PDF of $X_{\text{sum}}$ with Asymptotic Fitted qGaussian PDF

**System size:**
- **Size 2:** $q_{\infty} = -0.27112$, $\beta_{\infty} = 0.675$
- **Size 3:** $q_{\infty} = -0.042292$, $\beta_{\infty} = 0.39472$
- **Size 5:** $q_{\infty} = 0.15505$, $\beta_{\infty} = 0.18051$
- **Size 6:** $q_{\infty} = 0.19471$, $\beta_{\infty} = 0.13266$
- **Size 20:** $q_{\infty} = 0.31718$, $\beta_{\infty} = 0.014581$
- **Size 24:** $q_{\infty} = 0.32112$, $\beta_{\infty} = 0.010238$

Empirical PDF of $X_{\text{sum}}$ with Asymptotic Fitted qGaussian PDF: UNIFORM

system size=100  $\rho=0.2$  $q_\infty=0.8347$  $\beta_\infty=0.0024114$

Fitted Asymptotic $q_\infty$ versus Normal Correlations, $\rho$

System Size = 25  number deviates = 5e+006

$y = \frac{1 - \frac{5}{3}\rho}{1 - \rho}$

INFLUENCE OF THE RANGE OF CORRELATIONS DECAYING FAR FROM THE DIAGONAL OF THE COVARIANCE MATRIX:

$N \times N$ covariance matrix of the Normal distributions given by

$$
\begin{pmatrix}
1 & \rho(2) & \rho(3) & \ldots & \rho(N-1) & \rho(N) \\
\rho(2) & 1 & \rho(2) & \ldots & \rho(N-2) & \rho(N-1) \\
\rho(3) & \rho(2) & 1 & \ldots & \rho(N-3) & \rho(N-2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\rho(N-1) & \rho(N-2) & \ldots & \rho(2) & 1 & \rho(2) \\
\rho(N) & \rho(N-1) & \ldots & \rho(3) & \rho(2) & 1
\end{pmatrix}
$$

with

$$
\rho(r) = \frac{\rho}{r^\alpha}
$$

$(-1 \leq \rho \leq 1; \ \alpha \geq 0; \ r = 2, 3, ..., N)$

Fitted $q$ as a function of Normal Correlation ($\rho$) and Scaling Exponent ($\alpha$)

System Size = 35

$(N=35)$

$$
q \approx \frac{1 - \frac{\rho}{3}}{1 - \rho}
$$

Connections with Hamiltonian and more complex systems
\[ V(\vec{r}) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \]

\[ (A > 0, \quad \alpha \geq 0) \]
**BOLTZMANN-GIBBS STATISTICAL MECHANICS**
(Maxwell 1860, Boltzmann 1872, Gibbs ≤ 1902)

Entropy

\[ S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \]

Internal energy

\[ U_{BG} = \sum_{i=1}^{W} p_i E_i \]

Equilibrium distribution

\[ p_i = e^{-\beta E_i} / Z_{BG} \quad \left( Z_{BG} = \sum_{j=1}^{W} e^{-\beta E_j} \right) \]

Paradigmatic differential equation

\[
\begin{align*}
\frac{dy}{dx} &= ay \\
y(0) &= 1
\end{align*}
\]

\[ y = e^{ax} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( a )</th>
<th>( y(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium distribution</td>
<td>( E_i )</td>
<td>( -\beta )</td>
</tr>
<tr>
<td>Sensitivity to initial conditions</td>
<td>( t )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Typical relaxation of observable ( O )</td>
<td>( t )</td>
<td>(-1/\tau)</td>
</tr>
</tbody>
</table>

\( S_{BG} \rightarrow \) extensive, concave, Lesche-stable, finite entropy production
NONEXTENSIVE STATISTICAL MECHANICS


Entropy
\[ S_q = k \left( 1 - \sum_{i=1}^{W} p_i^q \right) / (q - 1) \]

Internal energy
\[ U_q = \sum_{i=1}^{W} p_i^q E_i / \sum_{j=1}^{W} p_j^q \]

Stationary state distribution
\[ p_i = e_q^{-\beta_q(E_i-U_q)} / Z_q \]
\[ Z_q \equiv \sum_{j=1}^{W} e_q^{-\beta_q(E_j-U_q)} \]

