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SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

31 July - 8 August 2006

Nonextensive Statistical Mechanics: Theoretical, Experimental, Observational and Computational Aspects

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NONEXTENSIVE STATISTICAL MECHANICS: THEORETICAL, EXPERIMENTAL, OBSERVATIONAL AND COMPUTATIONAL ASPECTS

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Santa Fe Institute, New Mexico, USA

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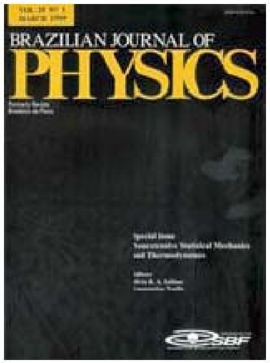
Trieste, July 2006

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS



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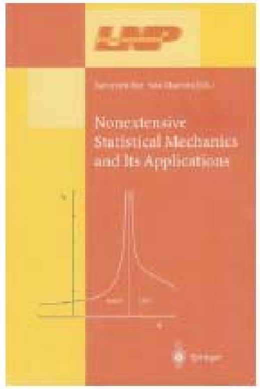

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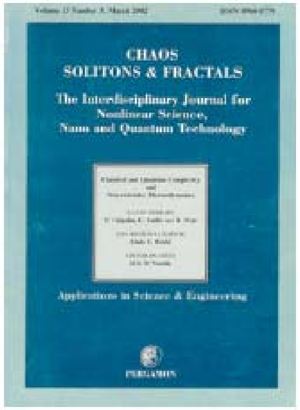

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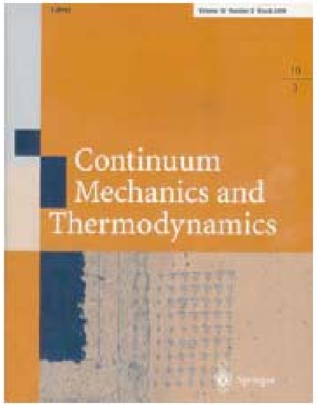

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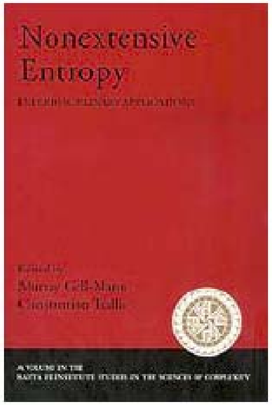

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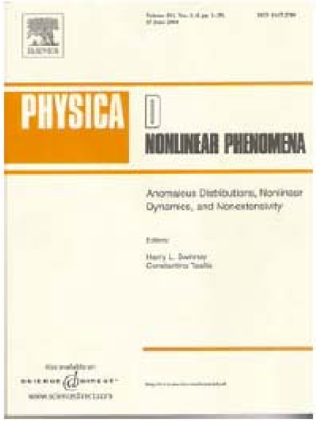

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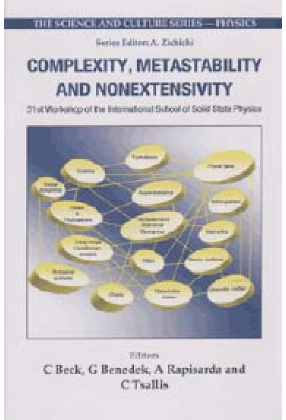

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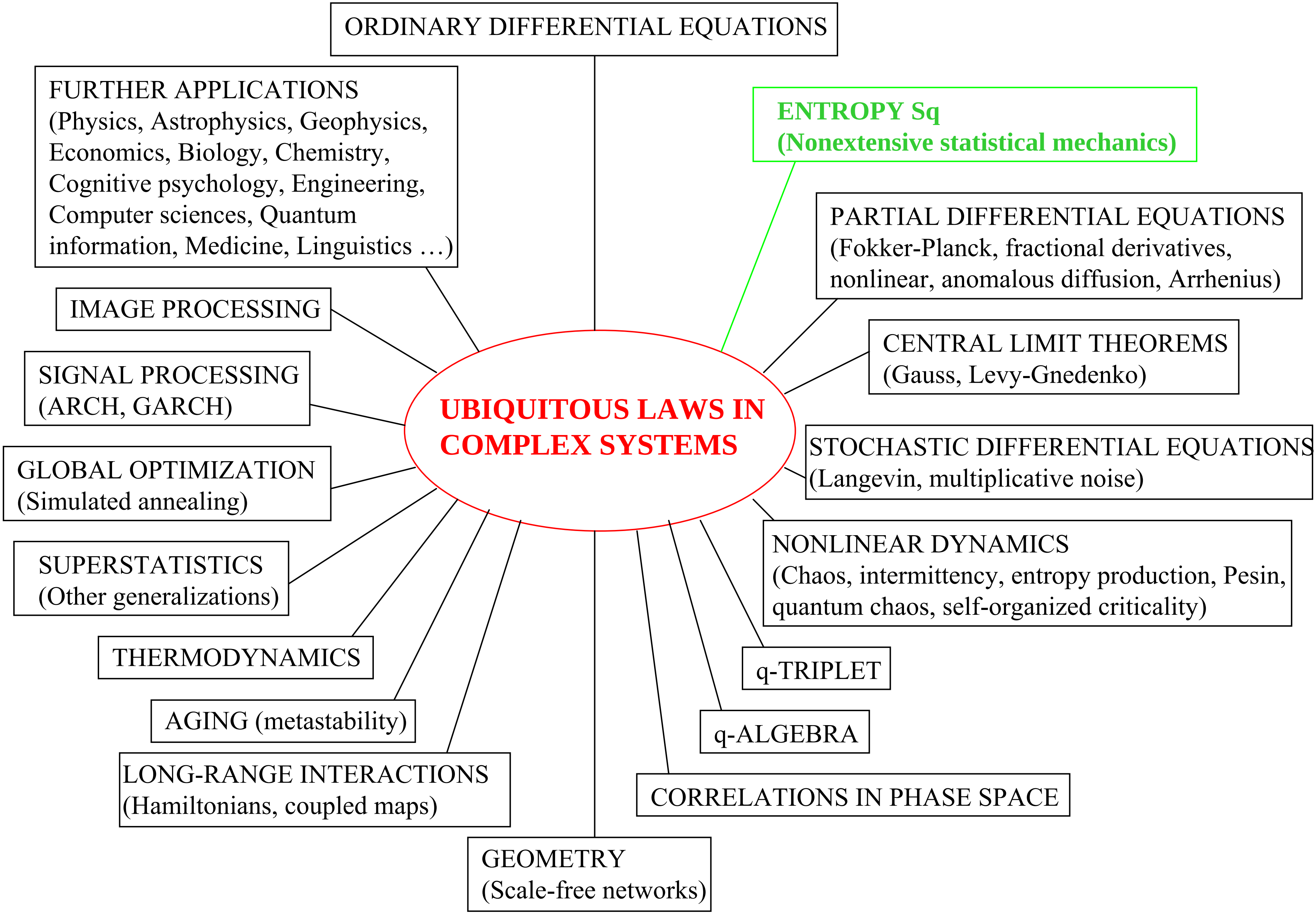
http://tsallis.cat.cbpf.br/biblio.htm


ORDINARY DIFFERENTIAL EQUATIONS
FURTHER APPLICATIONS (Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)
ENTROPY Sq (Nonextensive statistical mechanics)
PARTIAL DIFFERENTIAL EQUATIONS (Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)
IMAGE PROCESSING
CENTRAL LIMIT THEOREMS (Gauss, Levy-Gnedenko)
SIGNAL PROCESSING (ARCH, GARCH)
UBIQUITOUS LAWS IN COMPLEX SYSTEMS
STOCHASTIC DIFFERENTIAL EQUATIONS (Langevin, multiplicative noise)
GLOBAL OPTIMIZATION (Simulated annealing)
NONLINEAR DYNAMICS (Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)
SUPERSTATISTICS (Other generalizations)
THERMODYNAMICS
q-TRIPLET
AGING (metastability)
q-ALGEBRA
LONG-RANGE INTERACTIONS (Hamiltonians, coupled maps)
CORRELATIONS IN PHASE SPACE
GEOMETRY (Scale-free networks)

TRIADIC CANTOR SET:

$$d_f = \frac{\ln 2}{\ln 3} = 0.6309...$$

Hence the interesting measure is given by

$$(10\ cm)^{0.6309...} \cong 4.275\ cm^{0.6309}$$

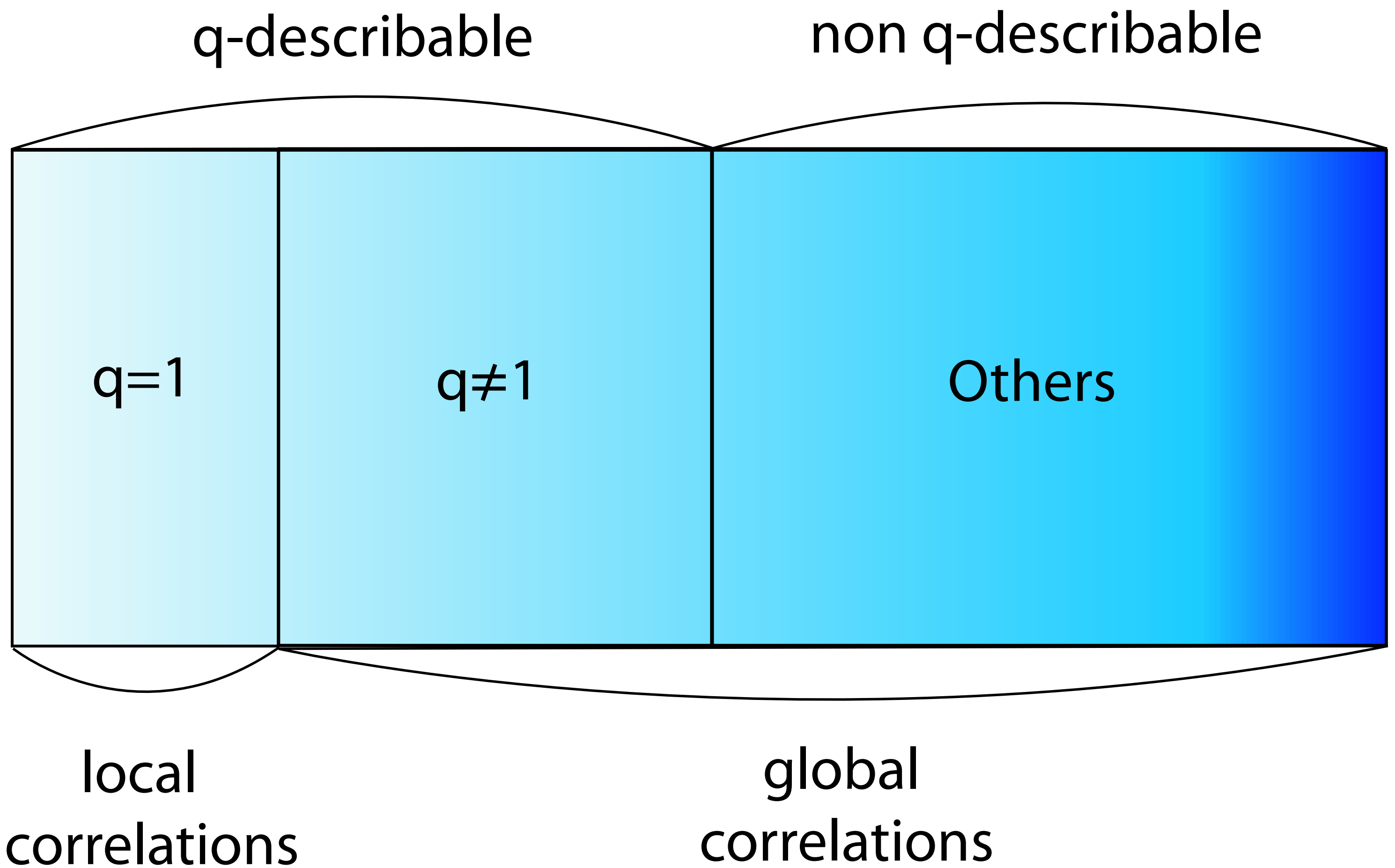
It is the natural (or artificial or social) system itself which, through its geometrical-dynamical properties, determines the specific informational tool --- entropy --- to be meaningfully used for the study of its (thermo) statistical properties.

ENTROPIC FORMS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \ (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG entropy $(q = 1)$	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Nonextensive entropy $(q = \Re)$ $(q \neq 1)$	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Possible generalization of
Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]



C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186 (2005)
[European Physical Society]

Ludwig BOLTZMANN

Vorlesungen uber Gastheorie (Leipzig, 1896)
Lectures on Gas Theory, transl. S. Brush
(Univ. California Press, Berkeley, 1964), page 13

The forces that two molecules impose one onto the other during an interaction can be completely arbitrary, only assuming that their sphere of action is very small compared to their mean free path.

J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics
(C. Scribner's Sons, New York, 1902;
Yale University Press, New Haven, 1948;
OX Bow Press, Woodbridge, Connecticut, 1981).

Page 35:

In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] *to have a finite value, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

Enrico FERMI

Thermodynamics (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

Laszlo TISZA

Generalized Thermodynamics
(MIT Press, Cambridge, Massachusetts, 1961)

The situation is different for the additivity postulate *Pa2*, the validity of which cannot be inferred from general principles. We have to require that the interaction energy between thermodynamic systems be negligible. This assumption is closely related to the homogeneity postulate *Pd1*. From the molecular point of view, additivity and homogeneity can be expected to be reasonable approximations for systems containing many particles, provided that the intramolecular forces have a short range character.

Peter LANDSBERG

Thermodynamics and Statistical Mechanics
(1978)

The presence of long-range forces causes important amendments to thermodynamics, some of which are not fully investigated as yet.

Is equilibrium always an entropy maximum?
J. Stat. Phys. **35**, 159 (1984).

[...] in the case of systems with long-range forces and which are therefore nonextensive (in some sense) some thermodynamic results do not hold. [...] The failure of some thermodynamic results, normally taken to be standard for black hole and other nonextensive systems has recently been discussed. [...] If two identical black holes are merged, the presence of long-range forces in the form of gravity leads to a more complicated situation, and the entropy is nonextensive.

L.G. TAFF

Celestial Mechanics
(John Wiley, New York, 1985)

This means that the total energy of any finite collection of self-gravitating mass points does not have a finite, extensive (e.g., proportional to the number of particles) lower bound. Without such a property there can be no rigorous basis for the statistical mechanics of such a system (Fisher and Ruelle 1966). Basically it is that simple. One can ignore the fact that one knows that there is no rigorous basis for one's computer manipulations; one can try to improve the situation, or one can look for another job.

David RUELLE

Thermodynamical Formalism -
The Mathematical Structures of Classical Equilibrium Statistical Mechanics
(page 1 of both 1978 and 2004 editions)

The formalism of equilibrium statistical mechanics -- which we shall call *thermodynamic formalism* -- has been developed since G.W. Gibbs to describe the properties of certain physical systems. [...] While the physical justification of the thermodynamic formalism remains quite insufficient, this formalism has proved remarkably successful at explaining facts.
The mathematical investigation of the thermodynamic formalism is in fact not completed: the theory is a young one, with emphasis still more on imagination than on technical difficulties. This situation is reminiscent of pre-classic art forms, where inspiration has not been castrated by the necessity to conform to standard technical patterns.

(page 3) The problem of why the Gibbs ensemble describes thermal equilibrium (at least for “large systems”) when the above physical identifications have been made is deep and incompletely clarified.

[The first equation is dedicated to define the *BG* entropy form. It is introduced after the words “we define its *entropy*” without any kind of justification or physical motivation.]

Floris TAKENS

Structures in Dynamics – Finite Dimensional Deterministic Studies

Eds. H.W. Broer, F. Dumortier, S.J. van Strien and F. Takens, p. 253 (North-Holland, Amsterdam, 1991)

The values of p_i are determined by the following dogma: if the energy of the system in the i-th state is E_i and if the temperature of the system is T then:

$$p_i = \frac{e^{-E_i/kT}}{Z(T)}, \text{ where } Z(T) = \sum_i e^{-E_i/kT}$$

(this last constant is taken so that $\sum_i p_i = 1$).

This choice of p_i is called the Gibbs distribution. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as " deep and incompletely clarified ".

ABOUT ORDINARY DIFFERENTIAL EQUATIONS:

$(i)\quad \frac{dy}{dx} = 0 \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = 1$

Inverse function: $x = 1$ ← (*in the sense of x = y symmetry*)

$(ii)\quad \frac{dy}{dx} = 1 \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = 1 + x$

Inverse function: $x = y - 1$

$(iii)\quad \frac{dy}{dx} = y \quad \text{with} \quad y(0) = 1$

$\Rightarrow y = e^x$ (EXPONENTIAL)

Inverse function: $x = \ln y$

Property: $\ln(y_A\, y_B) = \ln y_A + \ln y_B$

ABOUT ORDINARY DIFFERENTIAL EQUATIONS:

(i) $\frac{dy}{dx} = 0$ with $y(0) = 1$

$\Rightarrow y = 1$

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(iii) $\frac{dy}{dx} = y$ with $y(0) = 1$

$\Rightarrow y = e^x$ (EXPONENTIAL)

Inverse function: $x = \ln y$

Property: $\ln(y_A\, y_B) = \ln y_A + \ln y_B$

(iv) Unification:

$$\frac{dy}{dx} = y^q \quad (q \in \mathcal{R}) \quad \text{with} \quad y(0) = 1$$

$$\Rightarrow y = [1 + (1-q)x]^{\frac{1}{1-q}} \equiv e_q^x \quad \text{(POWER-LAW)}$$

Inverse function: $x = \frac{y^{1-q}-1}{1-q} \equiv \ln_q y$

Property: $\ln_q(y_A\, y_B) = \ln_q y_A + \ln_q y_B + (1-q)(\ln_q y_A)(\ln_q y_B)$

[$q = 1$, $q = 0$ and $q \to \infty$ recover the three previous cases]

ABOUT MEAN VALUES:

(i) Equiprobability: $p_i = \frac{1}{W} \quad (\forall i)$

$$S_{BG} = k \ln W \quad (k = \text{positive constant})$$

Naturally generalized into:

$$S_q = k \ln_q W$$

(ii) General: (not necessarily equiprobability)

$$S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i} \equiv \langle k \ln \frac{1}{p_i} \rangle$$

"surprise" or "unexpectedness"

[We verify that $p_i = \frac{1}{W} \quad (\forall i)$ recovers $S_{BG} = k \ln W$]

Naturally generalized into:

$$S_q(\{p_i\}) \equiv \langle k \ln_q \frac{1}{p_i} \rangle = k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$$

"q-surprise" or "q-unexpectedness"

hence: $S_q(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}$

[We verify that $p_i = \frac{1}{W} \quad (\forall i)$ recovers $S_q = k \ln_q W$]

Property: $p_{ij}^{A+B} = p_i^A p_j^B \Rightarrow$

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$$

ABOUT BIAS:

$[0 < p_i < 1 \quad (\forall i)]$

$$p_i^q \; > p_i \quad if \quad q < 1$$
$$< p_i \quad if \quad q > 1$$
$$= p_i \quad if \quad q = 1 \quad (BG)$$

(i) $S_q = (\{p_i\})$ *should be invariant under permutation.*
The simplest manner is to be

$$S_q(\{p_i\}) = f(\sum_{i=1}^{W} p_i^q)$$

(ii) *The simplest function* $f(x)$ *is*

$$S_q(\{p_i\}) = a + b \sum_{i=1}^{W} p_i^q$$

(iii) *Certainty must correspond to* $S_q = 0$
In other words $\quad p_i = 1 \quad for \; i = i_0$

$$= 0 \quad otherwise$$

hence $\quad a + b = 0$
hence $\quad S_q(\{p_i\}) = a(1 - \sum_{i=1}^{W} p_i^q)$

(iv) *For* $q \to 1$ *we must recover* $S_{BG}(\{p_i\})$.
Using $\quad p_i^{q-1} \sim 1 + (1-q)\ln p_i \quad$ we obtain

$$S_q(\{p_i\}) \sim -a(q-1)\sum_{i=1}^{W} p_i \ln p_i$$

consequently, by identifying $a(q-1) = k$ we obtain

$$S_q(\{p_i\}) = k\,\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$$

ABOUT REACTION UNDER BIAS:

[See Abe, Phys. Lett. A **224**, 326 (1997)]

(i) *Testing the function under* **translation** *of the bias x:*

$$S_{BG}(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i = -k \, [\frac{d}{dx} \sum_{i=1}^{W} p_i^x]_{x=1}$$

(ii) *Testing the function under* **dilatation** *of the bias x:*

We replace $\frac{d}{dx}$ by Jackson's 1909 generalized derivative

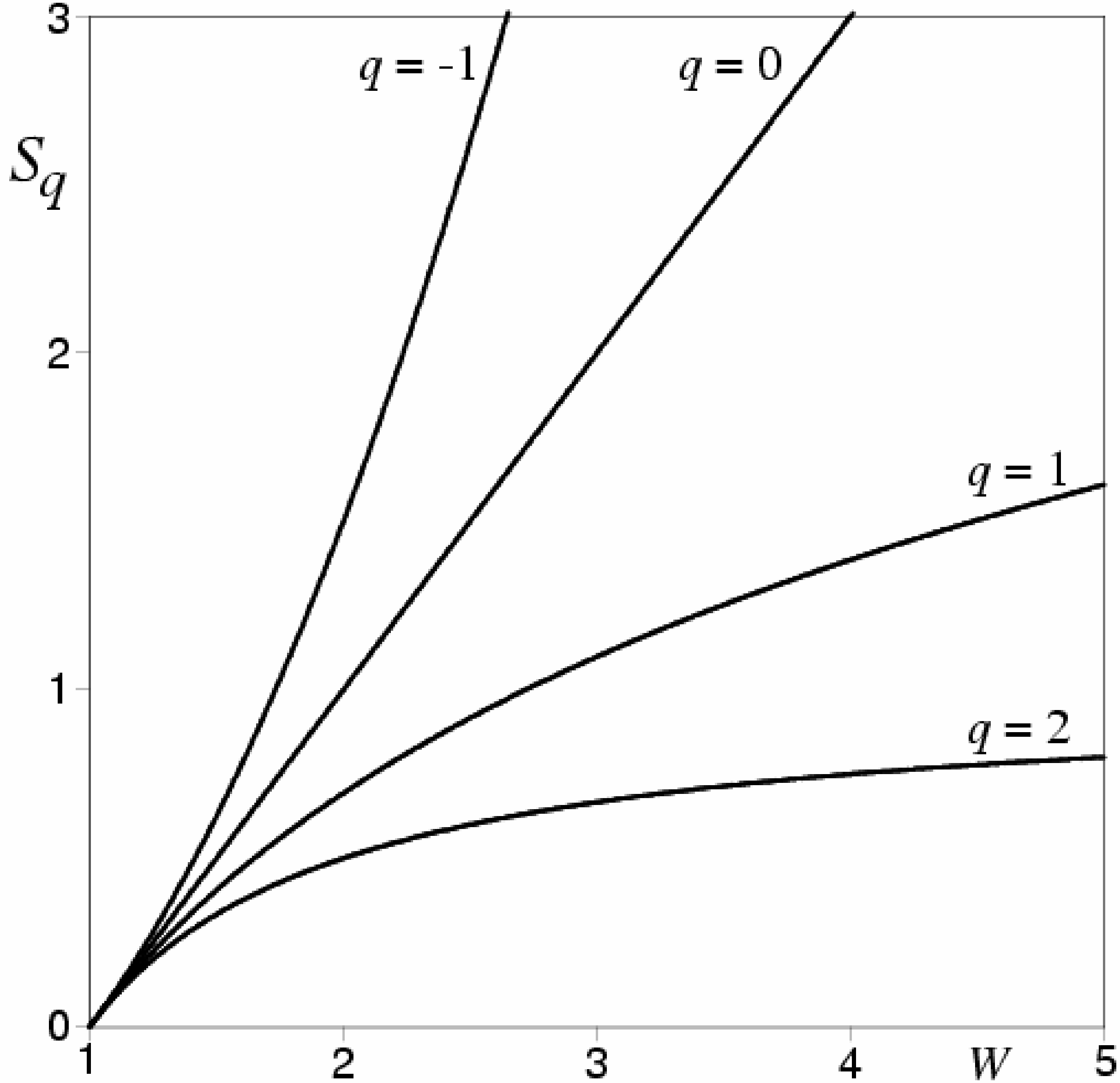
$$D_q \, h(x) \equiv \frac{h(qx)-h(x)}{qx-x} \quad [D_1 h(x) = \frac{dh(x)}{dx}]$$

and obtain

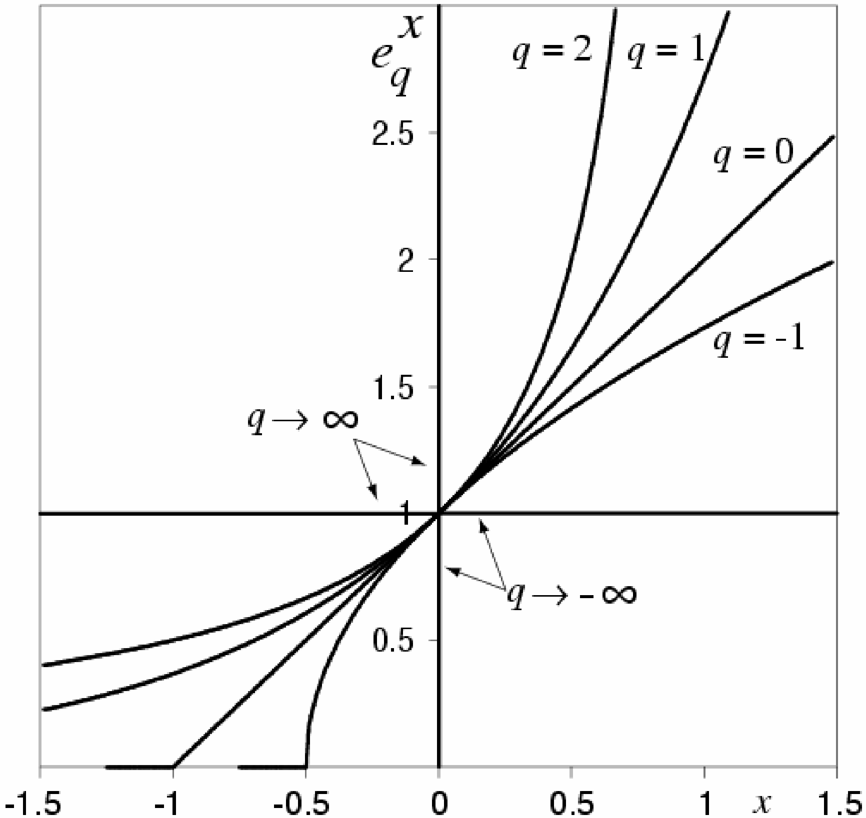
$$S_q(\{p_i\}) = -k \, [D_q \sum_{i=1}^{W} p_i^x]_{x=1}$$

hence

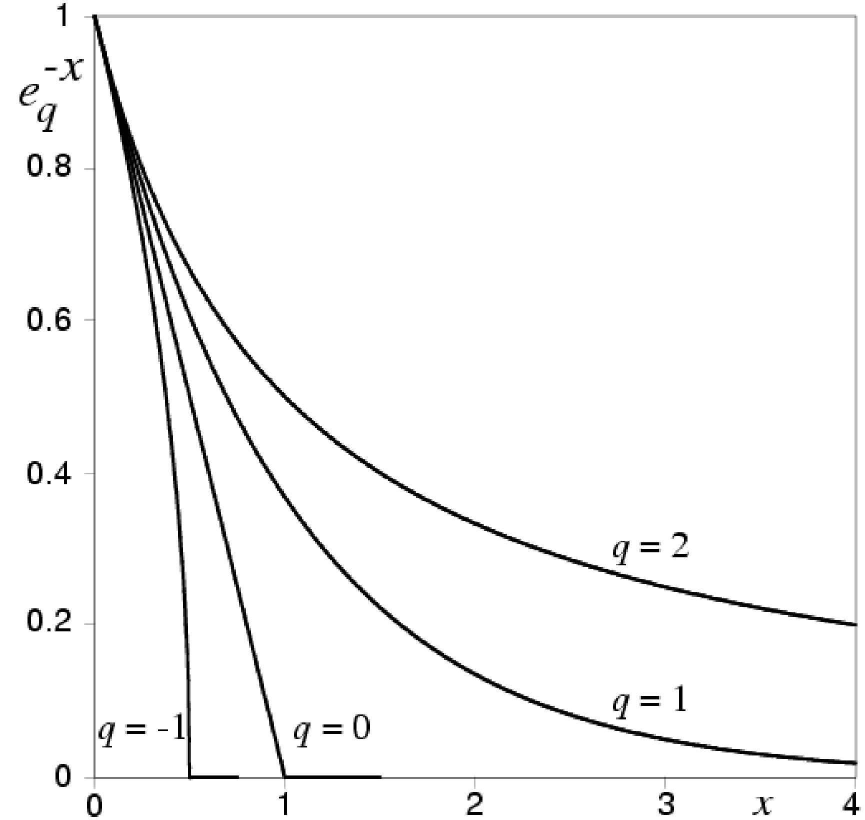
$$S_q(\{p_i\}) = k \, \frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$$



