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A GENERALIZATION OF THE CENTRAL LIMIT THEOREM CONSISTENT WITH NONEXTENSIVE STATISTICAL MECHANICS

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Abstract

As well known, the standard central limit theorem plays a fundamental role in Boltzmann-Gibbs (BG) statistical mechanics. This important physical theory has been generalized by one of us (CT) in 1988 by using the entropy $S_q = \frac{1-\sum_i p_i^q}{q-1}$ (with $q \in \mathcal{R}$) instead of its particular case $S_1 = S_{BG} = -\sum_i p_i \ln p_i$. The theory which emerges is usually referred to as *nonextensive statistical mechanics* and recovers the standard theory for $q = 1$. During the last two decades, this q-generalized statistical mechanics has been successfully applied to a considerable amount of physically interesting complex phenomena. Conjectures and numerical indications available in the literature were since a few years suggesting the possibility of q-generalizations of the standard central limit theorem by allowing the random variables that are being summed to be strongly correlated in some special manner, the case $q = 1$ corresponding to standard probabilistic independence. This is precisely what we prove in the present paper for some range of q which extends from below to above $q = 1$. The attractor, in the usual sense of a central limit theorem, is given by a distribution of the form $p(x) \propto [1-(1-q)\beta x^2]^{1/(1-q)}$ with $\beta > 0$. These distributions, sometimes referred to as q-Gaussians, are known to make, under appropriate constraints, extremal the functional S_q. Their $q = 1$ and $q = 2$ particular cases recover respectively Gaussian and Cauchy distributions.

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1 INTRODUCTION

Limit theorems and, particularly, the central limit theorems (CLT), surely are among the most important theorems in probability theory and statistics. They play an essential role in various applied sciences as well, including statistical mechanics. Historically A. de Moivre, P.S. de Laplace, S.D. Poisson and C.F. Gauss have first shown that Gaussian is the attractor of independent systems with a finite second variance. Chebyshev, Markov, Liapounov, Feller, Lindeberg, Levy have contributed essentially to the development of the central limit theorem.

It is well known in the classical Boltzmann-Gibbs (BG) statistical mechanics that the Gaussian maximizes, under appropriate constraints, the Boltzmann-Gibbs entropy $S_{BG} = -\sum_i p_i \ln p_i$. The

q-generalization of the classic entropy introduced in [1] as the basis for generalizing the BG theory, and given by $S_q = \frac{1-\sum_i p_i^q}{q-1}$ ($q \in \mathcal{R}$; $S_1 = S_{BG}$), reaches its maximum at the distributions usually referred to as q-Gaussian (see [2]). This fact, and a number of conjectures [3], numerical indications [4], and some other studies [1, 2, 5, 6] suggest that there should be a q-analog of the CLT as well.

In the classical central limit theorem, the random variables are required to be independent. Central limit theorems were established for weakly dependent random variables also. Not pretending completeness, we refer the reader to the works [7, 8, 9, 10, 11, 12] (see also references therein), where different types of dependence are considered, as well as the history of the developments. The central limit theorem does not hold if correlation between far-ranging random variables is not neglectable (see [13]). The theory of nonextensive statistical mechanics deals with strongly correlated random variables, whose correlation does not rapidly decrease with increasing 'distance' between random variables. This type of correlation is sometimes referred to as global correlation (see [14] for the definition).

In the present paper we will study globally correlated random variables and establish a generalization of the central limit theorem. Such a theorem is untractable if we rely on the classic algebra. But nonextensive statistical mechanics uses a construction based on a special algebra, which we call *q-mathematics.* We show that, in the framework of this theory, the corresponding q-generalization of the central limit theorem becomes possible and relatively simple.

The result that is obtained here is presented as a series of theorems depending on the type of correlations. Speaking on one particular element of these theorems we note that there is a dual index, q^* connected with q. The first index q defines the region of convergence, while the dual index q^* exhibits existence of q^*-Gaussians corresponding to the limits of sums. The arisen duality, in contrast with the classic CLT, is a specific feature of the q-theory, which comes from the specific definition of q-exponential. In general, the obtained q-generalization of the central limit theorem is connected with a triplet (q_{k-1}, q_k, q_{k+1}) (determined by a given $q \in (0,2)$), which is important for the description of the system under study. As we see in Section 3 for systems having correlation identified by q_k, the index q_{k-1} determines the attracting q-Gaussian, while the index q_{k+1} indicates the scaling rate. Note that, if $q = 1$, then the entire family of theorems reduces to one element, thus recovering the classic central limit theorem.

Now we recall briefly the basic operations of the q-mathematics [15, 16, 17, 18]. By definition, the *q-sum* of two numbers is defined as $x \oplus_q y = x + y + (1-q)xy$. The q-sum is commutative, associative, recovers the usual summing operation if $q = 1$ (i.e. $x \oplus_1 y = x+y$), and preserves 0 as the neutral element (i.e. $x \oplus_q 0 = x$). By inversion, we can define the *q-subtraction* as $x \ominus_q y = \frac{x-y}{1+(1-q)y}$. The *$q$-product* for x, y is defined by the binary relation $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$. This operation also commutative, associative, recovers the usual product when $q = 1$, and preserves 1 as the unity. It is defined only when $x^{1-q} + y^{1-q} \geq 1$. Again by inversion, it can be defined the *q-division:* $x \oslash_q y = (x^{1-q} - y^{1-q} + 1)^{\frac{1}{1-q}}$. Note that $x \otimes_q 0 \neq 0$, and that, for $q \neq 1$, division by zero is allowed.