Paradigmatic differential equation
\[ \frac{dy}{dx} = a y^q \]
\[ y(0) = 1 \] \[ \Rightarrow \]
\[ y = e_q^{ax} \equiv \left[ 1 + (1 - q) ax \right]^{1/(1-q)} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( a )</th>
<th>( y(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stationary state distribution</td>
<td>( E_i )</td>
<td>( -\beta_{q_{stat}} )</td>
</tr>
<tr>
<td>Sensitivity to initial conditions</td>
<td>( t )</td>
<td>( \lambda_{q_{sen}} )</td>
</tr>
<tr>
<td>Typical relaxation of observable ( O )</td>
<td>( t )</td>
<td>( -1 / \tau_{q_{rel}} )</td>
</tr>
</tbody>
</table>

\( S_q \rightarrow \) extensive, concave, Lesche-stable, finite entropy production


Fig. 2. The triangle of the basic values of $q$, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leq 1$, $q_{rel} \geq 1$ and $q_{stat} \geq 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent $\alpha$ and the dimension $d$, it could be that $q_{stat}$ decreases from a value above unity (e.g., 2 or $\frac{3}{2}$ ) to unity when $\alpha/d$ increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for $q_{sen}$ and $q_{rel}$, but only those associated with a Hamiltonian are expected to also have a well defined value for $q_{stat}$. 
SOLAR WIND: Magnetic Field Strength


[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; \textit{daily averages}]

\[ q_{\text{sen}} = -0.6 \pm 0.2 \]

\[ q_{\text{rel}} = 3.8 \pm 0.3 \]

\[ q_{\text{stat}} = 1.75 \pm 0.06 \]
Playing with additive duality \((q \to 2 - q)\)
and with multiplicative duality \((q \to 1/q)\)
(and using numerical results related to the \(q\)–generalized central limit theorem)

we conjecture
\[
q_{\text{rel}} + \frac{1}{q_{\text{sen}}} = 2 \quad \text{and} \quad q_{\text{stat}} + \frac{1}{q_{\text{rel}}} = 2
\]

hence
\[
1 - q_{\text{sen}} = \frac{1 - q_{\text{stat}}}{3 - 2q_{\text{stat}}}
\]

hence only one independent!

Burlaga and Vinas (NASA) most precise value of the \(q\)–triplet is
\[
q_{\text{stat}} = 1.75 = 7/4
\]

hence
\[
q_{\text{sen}} = -0.5 = -1/2 \quad \text{(consistent with } q_{\text{sen}} = -0.6 \pm 0.2 \text{ !)}
\]
and
\[
q_{\text{rel}} = 4 \quad \text{(consistent with } q_{\text{rel}} = 3.8 \pm 0.3 \text{ !)}
\]

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA 102, 15377 (2005)
Connections with asymptotically scale–free networks
(1) Locate site $i=1$ at the origin of say a plane

(2) Then locate the next site with

$$P_G \propto 1/r^{2+\alpha_G} \quad (\alpha_G \geq 0)$$

($r \equiv$ distance to the baricenter of the pre-existing cluster)

(3) Then link it to only one of the previous sites using

$$p_A \propto k_i/r_i^{\alpha_A} \quad (\alpha_A \geq 0)$$

($k_i \equiv$ links already attached to site $i$)

($r_i \equiv$ distance to site $i$)

4) Repeat
\((\alpha_G = 1; \alpha_A = 1; N = 250)\)
\[ \frac{P(k)}{P(0)} = e^{\frac{-k}{\kappa}} \]
\[ \equiv \frac{1}{\left[1 + (q - 1) \frac{k}{\kappa}\right]^{1/(q-1)}} \]

D.J.B. Soares, C. T., A.M. Mariz and L.R. Silva
Europhys Lett 70, 70 (2005)
Barabasi-Albert universality class

\[ q = 1 + \frac{1}{3} e^{-0.526 \alpha_A} \quad (\forall \alpha_G) \]

\[ \kappa = 0.083 + 0.092 \alpha_A \quad (\forall \alpha_G) \]