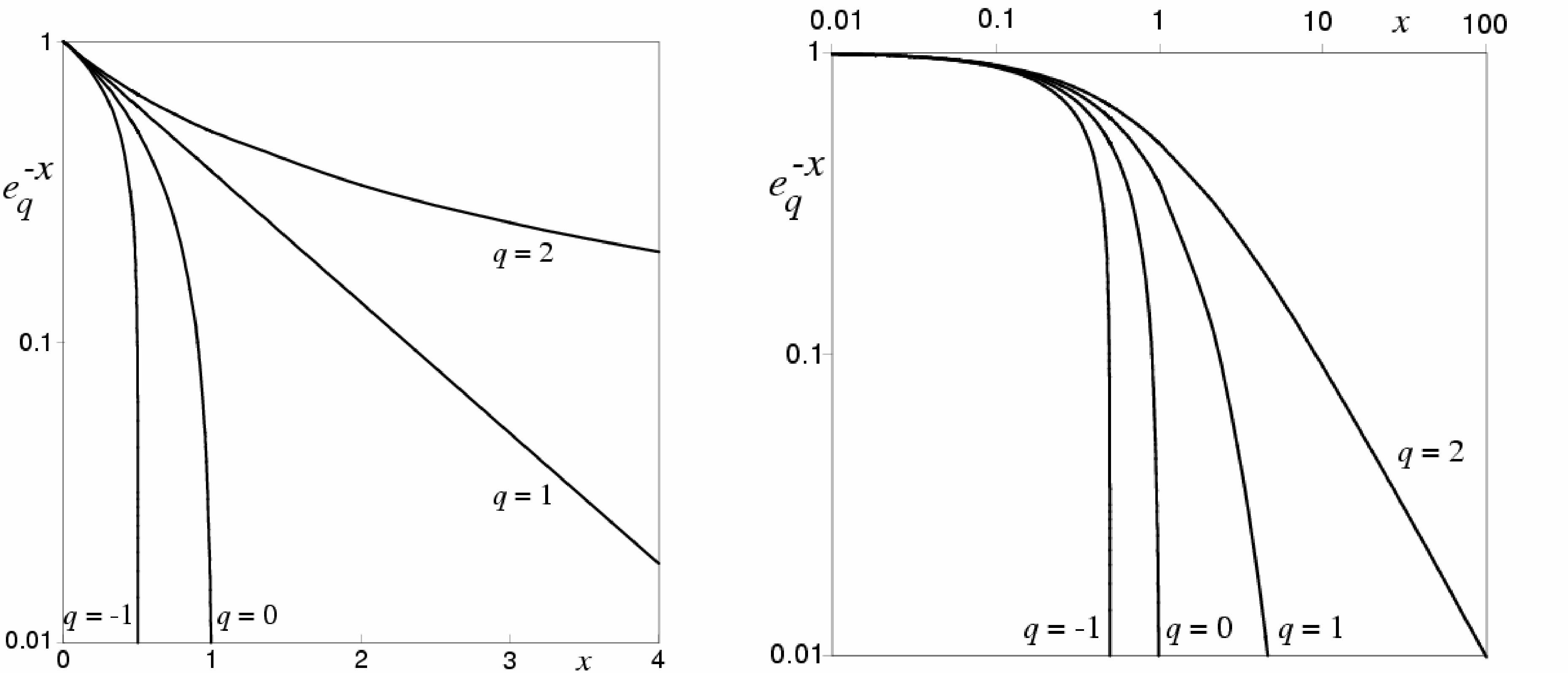
3
2
1
0
S_q
$q = -1$
$q = 0$
$q = 1$
$q = 2$
1
2
3
4
W
5



e_q^x
$q = 2$
$q = 1$
$q = 0$
$q = -1$
$q \to \infty$
$q \to -\infty$
2.5
2
1.5
1
0.5
-1.5
-1
-0.5
0
0.5
1
x
1.5



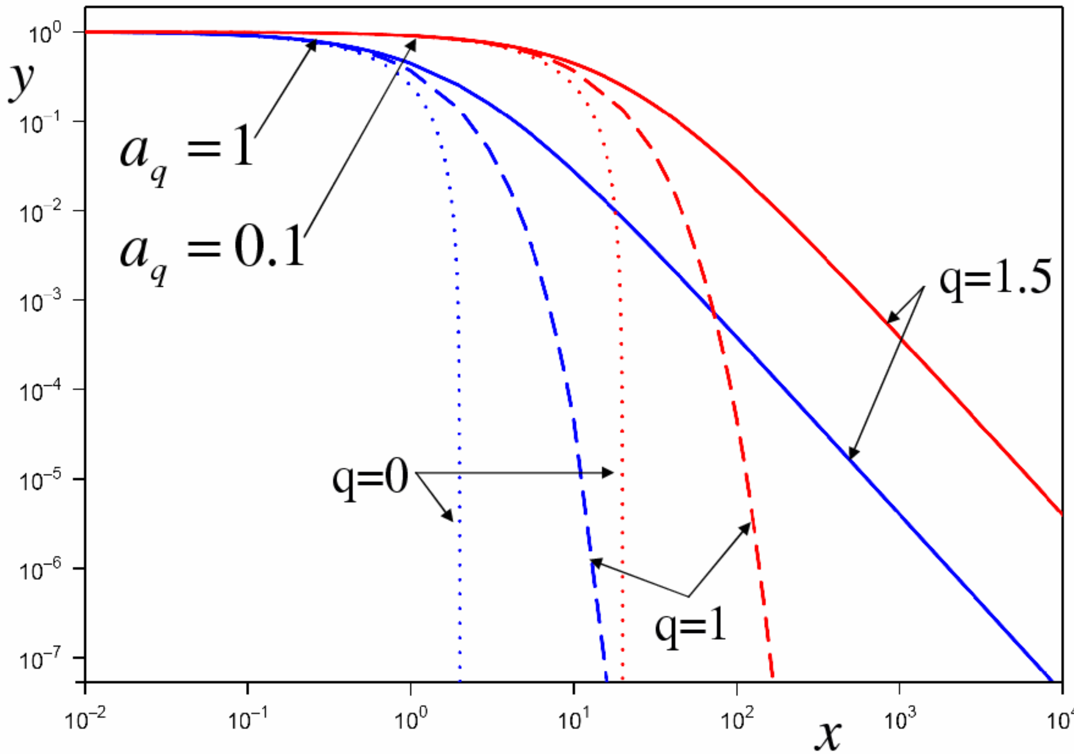
e_q^{-x}
1
0.8
0.6
0.4
0.2
0
$q = 2$
$q = 1$
$q = -1$
$q = 0$
0
1
2
3
x
4



1
0.1
0.01
e_q^{-x}
$q = 2$
$q = 1$
$q = -1$
$q = 0$
0
1
2
3
x
4
0.01
0.1
1
10
x
100
1
0.1
0.01
e_q^{-x}
$q = 2$
$q = -1$
$q = 0$
$q = 1$

$$\frac{dy}{dx} = -a_q y^q \quad with \quad y(0) = 1$$

$$\Rightarrow y = \frac{1}{[1+(q-1)a_q x]^{\frac{1}{q-1}}} \equiv e_q^{-a_q x}$$



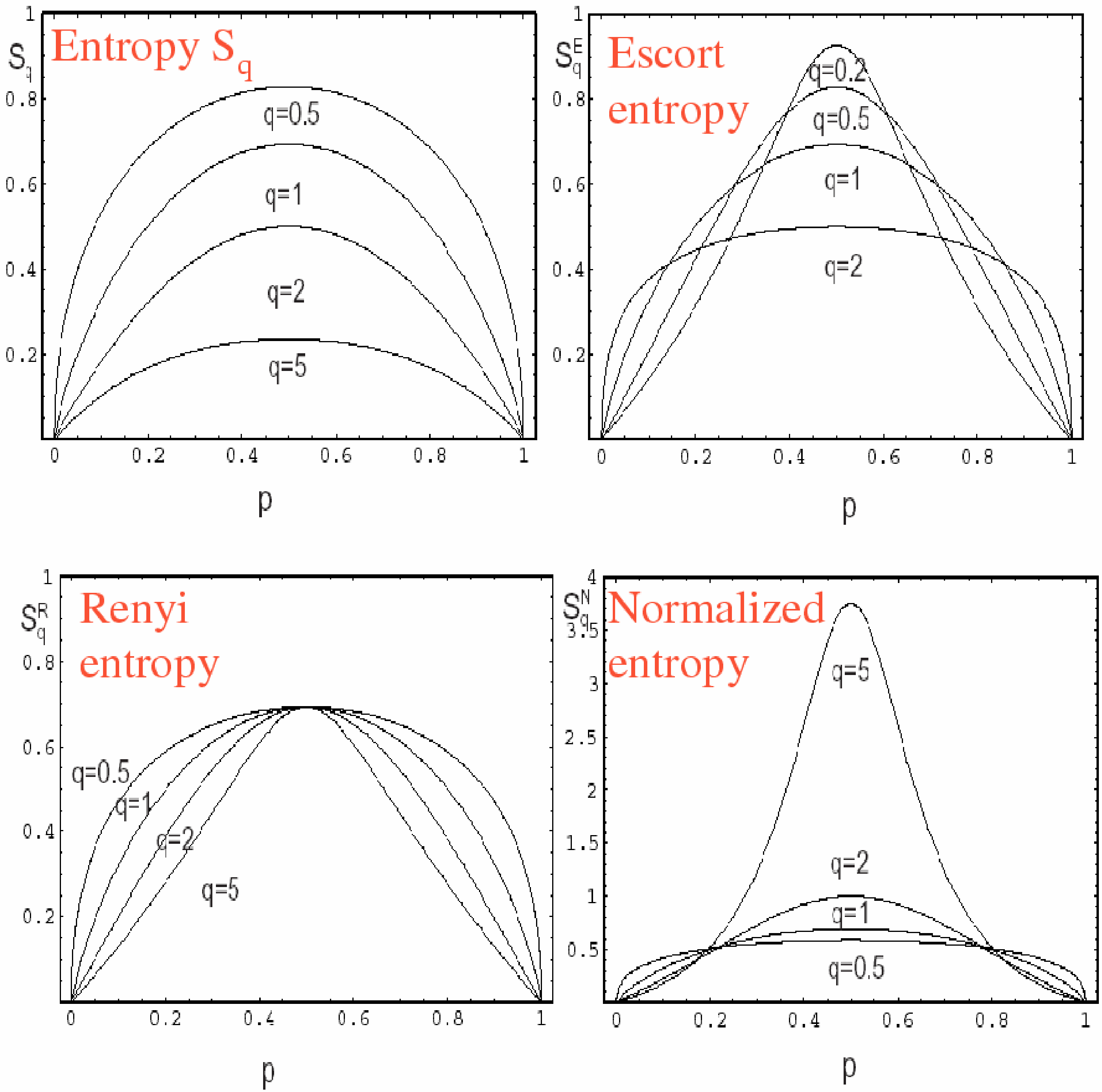
CONCAVITY:

Let us assume two arbitrary and different probability sets, namely $\{p_i\}$ and $\{p_i'\}$, associated with a single system having W states. We define an *intermediate* probability set as follows:

$$p_i'' = \lambda p_i + (1-\lambda)p_i' \quad (\forall i;\, 0 < \lambda < 1)$$

then

$$S_q(\{p_i''\}) > \lambda\, S_q(\{p_i\}) + (1-\lambda)\, S_q(\{p_i'\}) \quad (q > 0)$$



Entropy S q
q=0.5
q=1
q=2
q=5
p
Escort entropy
q=0.2
q=0.5
q=1
q=2
p
Renyi entropy
q=0.5
q=1
q=2
q=5
p
Normalized entropy
q=5
q=2
q=1
q=0.5
p

STABILITY
(or CONTINUITY or EXPERIMENTAL ROBUSTNESS)

B. Lesche, J Stat Phys **27**, 419 (1982)

The entropy S is said stable iff, for any given $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that, independently from W,

$$\sum_{i=1}^{W} | p_i - p'_i | \leq \delta_\varepsilon \quad \Rightarrow \quad \left| \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} \right| < \varepsilon$$

Hence
$$\lim_{\delta \to 0} \lim_{W \to \infty} \frac{S(\{p_i\}) - S(\{p_i'\})}{S_{\max}} = 0$$

S_{BG} and S_q $(\forall q > 0)$ are stable

S. Abe, Phys Rev E **66**, 046134 (2002)

$$S_q^R(\{p_i\}) \equiv \frac{\ln \sum_{i=1}^{W} p_i^q}{q-1} \qquad \text{(Renyi entropy)}$$

$$S_q^N(\{p_i\}) \equiv \frac{S_q(\{p_i\})}{\sum_{i=1}^{W} p_i^q} \qquad \text{(Normalized entropy)}$$

$$S_q^E(\{p_i\}) \equiv \frac{1 - \left(\sum_{i=1}^{W} p_i^{1/q} \right)^{-q}}{q-1} \qquad \text{(Escort entropy)}$$

are unstable

B. Lesche (1982); S. Abe (2002); C.T. and E. Brigatti (2003)

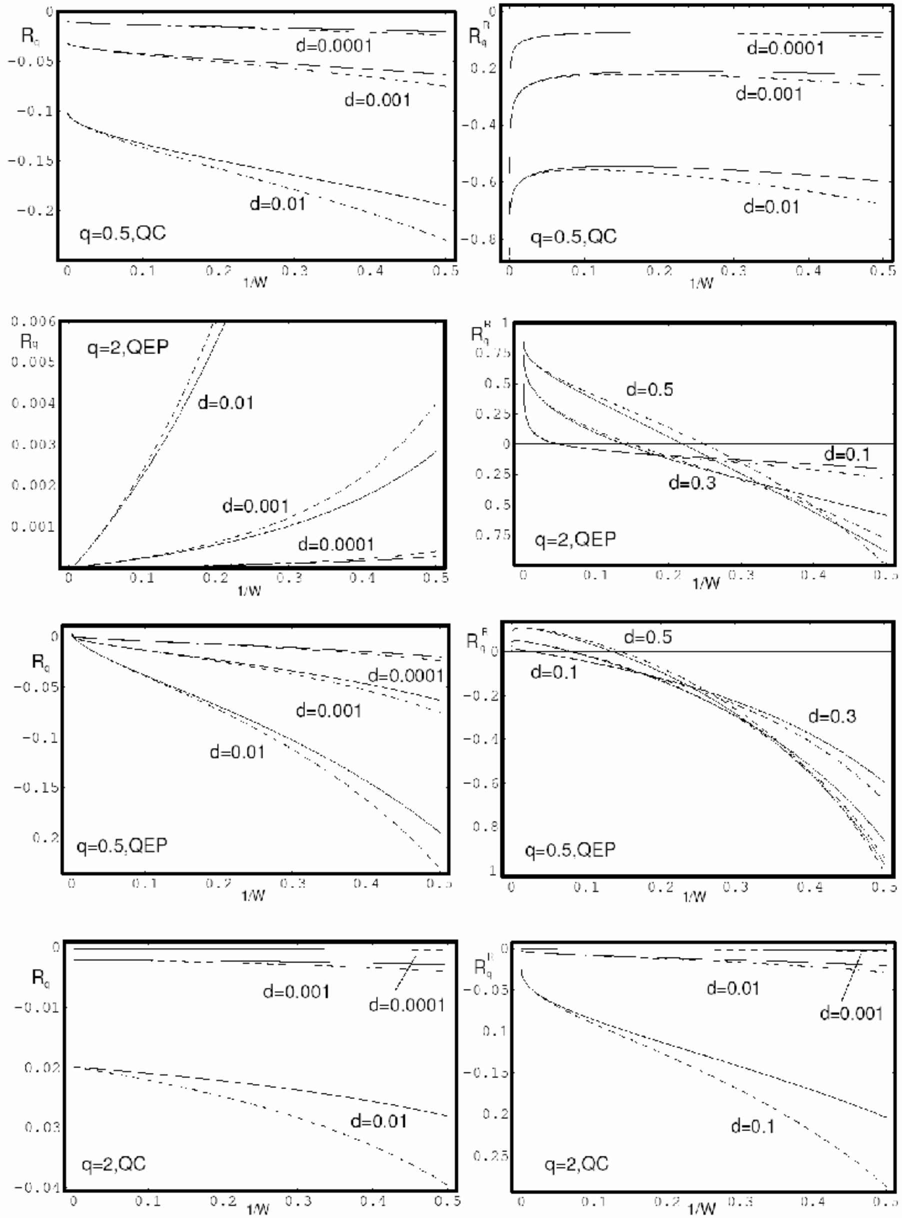
(S_q) (S_q^R)

q=0.5 (QC)

q=2 (QEP)

q=0.5 (QEP)

q=2 (QC)



QEP = quasi equal probabilities

QC = quasi certainty

C.T. and E. Brigatti (2003)

SANTOS THEOREM: RJV Santos, J Math Phys **38**, 4104 (1997)

(q - generalization of Shannon 1948 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W, \forall i)$ *monotonically increases with* W

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B)}{k}$ (*with* $p_{ij}^{A+B} = p_i^A p_j^B$)

AND $S(\{p_i\}) = S(p_L, p_M) + p_L^q \, S(\{p_l / p_L\}) + p_M^q \, S(\{p_m / p_M\})$ (*with* $p_L + p_M = 1$)

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad \left(q = 1 \Rightarrow S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i \right)$$

CE SHANNON (The Mathematical Theory of Communication):

"This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications."

ABE THEOREM: S Abe, Phys Lett A **271**, 74 (2000)

(q - generalization of Khinchin 1953 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W, \ \forall i)$ *monotonically increases with* W

AND $S(p_1, p_2, ..., p_W, 0) = S(p_1, p_2, ..., p_W)$

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B|A)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B|A)}{k}$

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i^{\,q}}{q-1} \quad \left(q=1 \Rightarrow S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i \right)$$

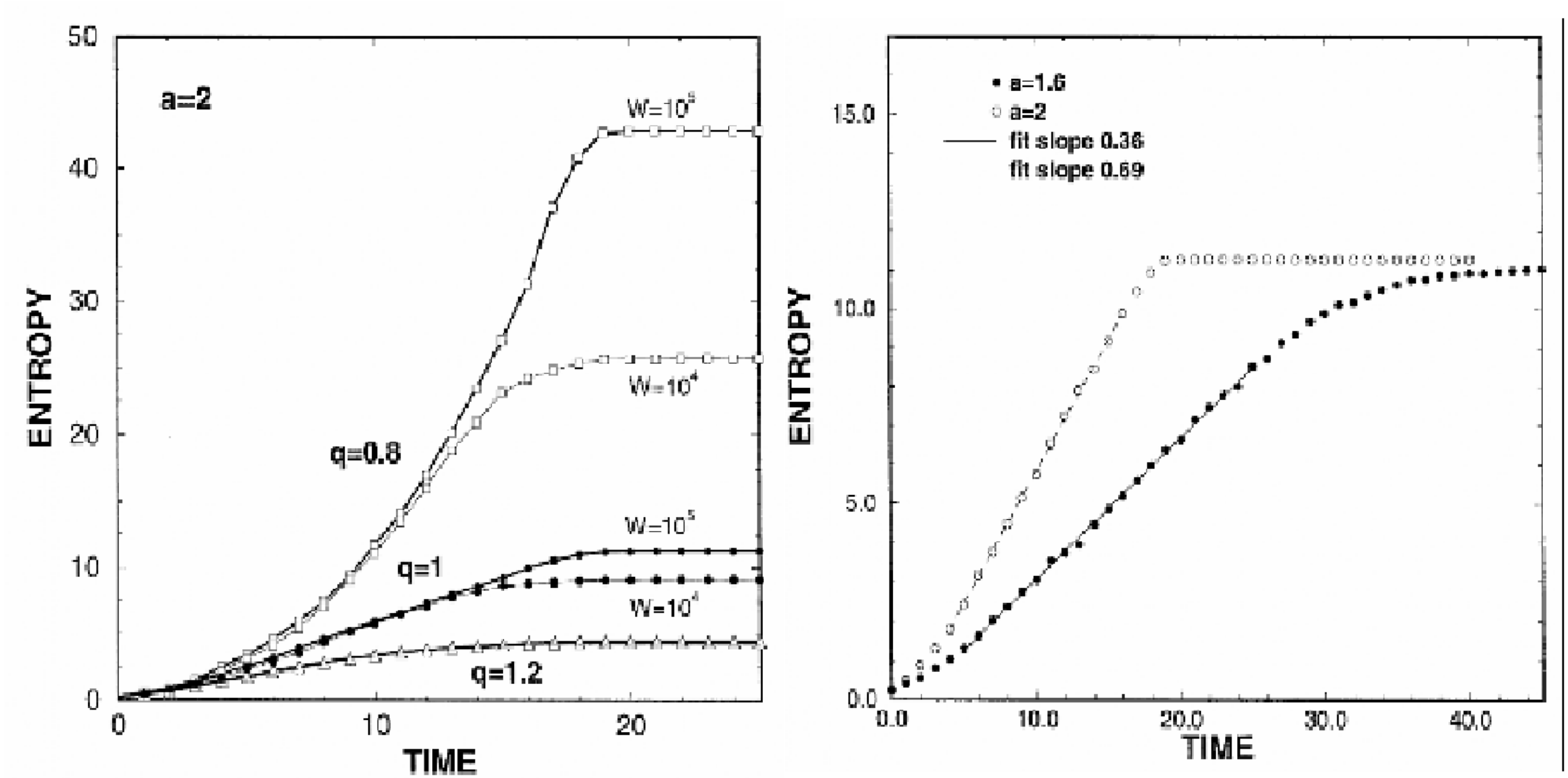
The possibility of such theorem was conjectured by AR Plastino and A Plastino (1996, 1999).

$$S_q(N,t)\ versus\ \ t$$

LOGISTIC MAP:

$$x_{t+1} = 1 - a\, x_t^2 \quad (0 \le a \le 2;\ -1 \le x_t \le 1;\ t = 0,1,2,\ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

We verify

$$K_1 = \lambda_1 \quad (Pesin-like\ identity)$$

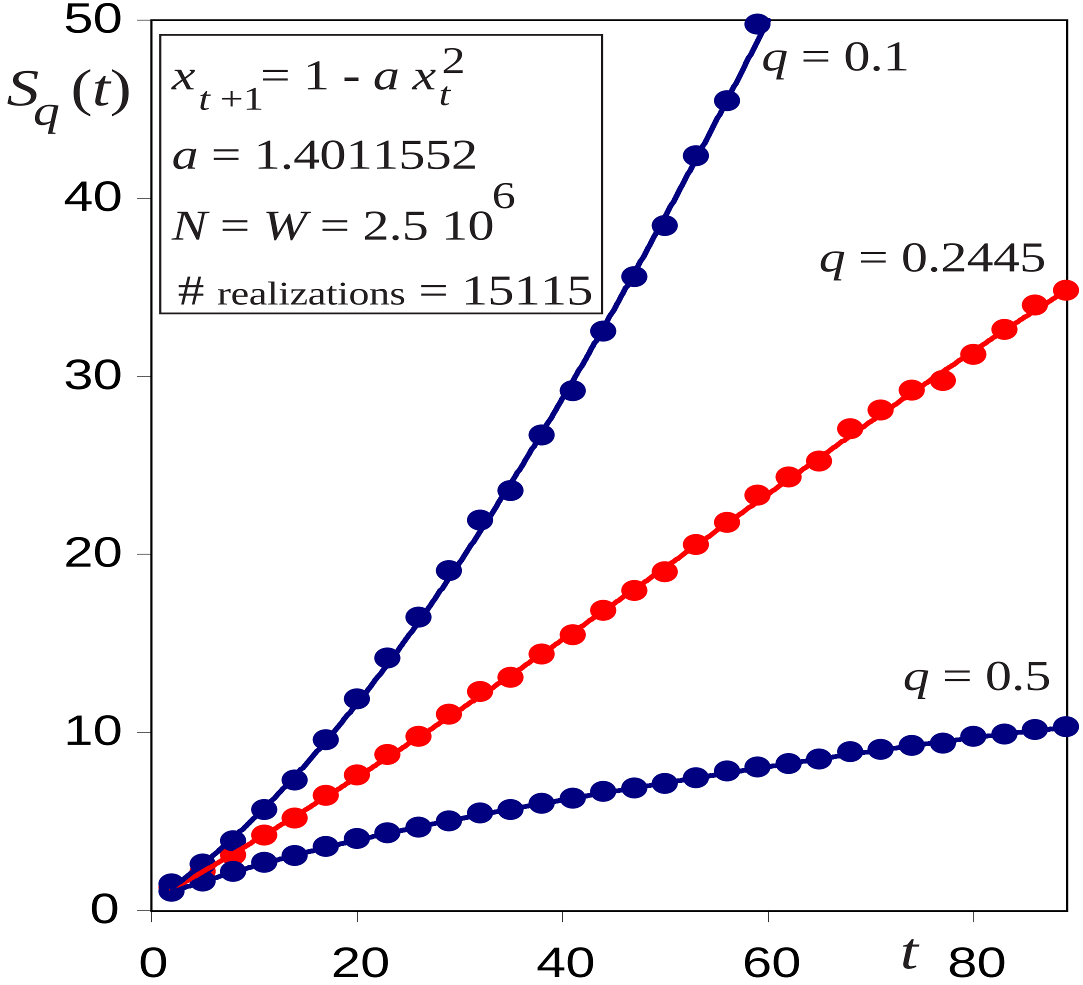
where

$$K_1 \equiv \lim_{t\to\infty} \frac{S_1(t)}{t}$$

and

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1\, t}$$

(weak chaos, i.e., **zero** Lyapunov exponent)



C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys. Rev. Lett. **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys. Lett. A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys. Rev. Lett. **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys. Rev. E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys. Rev. Lett. **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys. Rev. E **72**, 026209 (2005), and references therein

We verify

$$K_q = \lambda_q \quad (q-generalized \;\; Pesin-like\ identity)$$

where

$$K_q \equiv \lim_{t\to\infty} \sup\left\{\frac{S_q(t)}{t}\right\}$$

and

$$\xi(t) \equiv \sup\left\{\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)}\right\} = e_q^{\lambda_q\, t}$$

with

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}} = \frac{\ln \alpha_F}{\ln 2} \quad and \quad \lambda_q = \frac{1}{1-q}$$

$$\left[x_{t+1} = 1 - a\,|\,x_t\,|^z \;\;\Rightarrow\;\; \frac{1}{1-q(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = (z-1)\frac{\ln \alpha_F(z)}{\ln 2} \right]$$

THE CASATI-PROSEN TRIANGLE MAP:

G. Casati and T. Prosen,
Phys. Rev. Lett. **83**, 4729 (1999) and **85**, 4261 (2000)

"While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary."

$$y_{t+1} = y_t + \alpha \ \mathrm{sgn}(x_t) + \beta \quad (\mathrm{mod}\, 2)$$

$$x_{t+1} = x_t + y_{t+1} \quad (\mathrm{mod}\, 2)$$

(α and β independent irrationals)

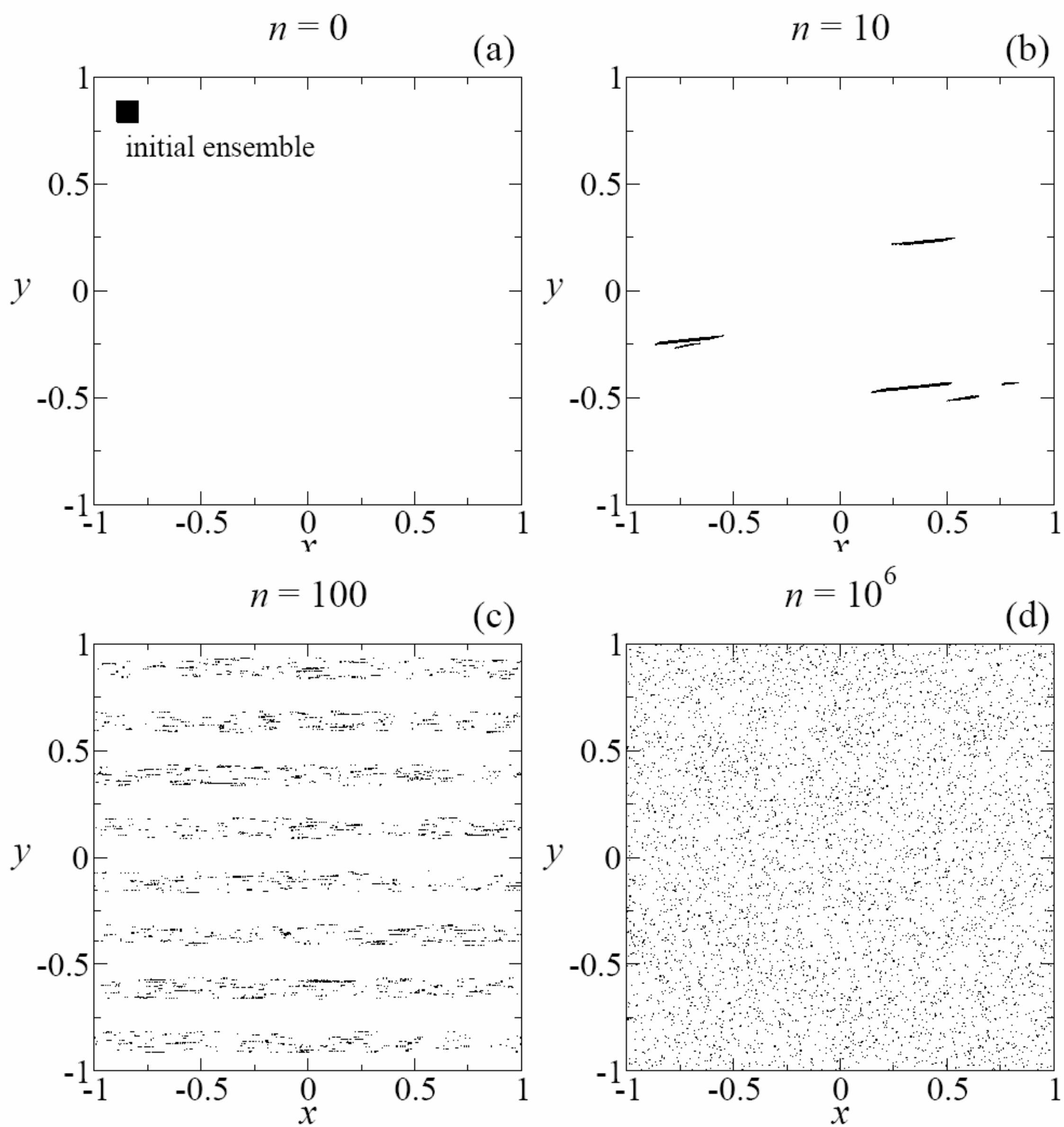
e.g., $(\alpha,\beta)=\left((1/2)(\sqrt{5}-1)-(1/e),\ (1/2)(\sqrt{5}-1)+(1/e)\right)$

This map is conservative, mixing, ergodic and nevertheless with zero Lyapunov exponent!

Furthermore $\xi \equiv \lim_{\Delta X(0)\to 0} \dfrac{\Delta X(t)}{\Delta X(0)} \propto t$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

NONEXTENSIVITY OF THE CASATI-PROSEN MAP:

Answer to the above equation:

[G. Casati, C.T. and F. Baldovin, Europhys Lett **72**, 355 (2005)]

It is not necessary: a meaningful statistical description is possible with zero Lyapunov exponent!