The paper is organized as follows. In Section 2 we start recalling the definitions of *q-exponential* and *q-logarithm.* Then we introduce the notion of the q-Fourier transform F_q and study its basic properties. Note that F_q coincides with the classic Fourier transform if $q = 1$. For $q \neq 1$ F_q is not a linear operator. Lemma 2.6 implies that F_q is invertible in the class of q-Gaussians. An important property of F_q is that it maps a q-Gaussian into a q_*-Gaussian, where $q_* \neq q$ if $q \neq 1$. In Section 3 we prove the main result of this paper, i.e., the q-version of the central limit theorem. We introduce the notion of q-correlated random variables, which classify globally correlated random variables. Only in the case $q = 1$ the correlation disappears, thus recovering the classic notion of independence of random variables.

2 q-FOURIER TRANSFORM AND ITS PROPERTIES

2.1 q-exponential and q-logarithm

The q-analysis relies essentially on the analogs of exponential and logarithmic functions, which are called q-exponential and q-logarithm [15]. In this paper we introduce and essentially use a new analog of the Fourier transform, which we call q-Fourier transform. The q-Fourier transform is defined based on the q-product and the q-exponential, and, in contrast to the usual Fourier transform, is a nonlinear transform.

Now we recall briefly the definitions and some properties of the q-exponential and q-logarithm. These functions are denoted by e_q^x and $ln_q x$ and are respectively defined as $e_q^x = [1+(1-q)x]_+^{\frac{1}{1-q}}$ and $\ln_q x = \frac{x^{1-q}-1}{1-q}$, $(x>0)$. The symbol $[x]_+$ means that $[x]_+ = x$, if $x \geq 0$, and $[x]_+ = 0$, if $x < 0$. We mention the main properties of these functions, which we will use essentially in this paper. For q-exponential the relations $e_q^{x \oplus_q y} = e_q^x e_q^y$ and $e_q^{x+y} = e_q^x \otimes_q e_q^y$ hold true. These relations can be written equivalently as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ [1] and $\ln_q(xy) = \ln_q x \oplus_q \ln_q y$. The q-exponential and q-logarithm have asymptotics $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$, $x \to 0$ and $\ln_q(1+x) = x - \frac{q}{2}x^2 + o(x^2)$, $x \to 0$. If $q < 1$, then for real x, $|e_q^{ix}| \geq 1$ and $|e_q^{ix}| \sim (1+x^2)^{\frac{1}{2(1-q)}}$, $x \to \infty$. Similarly, if $q > 1$, then $0 < |e_q^{ix}| \leq 1$ and $|e_q^{ix}| \to 0$ if $|x| \to \infty$.

2.2 q-Gaussian

Let β be a positive number. We call the function

$$G_q(\beta; x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \tag{1}$$

a q-Gaussian. The constant C_q is the normalizing constant, namely $C_q = \int_{-\infty}^{\infty} e_q^{-x^2} dx$. It is not difficult to verify that

$$C_q = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1}{1-q}\right)}{(3-q)\sqrt{1-q}\,\Gamma\left(\frac{3-q}{2(1-q)}\right)}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^{\infty} (1+y^2)^{\frac{-1}{q-1}} dy = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\frac{1}{q-1}\right)}, & 1 < q < 3. \end{cases} \tag{2}$$

For $q < 1$, the support of $G_q(\beta; x)$ is compact since this density vanishes for $|x| > 1/\sqrt{(1-q)\beta}$. Notice also that, for $q < 5/3$ $(5/3 \leq q < 3)$, the variance is finite (diverges). Finally, we can easily check that there are relationships between different values of q. For example, $e_q^{-x^2} = \left[e_{2-\frac{1}{q}}^{-qx^2}\right]^{\frac{1}{q}}$.

The following lemma establishes a general relationship (which contains the previous one as a particular case) between different q-Gaussians.

Lemma 2.1 *For any real q_1, $\beta_1 > 0$ and $\delta > 0$ there exist uniquely determined $q_2 = q_2(q_1, \delta)$ and $\beta_2 = \beta_2(\delta, \beta_1)$, such that*

$$(e_{q_1}^{-\beta_1 x^2})^{\delta} = e_{q_2}^{-\beta_2 x^2}.$$

Moreover, $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$.

[1] This property reflects the possible extensivity of S_q in the presence of special correlations [14, 19, 20, 21].

Proof. Let $q_1 \in R^1, \beta_1 > 0$ and $\delta > 0$ be any fixed real numbers. For the equation,

$$(1-(1-q_1)\beta_1 x^2)^{\frac{\delta}{1-q_1}} = (1-(1-q_2)\beta_2 x^2)^{\frac{1}{1-q_2}}$$

to be an identity it is needed $(1-q_1)\beta_1 = (1-q_2)\beta_2$, $1-q_1 = \delta(1-q_2)$. These equations have a unique solution $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$.

The set of all q-Gaussians will be denoted by $\mathcal{G}_q$, i.e.,

$$\mathcal{G}_q = \{b\, G_q(\beta, x) : b > 0, \beta > 0\}.$$

2.3 q-Fourier transform and q-characteristic function

Introduce the *q-Fourier transform* for a given function $f(x)$ by the formal formula [2]

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx\,. \tag{3}$$

For discrete functions $f_k, k = 0, \pm 1, \ldots$, this definition takes the form

$$F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e_q^{ik\xi} \otimes_q f_k\,. \tag{4}$$

In the future we use the same notation in both cases. We also call (3) or (4) the *q-characteristic function* of a given random variable X with an associated density $f(x)$, using the notations $F_q(X)$ or $F_q(f)$ equivalently. The following lemma establishes the expression of the q-Fourier transform without using the q-product.