D.J.B. Soares, C. T., A.M. Mariz and L.R. Silva
Europhys Lett 70, 70 (2005)
**GAS-LIKE (NODE COLLAPSING) NETWORK:**


Number $N$ of nodes fixed (*chemostat*); $i=1, 2, \ldots, N$

Merging probability $p_{ij} \propto \frac{1}{d_{ij}^\alpha}$ ($\alpha \geq 0$)

$d_{ij} \equiv$ shortest path (chemical distance) connecting nodes $i$ and $j$ on the network

$\alpha = 0$ and $\alpha \to \infty$ recover the random and the neighbor schemes respectively


$(N = 2^7; \alpha = 0; r = 2)$

Degree of the most connected node

Degree of a randomly chosen node
**Fig. 1:** Snapshot of a non-growing dynamic network with $q$-exponential degree distribution for $N = 256$ nodes and a linking rate of $\bar{r} = 1$, for details see [8, 9]. The shown network is small to make connection patterns visible.

S. Thurner, Europhys News 36, 218 (2005)
\[(\alpha \rightarrow \infty ; \quad < r >= 8)\]

\[Z_q(k) \equiv \ln_q \left[ P(> k) \right] = \frac{[P(> k)]^{1-q} - 1}{1 - q}\]

\[(\text{optimal } q_c = 1.84)\]

\( (N = 2^9; \ r = 2) \)

\[
\begin{align*}
P(\geq k) &= e^{-(k-2)/k} \quad (k = 2, 3, 4, \ldots) \\
linear \ correlation \ &\in [0.999901, 0.999976] \\
\end{align*}
\]

\((r = 2)\)

\[ S. \text{Thurner and C. T., Europhys Lett 72, 197 (2005)} \]
The solution of
\[ \frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2[p(x,t)]^{2-q}}{\partial x^2} \quad [p(x,0) = \delta(0)] \quad (q < 3) \]
is given by
\[ p(x,t) \propto \left[ 1 + (1-q) x^2 / (\Gamma t)^{2/(3-q)} \right]^{1/(1-q)} = e_q^{-x^2 / (\Gamma t)^{2/(3-q)}} \quad (\Gamma \propto D) \]
hence
\[ x^2 \text{ scales like } t^\gamma \quad (\text{e.g., } \langle x^2 \rangle \propto t^\gamma) \]
with
\[ \gamma = \frac{2}{3-q} \]
Hydra viridissima: A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada
Physica A 293, 549 (2001)

![Graph showing histogram of velocities with a power law fit, marked q=1.5]
slope $\gamma = 1.24 \pm 0.1$

hence $\gamma = \frac{2}{3-q}$ is satisfied
Defect turbulence:

\( q \approx 1.5 \) and \( \gamma \approx 4/3 \) are consistent with \( \gamma = \frac{2}{3-q} \)

XY FERROMAGNET WITH LONG-RANGE INTERACTIONS:

$W_t = 2.35$ ?

$q_{rel} \approx 2.35$ ?

$q_{stat} \approx 1.5$ ?
COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:


(i) The distribution of atomic velocities is a $q$-Gaussian;

(ii) $q = 1 + \frac{44E_R}{U_0}$ where $E_R \equiv$ recoil energy

$U_0 \equiv$ potential depth
Experimental and computational verifications

(Computational verification: quantum Monte Carlo simulations)

(Experimental verification)
EARTHQUAKES
Earthquakes

Data from

P. Bak, K. Christensen, L. Danon and T. Scanlon,

\[ y = y_0[1+(q-1)x/T]^{(1/(1-q))} \]

\[ q = 1 + 1/b \]
TIME INTERVALS BETWEEN EARTHQUAKES
Southern California data [S. Abe and N. Suzuki (2004)]

Calm periods (stationary states) between major earthquakes,
i.e., excluding the Omori-regime periods (nonstationary states)

\[ P(t > \tau) = \frac{1}{[1 + (q - 1)\tau / \tau_0]^{1/(q-1)}} \]

\[ q = 1.10 \]
\[ \tau_0 = 1830 \text{ s} \]

\[ q = 1.08 \]
\[ \tau_0 = 2410 \text{ s} \]

\[ q = 1.05 \]
\[ \tau_0 = 2330 \text{ s} \]
AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:

Aging:
AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:

Aging:
\[ C(n + n_w, n_w) \equiv \frac{\langle t_{n+n_w} t_{n_w} \rangle - \langle t_{n+n_w} \rangle \langle t_{n_w} \rangle}{(\sigma_{n+n_w}^2 \sigma_{n_w}^2)} \]

“Natural time” suggested in
P.A. Varotsos, N.V. Sarlis and E.S. Skordas,
$N=10000$

$a=0.2 ; f=0.01 ; s_1=1$

$C(n+n_w, n_w)$

- $n_w=250$
- $n_w=500$
- $n_w=1000$
- $n_w=2000$
- $n_w=5000$

natural time $n$
MODEL FOR EARTHQUAKES (OMORI REGIME):

\[ D(n+n_w, n_w) \]

\[ \ln_q [D(n+n_w, n_w)] \]

\[ (q=2.98) \]

S. Abe, U. Tirnakli and P.A. Varotsos
Europhysics News 36 (6), 206 (2005) [European Physical Society]
$N=10000$

$a=0.2 ; f=0.01 ; s_1=1$

$\ln_q \left[ C(n+n_w,n_w) \right]$

$n_w=1000$

$n_w=2000$

$n_w=5000$

$n / n_w^{1.05}$
EARTHQUAKES:

NEWMAN MODEL (average over 100,000 realizations)

\[ e_{2.98}(0.7 \, n / n_w^{1.05}) \]

OLAMI-FEDER-CHRISTENSEN MODEL (average over 20,000 realizations)

\[ e_{2.9}(0.6 \, n / n_w^{1.05}) \]

ASTROPHYSICS
FLUXES OF COSMIC RAYS
(Band Q: 22.8 GHz) (Band V: 60.8 GHz) (Band W: 93.5 GHz)

(Data after using Kp0 mask) \[ q = 1.045 \pm 0.005 \quad (99 \% \text{ confidence level}) \]

$q = 1.045 \pm 0.005$ (99\% confidence level)

\[ dv = -\gamma \left( v - \frac{\alpha + 1}{\beta} \right) dt + \sqrt{2v \frac{\gamma}{\beta}} dW_t, \]

\[ P(v) = \frac{1}{Z} \left( \frac{v}{\theta} \right)^\alpha \exp_q \left( -\frac{v}{\theta} \right) \]

\[ e_q^x \equiv [1 + (1 - q) x]^{\frac{1}{1-q}} \quad (e_1^x \equiv e^x) \]

**Figure 6.** In panel (a) open symbols represent the PDF for the ten-high 1 minute traded volume stocks in NYSE exchange; solid symbols represent the PDF obtained for the numerical realization depicted in panel (b) and line the theoretical PDF Eq. (28). Parameters are \( q = 1.17, \alpha = 1.79, \lambda = 1.42 \) and \( \delta = 3.09 \).
STOCK VOLUMES:

\[ P(v) = \frac{1}{Z} \left( \frac{v}{\theta} \right)^{-\alpha - 2} \exp_q \left[ -\frac{\theta}{v} \right] \]

J de Souza, SD Queiros and LG Moyano, physics/0510112 (2005)
**q-GENERALIZED BLACK-SCHOLES EQUATION:**

L Borland and J-P Bouchaud, cond-mat/0403022 (2004)
L Borland, Europhys News **36**, 228 (2005)
See also H Sakaguchi, J Phys Soc Jpn **70**, 3247 (2001)

**Fig.2:** The empirical distribution of daily returns from the stocks comprising the SP 100 (red) is fit very well by a $q$-Gaussian with $q = 1.4$ (blue).

**Fig.3:** Theoretical implied Black-Scholes volatilities from the $q = 1.4$ model (triangles) match empirical ones (circles) very well, across all strikes and for different times to expiration.