[Essentially because an integrable system has zero Lyapunov exponent but the opposite is not true]

In general, $\xi = [1 + (1-q)\lambda_q t]^{1/(1-q)}$

hence, $\xi \propto t \Rightarrow q = 0$

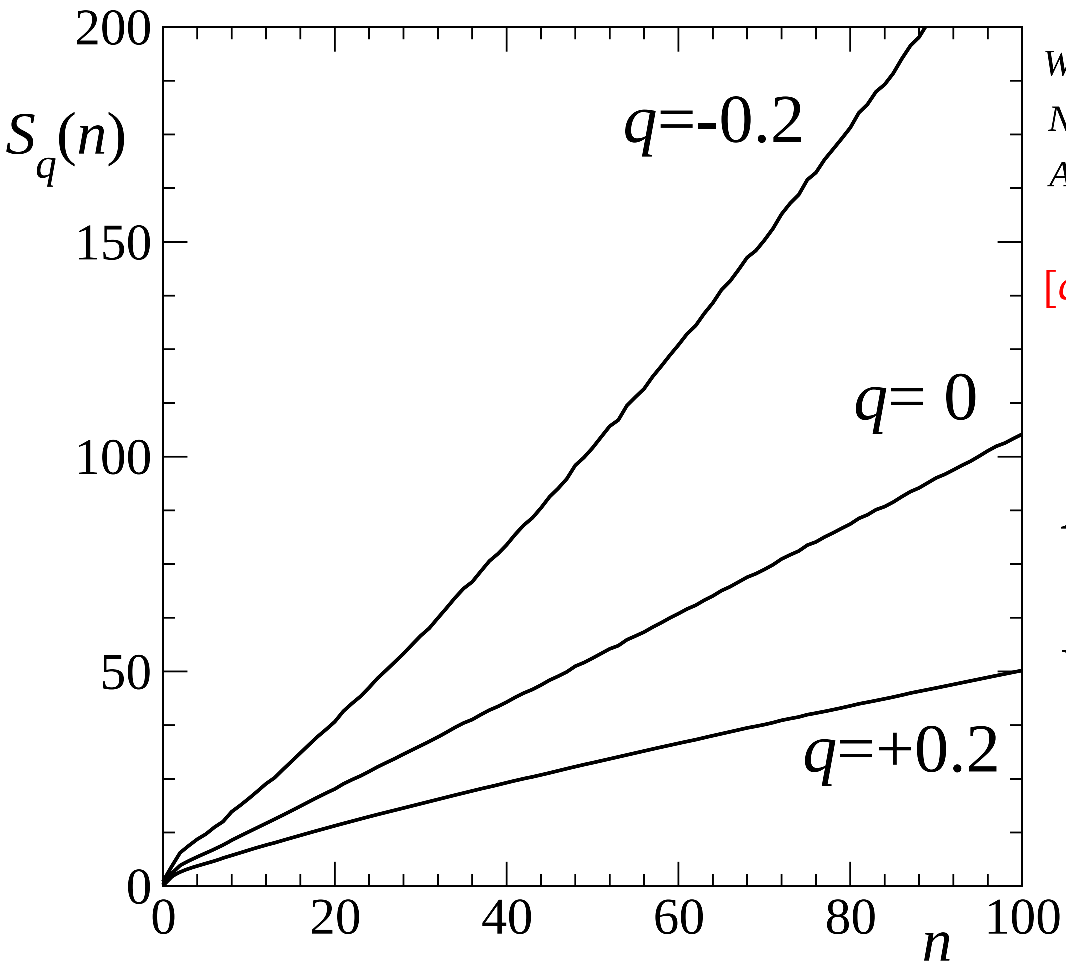
Consistently, we expect

$$(i)\ S_q(t) \equiv \frac{1 - \sum_{i=1}^{W} [p_i(t)]^q}{q-1} \propto t \text{ only for } q = 0$$

$$(ii) K_q \equiv \lim_{t\to\infty} \frac{S_q(t)}{t} = \lambda_q \qquad \text{for } q = 0$$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



$W = 4000 \times 4000$ *cells*

$N = 1000$ *initial conditions randomly chosen in one cell*

Average done over 100 *initial cells*

[$q = 0 \;\rightarrow\;$ *linear correlation* $= 0.99993$]

Also $\quad \xi = e_0^{\lambda_0 t}$

with $\quad \lambda_0 = \lim_{n\to\infty} \dfrac{S_0(n)}{n} = 1$

q - generalization of Pesin (- like) theorem

G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

$$S_q(N,t) \text{ versus } N$$

HYBRID PASCAL - LEIBNITZ TRIANGLE

(N = 0) $1 \times \frac{1}{1}$

(N = 1) $1 \times \frac{1}{2}$ $1 \times \frac{1}{2}$

(N = 2) $1 \times \frac{1}{3}$ $2 \times \frac{1}{6}$ $1 \times \frac{1}{3}$

(N = 3) $1 \times \frac{1}{4}$ $3 \times \frac{1}{12}$ $3 \times \frac{1}{12}$ $1 \times \frac{1}{4}$

(N = 4) $1 \times \frac{1}{5}$ $4 \times \frac{1}{20}$ $6 \times \frac{1}{30}$ $4 \times \frac{1}{20}$ $1 \times \frac{1}{5}$

(N = 5) $1 \times \frac{1}{6}$ $5 \times \frac{1}{30}$ $10 \times \frac{1}{60}$ $10 \times \frac{1}{60}$ $5 \times \frac{1}{30}$ $1 \times \frac{1}{6}$

$\Sigma = 1 \quad (\forall \; N)$

Blaise **Pascal** (1623-1662)
Gottfried Wilhelm **Leibnitz** (1646-1716)
Daniel **Bernoulli** (1700-1782)

$(N=2)$

A \ B	1	2	
1	$p^2+\kappa$	$p(1-p)-\kappa$	p
2	$p(1-p)-\kappa$	$(1-p)^2+\kappa$	1- p
	p	1- p	1

EQUIVALENTLY:

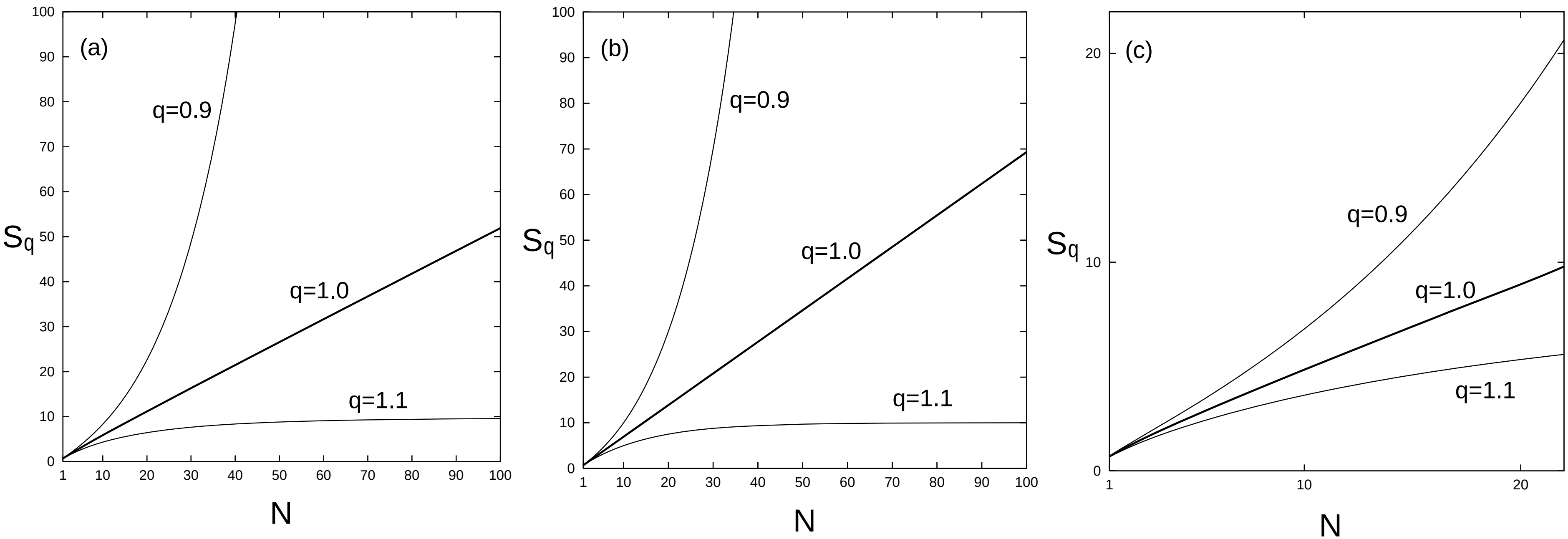
$(N=0)$ 1×1

$(N=1)$ $1\times p$ $1\times(1-p)$

$(N=2)$ $1\times[p^2+\kappa]$ $2\times[p(1-p)-\kappa]$ $1\times[(1-p)^2+\kappa]$

$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$



Leibnitz triangle	*N independent coins*	*Stretched exponential*
$\left(p_{N,0} = \frac{1}{N+1} \right)$	$\left(p_{N,0} = p^N \quad \text{with } p = 1/2 \right)$	$\left(p_{N,0} = p^{N^\alpha} \quad \text{with } p = \alpha = 1/2 \right)$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$(N=0)$	1				
$(N=1)$	$1/2$	$1/2$			
$(N=2)$	$1/3$	$1/6$	$1/3$		
$(N=3)$	$3/8$	$5/48$	$5/48$	0	
$(N=4)$	$2/5$	$3/40$	$1/20$	0	0

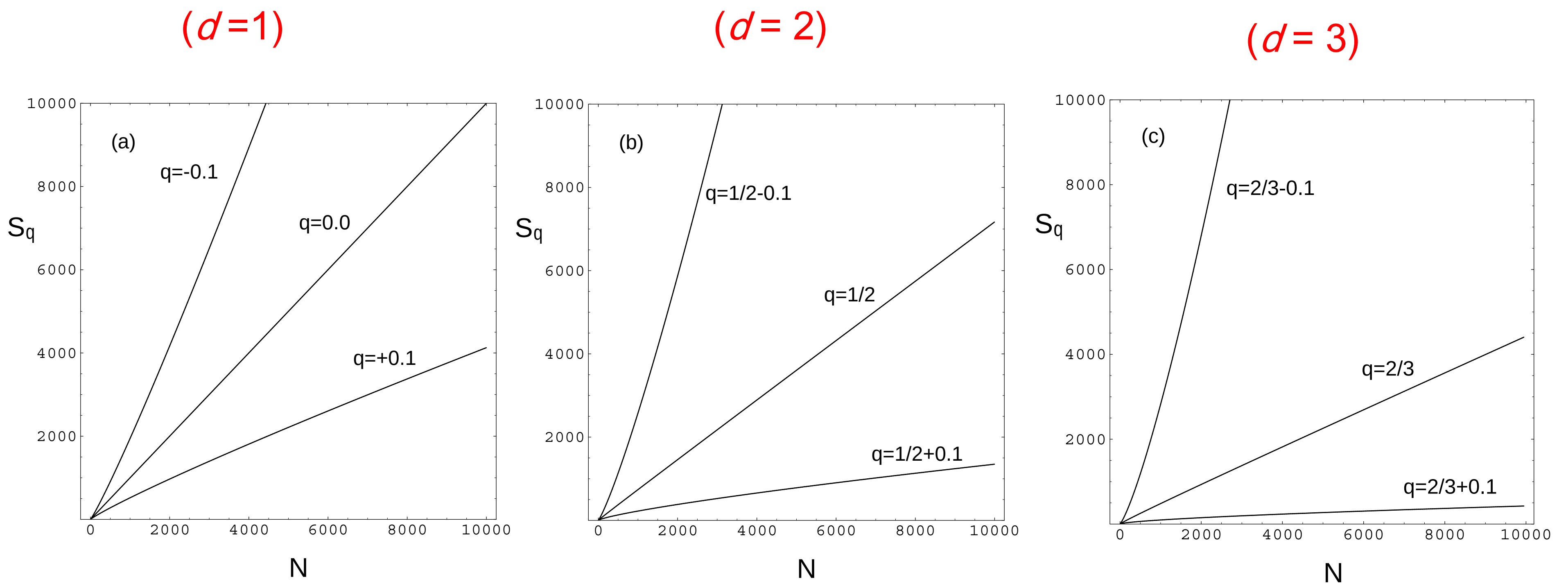
d+1

(It asymptotically satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

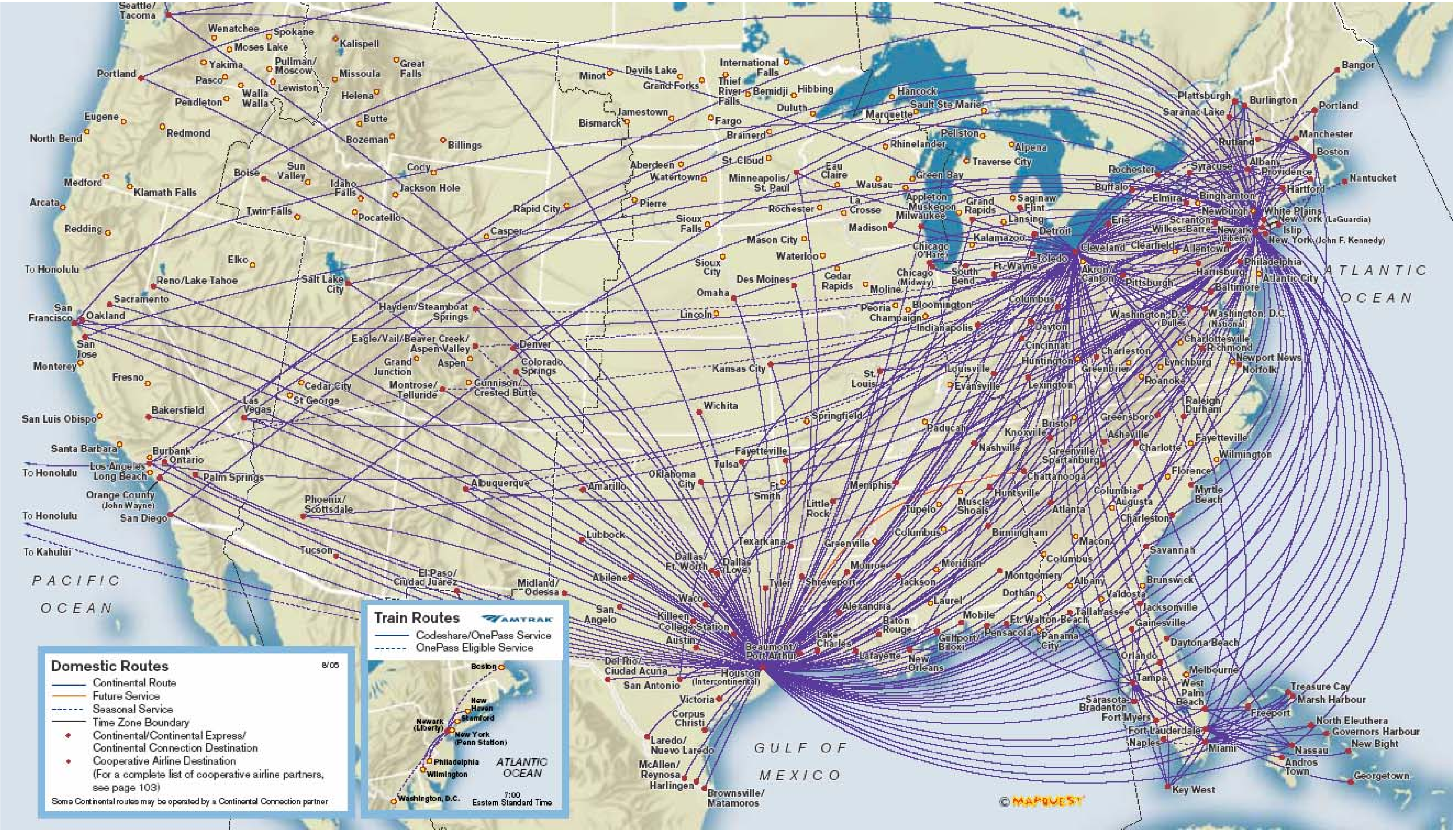
i.e., such that $S_q(N) \propto N \quad (N \to \infty)$



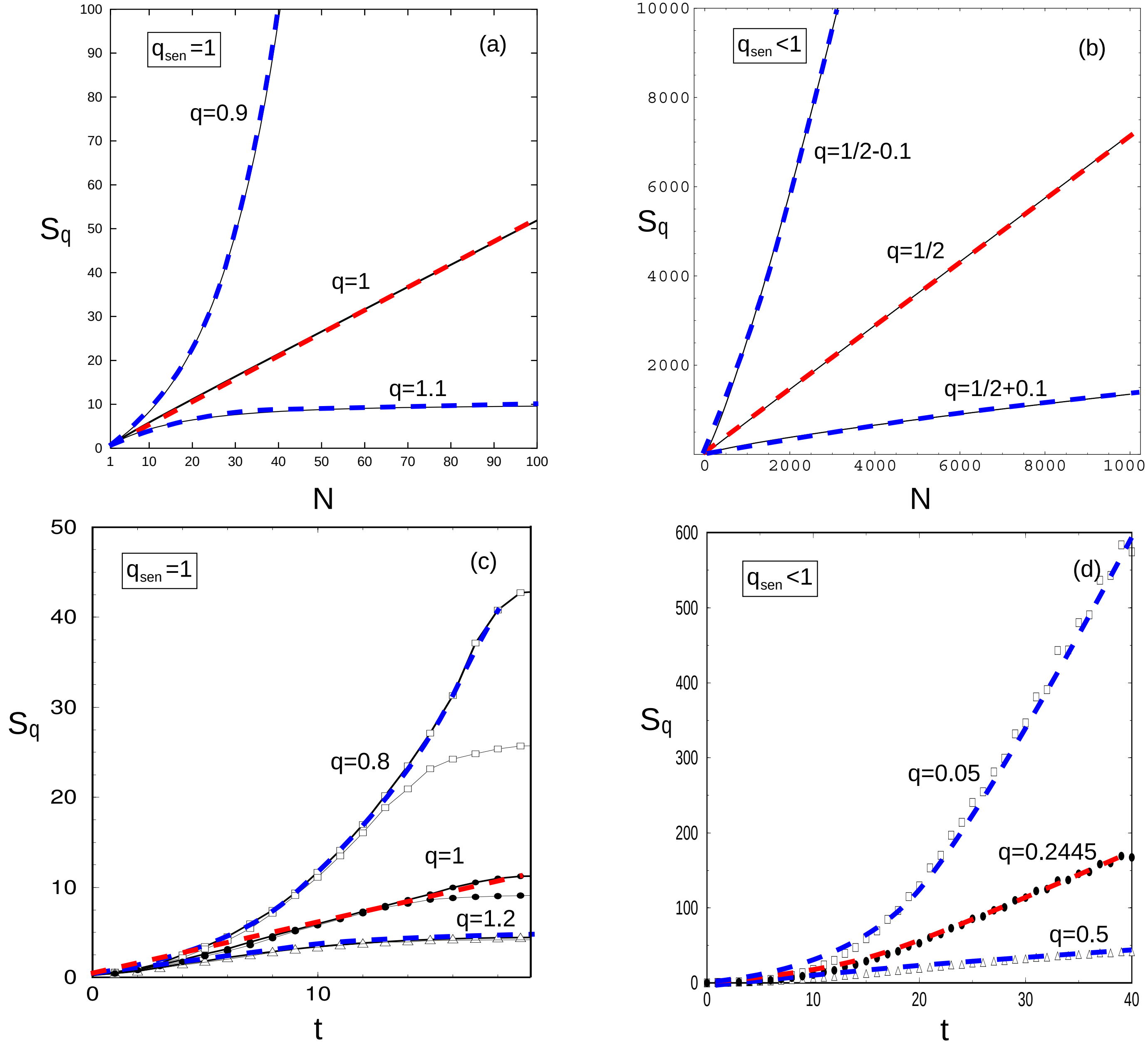
$$q = 1 - \frac{1}{d}$$

(All three examples asymptotically satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)



Continental Airlines



C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186 (2005) [European Physical Society]

A conjecture for $S_q(N,t)$:

For $q = q_{sen}$, $N \to \infty$ *and* $t \to \infty$ *play essentially the same role.*
In particular,

i) Under conditions of ***infinitely fine*** *graining in phase space,*

$$S_{q_{sen}}(N,t) \sim K_{q_{sen}}(N)\ t \ \propto \ N\ t$$

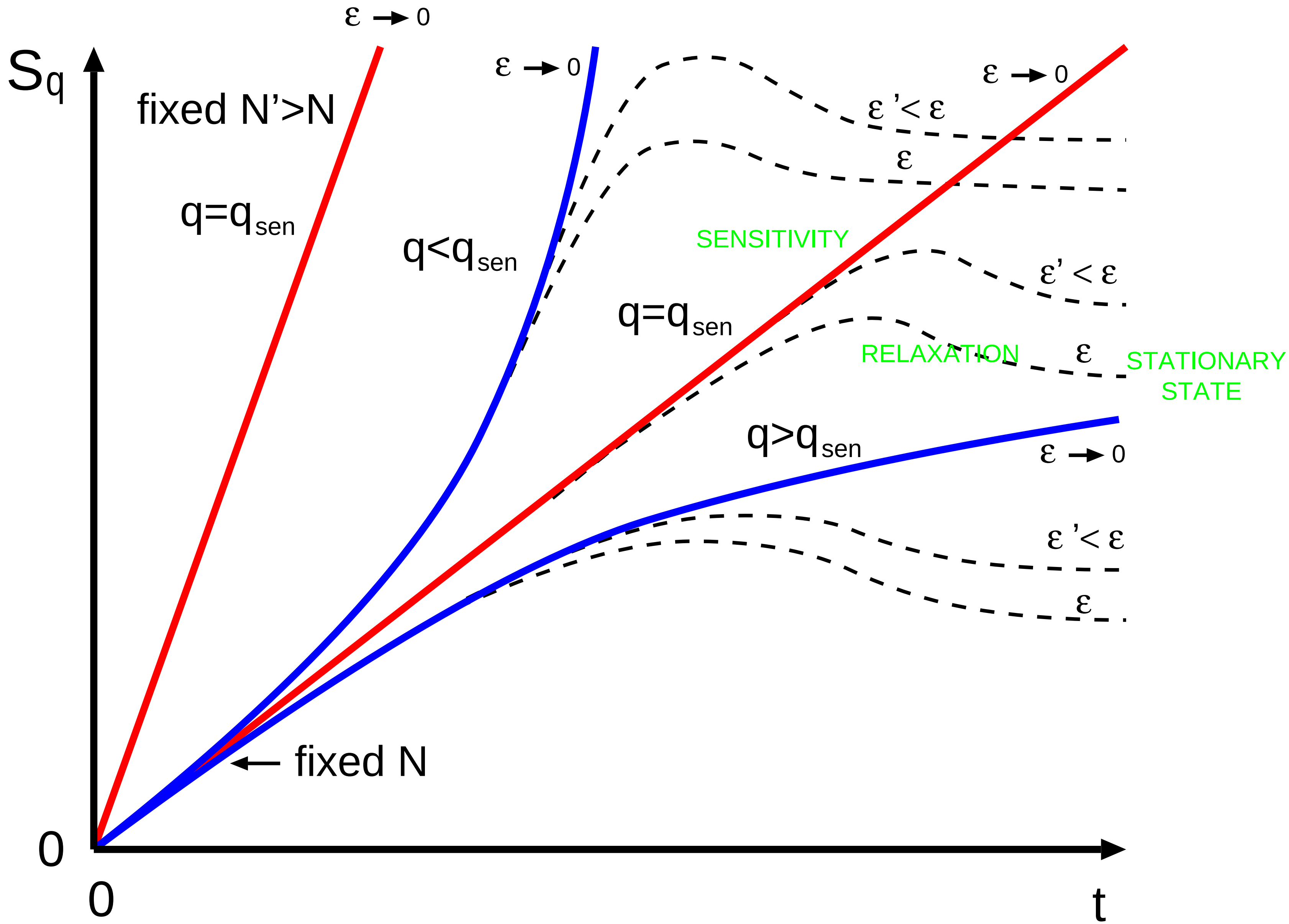
$$\left(q_{sen} = 1 \ \Rightarrow \ K_1 = \sum_{j/\lambda_1^{(j)}>0} \lambda_1^{(j)},\ \textit{i.e., } \text{Pesin}-\textit{like identity for finite } N\right)$$

ii) Under conditions of ***finite*** *graining in phase space,*

$$\lim_{t\to\infty} S_{q_{sen}}(N,t) \propto N$$

(Clausius)

C.T., M. Gell-Mann and Y. Sato
Europhysics News **36**, 186 (2005)



C.T., M. Gell-Mann and Y. Sato
Europhysics News Special Issue *Nonextensive Statistical Mechanics – New Trends, New Perspectives*
(European Physical Society, Nov-Dec 2005), in press

If A *and* B *are independent*,

i.e., if $p_{ij}^{A+B} = p_i^{A} \, p_j^{B}$,

then

$$S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B)$$

whereas

$$S_q(A+B) = S_q(A) + S_q(B) + \frac{1-q}{k_B} S_q(A)\, S_q(B)$$

$$\neq S_q(A) + S_q(B) \quad (if \;\; q \neq 1)$$

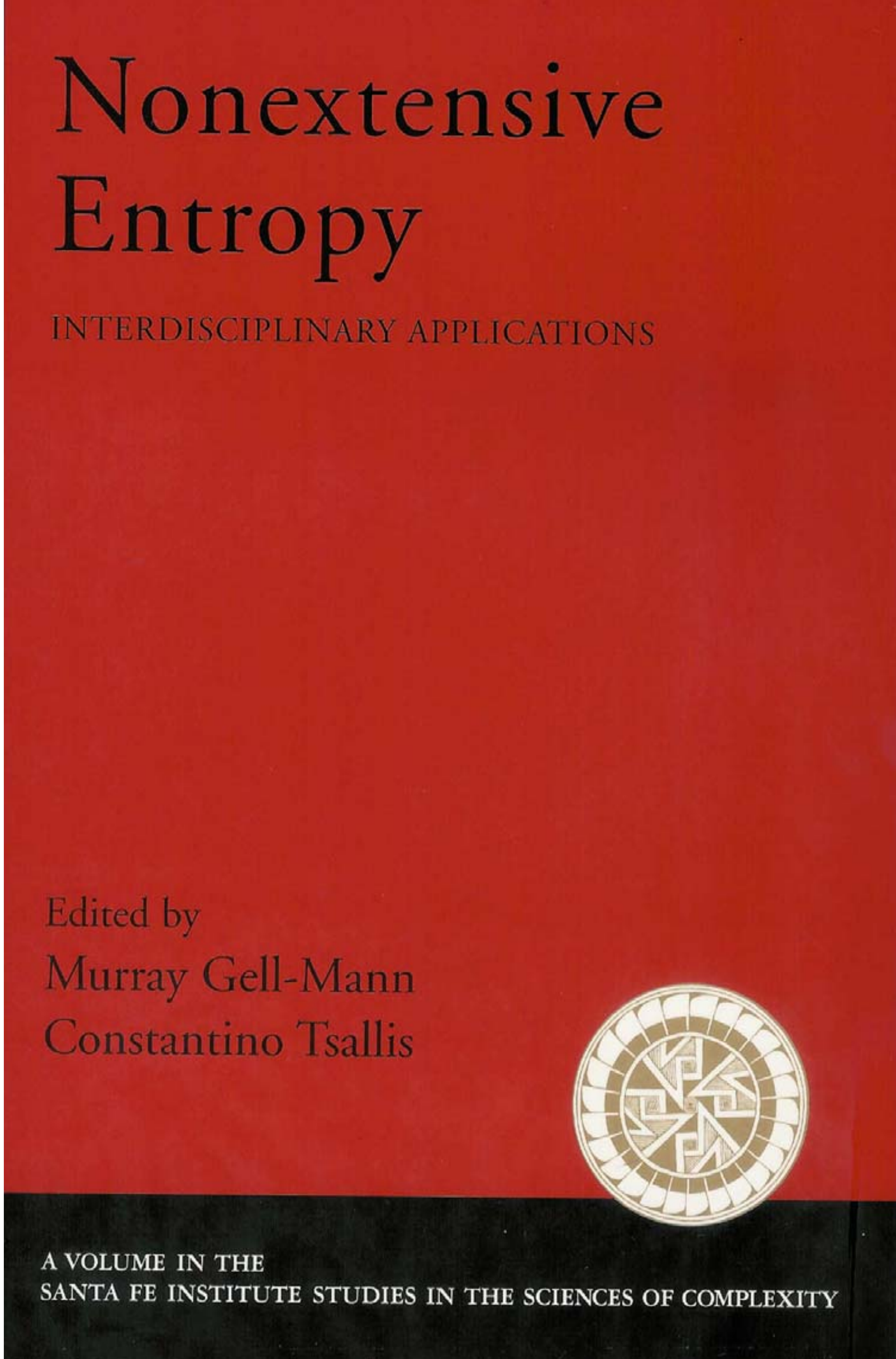
But if A *and* B *are especially (globally) correlated*,

then

$$S_q(A+B) = S_q(A) + S_q(B)$$

whereas

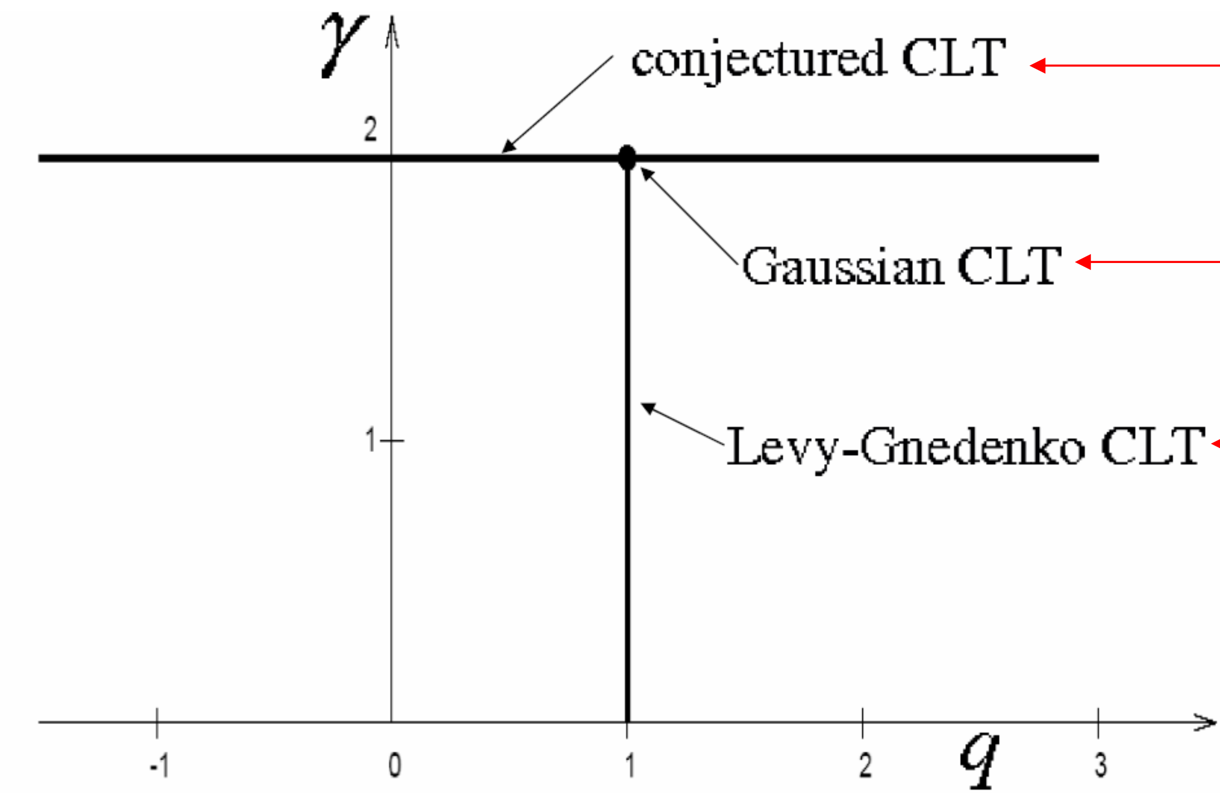
$$S_{BG}(A+B) \neq S_{BG}(A) + S_{BG}(B)$$