Lemma 2.2 *The q-Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} dx. \tag{5}$$

Proof. We have

$$e_q^{ix\xi} \otimes_q f(x) = [1 + (1-q)ix\xi + f(x)^{1-q} - 1]_+^{\frac{1}{1-q}} =$$
$$f(x)[1 + (1-q)ix\xi f(x)^{q-1}]_+^{\frac{1}{1-q}}. \tag{6}$$

Integrating both sides of Eq. (6) we obtain (5).

Remark 2.3 *It should be noted that if the q-Fourier transform of a given function $f(x)$ defined by the formal definition in (3) exists, then it coincides with the expression in (5). The q-Fourier transform determined by the formula (5) has an advantage if compared to the formal definition: it does not use the q-product, which is, in general, restrictive in use. From now on we refer to (5) when we speak of the q-Fourier transform.*

Corollary 2.4 *The q-Fourier transform exists for any $f \in L_1(R)$ if $q \geq 1$. For $q < 1$ the q-Fourier transform exists if f additionally satisfies the condition $|f| \sim \frac{1}{|x|^\gamma}$, $\gamma > \frac{2-q}{1-q}$. Moreover, $|F_q[f](\xi)| \leq \|f\|_{L_1}$ [3], for $q \geq 1$, and $|F_q[f](\xi)| \leq \|f(x)(1+|x|)^{\frac{1}{1-q}}\|_{L_1}$ for $q < 1$.*

[2]Note, if f has compact support, then integration should be taken over this support, otherwise the integral does not converge.

[3]Here, and elsewhere, $\|f\|_{L_1} = \int_{\mathcal{R}} f(x)dx$, and L_1 is the space of absolutely integrable functions.

Proof. This is a simple implication of Lemma 2.2 and of the asymptotics of e_q^{ix} for large $|x|$ mentioned above.

Corollary 2.5 *Assume $f(x) \geq 0$, $x \in R$ and $F_q[f](\xi) = 0$ for all $\xi \in R$. Then $f(x) = 0$ for almost all $x \in R$.*

Lemma 2.6 *Let $q < 3$. For the q-Fourier transform of the q-Gaussian, the following formula holds:*

$$F_q[G_q(\beta;x)](\xi) = \Big(e_q^{-\frac{\xi^2}{4\beta^{2-q}C_q^{2(q-1)}}}\Big)^{\frac{3-q}{2}}. \tag{7}$$

Proof. Denote $a = \frac{\sqrt{\beta}}{C_q}$ and write

$$F_q[a\,e_q^{-\beta x^2}](\xi) = \int_{-\infty}^{\infty} (a\,e_q^{-\beta x^2}) \otimes_q (e_q^{ix\xi})dx$$

using the property $e_q^{x+y} = e_q^x \otimes_q e_q^y$ of the q-exponential, in the form

$$F_q[ae_q^{-\beta x^2}](\xi) = a\int_{-\infty}^{\infty} e_q^{-\beta x^2 + ia^{q-1}x\xi}dx = a\int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}})^2 - \frac{a^{2(q-1)}\xi^2}{4\beta}}dx =$$

$$a\int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}})^2} \otimes_q e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}}dx.$$

The substitution $y = \sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}}$ yields the equation

$$F_q[ae_q^{-\beta x^2}](\xi) = \frac{a}{\sqrt{\beta}}\int_{-\infty+i\eta}^{\infty+i\eta} e_q^{-y^2} \otimes_q e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}}dy\,,$$

where $\eta = \frac{\xi a^{q-1}}{2\sqrt{\beta}}$. Further using the Cauchy theorem on integrals over closed curves, which is applicable because of a power law decay of q-exponential for any $q < 3$, we can transfer the integration from $R + i\eta$ to R. Hence, applying again Lemma 2.2, we have

$$F_q[G_q(\beta;x)](\xi) = \frac{ae_q^{-\frac{a^{2(q-1)}}{4\beta}\xi^2}}{\sqrt{\beta}}\int_{-\infty}^{\infty} e_q^{-y^2\left(e_q^{-\frac{a^{2(q-1)}}{4\beta}\xi^2}\right)^{q-1}}dy =$$

$$\frac{aC_q}{\sqrt{\beta}}\Big(e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}}\Big)^{1-\frac{q-1}{2}}.$$

Simplifying the last expression, we arrive at (7).

Introduce the function $z(s) = \frac{1+s}{3-s}$ for $s \in (-\infty, 3)$, and denote its inverse $z^{-1}(t)$, $t \in (-1, \infty)$. It can be easily verified that $z(\frac{1}{z(s)}) = \frac{1}{s}$ and $z(\frac{1}{s}) = \frac{1}{z^{-1}(s)}$. Let $q_1 = z(q)$ and $q_{-1} = z^{-1}(q)$. It follows from the mentioned properties of $z(q)$ that

$$z\Big(\frac{1}{q_1}\Big) = \frac{1}{q} \quad \text{and} \quad z\Big(\frac{1}{q}\Big) = \frac{1}{q_{-1}}. \tag{8}$$

The function $z(s)$ also possess the following two important properties

$$z(s)\,z(2-s) = 1 \quad \text{and} \quad z(2-s) + z^{-1}(s) = 2. \tag{9}$$

It follows from these properties that $q_{-1} + \frac{1}{q_1} = 2$.