**REMARK:** Student t-distributions are the particular case of $q$-Gaussians when $q = \frac{n+3}{n+1}$ with $n$ integer
LONDON STOCK EXCHANGE (Block market):


VODAPHONE stocks (31 May 2000 to 31 December 2002)

Cumulative distribution

$q = 3.28 \ ; \ \beta_q = 1.1 \times 10^{-4}$

$q' = 1.45 \ ; \ \beta_{q'} = 1.1 \times 10^{-6}$

Daily net exchange of shares (between all pairs of two institutions)
LAND PRICES IN JAPAN
(cumulative distribution)


\[ P(x) = \frac{1}{[1+(q-1)\beta_q x^2]^{\frac{1}{q-1}}} \]
\[ q = 2.136, \sqrt{1/\beta_q} = 188,982 \, ¥ \]
FINGERING
P. Grosfils and J.P. Boon, 2005
GENERALIZED SIMULATED ANNEALING
AND RELATED ALGORITHMS
HYBRID LEARNING OF NEURAL NETWORKS


(Hybrid Learning Scheme = HLS; \( q > 1 \))

Fig. 1. The weights trajectory of the Hybrid Learning Scheme converges to the global minimum (left), whilst the trajectory of Rprop to a local minimizer (right).
Fig. 2. Typical learning error curve for the Parity-3 problem.
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<td>SARprop</td>
<td>1430 (+)</td>
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<td>HLS</td>
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</table>

Anastasiadis and Magoulas (2004)
ELECTROENCEPHALOGRAMS (tonic-clonic transition in epilepsy):

A. Plastino and O.A. Rosso
Europhysics News 36 (6), 224 (2005) [European Physical Society]
Fig. 3. Example of entropic segmentation for mammography image with an inhomogeneous spatial noise. Two image segmentation results are presented for $q = 1.0$ (classic entropic segmentation) and $q = 4.0$. 
Fig. 4. Influence of parameter $q$ in natural images: $q = 0.5$, $q = 1.0$ (classical entropic segmentation) and $q = 3.0$. 
**IMAGE EDGE DETECTION** [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]

Original image

Canny edge detector

q = 1.5

q = 1 (Jensen-Shannon)
IMAGE EDGE DETECTION  [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]

Original image

Canny edge detector

q = 1.5

q = 1
(Jensen-Shannon)
IMAGE EDGE DETECTION  [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]

Original image

Canny edge detector

\( q = 1.5 \)

\( q = 1 \)
(Jensen-Shannon)
IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]
q-GENERALIZED SIMULATED ANNEALING (GSA):


Visiting algorithm:

Boltzmann machine $\rightarrow$ Gaussian

Generalized machine $\rightarrow q_V$ – Gaussian

Acceptance algorithm:

Boltzmann machine $\rightarrow$ Boltzmann weight

Generalized machine $\rightarrow q_A$ – exponential weight

Cooling algorithm:

Boltzmann machine $\rightarrow \frac{T(t)}{T(1)} = \frac{\ln 2}{\ln(1+t)}$

Generalized machine $\rightarrow \frac{T(t)}{T(1)} = \frac{2^{q_V^{-1}} - 1}{(1+t)^{q_V^{-1}} - 1}$

[Typical values: $1 < q_V < 3$ and $q_A < 1$]
q-GENERALIZED PIVOT METHOD:


(Branin function)

Genetic algorithm

(Lennard-Jones clusters)

Number of function calls

Recently: M.A. Moret, P.G. Pascutti, P.M. Bisch, M.S.P. Mundim and K.C. Mundim

Classical and quantum conformational analysis using Generalized Genetic Algorithm

Physica A (2006), in press  (presumably better than both!)
Illustration: \[ E(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} (x_i^2 - 8)^2 + 5 \sum_{i=1}^{4} x_i \]

(15 local minima and one global minimum)

\((q_v = 1 \Rightarrow \text{mean convergence time} \approx 50000)\)
than \_q\_
6 August ROUND TABLE

(PANELISTS: Nauenberg, Rapisarda, Robledo, Ruffo)
FIG. 1. Temperature evolution of an isolated $N$-rotor system (Eq. (1)) in grey line, and cold (hot) $M$-rotor subsystem in black line (circles). Inset: magnification of the crossover between $T_{QSS}$ and $T_{BG}$.