PROPERTY \ ENTROPY	[a] $S_{BG} \equiv -\sum_i p_i \ln p_i = S_1 = S_1^R = S_1^{LVRA} = S_1^E$	[b] $S_q \equiv \frac{1-\sum_i p_i^q}{q-1}$	[c] $S_q^R \equiv \frac{\ln \sum_i p_i^q}{1-q} = \frac{\ln\left[1+(1-q)S_q\right]}{1-q}$	[d] $S_q^{LVRA} \equiv \frac{S_q}{\sum_i p_i^q} = \frac{1-\left[\sum_i p_i^q\right]^{-1}}{1-q} = \frac{S_q}{1+(1-q)S_q}$	[e] $S_q^E \equiv \frac{1-\left[\sum_i p_i^{1/q}\right]^{-q}}{q-1}$
Extensive $(\forall q)$ $\left[p_{ij}^{A+B} = p_i^A p_j^B\right]$	YES	NO	YES	NO	NO
Extensive $(q \neq 1)$ [special global correlations]	NO	YES	NO	NO	NO
Concave $(\forall q > 0)$	YES	YES	NO	NO	NO
Lesche-stable $(\forall q > 0)$	YES	YES	NO	NO	
Jackson q-derivative application on $-\sum_i p_i^x$ $(\forall q)$	YES	YES	NO	NO	
Finite entropy production per unit time $(\forall q = q_{sen} \leq 1)$	YES	YES	NO	NO	
$\exists$ $\hat{S}$ / $\hat{S}$ and $S \equiv \langle \hat{S} \rangle$ obey same composition law $(\hat{\rho}^{A+B} = \hat{\rho}^A \otimes \hat{\rho}^B)$ $(\forall q)$	YES	YES	NO	NO	
$\exists$ $\hat{S}$ / $\hat{S}(\hat{\rho}^{-1})$ has same functional form as $S(p_i = 1/W)$ $(\forall q)$	YES	YES	NO	NO	
Same functional form for $Z_q(\beta F_q)$ and $Z_q p(\beta E_i)$ $(\forall q)$	YES	YES	NO	NO	
Energy eigenvalues scaling temperature coincides with inverse Lagrange parameter	YES	NO	YES	NO	
Optimizing distribution $(\forall q)$	Exponential	Power-law	Power-law	Power-law	Power-law

q - CENTRAL LIMIT THEOREM: (conjecture)

$$\frac{\partial\, p(x,t)}{\partial t} = D\,\frac{\partial^{\gamma}[p(x,t)]^{2-q}}{\partial |x|^{\gamma}} \quad (0 < \gamma \le 2;\; q < 3)$$



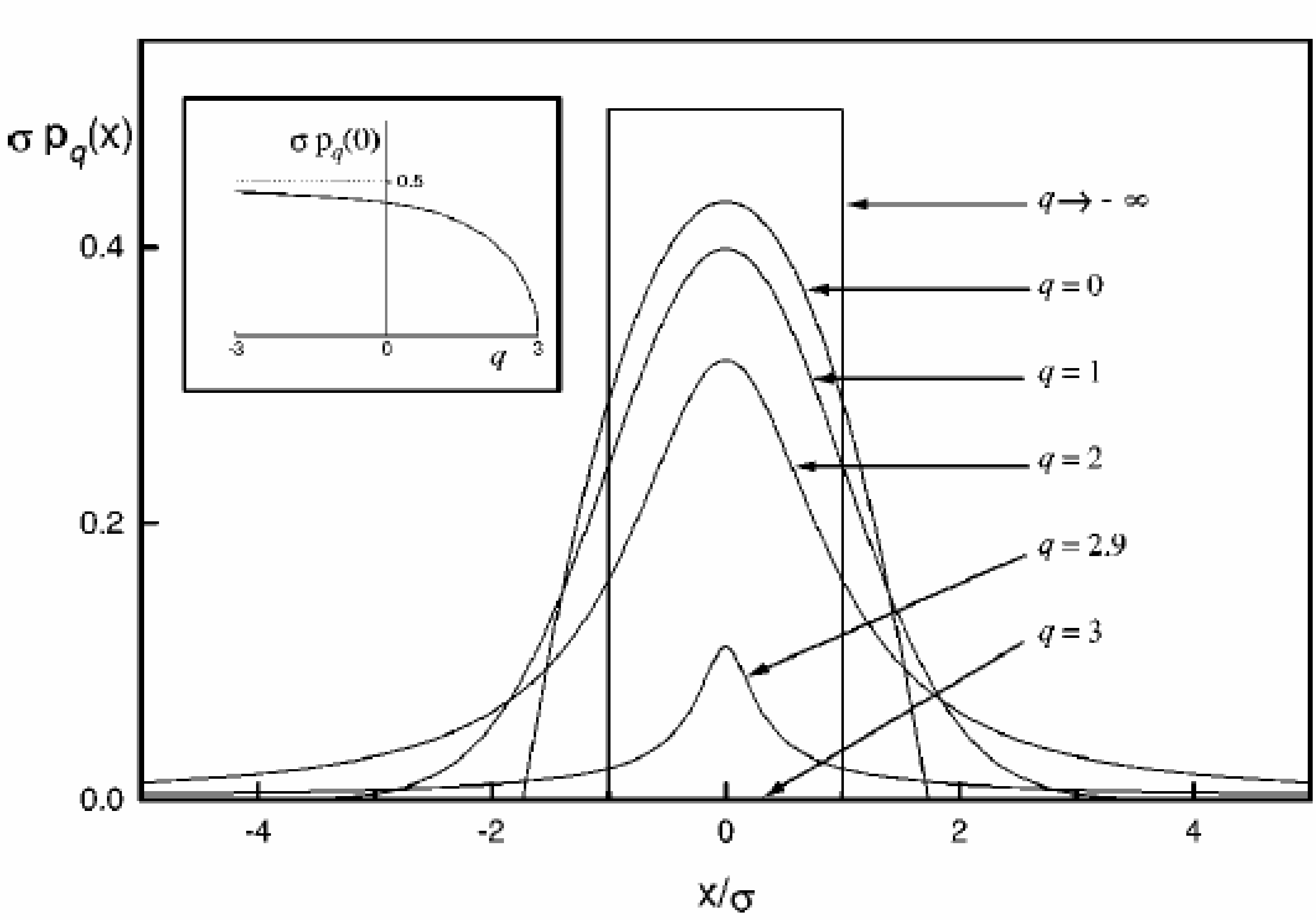
globally correlated variables;
finite q-variance;
q-Gaussian attractor

independent variables;
finite variance;
Gaussian attractor

independent variables;
divergent variance;
Levy attractor

C.T., Milan J. Math. **73**, 145 (2005)

q-GAUSSIANS:



D. Prato and C. T, Phys Rev E **60**, 2398 (1999)

q - CENTRAL LIMIT THEOREM (q-product and de Moivre-Laplace theorem):

The q- product is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

$Properties:$

$i) \quad x \otimes_1 y = x\, y$

$ii) \quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$

$[whereas \;\; \ln_q (x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$

[L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003); E.P. Borges, Physica A **340**, 95 (2004)]

The de Moivre-Laplace theorem can be constructed with

$$p_{N,0} = p^N \quad with \quad p = 1/2$$

and

Leibnitz rule

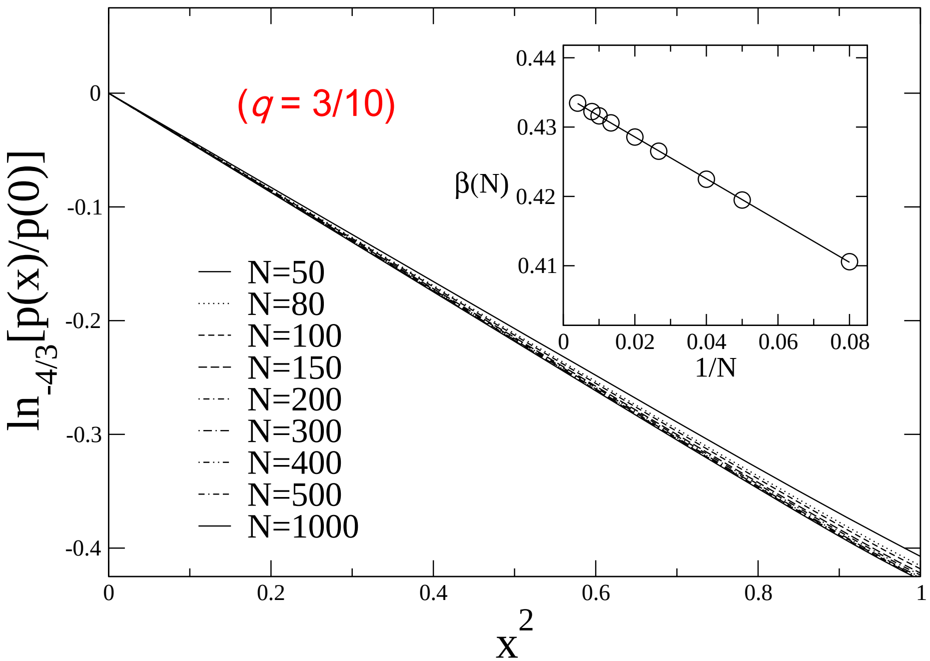
q - CENTRAL LIMIT THEOREM: (numerical indications)

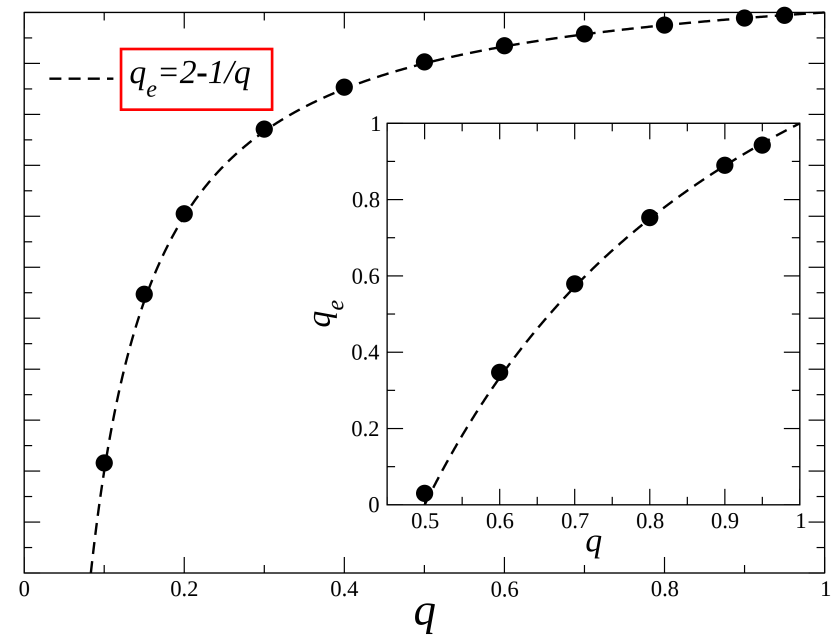
We q – generalize the de Moivre – Laplace theorem with

$$\frac{1}{p_{N,0}} = \left(\frac{1}{p}\right) \otimes_q \left(\frac{1}{p}\right) \otimes_q \ldots \left(\frac{1}{p}\right) \quad (N \ terms)$$

i.e.,

$$p_{N,0} = \left[N\, p^{q-1} - (N-1)\right]^{\frac{1}{q-1}} \quad (with\ p = 1/2)$$





[Hence q → 2 – q (additive duality) and q →1/q (multiplicative duality) are involved]

L.G. Moyano, C. T. and M. Gell-Mann, Europhys. Lett. **73**, 813 (2006)

q- GENERALIZED CENTRAL LIMIT THEOREM: (mathematical proof)

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} f(x)\, dx$$ (nonlinear!)

q-correlation:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$]

are said q*-correlated if*

$$F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi)\,,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\; e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\; e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_q \left[\int_{-\infty}^{\infty} dy\; e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with $f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; h(x,y)\; \delta(x+y-z) = \int_{-\infty}^{\infty} dx\; h(x, z-x) = \int_{-\infty}^{\infty} dy\; h(z-y, y)$

where $h(x,y)$ *is the joint density.*

$$\left(\begin{array}{llll} q\text{-correlation means} & \text{independence} & \text{if } q=1\,,\ \text{i.e.,} & h(x,y) = f_X(x)\, f_Y(y) \\ & \text{global correlation} & \text{if } q \neq 1\,,\ \text{hence} & h(x,y) \neq f_X(x)\, f_Y(y) \end{array} \right)$$

Closure:

The q-Fourier transform of a q-Gaussian is a z(q)-Gaussian with

$$z(q)=\frac{1+q}{3-q}\in(-\infty,3)$$

Iteration:

$$q_n \equiv z_n(q) \equiv z(z_{n-1}(q)) = \frac{2q+n(1-q)}{2+n(1-q)} \quad (n=0,\ \pm1,\ \pm2,\ldots;\ q_0=q)$$

(the same as in R.S. Mendes and C.T. [Phys Lett A 285, 273 (2005)] when calculating marginal probabilities!)

hence

(i) $q_n(1)=1\ (\forall n), \quad q_{\pm\infty}(q)=1\ (\forall q)$,

(ii) $q_{n-1}=2-\frac{1}{q_{n+1}}$,

(the same as in L.G. Moyano, C.T. and M. Gell-Mann (2005)!)

(the same as in A. Robledo [Physica D 193, 153 (2004)] for pitchfork and tangent bifurcations!)

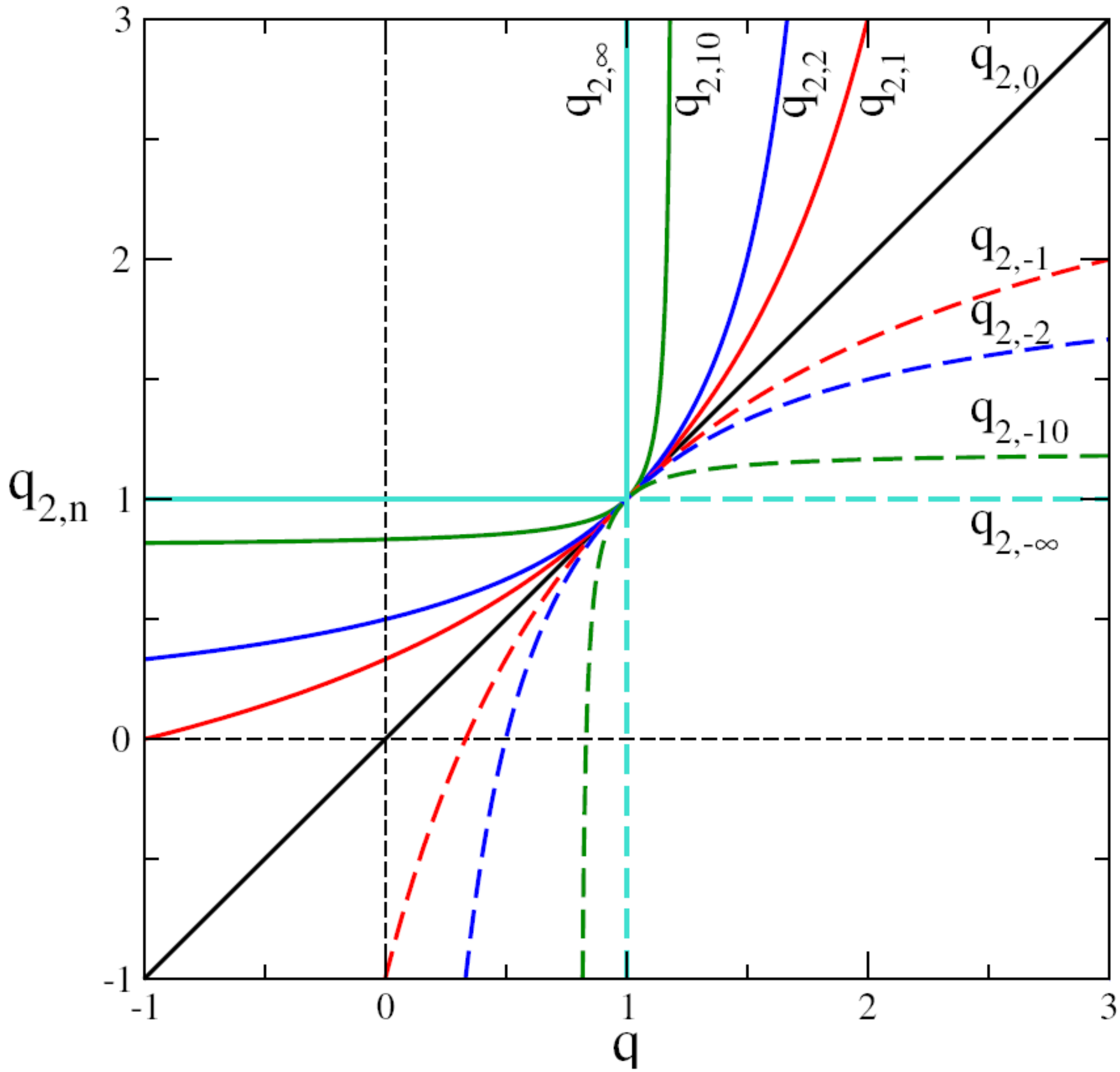
(iii) $n=2m=0,\ \pm2,\ \pm4,\ldots$ *yields* $q_{(m)} \equiv q_{2m} = \frac{q+m(1-q)}{1+m(1-q)}$

(the same obtained in C.T., M. Gell-Mann and Y. Sato [Proc Natl Acad Sci (USA) 102, 15377 (2005)], by combining *only* additive and multiplicative dualities, and which was conjectured to be a possible explanation for the NASA-detected q-triangle for m = 0, ±1!)

ALGEBRA ASSOCIATED WITH q-GENERALIZED CENTRAL LIMIT THEOREMS:

$$\frac{\alpha}{1-q_{\alpha,n}} = \frac{\alpha}{1-q} + n$$

$$(n = 0, \ \pm 1, \ \pm 2, \ldots)$$



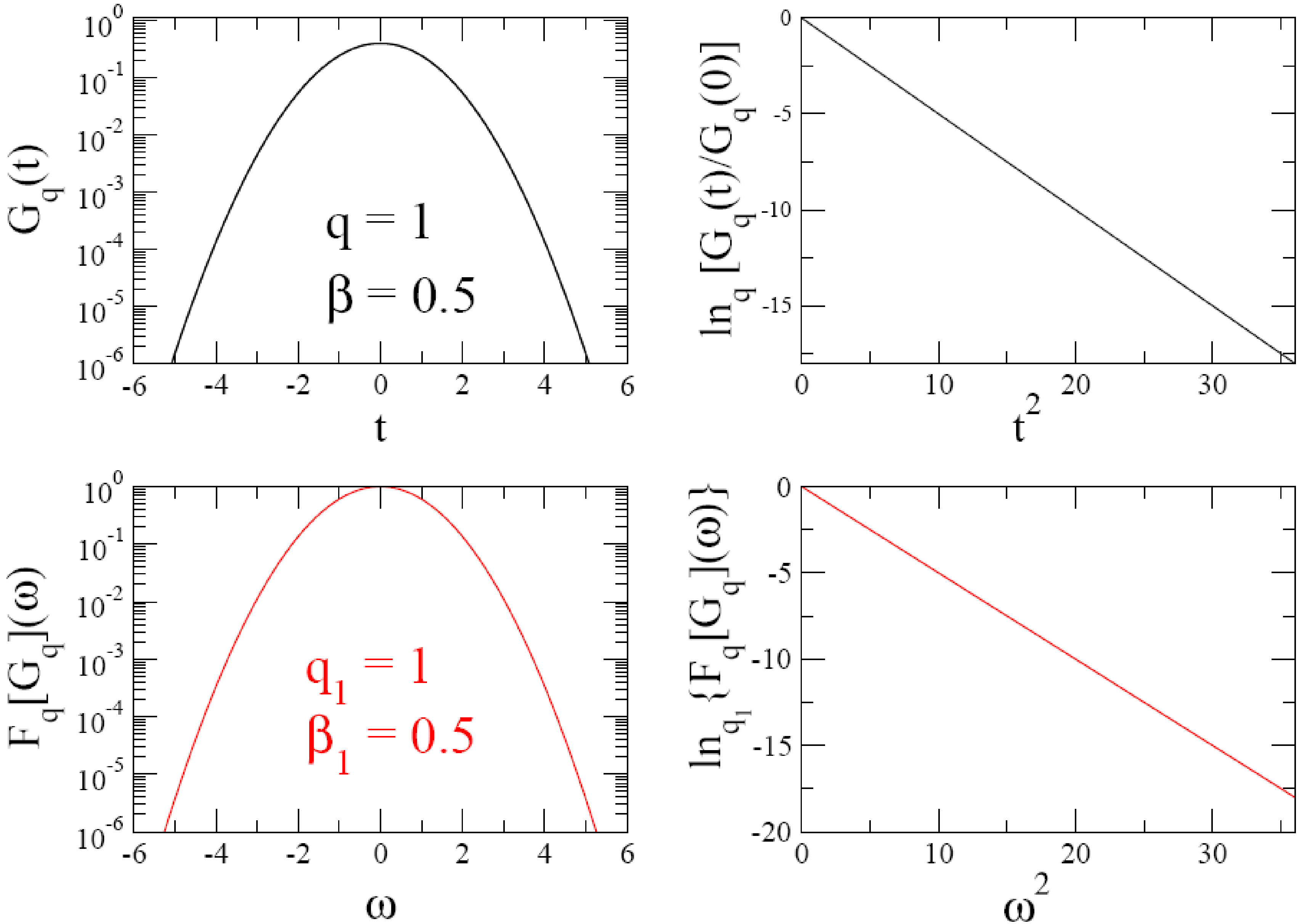
S. Umarov, C.T., M. Gell-Mann and S. Steinberg (2006), cond-mat/0606040

$$q-\mathrm{FourierTransform}\left[\frac{\sqrt{\beta}}{C_q}e_q^{-\beta\, t^2}\right]=e_{q_1}^{-\beta_1\,\omega^2}$$

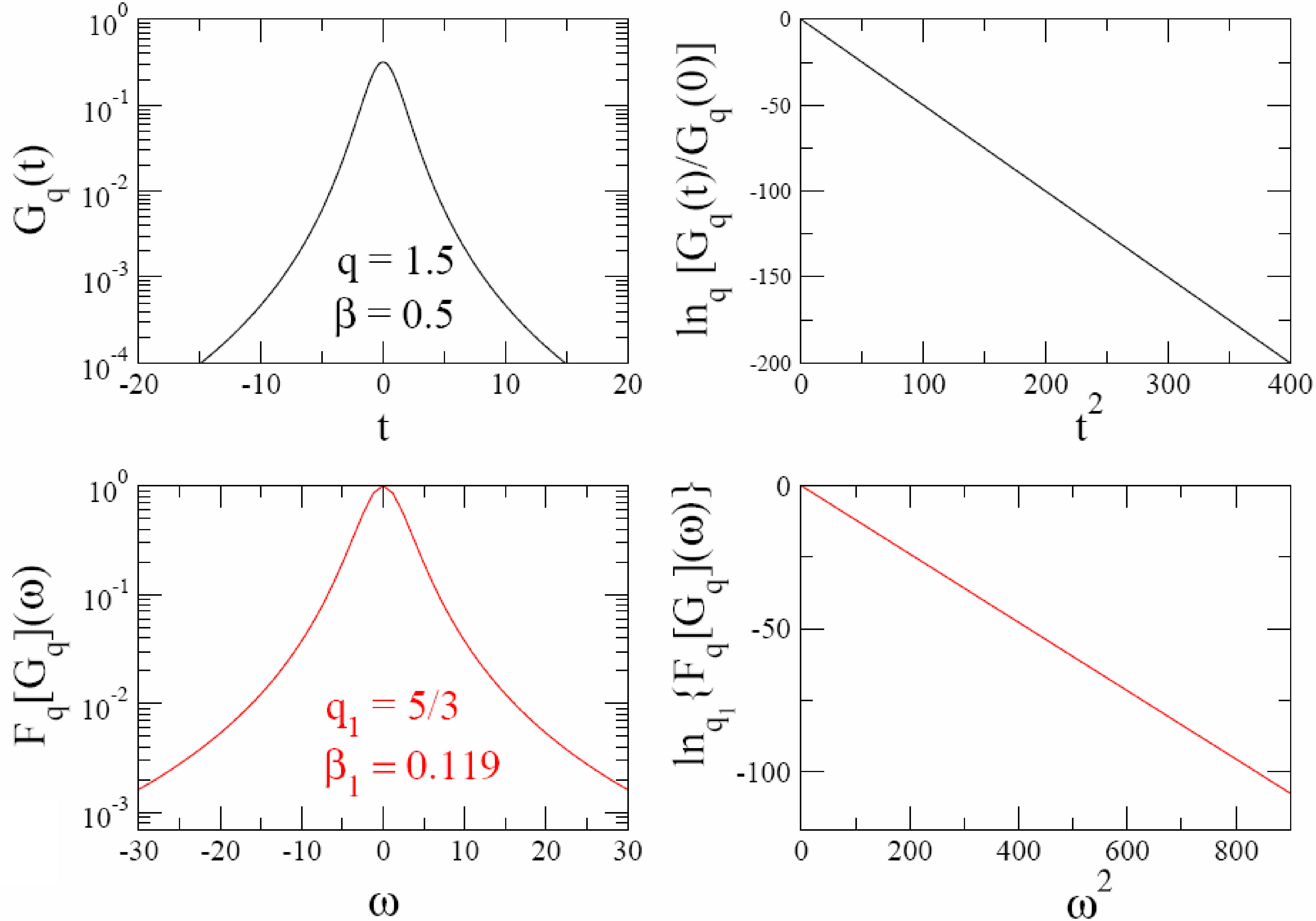
where $$q_1=\frac{1+q}{3-q}$$

and $$\beta_1=\frac{3-q}{8\beta^{2-q}C_q^{2(1-q)}}$$

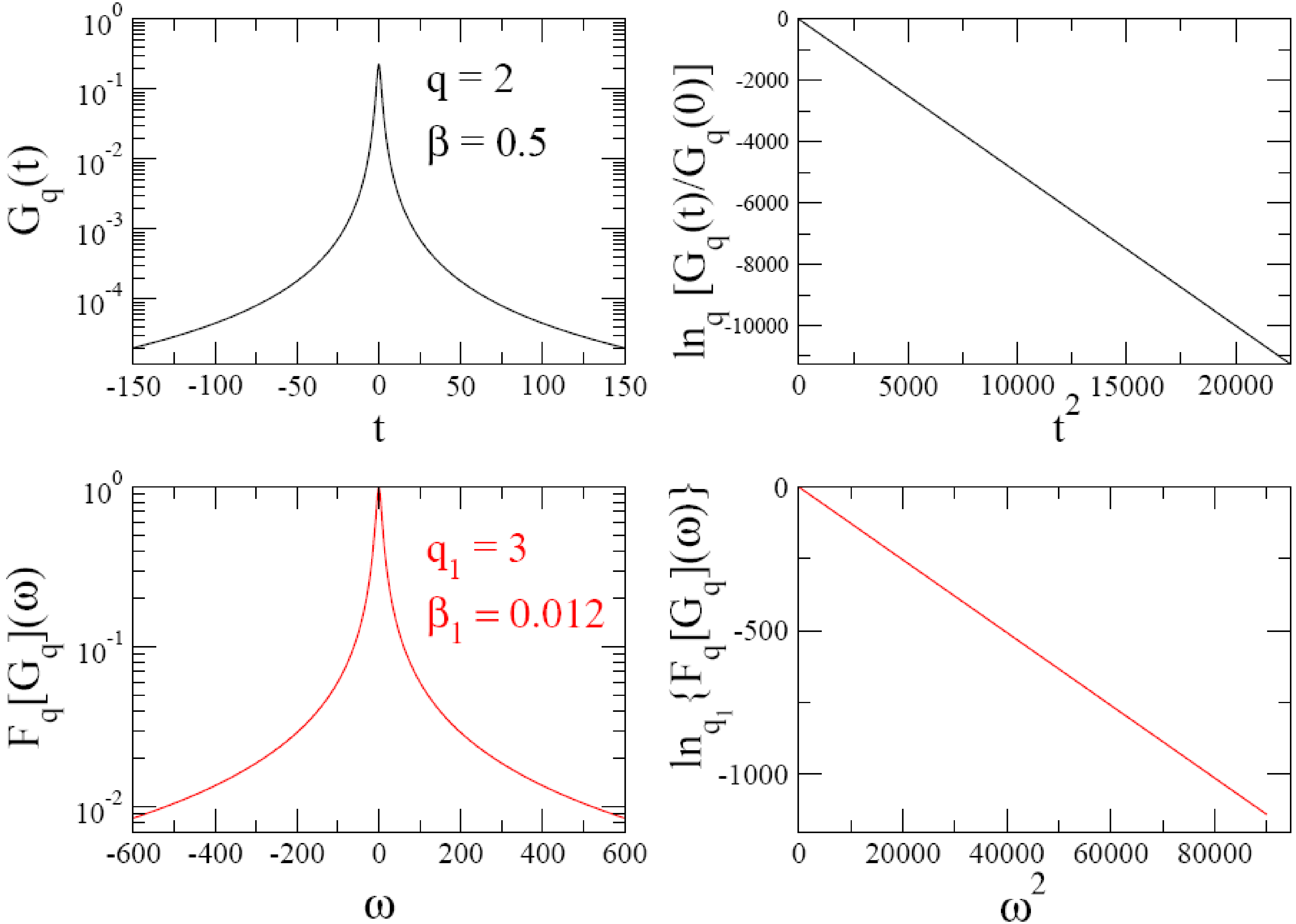
with $$C_q=\begin{cases}\dfrac{2\sqrt{\pi}\,\Gamma\left(\dfrac{1}{q-1}\right)}{(3-q)\sqrt{(1-q)}\,\Gamma\left(\dfrac{3-q}{2(1-q)}\right)} & \text{if}\quad q<1\\ \sqrt{\pi} & \text{if}\quad q=1\\ \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\dfrac{1}{q-1}\right)} & \text{if}\quad 1<q<3\end{cases}$$