Corollary 2.7 *For q-Gaussians the following q-Fourier transforms hold*

$$F_q[G_q(\beta;x)](\xi) = e_{q_1}^{-\beta_*(q)\xi^2},\ q_1 = z(q),\ q < 3; \tag{10}$$

$$F_{q_{-1}}[G_{q_{-1}}(\beta;x)](\xi) = e_q^{-\beta_*(q_{-1})\xi^2},\ q_{-1} = z^{-1}(q),\ q > -1, \tag{11}$$

where $\beta_*(s) = \frac{3-s}{8\beta^{2-s}C_s^{2(s-1)}}$ *(or, more symmetrically,* $\beta_*^{\frac{1}{\sqrt{2-s}}}\beta^{\sqrt{2-s}} = K(s)$ *with* $K(s) = \left[\frac{3-s}{8C_s^{2(s-1)}}\right]^{\frac{1}{\sqrt{2-s}}}$*;* $0 \le K(s) < 1$ *for* $s \le 2$*, with* $\lim_{s\to-\infty} K(s) = K(2) = 0$*).*

Remark 2.8 *Note that* $\beta_*(s) > 0$ *if* $s < 3$.

Corollary 2.9 *The following mappings*

$$F_q : \mathcal{G}_q \to \mathcal{G}_{q_1}, q_1 = z(q),\ q < 3,$$

$$F_{q_{-1}} : \mathcal{G}_{q_{-1}} \to \mathcal{G}_q,\ q_{-1} = z^{-1}(q),\ q > -1,$$

hold and they are injective.

Corollary 2.10 *There exist the following inverse q-Fourier transforms*

$$F_q^{-1} : \mathcal{G}_{q_1} \to \mathcal{G}_q,\ q_1 = z(q),\ q < 3,$$

$$F_{q_{-1}}^{-1} : \mathcal{G}_q \to \mathcal{G}_{q_{-1}}, q_{-1} = z^{-1}(q),\ q > -1.$$

Lemma 2.11 *The following mappings*

$$F_{\frac{1}{q_1}} : \mathcal{G}_{\frac{1}{q_1}} \to \mathcal{G}_{\frac{1}{q}},\ q_1 = z(q),\ q < 3,$$

$$F_{\frac{1}{q}} : \mathcal{G}_{\frac{1}{q}} \to \mathcal{G}_{\frac{1}{q_{-1}}},\ q_{-1} = z^{-1}(q),\ q > -1.$$

hold.

Proof. The assertion of this lemma follows from Corollary 2.9 if we take into account the properties (8).

Let us introduce the sequence $q_n = z_n(q) = z(z_{n-1}(q)), n = 1, 2, ...,$ with a given $q = z_0(q)$, $q < 3$. We can extend the sequence q_n for negative integers $n = -1, -2, ...$ as well putting $q_{-n} = z_{-n}(q) = z^{-1}(z_{1-n}(q)), n = 1, 2, ...$ It is not hard to verify that [4]

$$q_n = \frac{2q + n(1-q)}{2 + n(1-q)},\ n = 0, \pm1, \pm2, ... \tag{12}$$

In Eq. (12) we require $q < 1 + \frac{2}{n}$ for $n > 0$ and $q > 1 + \frac{2}{n}$ for $n < 0$. Note $q_n \equiv 1$ for all $n = 0, \pm1, \pm2, ...$, if $q = 1$ and $lim_{n\to\pm\infty} z_n(q) = 1$ for all $q \neq 1$. Let us note also that the definition of the sequence q_n can be given through the series of mappings which follow.

[4]Essentially the same mathematical structure has already appeared in a quite different, though possibly related, context: see Footnote of page 15378 of [14].

Definition 2.12

$$\ldots \xrightarrow{z} q_{-2} \xrightarrow{z} q_{-1} \xrightarrow{z} q_0 = q \xrightarrow{z} q_1 \xrightarrow{z} q_2 \xrightarrow{z} \ldots \tag{13}$$

$$\ldots \xleftarrow{z^{-1}} q_{-2} \xleftarrow{z^{-1}} q_{-1} \xleftarrow{z^{-1}} q_0 = q \xleftarrow{z^{-1}} q_1 \xleftarrow{z^{-1}} q_2 \xleftarrow{z} \ldots \tag{14}$$

Further, we set for $k = 1, 2, \ldots$ and $n = 0, \pm 1, \ldots,$

$$F_{q_n}^k = F_{q_{n+k-1}} \circ \ldots \circ F_{q_n},$$

and

$$F_{q_n}^{-k} = F_{q_{n-k}}^{-1} \circ \ldots \circ F_{q_{n-1}}^{-1}.$$

Additionally for $k = 0$ we let $F_q^0[f] = f$. Summarizing the above mentioned relationships related to $z_n(q)$, we obtain the following assertions.

Lemma 2.13 *There holds the following duality relations*

$$q_{n-1} + \frac{1}{q_{n+1}} = 2, \; n = 0, \pm 1, \pm 2, \ldots \tag{15}$$

Proof. Making use the properties (9), we obtain

$$q_{n-1} = z^{-1}(q_n) = 2 - z(2 - q_n) = 2 - \frac{1}{z(q_n)} = 2 - \frac{1}{q_{n+1}}.$$

Lemma 2.14 *The following mappings hold:*

$$F_q^k : \mathcal{G}_{q_n} \to \mathcal{G}_{q_{k+n}}, \;\; k, \, n = 0, \pm 1, \pm 2, \ldots$$

$$\lim_{k \to \pm\infty} F_q^k \mathcal{G}_q = \mathcal{G}$$

where $\mathcal{G}$ is the set of classic Gaussians.