FIG. 2. Temperature evolution of an $N$-rotor thermostat (Eq. (1)) in grey line, and of an $M$-rotor thermometer (Eq. (3)) in black line. After $t_{\text{contact}}$ the systems interact through $H_{\text{int}}$. Inset: magnification of the thermostat temperature minimum (see text for details).

L.G. Moyano, F. Baldovin and C. T., cond-mat/0305091
Facts that make think of q-statistics for the HMF and similar models:

1) For \( u = 5 \), and \( d=1, 2, 3 \) : maximal Lyapunov exponent vanishes like \( 1 / N^{\delta} \) with \( \delta(\alpha/d) \) decreasing from \( 1/3 \) to zero for \( \alpha/d \) increasing from zero to 1, and remaining zero for \( \alpha/d > 1 \);

2) For \( u = 0.69 \), at QSS, for \( d = 1 \): maximal Lyapunov exponent vanishes like \( 1 / N^{\delta'} \) with \( \delta'(\alpha/d) \) decreasing from \( 1/9 \) to zero for \( \alpha/d \) increasing from zero to 1, and remaining zero for \( \alpha/d > 1 \);

3) For \( u = 0.69 \), for \( \alpha = 0 \) and \( d=1 \): \( T(t) - T(\text{infinity}) \sim \exp_q (\ - t / \tau) \), with \( q \) different from unity;

4) For \( u = 0.69 \) and \( d = 1 \): \( t_{\text{QSS}} \sim \ln(\alpha / d) N \) for \( 0 < \alpha < 1 \);

5) For \( u = 0.69 \), at QSS, for \( \alpha = 0 \), and \( d = 1 \): the marginal probability of the velocity of one particle is not Maxwellian, and its central part decays like \( \exp_q (\ - B p^2) \) for \( M_0 =1 \), with \( q \) different from unity;

6) For \( u = 0.69 \), at QSS, for \( \alpha = 0 \) and \( d = 1 \): The autocorrelation function of velocities presents scalable aging and decays like \( \exp_q [\ - A t / (t_W^{\rho})] \) with different from unity;
7) For $u=5$, $\alpha = 0$ and $d = 1$: The autocorrelation function of velocities presents no aging and decays like $\exp_q [- A t]$ with $q$ different from unity which coincides with that of point (6);

8) For various $N$, various $\alpha$, various $M_0$, and $d=1$: $\gamma = 2 / (3-q)$

9) For $u = 0.69$, $N >> 1$, and $t >>> 1$, $\alpha = 0$ and $d = 1$: marginal probability for the angles of one particle $\sim \exp_q (- C \theta^2)$ with $q$ different from unity;

10) For $u = 0.69$, $\alpha = 0$, $d=1$ and finite $N$: the system has long memory as exhibited by the relevant influence of the initial conditions (dependence on $M_0$, and dependence on initial condition Catania-type or Rio de Janeiro-type);

11) For $u = 0.69$, $\alpha = 0$, $d=1$ and finite $N$: the system has long memory as exhibited by the nonvanishing glassy polarization versus $N$ along some time regime;

12) There is a manner of presenting the recent results by Baldovin and Orlandini in the canonical ensemble (instead of microcanonical) which enables them to be consistent with any value of $q$ between unity and say 2. This is a consequence of the fact that their variation of computational total energy is only of 8 %, and of the fact that, in first order, the $q$-exponential function does not depend on $q$;

13) Werner Braun is not sure whether one can take all those derivatives in the Braun-Hepp theorem, in the case $0 < \alpha / d < 1$. This suggests that his intuition tells him that something quite unusual might occur in such a case.
Forse mi è scappata qualche altra ragione.

Posso qualificare con più dettagli (valori di N, valori di t, condizioni iniziali precise, algoritmo di calcolo in dinamica molecolare, etc) le condizioni sui quali ogni una di queste ragioni è valida. Ho tutte le referenze alla tua disposizione.

Si come non abbiamo ancora una prova irrefutabile, il mio argomento e temporariamente che quello che ha il sapore di pizza, odore di pizza, e rotondo come pizza, ha pomodoro come pizza, ha muzarella come pizza, e venduto nelle pizzerie ... probabilmente è pizza!

[Fragmenti della lettera di Constantino a Stefano sul tema]