$G_q(t)$
$q = 1$
$\beta = 0.5$
t
$\ln_q[G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
$q_1 = 1$
$\beta_1 = 0.5$
ω
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2



$G_q(t)$
t
q = 1.5
β = 0.5
$\ln_q[G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
ω
q_1 = 5/3
β_1 = 0.119
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2



$G_q(t)$
$q = 2$
$\beta = 0.5$
t
$\ln_q [G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
$q_1 = 3$
$\beta_1 = 0.012$
ω
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2

$A\ random\ \ variable\ X\ \ is\ said\ \ to\ have\ a\ \ (q,\alpha)\text{-}stable\ distribution\ \ L_{q,\alpha}(x)$

$if\ \ its\ q\text{-}Fourier\ transform\ has\ the\ form\ a\ e_q^{-b\ |\xi|^{\alpha}}\ \ (a>0,\ b>0,\ 0<\alpha\le 2)$

i.e., if

$$F_q[L_{q,\alpha}](\xi)\ \equiv\ \int_{-\infty}^{\infty} e_q^{ix\xi}\otimes_q L_{q,\alpha}(x)\ dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[L_{q,\alpha}(x)]^{1-q}}} L_{q,\alpha}(x)\ dx\ = a\ e_q^{-b\ |\xi|^{\alpha}}$$

$L_{1,2}(x)\equiv G(x)\quad (Gaussian)$

$L_{1,\alpha}(x)\equiv L_{\alpha}(x)\ \ (\alpha-stable\ Levy\ distribution)$

$L_{q,2}(x)\equiv G_q(x)\ \ (q-Gaussian)$

S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006)

cond-mat/0606038

cond-mat/0606040

CENTRAL LIMIT THEOREMS: $N^{1/[\alpha(2-q)]}$ - SCALED ATTRACTOR $\mathbb{F}(x)$ WHEN SUMMING $N \to \infty$ q-CORRELATED IDENTICAL RANDOM VARIABLES WITH SYMMETRIC DISTRIBUTION $f(x)$

	$q=1$ [independent]	$q \neq 1$ (i.e., $Q \equiv 2q-1 \neq 1$) [globally correlated]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) =$ Gaussian $G(x)$, with same σ_1 of $f(x)$ Classic CLT	$\mathbb{F}(x) = G_{\frac{3q-1}{q+1}}(x) \equiv \frac{3q-1}{q+1}$-Gaussian, with same $\sigma_Q \left[\equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q\right]$ of $f(x)$ $G_{\frac{3q-1}{q+1}}(x) \begin{cases} \simeq G(x) & \text{if } \lvert x\rvert \ll x_c(q,2) \\ \sim f(x) \sim C_q / \lvert x\rvert^{(q+1)/(q-1)} & \text{if } \lvert x\rvert \gg x_c(q,2) \end{cases}$ with $\lim_{q\to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg (2006) [cond-mat/0603593]
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) =$ Levy distribution $L_\alpha(x)$, with same $\lvert x\rvert \to \infty$ asymptotic behavior $L_\alpha(x) \begin{cases} \simeq G(x) & \text{if } \lvert x\rvert \ll x_c(1,\alpha) \\ \sim f(x) \sim C_\alpha / \lvert x\rvert^{1+\alpha} & \text{if } \lvert x\rvert \gg x_c(1,\alpha) \end{cases}$ with $\lim_{\alpha\to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{\frac{2\alpha q-\alpha+3}{\alpha+1},\alpha}$ stable distribution, with $L_{\frac{2\alpha q-\alpha+3}{\alpha+1},\alpha} \sim f(x) \sim C^{(L)}_{q,\alpha} / \lvert x\rvert^{(1+\alpha)/(1+\alpha q-\alpha)}$ or $\mathbb{F}(x) = L_{\frac{\alpha q+q-1}{\alpha+q-1},\alpha}$ stable distribution, with $L_{\frac{\alpha q+q-1}{\alpha+q-1},\alpha} \sim f(x) \sim C^{(*)}_{q,\alpha} / \lvert x\rvert^{2(\alpha+q-1)/\alpha(q-1)}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006) [cond-mat/0606038] and [cond-mat/0606040]

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS (CANONICAL ENSEMBLE):

Extremization of the functional

$$S_q[p_i] \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$$

with the constraints

$$\sum_{i=1}^{W} p_i = 1 \quad \text{and} \quad \frac{\sum_{i=1}^{W} p_i^q E_i}{\sum_{i=1}^{W} p_i^q} = U_q$$

yields

$$p_i = \frac{e_q^{-\beta_q (E_i - U_q)}}{\mathbb{Z}_q}$$

with $\beta_q \equiv \frac{\beta}{\sum_{i=1}^{W} p_i^q}$, $\beta \equiv$ *energy Lagrange parameter*, *and* $\mathbb{Z}_q \equiv \sum_{i=1}^{W} e_q^{-\beta_q (E_i - U_q)}$

We can rewrite $$p_i = \frac{e_q^{-\beta_q' E_i}}{Z_q'}$$

with $$\beta_q' \equiv \frac{\beta_q}{1+(1-q)\beta_q U_q}\,, \quad \textit{and} \quad Z_q' \equiv \sum_{i=1}^{W} e_q^{-\beta_q' E_i}$$

And we can prove

(i) $$\frac{1}{T} = \frac{\partial S_q}{\partial U_q} \quad \textit{with} \quad T \equiv \frac{1}{k\beta}$$

(ii) $$F_q \equiv U_q - TS_q = -\frac{1}{\beta}\ln_q Z_q \quad \textit{where} \quad \ln_q Z_q = \ln_q \mathbb{Z}_q - \beta U_q$$

(iii) $$U_q = -\frac{\partial}{\partial \beta}\ln_q Z_q$$

(iv) $$C_q \equiv T\frac{\partial S_q}{\partial T} = \frac{\partial U_q}{\partial T} = -T\frac{\partial^2 F_q}{\partial T^2}$$

(*i.e., the Legendre structure of Thermodynamics is q-invariant!*)

SOME FORM-INVARIANT RELATIONS (arbitrary q)

CLAUSIUS INEQUALITY AND BOLTZMANN H-THEOREM
(macroscopic time irreversibility)

Mariz, Phys Lett A **165** (1992) 409; Ramshaw, Phys Lett A **175** (1993) 169;
Abe and Rajagopal, Phys Rev Lett **91** (2003)

$$\beta\,\delta Q_q \le \delta S_q \quad ; \qquad q\,\frac{dS_q}{dt} \ge 0$$

EHRENFEST THEOREM (correspondence principle)

Plastino and Plastino, Phys Lett A **177** (1993) 177

$$\frac{d}{dt}\left\langle \hat{O} \right\rangle_q = \frac{i}{\hbar}\left\langle \left[\hat{H}\,, \hat{O} \right] \right\rangle_q$$

FACTORIZATION OF LIKELIHOOD FUNCTION
(Einstein's 1910 reversal of Boltzmann's formula; thermodynamically independent systems)

Caceres and Tsallis, unpublished (1993); Chame and Mello,
J Phys A **27** (1994) 3663; Tsallis, Chaos, Solitons and Fractals **6** (1995) 539

$$W_q(A+B) = W_q(A)\,W_q(B)$$

ONSAGER RECIPROCITY THEOREM
(microscopic time reversibility)

Caceres, Physica A **218** (1995) 471; Rajagopal, Phys Rev Lett **76** (1996) 3469;
Chame and Mello, Phys Lett A **228** (1997) 159

$$L_{jk} = L_{kj}$$

KRAMERS AND KRONIG RELATION (causality)

Rajagopal, Phys Rev Lett **76** (1996) 3469

PESIN EQUALITY
(mixing; Kolmogorov-Sinai entropy and Lyapunov exponent)

Tsallis, Plastino and Zheng, Chaos, Solitons and Fractals **8** (1997) 885;
Baldovin and Robledo, Phys. Rev. E **69**, 045202(R) (2004).

$$K_q = \begin{cases} \lambda_q & \text{if } \lambda_q > 0 \\ 0 & \text{otherwise} \end{cases}$$

ENTROPIC FORM AND EQUILIBRIUM STATISTICS: FOUNDATIONS

	BG (thermal equilibrium)	$q \neq 1$ (thermal metaequilibrium, nonequilibrium)
Distribution of velocities at equilibrium	Maxwell 1860	R. Silva, A.R. Plastino, J.A.S. Lima Phys Lett A 249, 401 (1998) R.S. Mendes and C. Tsallis Phys Lett A 285, 273 (2001)
Kinetic equation Molecular chaos hypothesis (*Stosszahlansatz*)	Boltzmann 1872	J.A.S. Lima, R. Silva, A.R. Plastino Phys Rev Lett 86, 2938 (2001) G. Kaniadakis Physica A 296, 405 (2001)
Optimization of entropy with constraints	Gibbs 1902	C.Tsallis J Stat Phys 52, 479 (1988) E.M.F. Curado and C.Tsallis J Phys A 24, L69 (1991) C.Tsallis, R.S. Mendes, A.R. Plastino Physica A 261, 534 (1998)
Steepest descent	Darwin-Fowler 1922	S. Abe and A.K. Rajagopal J Phys A 33, 8733 (2000)
Conditions of uniqueness of S	Shannon 1948	R. J. V. Santos J Math Phys 38, 4104 (1997)
Law of large numbers	Khinchin 1949	S. Abe and A.K. Rajagopal Europhys Lett 52, 610 (2000)
Compact conditions of uniqueness of S	Khinchin 1953	S. Abe Phys Lett A 271, 74 (2000)
Counting in the microcanonical ensemble	Balian-Balazs 1987 Kubo et al 1988	S. Abe and A.K. Rajagopal Phys Lett A 272, 341 (2000) Europhys Lett 55, 6 (2001)

ON THE NATURE OF q-CORRELATIONS:

W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)

Let us consider N correlated Normal random variables that generate N correlated uniform distributions

$$f(x) = \begin{cases} 1 & \text{if} \quad -1/2 \le x \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

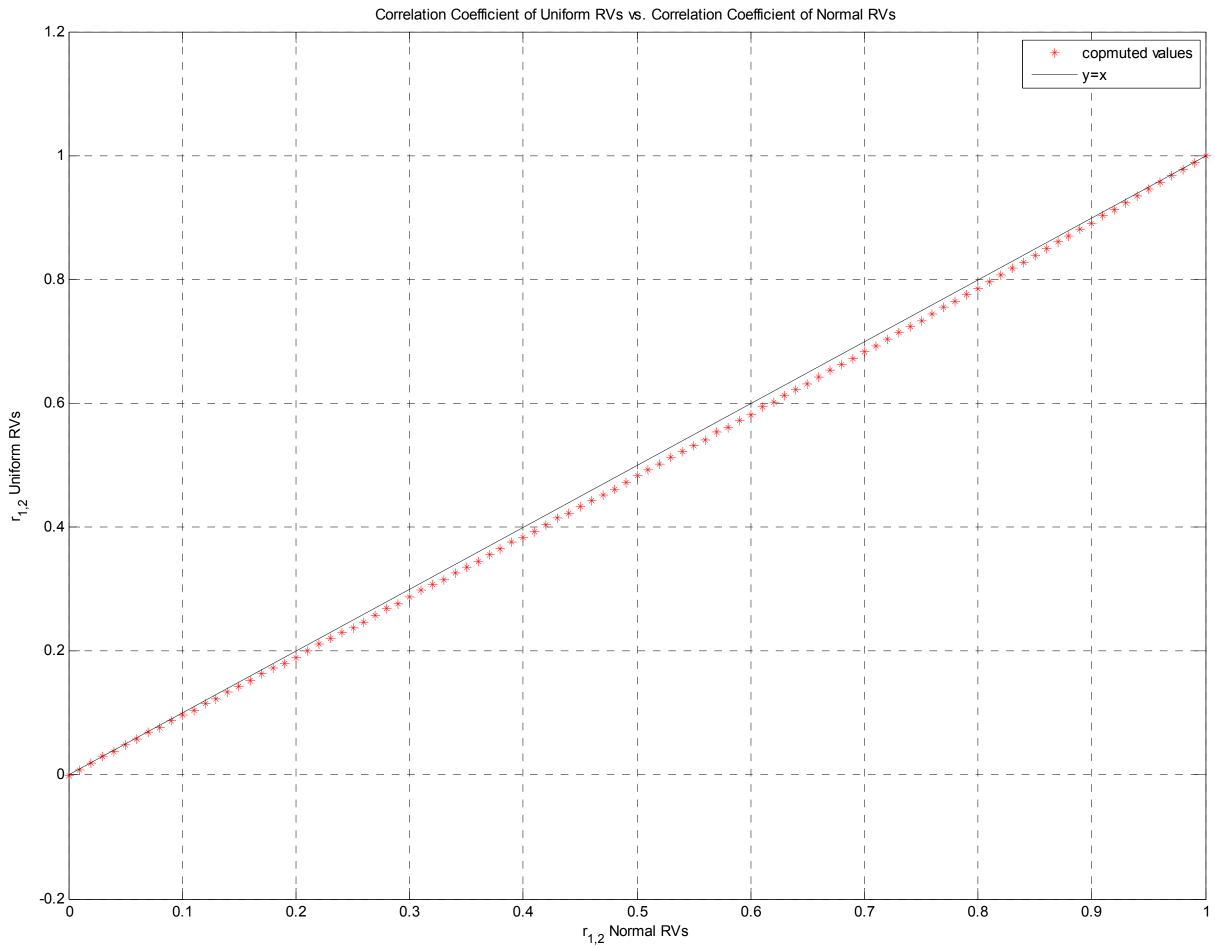
The $N \times N$ *covariance matrix of the N Normal distributions is given by*

$$\Sigma_Z = \begin{bmatrix} 1 & \rho & \cdots & \cdots & \rho \\ \rho & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \rho \\ \rho & \cdots & \cdots & \rho & 1 \end{bmatrix} \qquad (-1 \le \rho \le 1)$$

$\rho = 0 \quad \Rightarrow$ *independence*

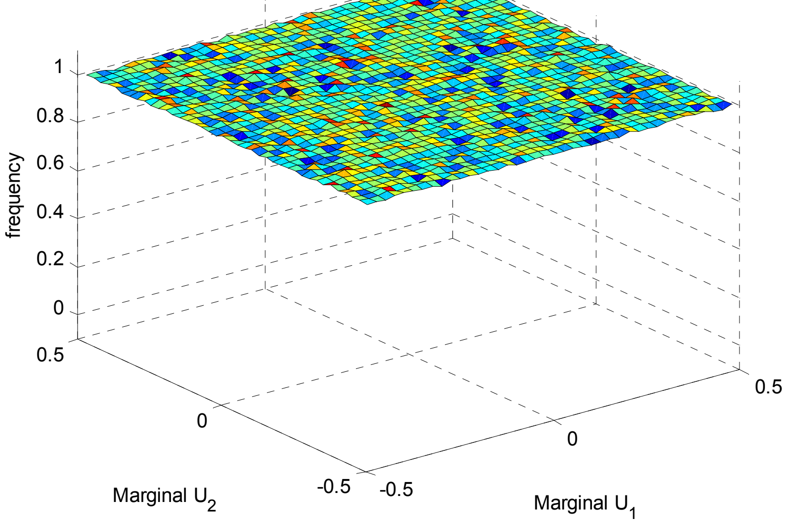
$\rho = 1 \quad \Rightarrow$ *full correlation*

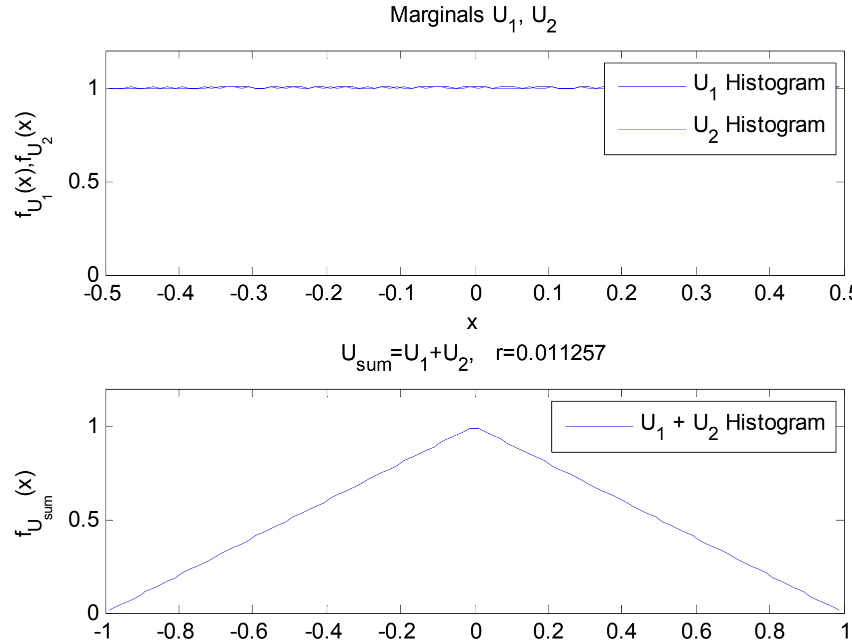
(N=2)



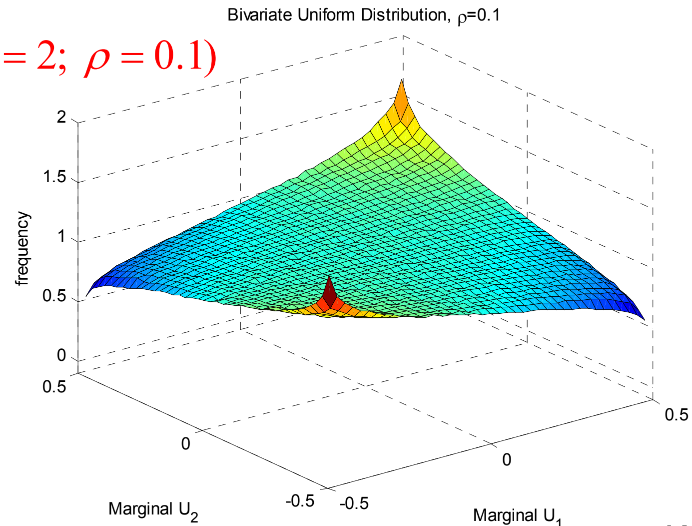
Bivariate normal data with a specified correlation coefficient is generated. The data (red dots) corresponding to the resulting marginal distributions are transformed to uniformly distributed random variables on (-1/2,1/2). The correlation coefficient of the bivariate uniform marginal data is presented as a function of the normal correlation coefficient. The bisector (continuous black line) is indicated as well as a guide to the eye.

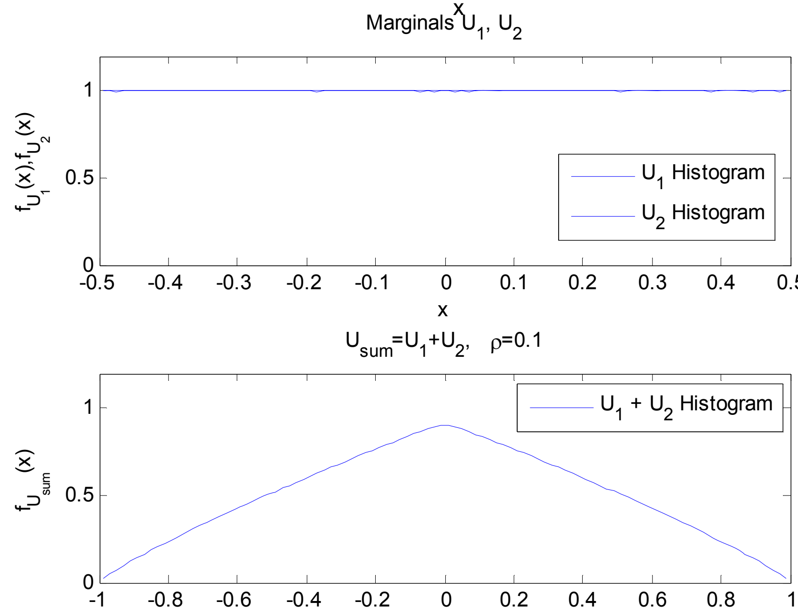
$(N = 2;\ \rho = 0)$





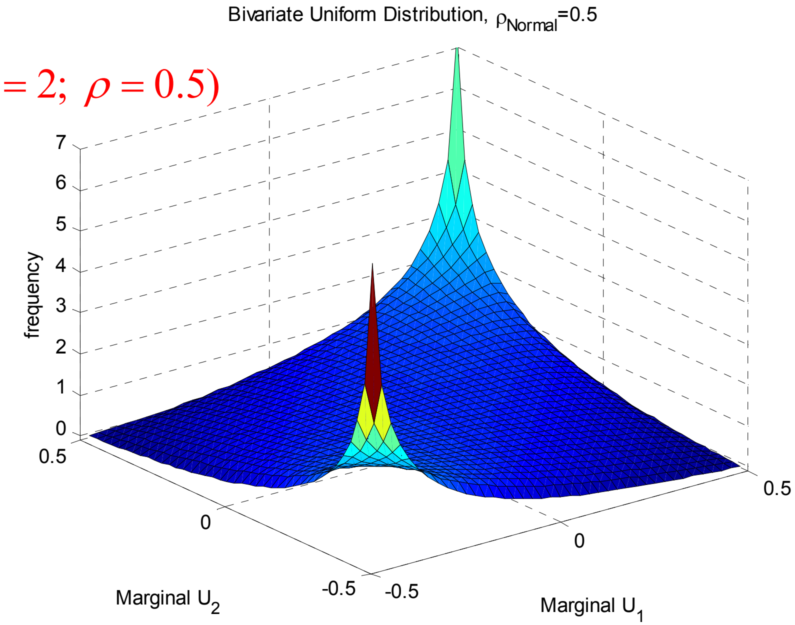
$(N = 2;\ \rho = 0.1)$

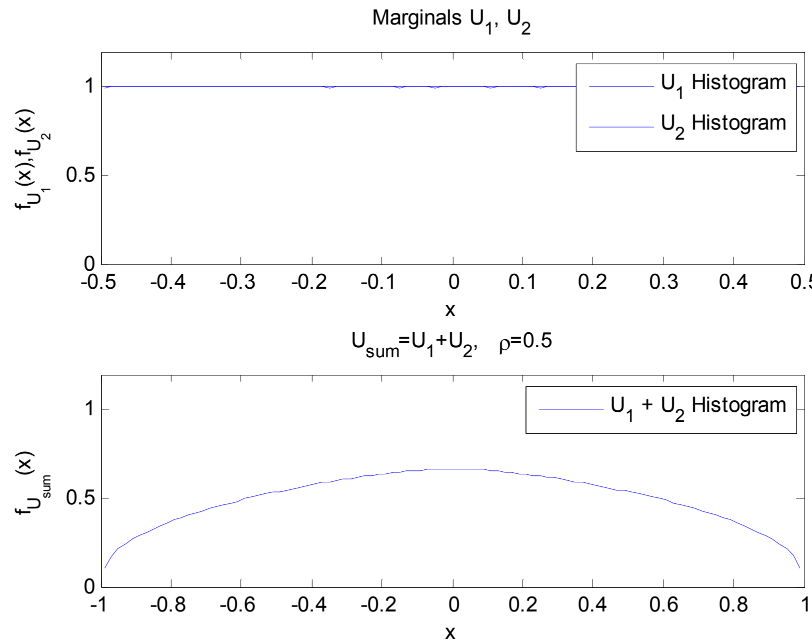




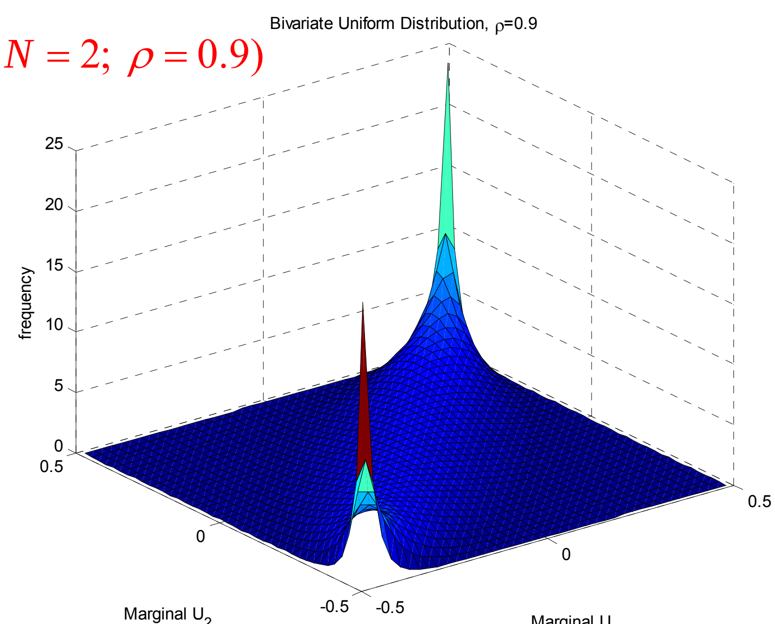
W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)