Lemma 2.15 *The series of mappings hold:*

$$\ldots \overset{F_{q_{-3}}}{\longrightarrow} \mathcal{G}_{q_{-2}} \overset{F_{q_{-2}}}{\longrightarrow} \mathcal{G}_{q_{-1}} \overset{F_{q_{-2}}}{\longrightarrow} \mathcal{G}_q \overset{F_q}{\longrightarrow} \mathcal{G}_{q_1} \overset{F_{q_1}}{\longrightarrow} \mathcal{G}_{q_2} \overset{F_{q_2}}{\longrightarrow} \ldots \tag{16}$$

$$\ldots \overset{F_{q_{-3}}^{-1}}{\longleftarrow} \mathcal{G}_{q_{-2}} \overset{F_{q_{-2}}^{-1}}{\longleftarrow} \mathcal{G}_{q_{-1}} \overset{F_{q_{-1}}^{-1}}{\longleftarrow} \mathcal{G}_q \overset{F_q^{-1}}{\longleftarrow} \mathcal{G}_{q_1} \overset{F_{q_1}^{-1}}{\longleftarrow} \mathcal{G}_{q_2} \overset{F_{q_2}^{-1}}{\longleftarrow} \ldots \tag{17}$$

3 MAIN RESULTS

3.1 q-correlated random variables

In this section we establish a q-generalization of the classical CLT. First we introduce some notions necessary to formulate the corresponding results. Let X be a random variable and $f(x)$ be an associated density. Denote

$$f_q(x) = \frac{[f(x)]^q}{\nu_q(f)},$$

where $\nu_q(f) = \int_{-\infty}^{\infty} [f(x)]^q \, dx$. The density $f_q(x)$ is commonly referred to as the *escort density* [22]. Further, introduce for X the notions *q-mean*, $\mu_q = \mu_q(X) = \int_{-\infty}^{\infty} x f_q(x) \, dx$, and *$q$-variance* $\sigma_q^2 = \sigma_q^2(X - \mu_q) = \int_{-\infty}^{\infty} (x - \mu_q)^2 f_q(x) \, dx$, and *$q$-moment of order k*, $M_{q,k} = M_{q,k}(X) = \int_{-\infty}^{\infty} x^k f_q(x) \, dx$, subject to all integrals used in these definitions to converge. We also use the notation $\nu_{q,k} = \int_{-\infty}^{\infty} x^k f^q(N; x) dx$, where $f(N; x)$ is the density associated with the sum $X_1 + \ldots + X_N$.

The formulas below can be verified directly.

Lemma 3.1 *The following formulas hold true*

1. $\mu_q(aX) = a\mu_q(X)$;
2. $\mu_q(X - \mu_q(X)) = 0$;
3. $\sigma_q^2(aX) = a^2\sigma_q^2(X)$;
4. $\mu_q(X_1 + ... + X_N) = \sum_{i=1}^N \mu_q(X_i)$;

Further, we introduce the notions of *q-correlation*, *q-convergence* and *q-normality*.

Definition 3.2 *Two random variables X and Y are said q-correlated if*

$$F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi). \tag{18}$$

The relation (18) can be rewritten as follows. Let f_X and f_Y be densities of X and Y respectively, and let f_{X+Y} be the density of $X+Y$. Then

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f_{X+Y}(x)dx = F_q[f_X](\xi) \otimes_q F_q[f_Y](\xi). \tag{19}$$

For $q = 1$ the condition (19) turns into the well known relation

$$F[f_X * f_Y] = F[f_X] \cdot F[f_Y]$$

between the convolution (noted $*$) of two densities and the multiplication of their (classical) Fourier images, and holds for independent X and Y. If $q \neq 1$, then q-correlation describes a special type of global correlation.

Definition 3.3 *A sequence of random variables X_N is said to be q-convergent if* $\lim_{N\to\infty} F_q[X_N](\xi) \in \mathcal{G}_q$ *locally uniformly by ξ for some $q < 3$. Further, we will say that q-limit of the sequence X_N is q_*-normal, if there are some $q_* < 3$ and $\beta > 0$ such that* $\lim_{N\to\infty} F_q(X_N) = F_{q_*}(G_{q_*}(\beta; x))$.

Remark 3.4 *In other words the q-limit of a sequence X_N is q_*-normal, if for some $q_* < 3$ and $\beta > 0$,* $\lim_{N\to\infty} X_N \in F_q^{-1} \circ F_{q_*}(G_{q_*}(\beta; x))$.

We will study limits of sums

$$Z_N = \frac{1}{D_N(q)}(X_1 + ... + X_N - N\mu_q), N = 1, 2, ...$$

where $D_N(q)$, $N = 1, 2, ...$, are some reals (scaling parameter), in the sense of Definition 3.3, when $N \to \infty$. Namely, the question we are interested in: *Is there a q-normal distribution that attracts the sequence Z_N? If yes, what is the admissible range of values of q?* For $q = 1$ the answer is well known and it is the content of the classical central limit theorem.

The q-generalization of the central limit theorem, we are suggesting in the present paper, is formulated as follows.

Theorem 1. *Assume a sequence $\{..., q_{-2}, q_{-1}, q_0, q_1, q_2, ...\}$ is given as (13) with $q_0 = q \in (1/2, 2)$. Let $X_1, ..., X_N, ...$ be a sequence of q_k-correlated for some $k \in \mathcal{Z}$ and identically distributed random variables with a finite q_k-mean μ_{q_k} and a finite second $(2q_k - 1)$-moment $\sigma^2_{2q_k-1}$.*

Then $Z_N = \frac{X_1+...+X_N - N\mu_{q_k}}{D_N(q)}$, with $D_N(q) = (\sqrt{N\nu_{2q_k-1}}\sigma_{2q_k-1})^{\frac{1}{2-q_k}}$, is q_k-convergent to a q_{k-1}-normal distribution as $N \to \infty$.

Remark 3.5 *Note the corresponding attractor is $G_{q_{k-1}}(\beta_k; x)$, where*

$$\beta_k = \Big(\frac{3-q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}}\Big)^{\frac{1}{2-q_{k-1}}}. \tag{20}$$

The proof of this theorem follows from Theorem 2 proved below and Lemma 2.15. Theorem 2 represents one element ($k=0$) in the series of assertions contained in Theorem 1.

Theorem 2. *Assume $1/2 < q \le 2$, or equivalently $1/3 < q^* < 5/3$, $q^* = z^{-1}(q)$. Let $X_1, ..., X_N, ...$ be a sequence of q-correlated and identically distributed random variables with a finite q-mean μ_q and a finite second $(2q-1)$-moment σ^2_{2q-1}.*

Then $Z_N = \frac{X_1+...+X_N - N\mu_{2q-1}}{D_N(q)}$, with $D_N(q) = (\sqrt{N\nu_{2q-1}}\sigma_{2q-1})^{\frac{1}{2-q}}$, is q-convergent to a q_{-1}-normal distribution as $N \to \infty$. The corresponding q_{-1}-Gaussian is $G_{q_{-1}}(\beta; x)$, with $\beta = \Big(\frac{3-q_{-1}}{4qC_{q_{-1}}^{2q_{-1}-2}}\Big)^{\frac{1}{2-q_{-1}}}$.

Proof. Let f be the density associated with $X_1 - \mu_q$. First we evaluate $F_q(X_1-\mu_q) = F_q(f(x))$. Using Lemma 2.2 we have

$$F_q[f](\xi) = \int_{-\infty}^{\infty} (e_q^{ix\xi}) \otimes_q f(x)dx = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} dx. \tag{21}$$

Making use of the asymptotic expansion $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$, $x \to 0$, we can rewrite the right hand side of (21) in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x)\left(1 + \frac{ix\xi}{[f(x)]^{1-q}} - q/2\frac{x^2\xi^2}{[f(x)]^{2(1-q)}} + o(\frac{x^2\xi^2}{[f(x)]^{2(1-q)}})\right)dx =$$

$$1 + i\xi\mu_q\nu_q - (q/2)\xi^2\sigma^2_{2q-1}\nu_{2q-1} + o(\xi^2),\ \xi \to 0. \tag{22}$$

In accordance with the condition of the theorem and Lemma 3.1, $\mu_q = \mu_q(X_1 - \mu_q) = 0$. Denote $Y_j = D_N(q)^{-1}(X_j - \mu_q), j = 1, 2, ...$ Then $Z_N = Y_1 + ... + Y_N$. Further, it is readily seen that, for a given random variable X and real $a > 0$, there holds $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$. It follows from this relation that $F_q(Y_1) = F_q[f](\frac{\xi}{\sqrt{N\nu_{2q-1}\sigma_{2q-1}}})$. Moreover, it follows from the q-correlation of $Y_1, Y_2, ...$ (which is an obvious consequence of the q-correlation of $X_1, X_2, ...$) and the associativity of the q-product that

$$F_q[Z_N](\xi) = F_q[f](\frac{\xi}{\sqrt{N\nu_{2q-1}}\sigma_{2q-1}}) \otimes_q ... \otimes_q F_q[f](\frac{\xi}{\sqrt{N\nu_{2q-1}}\sigma_{2q-1}})\ (N \text{ factors}). \tag{23}$$

Hence, making use of properties of the q-logarithm, from (23) we obtain

$$\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f](\frac{\xi}{\sqrt{N\nu_{2q-1}}\sigma_{2q-1}}) = N\ln_q(1 - \frac{q}{2}\frac{\xi^2}{N} + o(\frac{\xi^2}{N})) =$$

$$-\frac{q}{2}\xi^2 + o(1),\ N \to \infty, \tag{24}$$

locally uniformly by ξ.

Consequently, locally uniformly by ξ,

$$\lim_{N\to\infty} F_q(Z_N) = e_q^{-(q/2)\xi^2} \in \mathcal{G}_q. \tag{25}$$

Thus, Z_N is q-convergent.

In accordance with Corollary 2.7 for q_{-1} and some β we have $F_{q_{-1}}(G_{q_{-1}}(\beta;x)) = e_q^{-(q/2)\xi^2}$. Let us now find β. It follows from Corollary 2.7 (see (11)) that $\beta_*(q_{-1}) = q/2$. Solving this equation with respect to β we obtain

$$\beta = \left(\frac{3-q_{-1}}{4qC_{q_{-1}}^{2(q_{-1}-1)}}\right)^{\frac{1}{2-q_{-1}}}, \tag{26}$$

where $q = z(q_{-1})$. The explicit form of the corresponding q_{-1}-Gaussian reads as

$$G_{q_{-1}}(\beta;x) = C_{q_{-1}}^{-1}\left(\frac{3-q_{-1}}{2C_{q_{-1}}^{q_{-1}-1}\sqrt{1+q_{-1}}}\right)^{\frac{1}{2-q_{-1}}} e_{q_{-1}}^{-\left(\frac{(3-q_{-1})^2}{4(1+q_{-1})C_{q_{-1}}^{2(q_{-1}-1)}}\right)^{\frac{1}{2-q_{-1}}}x^2}.$$

Obviously, $\frac{q+1}{3-q} = 1$ if and only if $q = 1$. This fact yields the following corollary.