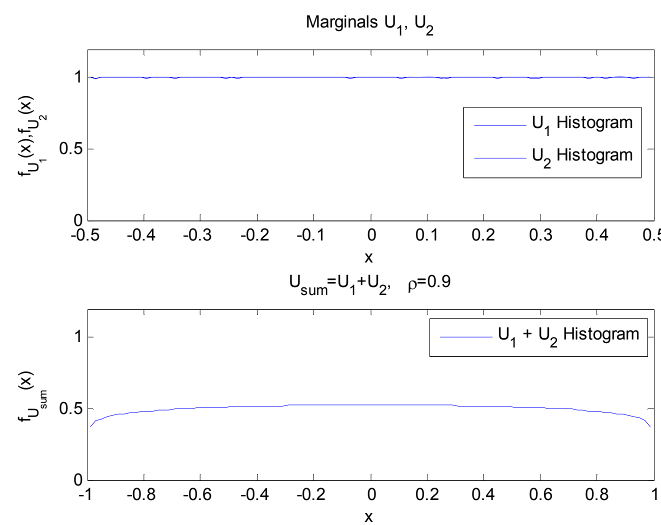
$(N=2;\ \rho=0.5)$



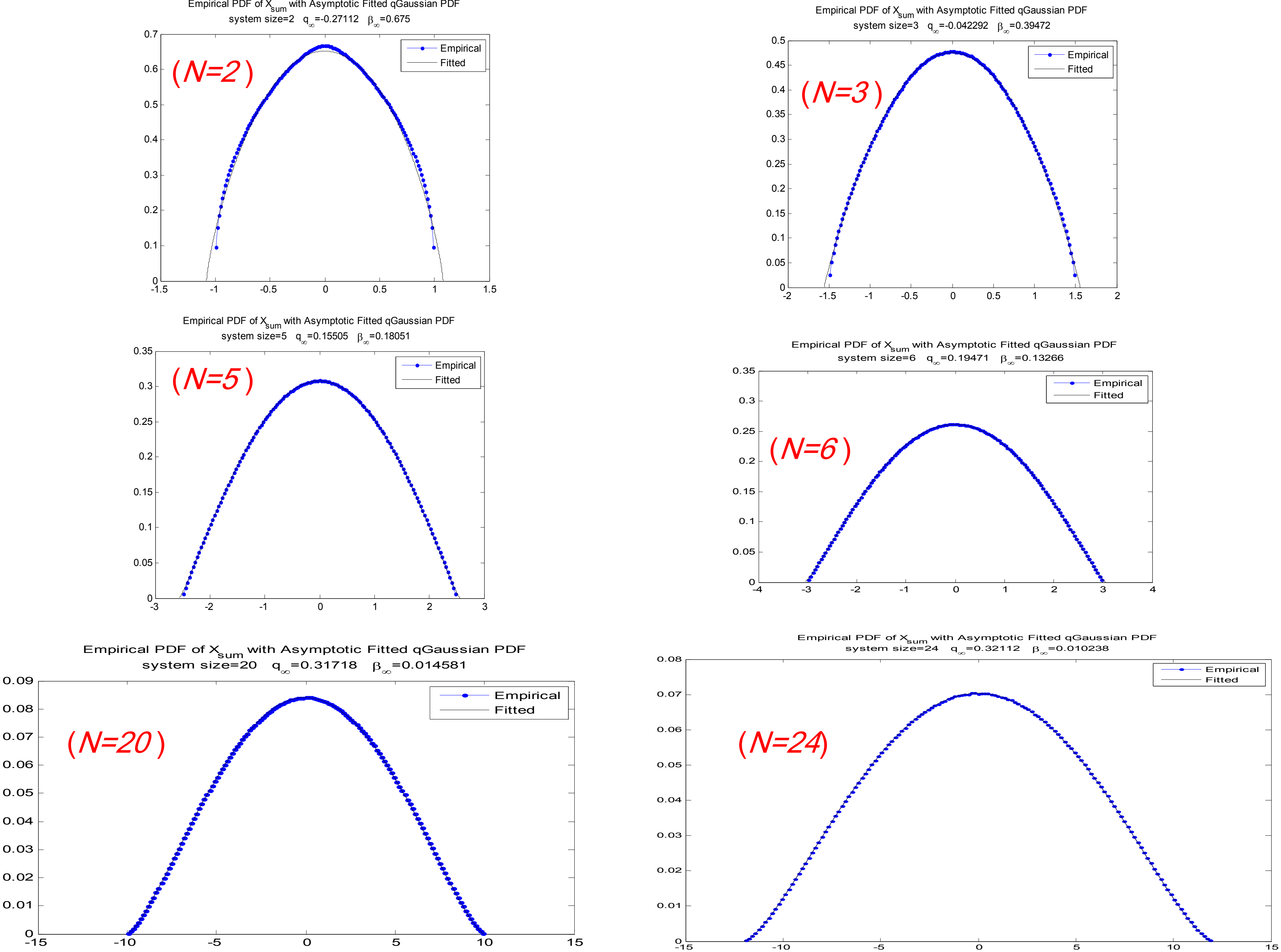


$(N=2;\ \rho=0.9)$

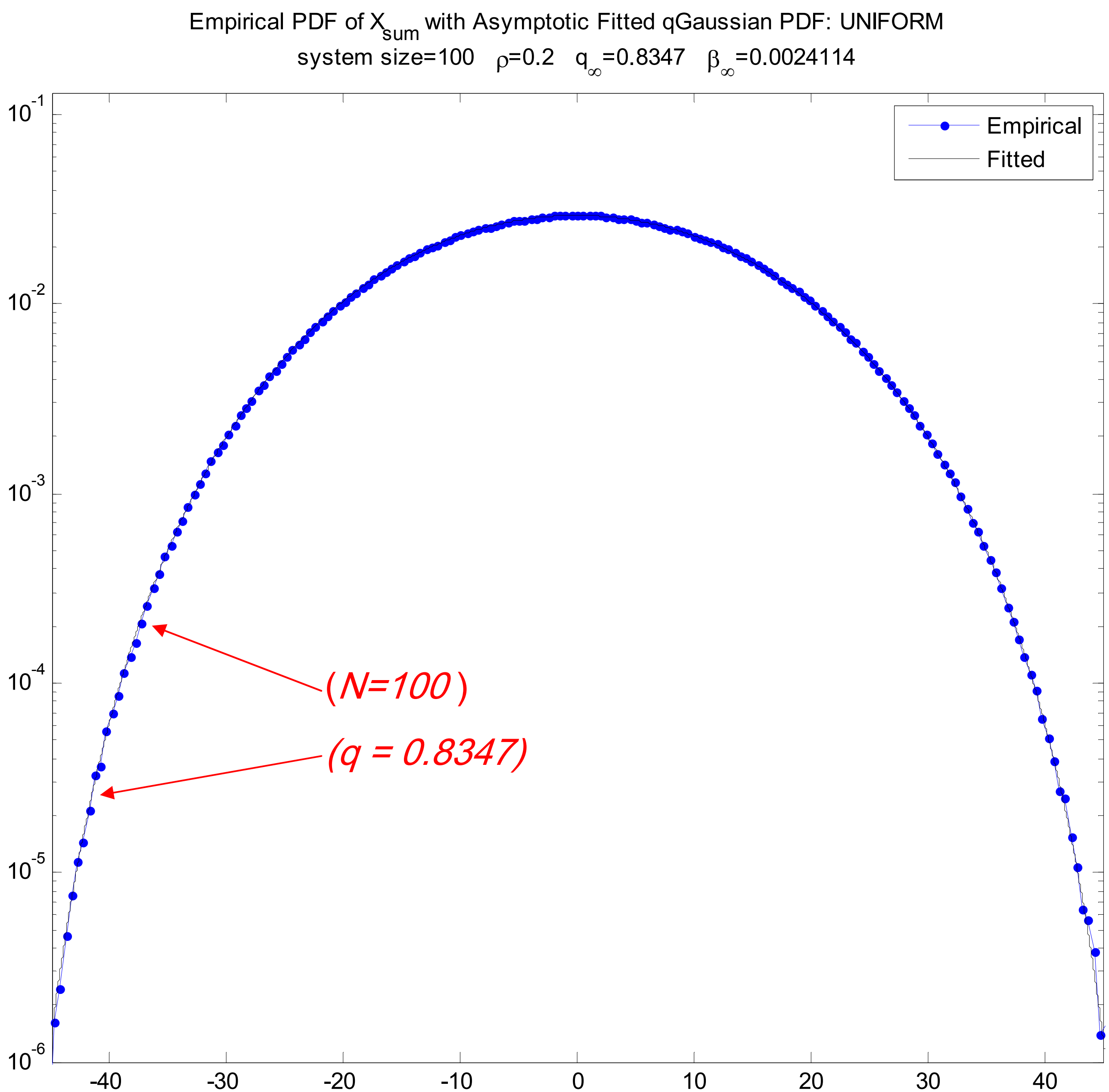




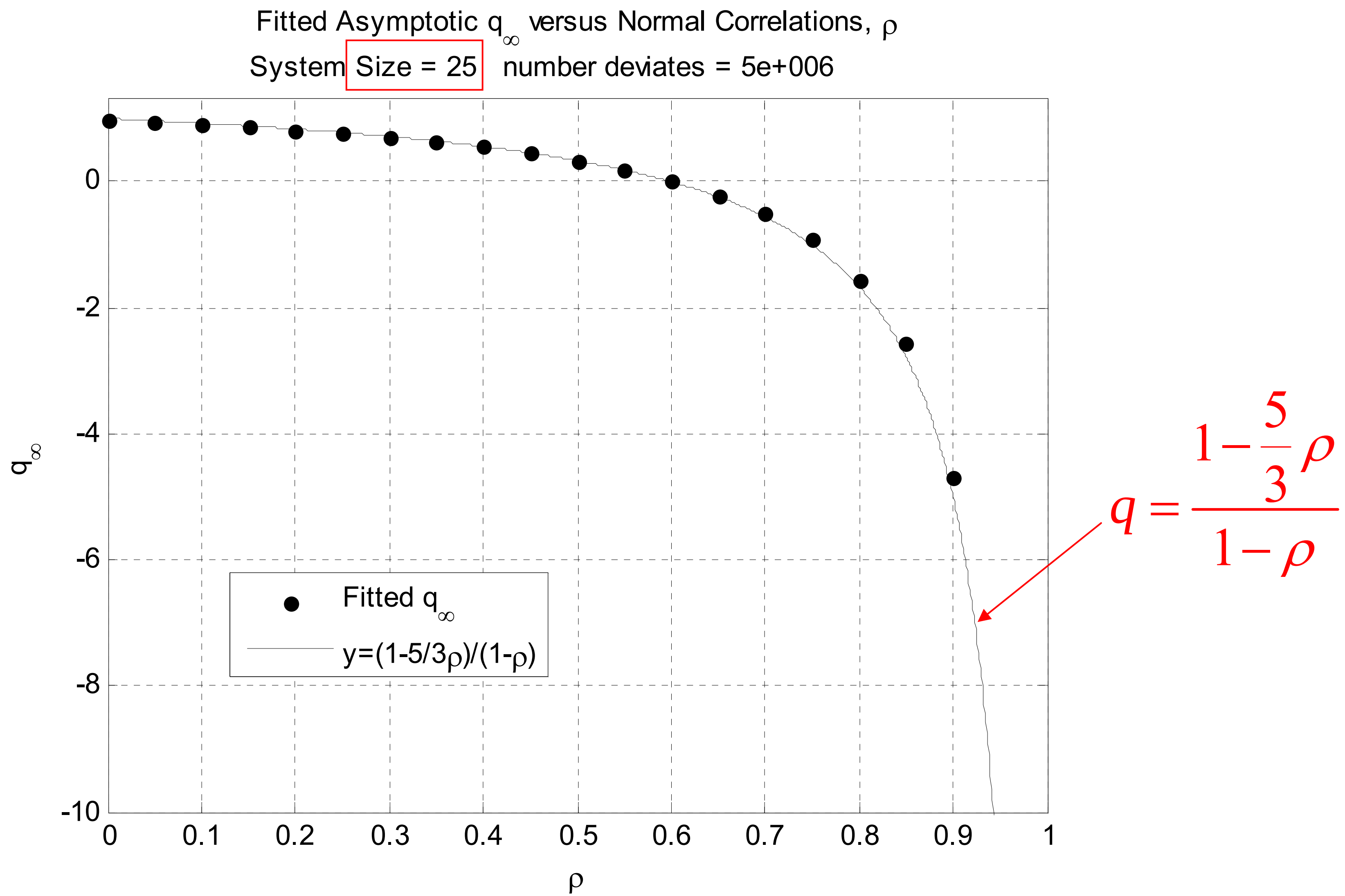
W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)



W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)



W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)



W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)

INFLUENCE OF THE RANGE OF CORRELATIONS DECAYING FAR FROM THE DIAGONAL OF THE COVARIANCE MATRIX:

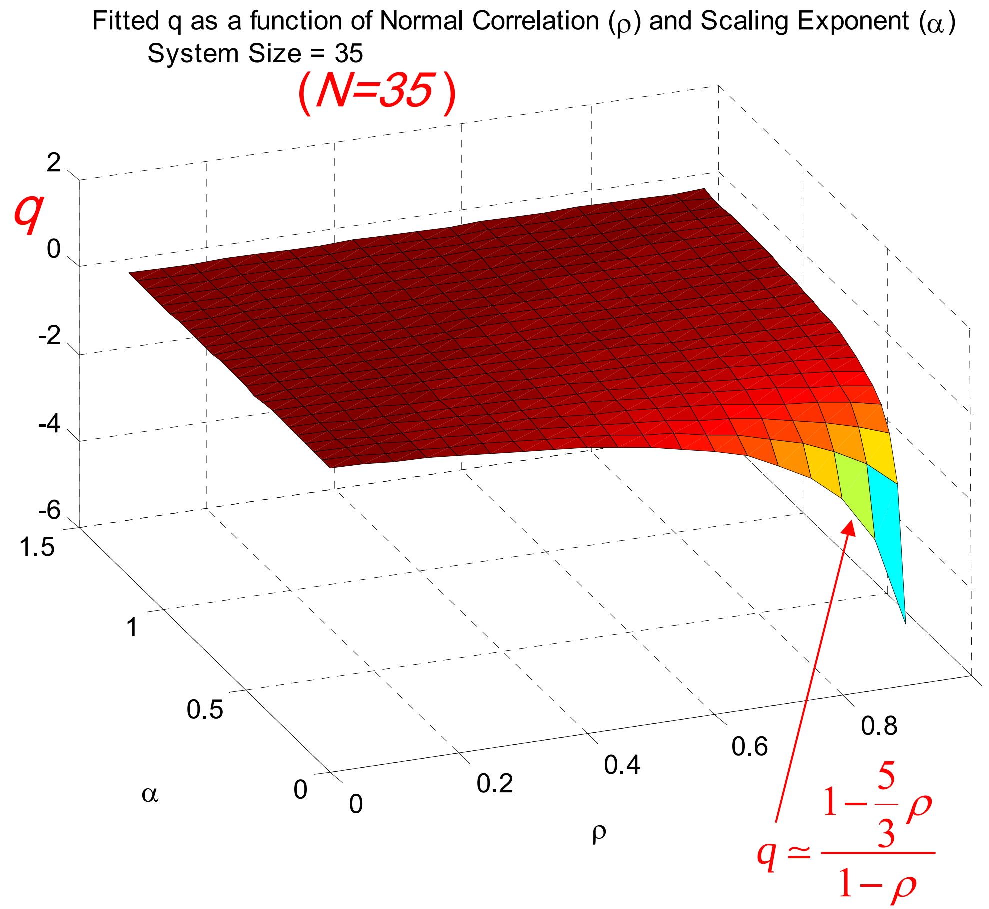
$N \times N$ covariance matrix of the Normal distributions given by

$$\begin{pmatrix} 1 & \rho(2) & \rho(3) & \dots & & \rho(N-1) & \rho(N) \\ \rho(2) & 1 & \rho(2) & \dots & & \rho(N-2) & \rho(N-1) \\ \rho(3) & \rho(2) & 1 & \dots & & \rho(N-3) & \rho(N-2) \\ \dots & \dots & \dots & \dots & & \dots & \dots \\ \rho(N-1) & \rho(N-2) & & \dots & \rho(2) & 1 & \rho(2) \\ \rho(N) & \rho(N-1) & & \dots & \rho(3) & \rho(2) & 1 \end{pmatrix}$$

with

$$\rho(r) = \frac{\rho}{r^{\alpha}}$$

$$(-1 \le \rho \le 1;\ \alpha \ge 0;\ r = 2,\ 3, \dots,\ N)$$

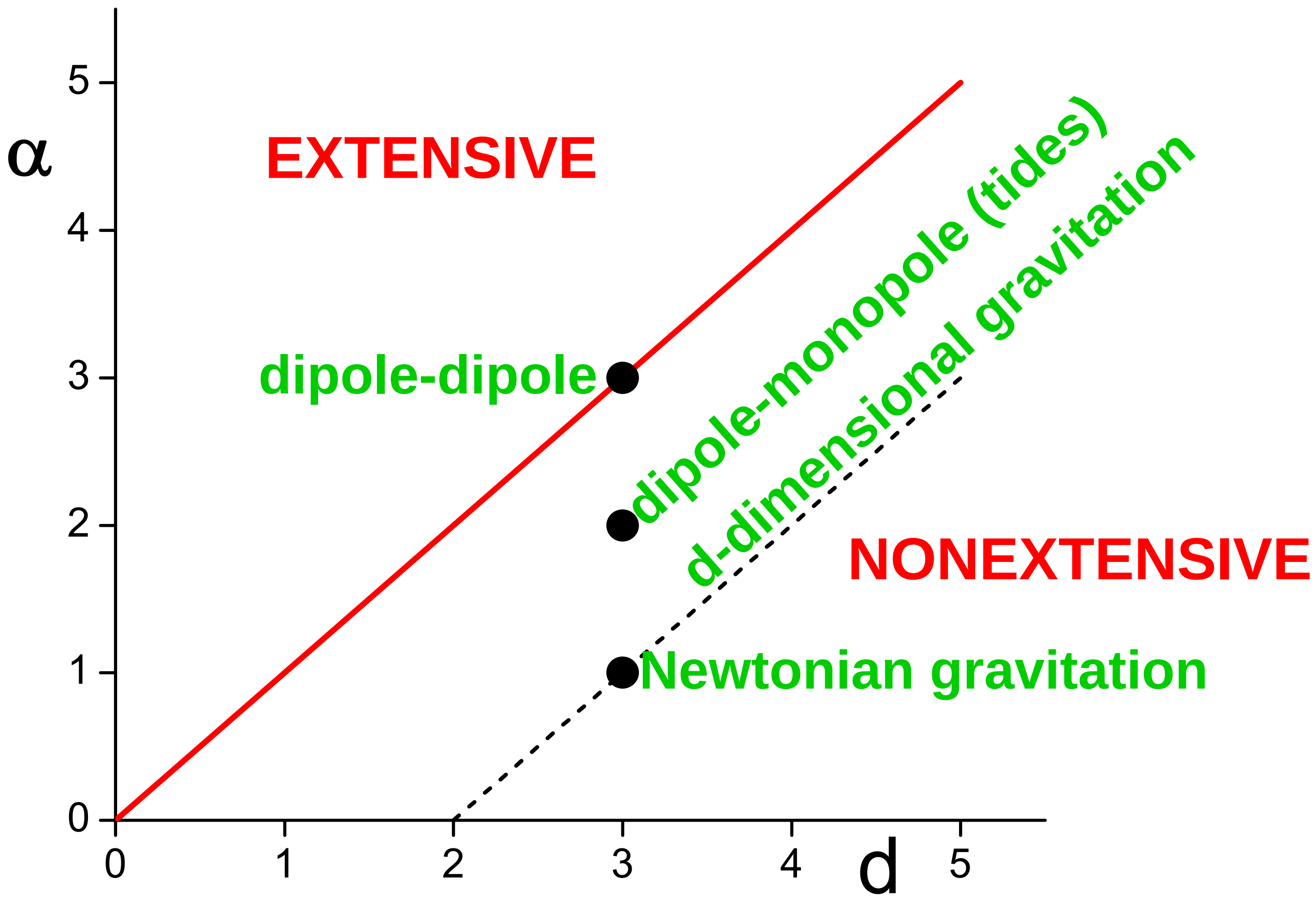


W. Thistleton, J.A. Marsh, K. Nelson, L.G. Moyano and C. T. (2006)

Connections with Hamiltonian and more complex systems

$$V(\vec{r}) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty)$$

$$(A > 0, \quad \alpha \geq 0)$$



BOLTZMANN-GIBBS STATISTICAL MECHANICS

(Maxwell 1860, Boltzmann 1872, Gibbs $\leq$ 1902)

Entropy $$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i$$

Internal energy $$U_{BG} = \sum_{i=1}^{W} p_i E_i$$

Equilibrium distribution $$p_i = e^{-\beta E_i} / Z_{BG} \qquad \left(Z_{BG} \equiv \sum_{j=1}^{W} e^{-\beta E_j} \right)$$

Paradigmatic differential equation $$\left. \begin{array}{l} \frac{dy}{dx} = a\,y \\ y(0) = 1 \end{array} \right\} \Rightarrow \qquad y = e^{ax}$$

	x	a	$y(x)$
Equilibrium distribution	E_i	$-\beta$	$Z\,p(E_i)$
Sensitivity to initial conditions	t	λ	$\xi \equiv \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda t}$
Typical relaxation of observable O	t	$-1/\tau$	$\Omega \equiv \frac{O(t) - O(\infty)}{O(0) - O(\infty)} = e^{-t/\tau}$

S_{BG} → extensive, concave, Lesche-stable, finite entropy production

NONEXTENSIVE STATISTICAL MECHANICS

(C. T. 1988, E.M.F. Curado and C. T. 1991, C. T., R.S. Mendes and A.R. Plastino 1998)

Entropy
$$S_q = k\left(1-\sum_{i=1}^{W} p_i^q\right)/(q-1)$$

Internal energy
$$U_q = \sum_{i=1}^{W} p_i^q E_i \Big/ \sum_{j=1}^{W} p_j^q$$

Stationary state distribution
$$p_i = e_q^{-\beta_q (E_i - U_q)} / Z_q \qquad \left(Z_q \equiv \sum_{j=1}^{W} e_q^{-\beta_q (E_j - U_q)} \right)$$

Paradigmatic differential equation
$$\left.\begin{array}{l} \dfrac{dy}{dx} = a\, y^{q} \\ y(0) = 1 \end{array}\right\} \Rightarrow \qquad y = e_q^{a x} \equiv [1 + (1-q)\, a x]^{\frac{1}{1-q}}$$

	x	a	$y(x)$
Stationary state distribution	E_i	$-\beta_{q_{stat}}$	$Z_{q_{stat}}\, p(E_i)$ (typically $q_{stat} \geq 1$)
Sensitivity to initial conditions	t	$\lambda_{q_{sen}}$	$\xi = e_{q_{sen}}^{\lambda_{q_{sen}} t}$ (typically $q_{sen} \leq 1$)
Typical relaxation of observable O	t	$-1/\tau_{q_{rel}}$	$\Omega = e_{q_{rel}}^{-t/\tau_{q_{rel}}}$ (typically $q_{rel} \geq 1$)

S_q → extensive, concave, Lesche-stable, finite entropy production

C. T., Physica A **340**,1 (2004)

Prediction of the q - triplet: C. T., Physica A **340**,1 (2004)

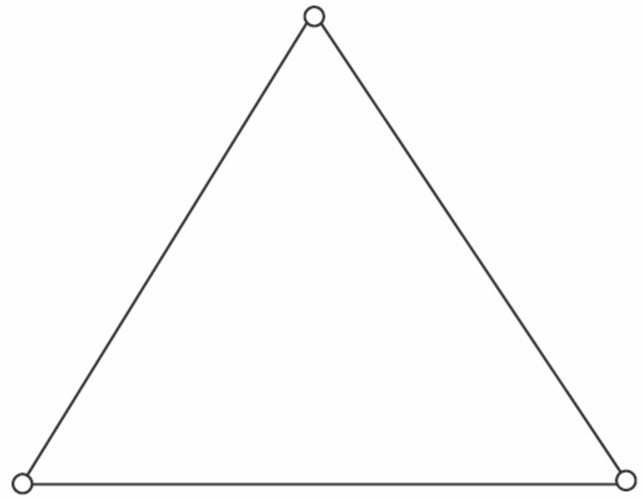
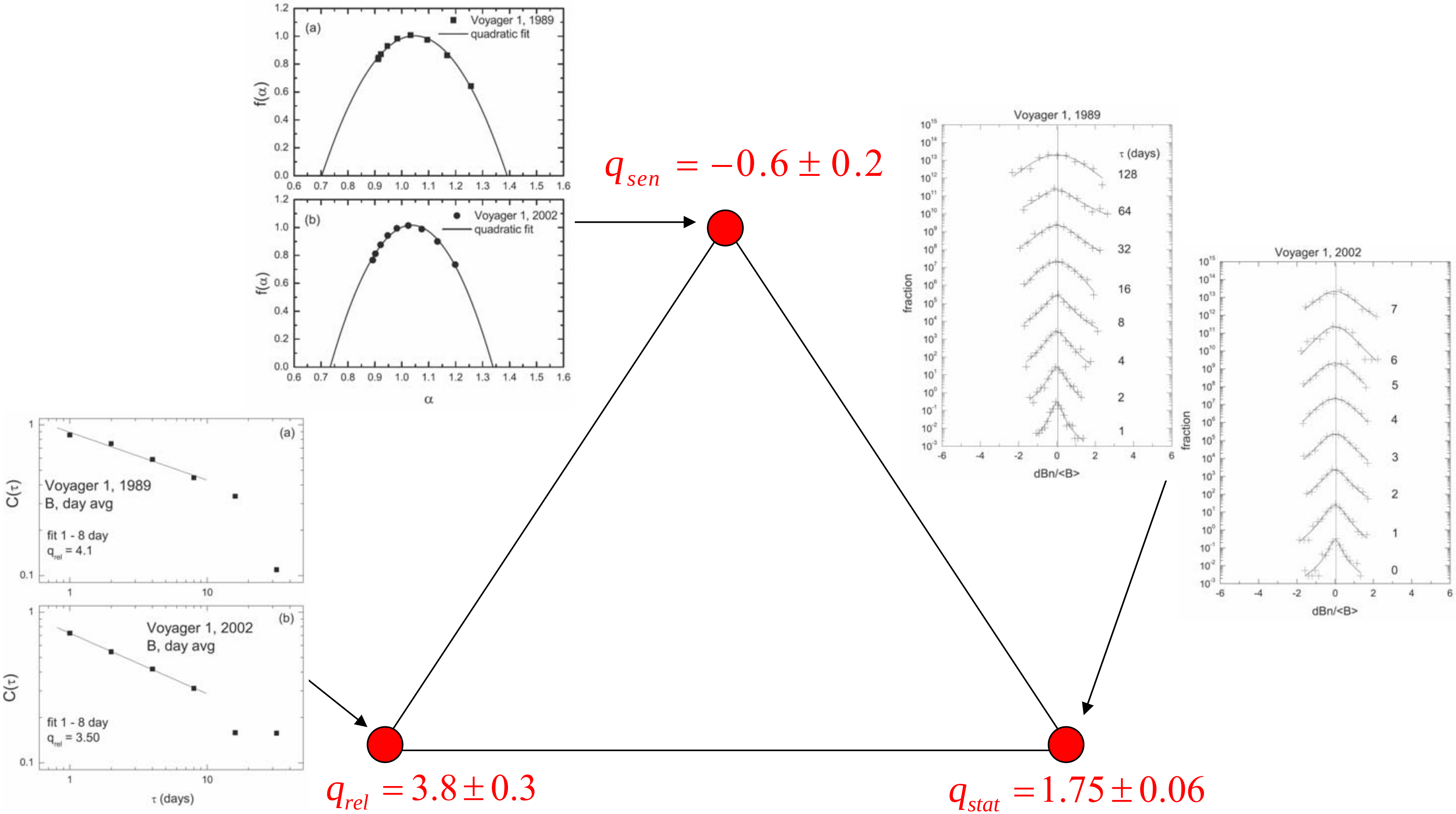


Fig. 2. The triangle of the basic values of q, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leqslant 1$, $q_{rel} \geqslant 1$ and $q_{stat} \geqslant 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent α and the dimension d, it could be that q_{stat} decreases from a value above unity (e.g., 2 or $\frac{3}{2}$) to unity when α/d increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for q_{sen} and q_{rel}, but only those associated with a Hamiltonian are expected to *also* have a well defined value for q_{stat}.

SOLAR WIND: Magnetic Field Strength

L.F. Burlaga and A. F.-Vinas (2005) / NASA Goddard Space Flight Center; Physica A **356**, 375 (2005)

[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]



$$Playing\ with\ additive\ duality \quad (q \to 2-q)$$

$$and\ with\ multiplicative\ duality \quad (q \to 1/q)$$

$$(and\ using\ numerical\ results\ related\ to\ the\ q-generalized\ central\ limit\ theorem)$$

we conjecture

$$q_{rel} + \frac{1}{q_{sen}} = 2 \qquad and \qquad q_{stat} + \frac{1}{q_{rel}} = 2$$

$$hence \qquad 1- q_{sen} = \frac{1-q_{stat}}{3-2\,q_{stat}}$$

hence only one independent!

$$Burlaga\ and\ Vinas\ (NASA)\ most\ precise\ value\ of\ the\ q-triplet\ is$$

$$q_{stat} = 1.75 = 7/4$$

$$hence \qquad q_{sen} = -\,0.5 = -\,1/2 \qquad (consistent\ with\ q_{sen} = -\,0.6\ \pm 0.2\ !)$$

$$and \qquad q_{rel} = 4 \qquad (consistent\ with\ q_{rel} = 3.8\ \pm 0.3\ !)$$

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

Connections with

asymptotically scale – free networks

GEOGRAPHIC PREFERENTIAL ATTACHMENT GROWING NETWORK:

THE NATAL MODEL

D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva, Europhys Lett **70**, 70 (2005)

(1) Locate site $i = 1$ at the origin of say a plane

(2) Then locate the next site with

$$P_G \propto 1/r^{2+\alpha_G} \quad (\alpha_G \geq 0)$$

$(r \equiv \textit{distance to the baricenter of the pre-existing cluster})$

(3) Then link it to only one of the previous sites using

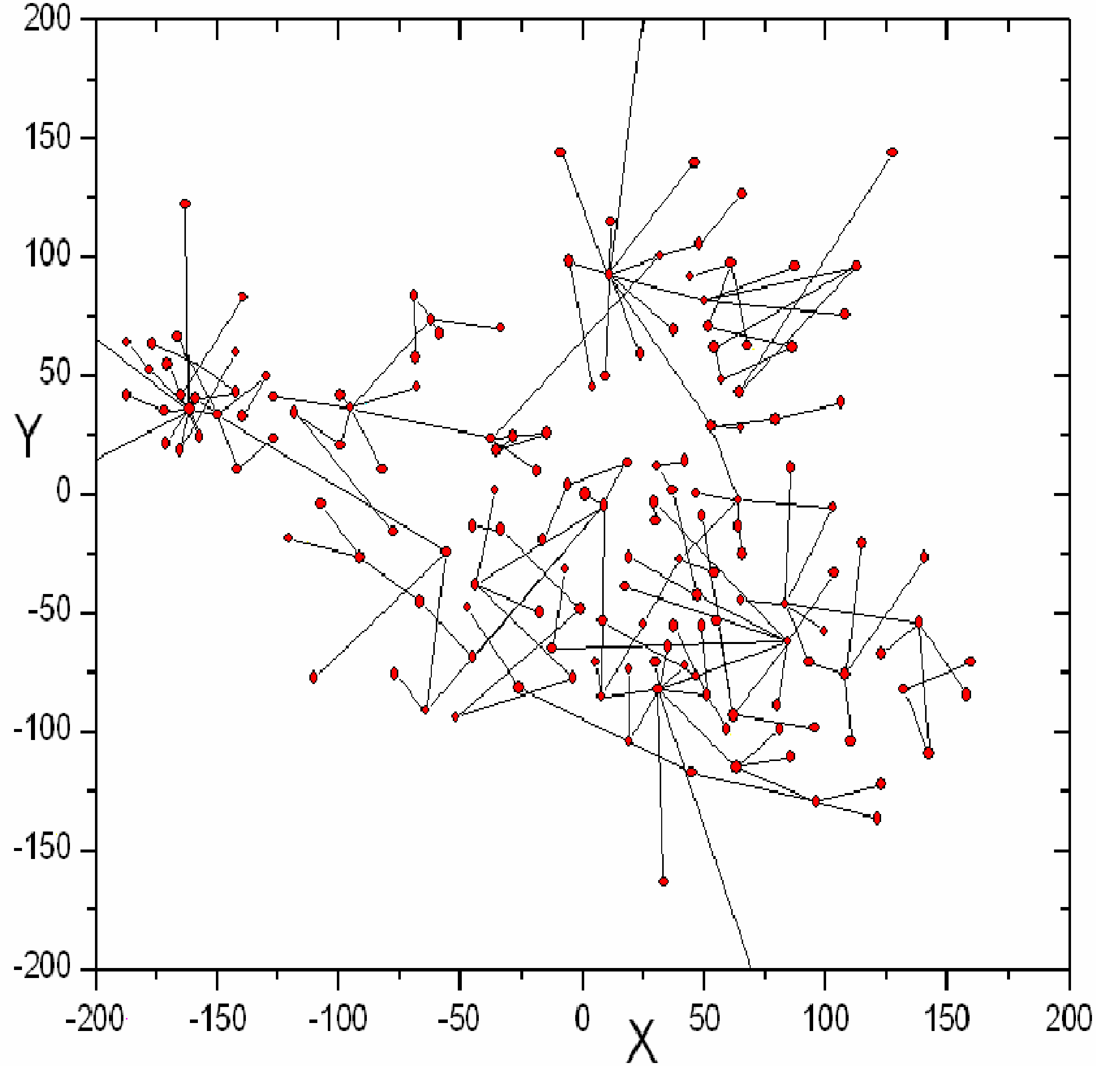
$$p_{\mathrm{A}} \propto k_i / r_i^{\alpha_A} \quad (\alpha_A \geq 0)$$