Corollary 3.6 *Let $X_1, ..., X_N, ...$ be a given sequence of q-correlated and identically distributed random variables with a q-mean μ_q and a finite second moment σ^2_{2q-1}. Then $Z_N = D_N^{-1}(q)\,(X_1 + ... + X_N - N\mu_q)$ is q-convergent to a q-normal distribution if and only if $q = 1$, that is, in the classic case.*

Remark 3.7 *As is seen from (25), if $q < 0$, then the corresponding q_{-1}-Gaussian is not normalizable. For densities with a finite support Theorem 1 and Theorem 2 can be extended for $q \in (0, 1/2]$ as well.*

3.2 Generalization of the previous theorem

Obviously, Theorem 1 is true if a sequence $X_1, X_2, ..., X_N$ is *asymptotically q-correlated*, i.e, if they are mutually q-correlated for all $N > N_0$ starting from a number $N_0 > 1$ [5]. We shall now extend the domain of validity of Theorem 1 by showing that, although the hypothesis used in Theorem 1 are sufficient, they are not necessary. We can somewhat relax them and the attractors still remain the same. In what follows, the particular case $\rho = 0$ (see definition just below) of Theorem 3 recovers Theorem 1. Note that in this section we use $q^* = z^{-1}(q)$ instead of q_{-1}.

Theorem 3. *Assume $1/2 < q \le 2$ or equivalently $1/3 < q^* < 5/3$, $q^* = z^{-1}(q)$. Let $X_1, ..., X_N, ...$ be a sequence of identically distributed and globally correlated random variables satisfying the conditions*

1. *$\mu_q = 0$ and $\nu_{2q-1,2} \sim N^{1+\rho}$, $0 \le \rho < 1$;*
2. *$\nu_{3q-2,3} \sim N^{\gamma}$, where $\gamma < \frac{3(1+\rho)}{2}$.*

Then $Z_N = \frac{X_1+...+X_N}{D_N(q)}$, with a scaling parameter $D_N(q) \sim N^{\frac{1+\rho}{2}}$, $N \to \infty$, is q-convergent to a q^*-normal distribution as $N \to \infty$. The corresponding q^*-Gaussian is $G_{q^*}(\beta;x)$, with $\beta = \left(\frac{3-q^*}{4qC_{q^*}^{2q^*-2}}\right)^{\frac{1}{2-q^*}}$.

[5] Such a strong cutoff might be relaxed into a softer one.

Proof. Let $f(N;x)$ and $g(N;x)$ be the density functions of the sums $X_1+...+X_N$ and $Y_1+...+Y_N$, respectively, where again $Y_j = D_N(q)^{-1}(X_j - \mu_q), j = 1, ...N$. Evaluate $F_q[f(N;x)]$. Using Lemma 2.2 we have

$$F_q[f(N;\cdot)](\xi) = \int_{-\infty}^{\infty}(e_q^{ix\xi}) \otimes_q f(N;x)dx =$$

$$\int_{-\infty}^{\infty} f(N;x)e_q^{\frac{ix\xi}{[f(N;x)]^{1-q}}}dx. \tag{27}$$

Again using the asymptotic expansion $e_q^x = 1 + x + \frac{q}{2}x^2 + O(x^3)$, $x \to 0$, we obtain

$$F_q[f(N;x)](\xi) = \int_{-\infty}^{\infty} f(N;x)\{1 + \frac{ix\xi}{(f(N;x))^{1-q}} -$$

$$q/2\frac{x^2\xi^2}{(f(N;x))^{2(1-q)}} + O(\frac{x^3\xi^3}{(f(N;x))^{3(1-q)}})\}dx =$$

$$= 1 - (q/2)\xi^2\,\nu_{2q-1,2}(f(N;\cdot)) + O(\xi^3\nu_{3q-2,3}(f(N;\cdot))),\ N \to \infty. \tag{28}$$

Taking into account the relationship between X_j and Y_j, we have

$$F_q[Z_N](\xi) = F_q[f(N;\cdot)](\frac{\xi}{D_N(q)}) = 1 - (q/2)\xi^2 + O(\frac{\nu_{3q-2,3}}{N^{3/2(\rho+1)}}),\ N \to \infty. \tag{29}$$

Taking into account Condition 2 of Theorem, the last term in (29) can be evaluated as $CN^{-(3/2(1+\rho)-\gamma)} = o(1)$, $N \to \infty$, where C is a constant. Hence,

$$F_q[Z_N](\xi) = 1 - (q/2)\xi^2 + o(1),\ N \to \infty, \tag{30}$$

locally uniformly by ξ. Further, taking q-logarithm of both sides of (30), we have

$$\lim_{N\to\infty} \ln_q(F_q(Z_N)) = -(q/2)\xi^2. \tag{31}$$

Consequently, locally uniformly by ξ,

$$\lim_{N\to\infty} F_q(Z_N) = e_q^{-(q/2)\xi^2} \in \mathcal{G}_q. \tag{32}$$

Thus, Z_N is q-convergent.

The rest of the proof follows in exactly the same way as in Theorem 1.

Remark 3.8 *Let us note that the conditions $\rho < 1$ and $\gamma < \frac{3(1+\rho)}{2}$ are necessary to guarantee that the appropriately scaled escort third moment of the sum $X_1 + ... + X_N$ vanishes in the $N \to \infty$ limit.*

4 CONCLUSION

In the present paper we studied a q-generalization of the classic central limit theorem. As is known, q-Gaussians extremize, under appropriate constraints, the entropy S_q. The classic analog of this fact is that the usual Gaussian maximizes the classic Boltzmann-Gibbs-Shannon entropy. Following this correspondence, it is expected that there exists an entire class of q- central limit theorems. In other words, normalized sums of sequences of identically distributed random variables with a finite