$(k_i \equiv \textit{links already attached to site } i)$

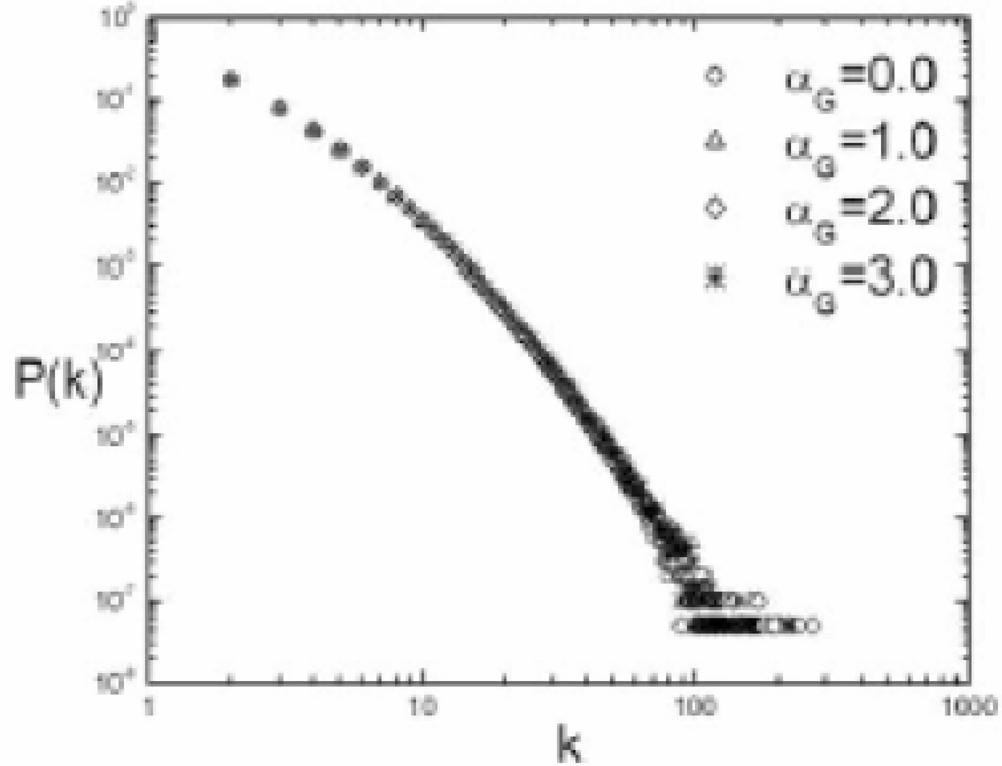
$(r_i \equiv \textit{distance to site } i)$

4) Repeat

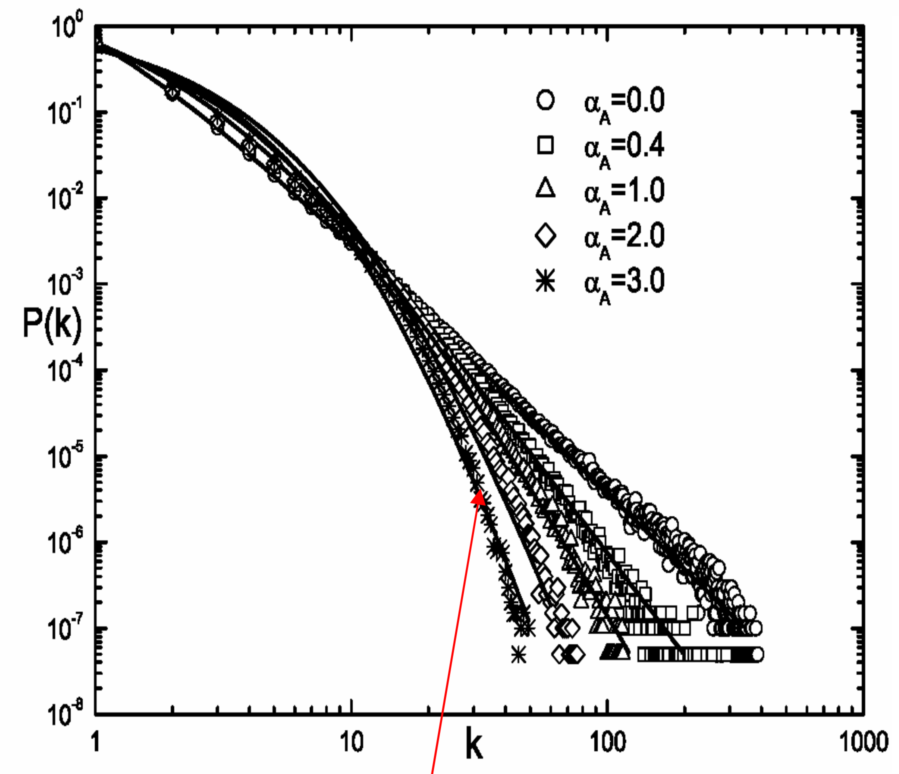
$$(\alpha_{\mathrm{G}} = 1; \alpha_A = 1; N = 250)$$

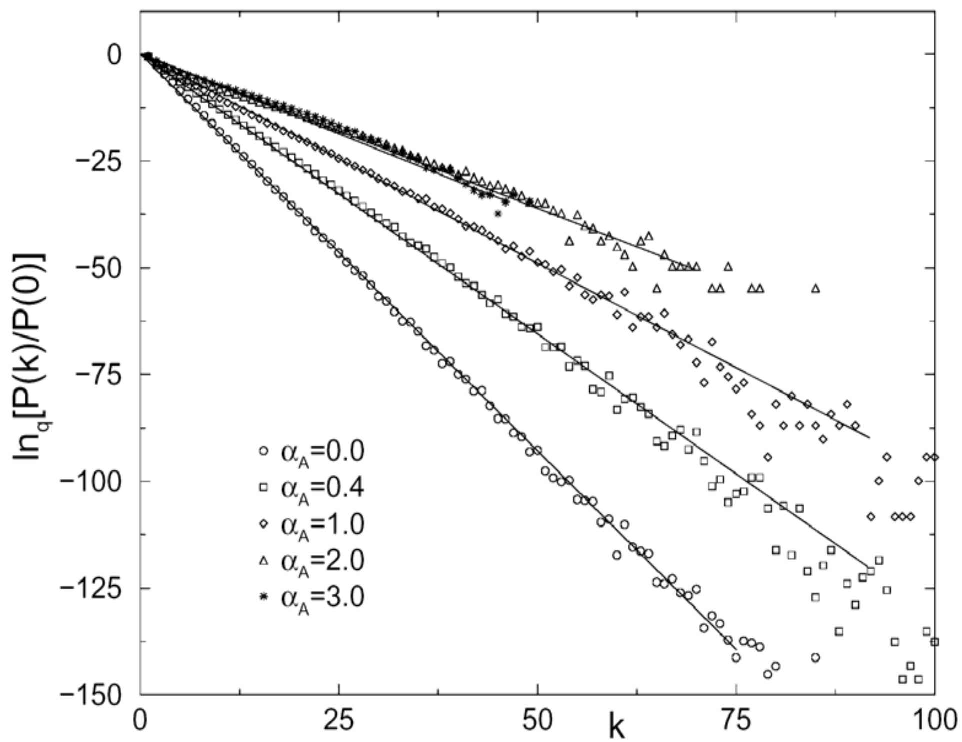


D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva
Europhys Lett **70**, 70 (2005)



αG=0.0
αG=1.0
αG=2.0
αG=3.0
P(k)
1
10
100
1000
k

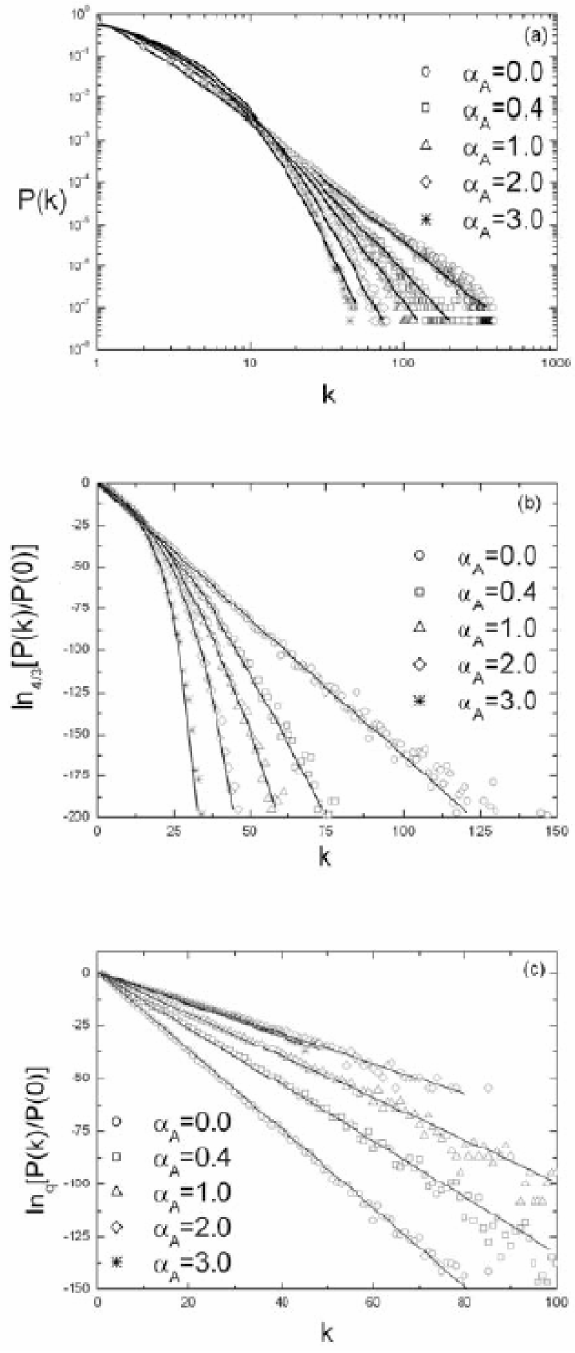




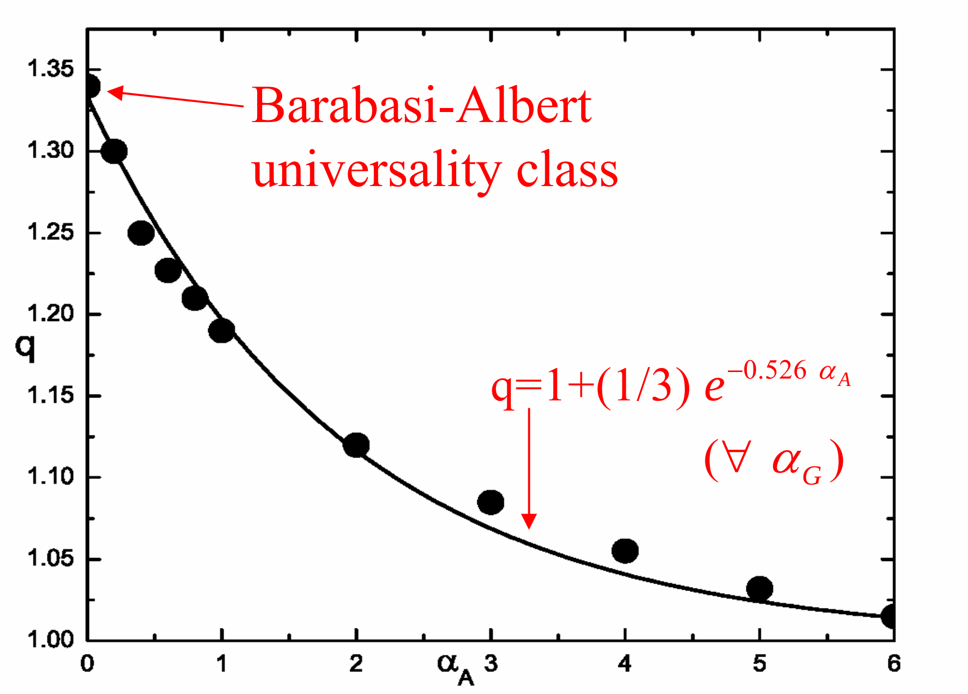
$$P(k)/P(0) = e_q^{-k/\kappa}$$

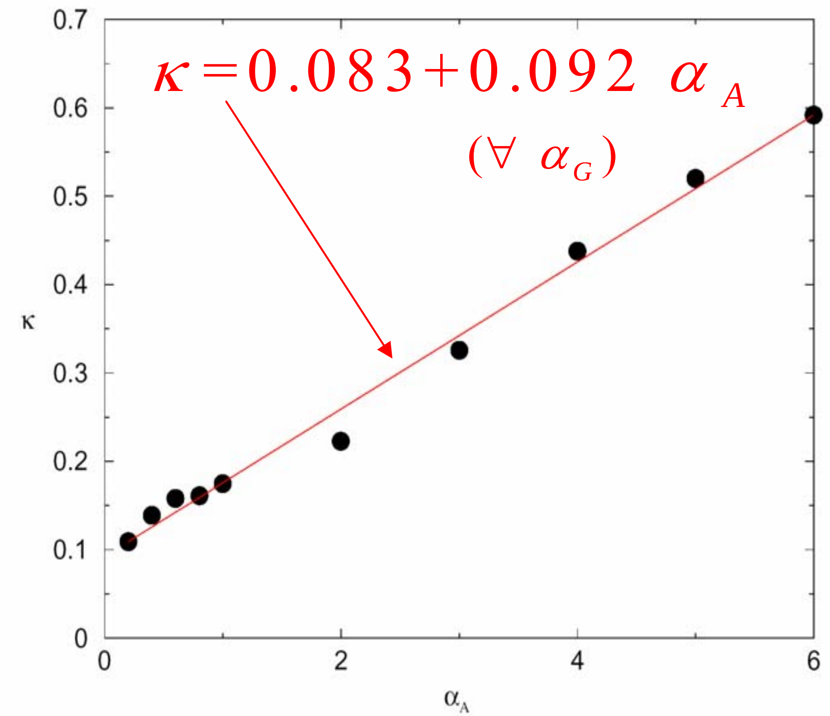
$$\equiv 1/[1+(q-1)k/\kappa]^{1/(q-1)}$$

D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva
Europhys Lett **70**, 70 (2005)



(a)
P(k)
k
α_A=0.0
α_A=0.4
α_A=1.0
α_A=2.0
α_A=3.0
1
10
100
1000
(b)
ln_{4/3}[P(k)/P(0)]
k
α_A=0.0
α_A=0.4
α_A=1.0
α_A=2.0
α_A=3.0
0
-25
-50
-75
-100
-125
-150
-175
-200
0
25
50
75
100
125
150
(c)
ln_q[P(k)/P(0)]
k
α_A=0.0
α_A=0.4
α_A=1.0
α_A=2.0
α_A=3.0
0
-25
-50
-75
-100
-125
-150
0
20
40
60
80
100





D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva
Europhys Lett **70**, 70 (2005)

GAS-LIKE (NODE COLLAPSING) NETWORK:

S. Thurner and C. T., Europhys Lett **72**, 197 (2005)

Number *N* of nodes fixed (*chemostat*); i=1, 2, …, *N*

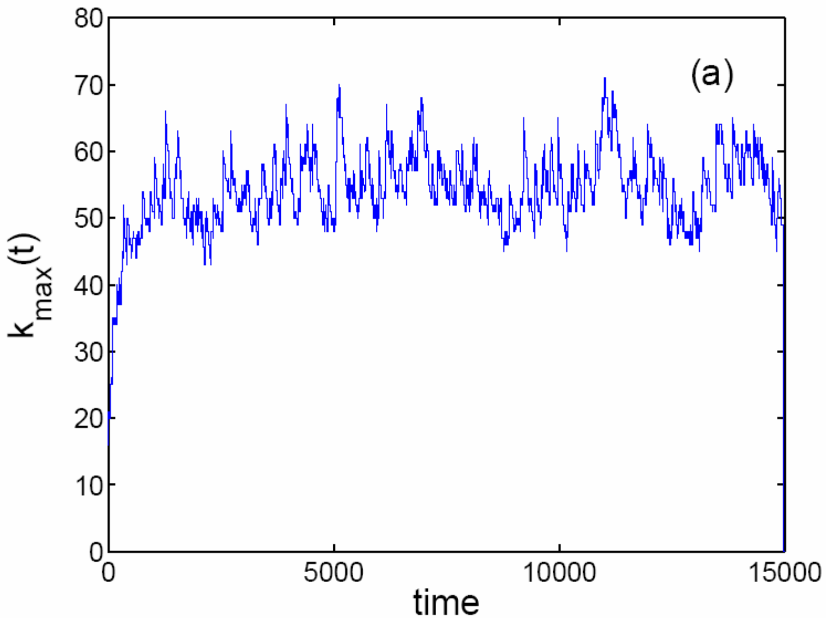
$$Merging\ probability\ \ p_{ij} \propto \frac{1}{d_{ij}{}^{\alpha}} \quad (\alpha \geq 0)$$

$$d_{ij} \equiv shortest\ path\ (chemical\ distance)\ connecting\ nodes\ i\ and\ j\ on\ the\ network$$

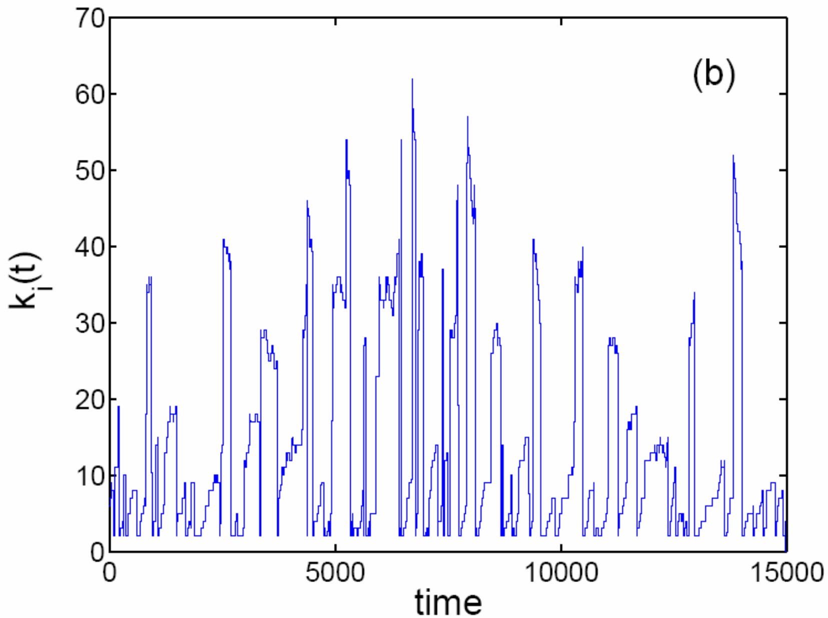
$\alpha = 0$ *and* $\alpha \to \infty$ *recover the random and the neighbor schemes respectively*

(Kim, Trusina, Minnhagen and Sneppen, *Eur. Phys. J. B* 43 (2005) 369)

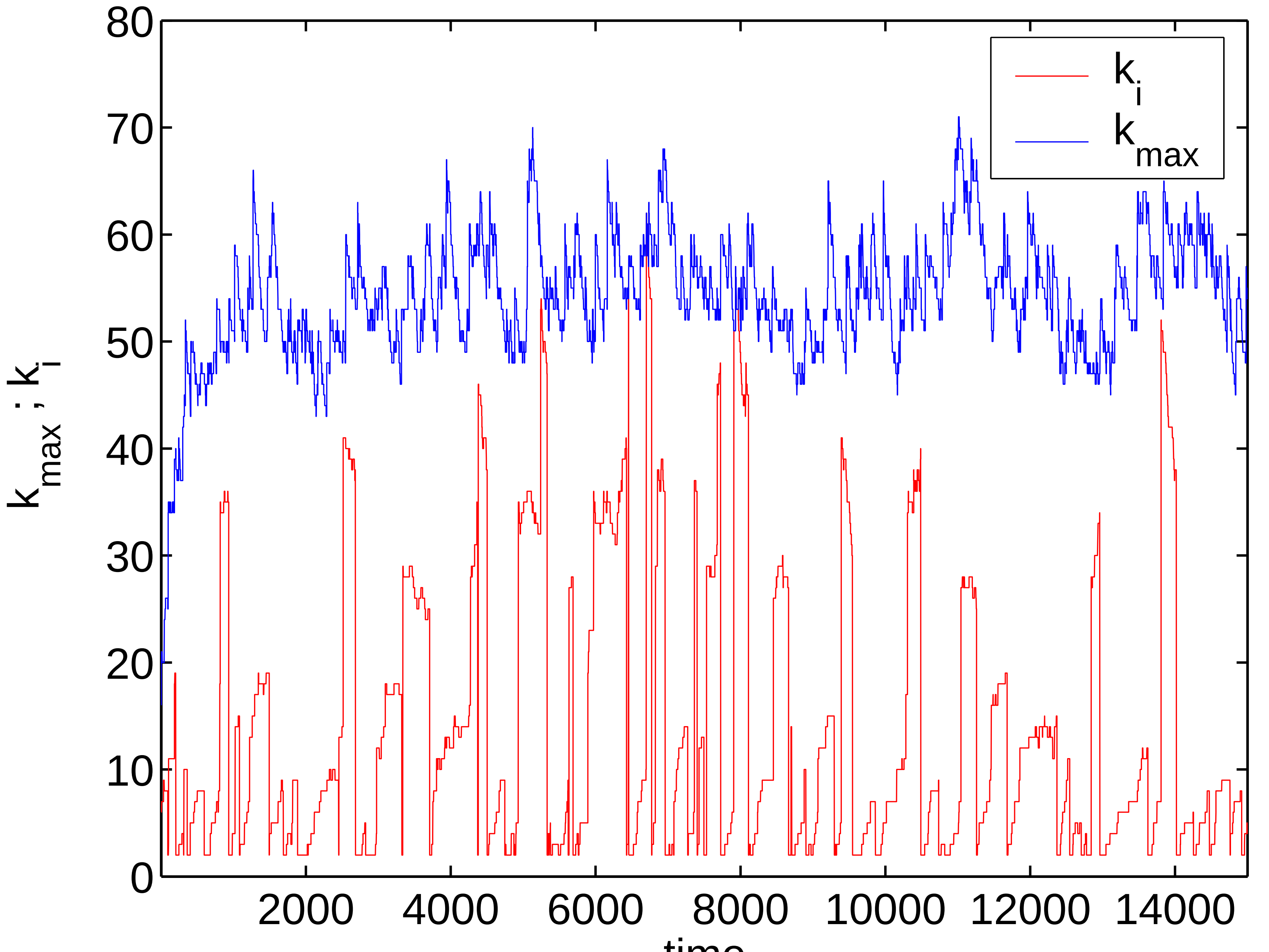
$(N = 2^7;\ \alpha = 0;\ r = 2)$



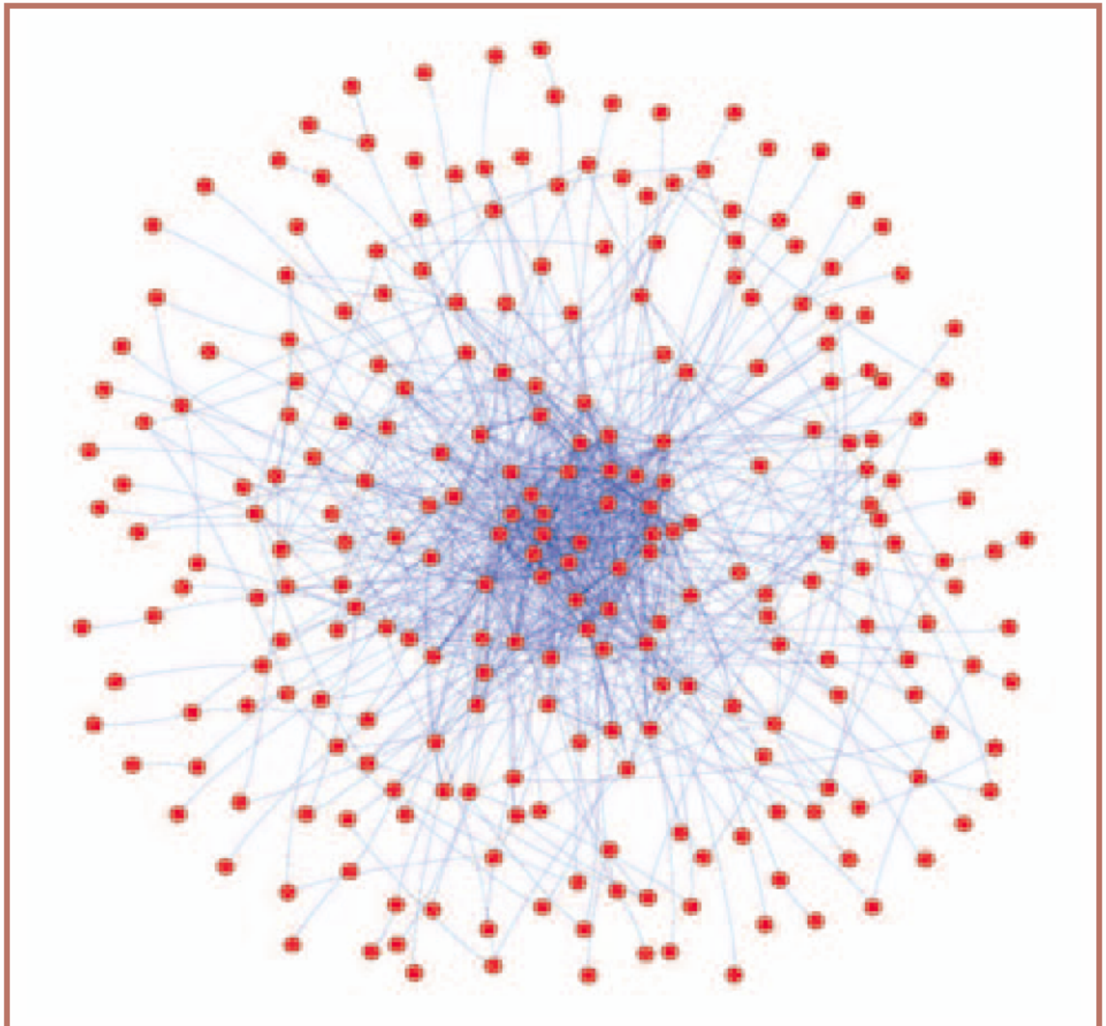
Degree of the most connected node



Degree of a randomly chosen node



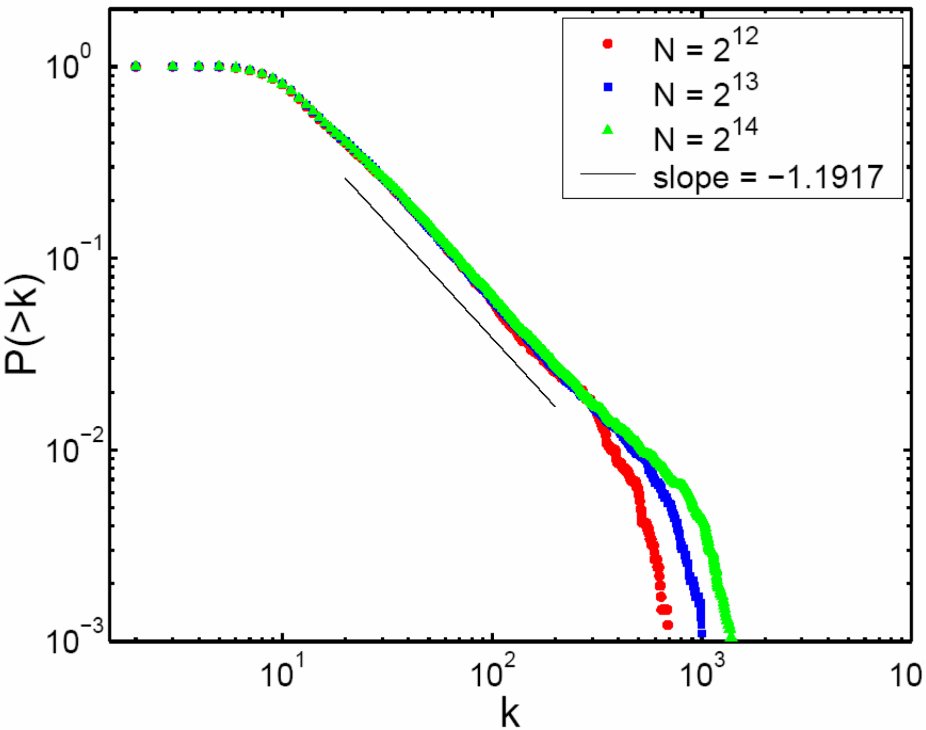
80
70
60
50
40
30
20
10
0
2000
4000
6000
8000
10000
12000
14000
time
k_{max} ; k_i
k_i
k_{max}

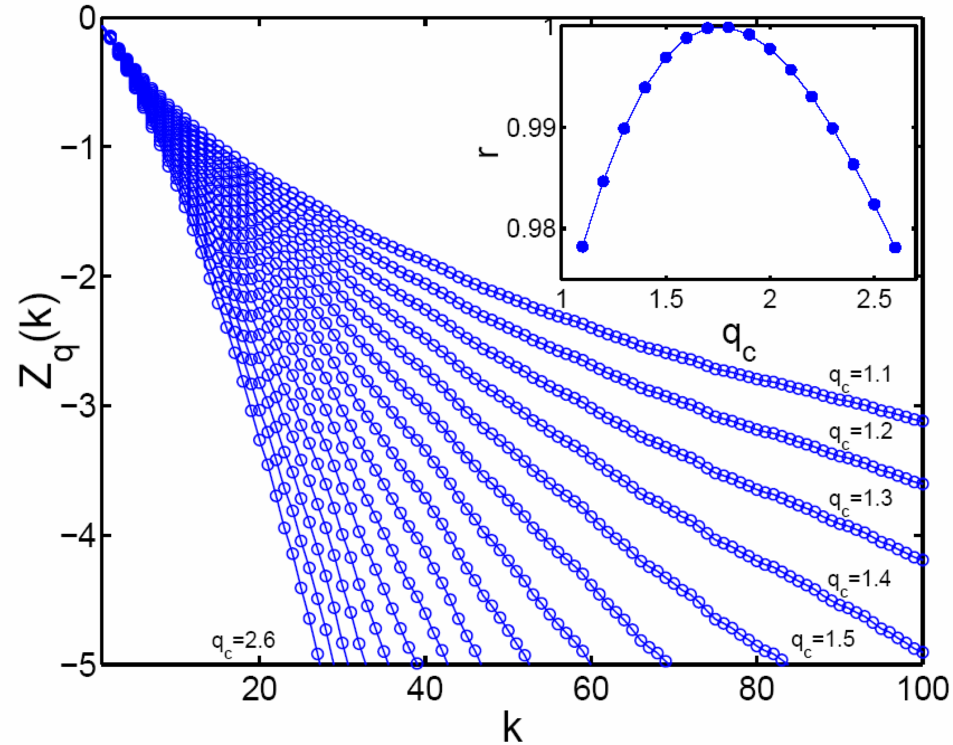


▲ **Fig. 1:** Snapshot of a non-growing dynamic network with q-exponential degree distribution for $N = 256$ nodes and a linking rate of $\bar{r} = 1$, for details see [8, 9]. The shown network is small to make connection patterns visible.