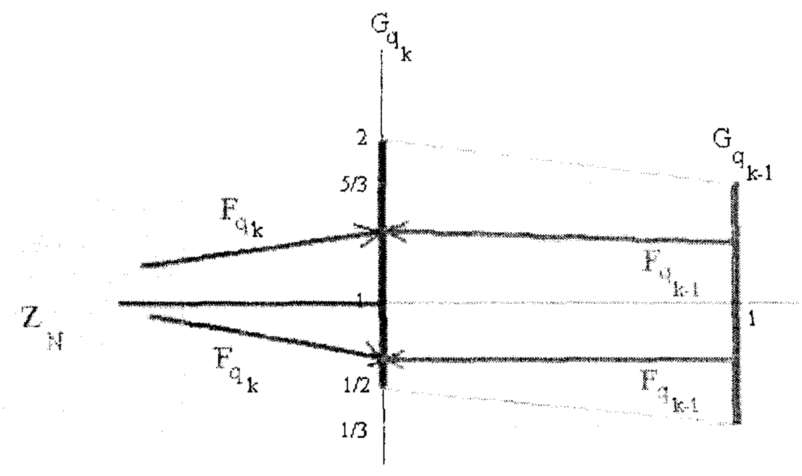

Figure 1: Schematic representation of the q-CLT: Z_N represents the set of rescaled sums of all q_k-correlated random variables. The q_k-Fourier transforms of these sums belong to $\mathcal{G}_{q_k}$, which in turn is the $F_{q_{k-1}}$ image of $\mathcal{G}_{q_{k-1}}$. The process described in this scheme reflects the q_k-convergence of Z_N to a q_{k-1}-Gaussian. These transformations admit only one fixed point, namely $q_k = 1$, corresponding to the classical CLT (represented here as a horizontal straight line).

q-variance must converge to q-Gaussians. Theorem 1 represents one of the possible generalizations of the classic central limit theorem for a sequence of q-correlated random variables. The notion of q-correlation coincides with the classic notion of independence if $q = 1$, and characterizes a specific type of global correlations otherwise. Theorem 3 considers more general sequences of correlated random variables, which are nevertheless attracted by the same q-Gaussians.

At the same time the corresponding q-normal distribution is described exactly by the $z^{-1}(q)$-Gaussian, see Figure 1. In the classic case q- or $z^{-1}(q)$-Gaussians do not differ. So, Corollary 3.6 says that such duality is a specific feature of the statistical q-theory, which comes from the specific definition of the q-exponential.

We conclude the paper by making an important remark. The classical CLT may in principle be generalized in various manners, each of them referring to global correlations of specific kinds. A first example is the model numerically discussed in [4]. The correlations were introduced, in a scale-invariant manner, through a q-product in the space of the joint probabilities of N binary variables, with $0 \leq q \leq 1$. It was numerically shown that the attractors are (double-branched) Q-Gaussians, with $Q = 2 - \frac{1}{q} \in (-\infty, 1]$, and that the model is superdiffusive [21] (i.e., space x scales with time t, for large values of t, as $x \propto t^{\delta/2}$, with $1 \leq \delta \leq 2$). The relation $Q = 2 - \frac{1}{q}$ corresponds to the particular case $k = -1$ of the present Theorem 1. It comes from Lemma 2.13, with $q_{-2} = 2 - \frac{1}{q_0} = 2 - \frac{1}{q}$, which holds when k=-1. Notice, however, that these two models differ in other aspects. Indeed, although they share the relation $Q = 2 - \frac{1}{q}$, there δ's are different. In the model introduced in [4] only superdiffusion occurs, with δ monotonically decreasing from 2 to 1 when Q increases from $-\infty$ to 1 [21]. In contrast, in the $k = -1$ model associated with the present Theorem 1, we have $\delta = q = \frac{1}{2-Q}$, with $q \in (1/2, 2)$, hence $\delta \in (1/2, 2)$.

A second example is suggested by the exact stable solutions of a nonlinear Fokker-Planck equation [6]. The correlations are introduced through a $q = 2 - Q$ exponent in the spatial member of the equation (the second derivative term). The solutions are Q-Gaussians with $Q \in (-\infty, 3)$, and $\delta = 2/(3-Q) \in [0, \infty]$, hence both superdiffusion and subdiffusion can exist in addition to normal diffusion. This model is particularly interesting because the scaling $\delta = 2/(3-Q)$ was conjectured in [23], and it was verified in various experimental and computational studies [24, 25, 26].

A third example is the family of models presented here. The correlations are introduced through q_k-products of q_k-Fourier transforms, where $q_k = \frac{2q+k(1-q)}{2+k(1-q)}$, $q \in (1/2,\, 2)$. The attractors are q_{k-1}-Gaussians and $\delta = 1/(2-q_{k-1})$, as can be seen from Eq. (20). Applying Lemma 2.13 we obtain $2 - q_{k-1} = 1/q_{k+1}$, henceforce $\delta = q_{k+1}$. Thus the triplet $(q_{k-1},\, q_k,\, q_{k+1})$ characterises features of the system under study identifying the type of correlation, the corresponding attractor, and the scaling rate.

In the particular case, $k = 1$, we have $\delta = 1/(2-q)$. This coincides with the nonlinear Fokker-Planck equation mentioned above. Indeed, in our theorem we required the finitness of $(2q-1)$-variance. Denoting $2q - 1 = Q$, we get $\delta = 1/(2-q) = 2/(3-Q)$. Notice, however, that this example differs from the nonlinear Fokker-Planck above. Indeed, although we do obtain, from the finiteness of the second momentum, the same expression for δ, the attractor is not a Q-Gaussian, but rather a q-Gaussian, with $q = (Q+1)/2$.

Summarizing, the present Theorems 1 and 3 suggest a quite general and rich structure at the basis of nonextensive statistical mechanics. Moreover, they recover, as particular instances, central relations emerging in the above first and second examples. The structure we have presently shown might pave a deep understanding of the so-called q-triplet $(q_{sen}, q_{rel}, q_{stat})$, where *sen*, *rel* and *stat* respectively stand for *sensitivity to the initial conditions*, *relaxation*, and *stationary state* [27, 28] in nonextensive statistics. This remains however as a challenge at the present stage.

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