S. Thurner, Europhys News **36**, 218 (2005)

$$(\alpha \to \infty \; ; \; <r> = 8)$$



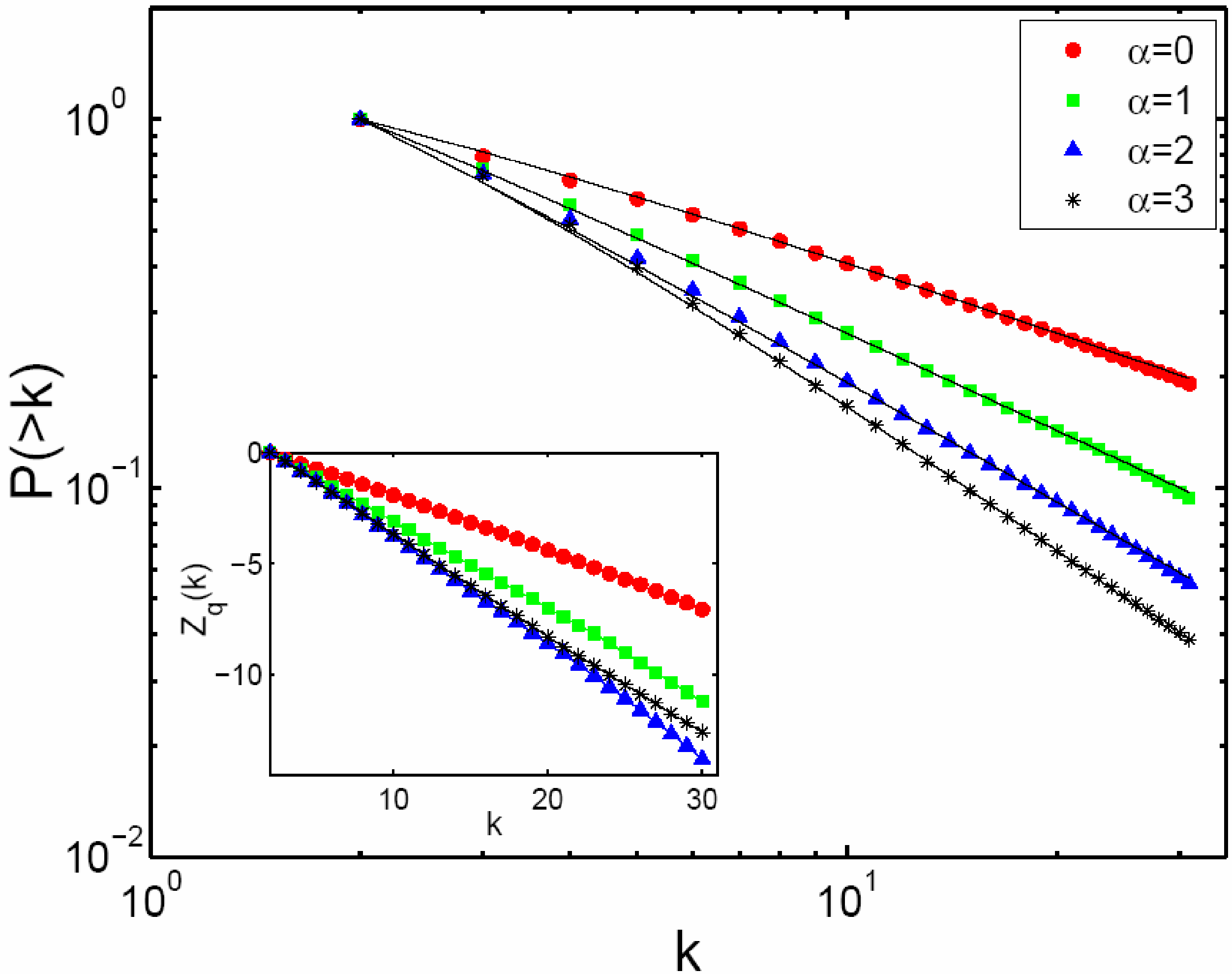


$$Z_q(k) \equiv \ln_q \left[P(>k) \right] \equiv \frac{\left[P(>k) \right]^{1-q} - 1}{1-q}$$

$$(optimal \;\; q_c = 1.84)$$

S. Thurner and C. T., Europhys Lett **72**, 197 (2005)

$$(N = 2^9;\ r = 2)$$

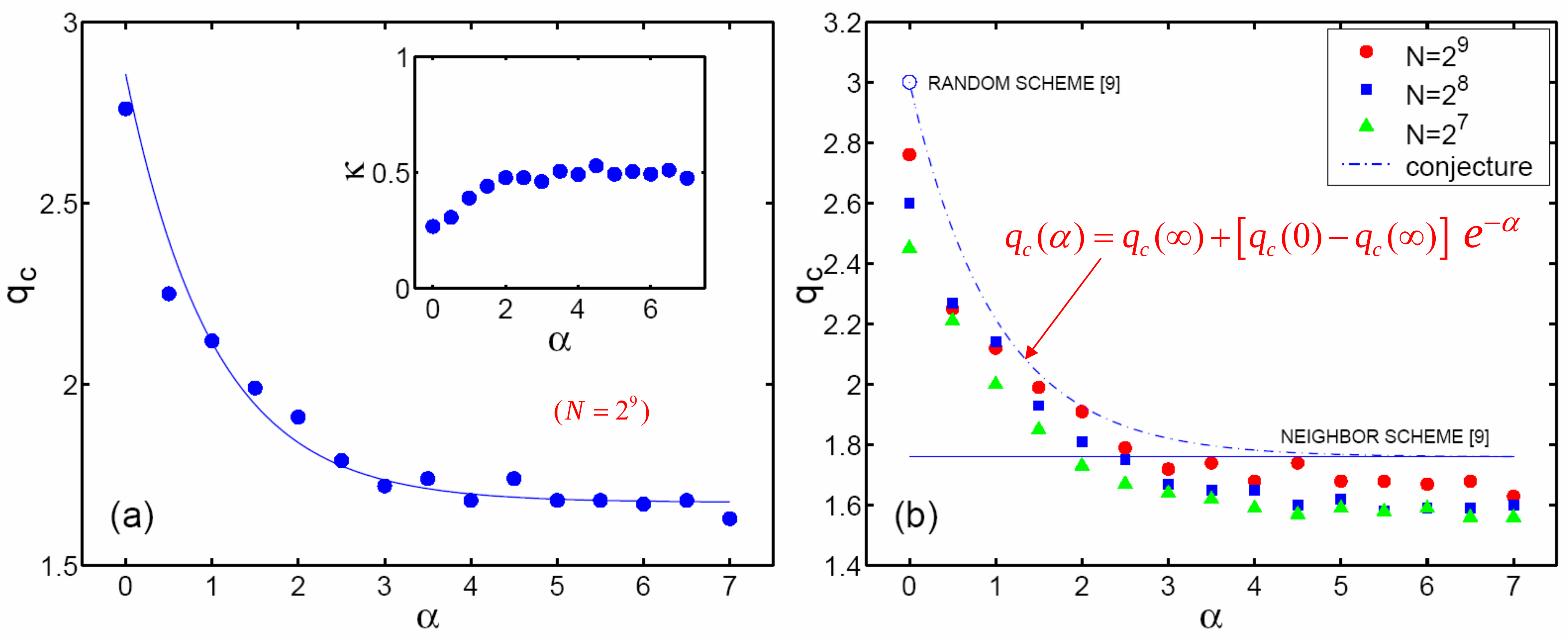


$$P(\geq k) = e_{q_c}^{-(k\text{-}2)/\kappa} \quad (k = 2,\ 3,\ 4, \ldots)$$

$$linear\ correlation \in [0.999901, 0.999976]$$

S. Thurner and C. T., Europhys Lett **72**, 197 (2005)

$(r = 2)$



S. Thurner and C. T., Europhys Lett **72**, 197 (2005)

PREDICTION:

The solution of

$$\frac{\partial\, p(x,t)}{\partial t} = D\, \frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} \qquad [p(x,0) = \delta(0)] \qquad (q < 3)$$

is given by

$$p(x,t) \propto \left[1 + (1-q)\, x^2 / \; (\Gamma t)^{2/(3-q)}\right]^{1/(1-q)} \equiv e_q^{-x^2 / \; (\Gamma t)^{2/(3-q)}} \qquad (\Gamma \propto D)$$

hence

x^2 *scales like* t^{γ} (*e.g.,* $\langle x^2 \rangle \propto t^{\gamma}$)

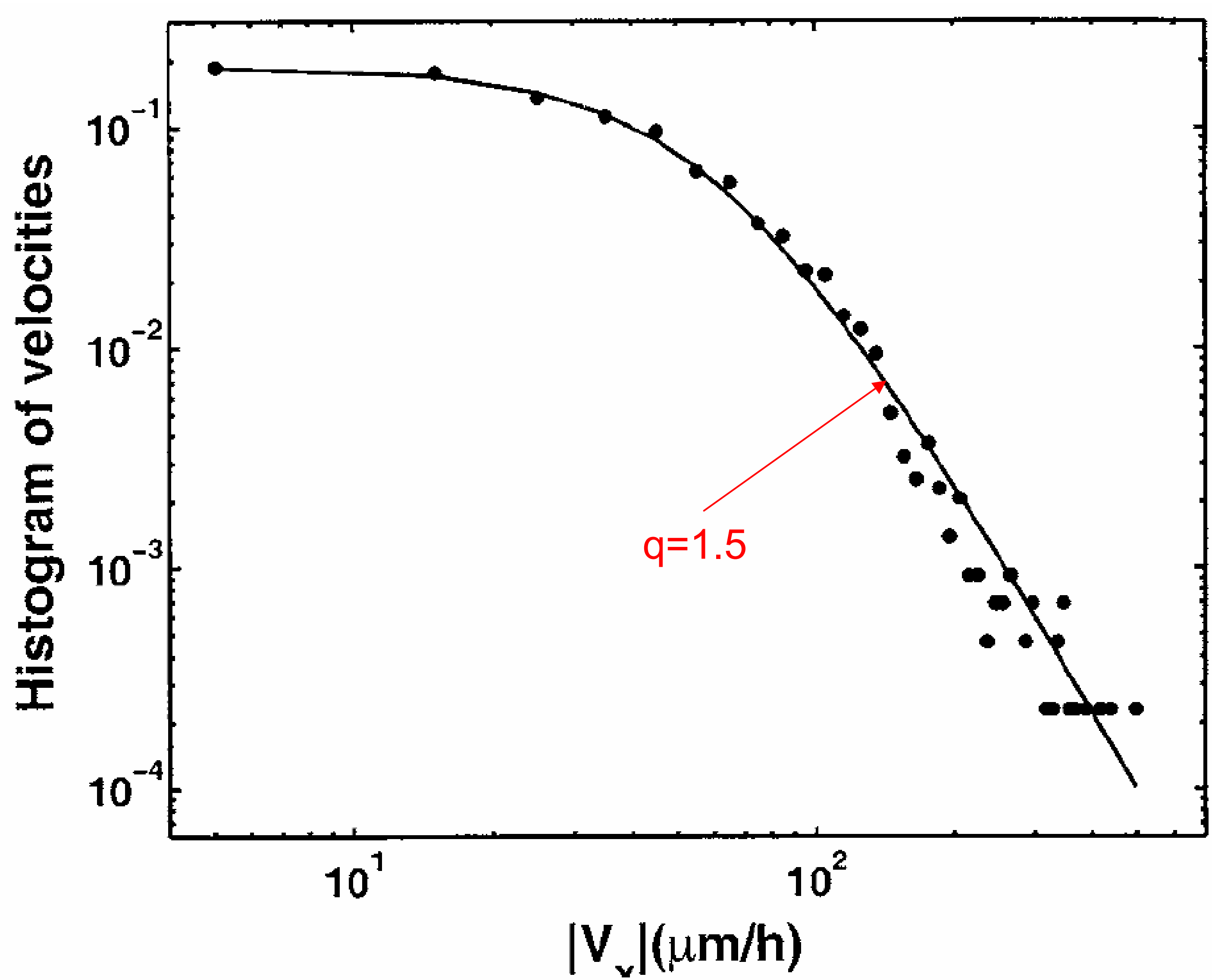
with

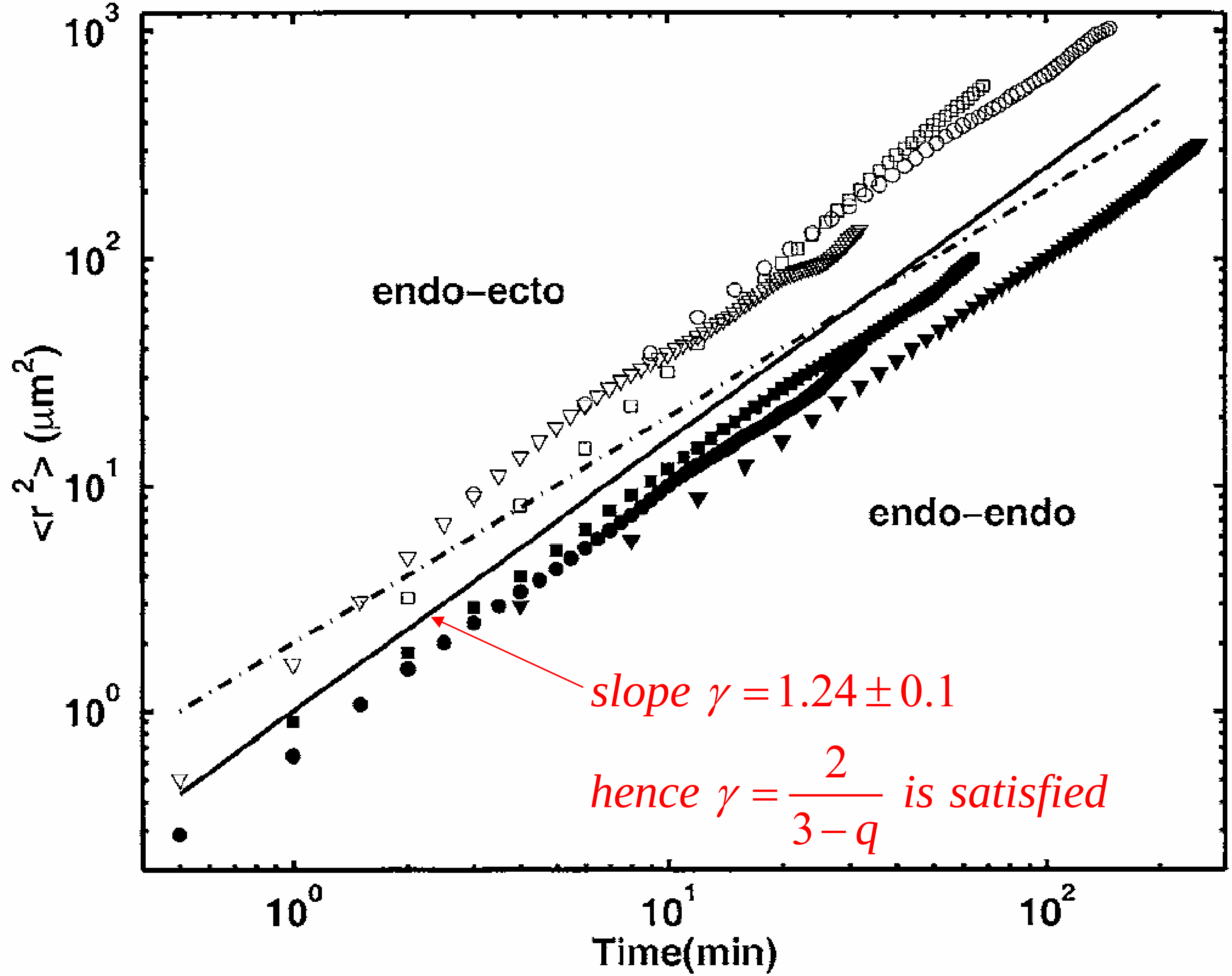
$$\gamma = \frac{2}{3-q}$$

C.T. and D.J. Bukman, Phys Rev E **54**, R2197 (1996)

Hydra viridissima:

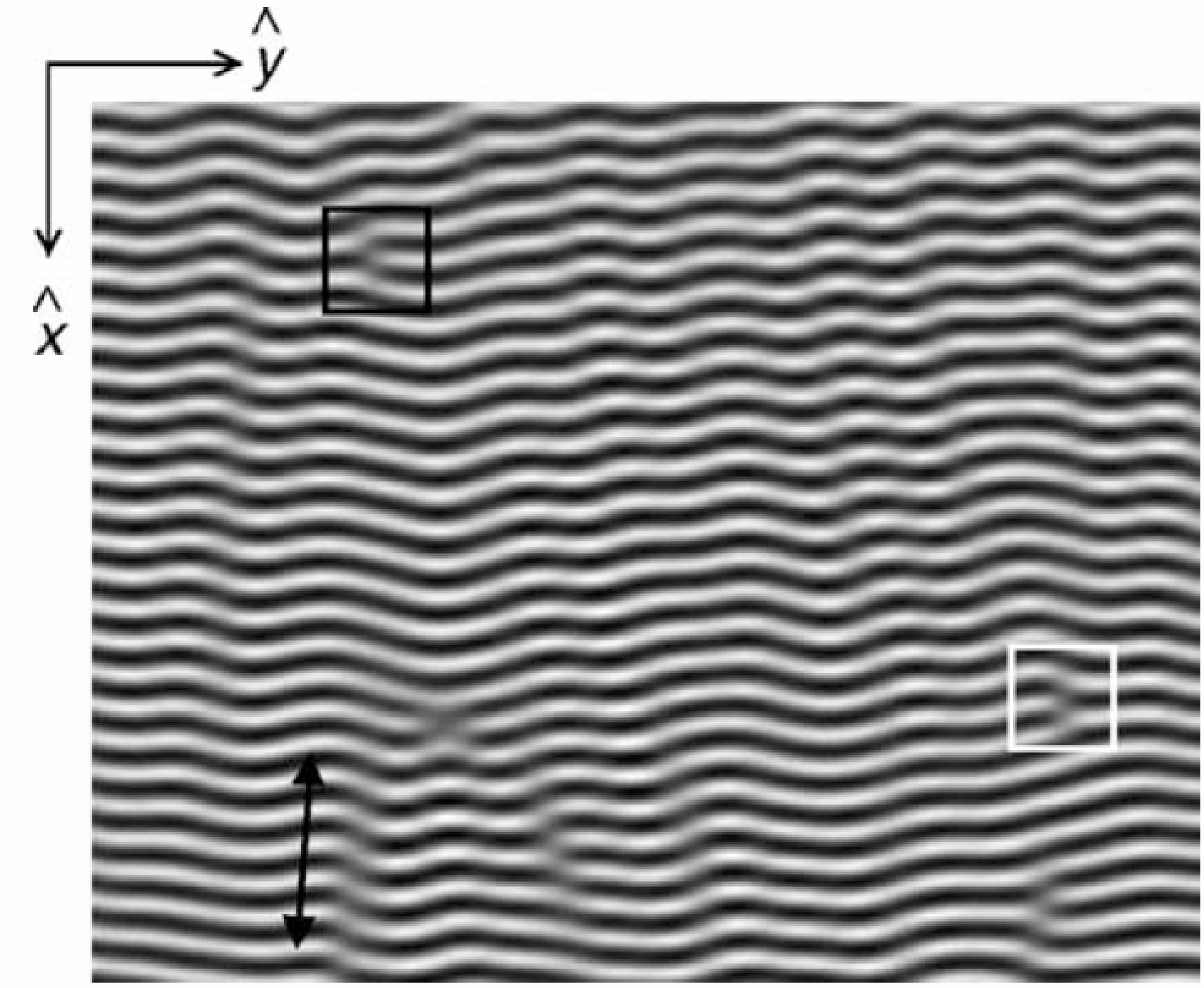
A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada
Physica A **293**, 549 (2001)

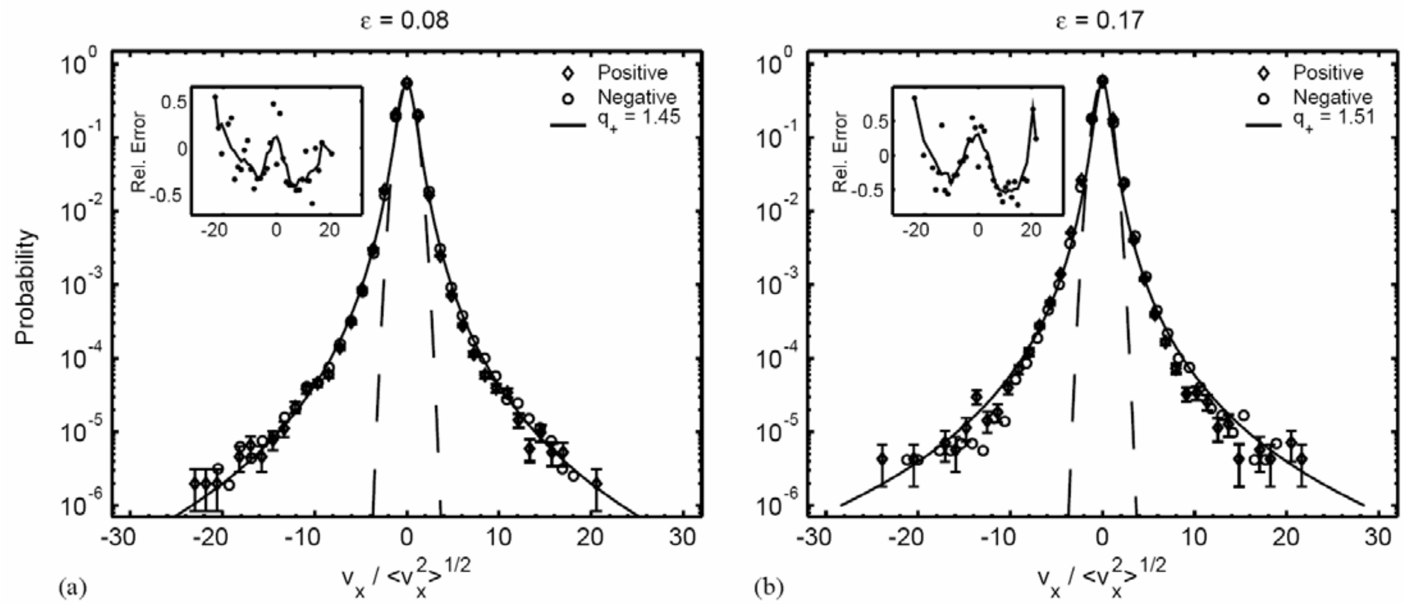


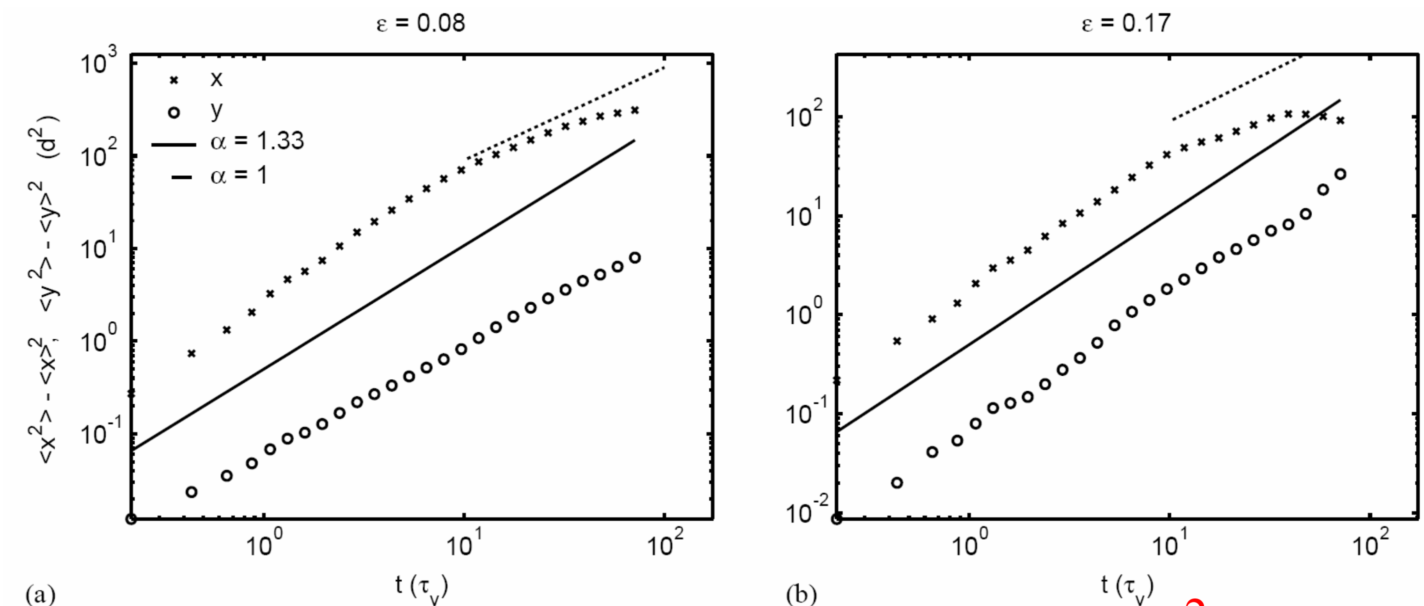

endo–ecto
endo–endo
$\langle r^2 \rangle$ (μm^2)
Time(min)
10^0
10^1
10^2
10^3
slope $\gamma = 1.24 \pm 0.1$
hence $\gamma = \frac{2}{3-q}$ is satisfied

Defect turbulence:

K.E. Daniels, C. Beck and E. Bodenschatz, Physica D **193**, 208 (2004)



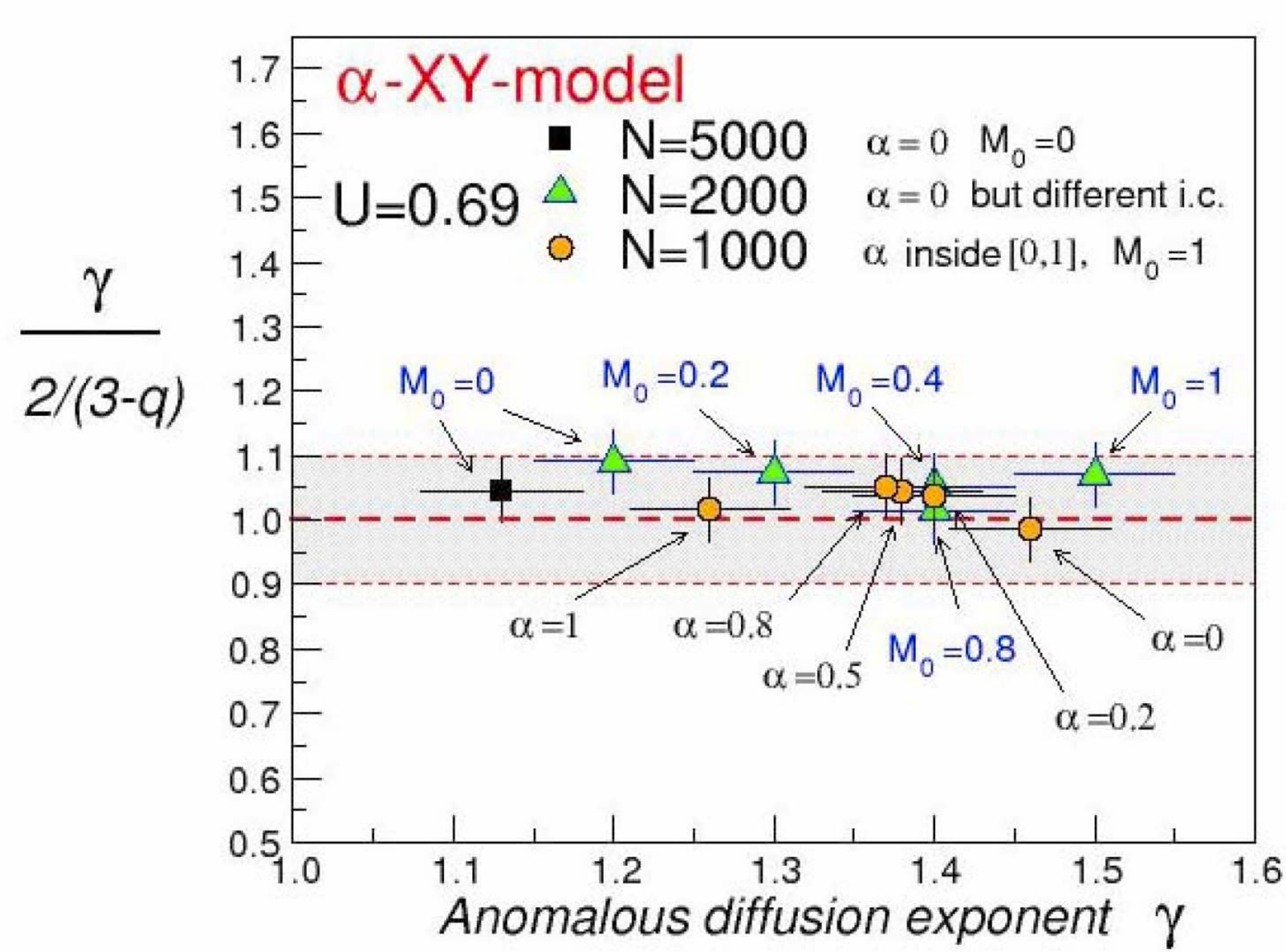




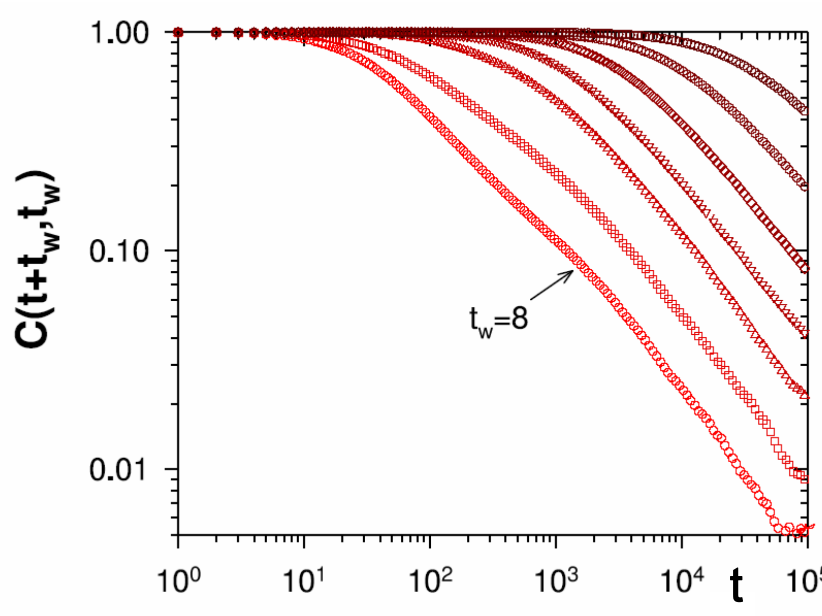
$q \approx 1.5$ *and* $\gamma \approx 4/3$ *are consistent with* $\gamma = \dfrac{2}{3-q}$

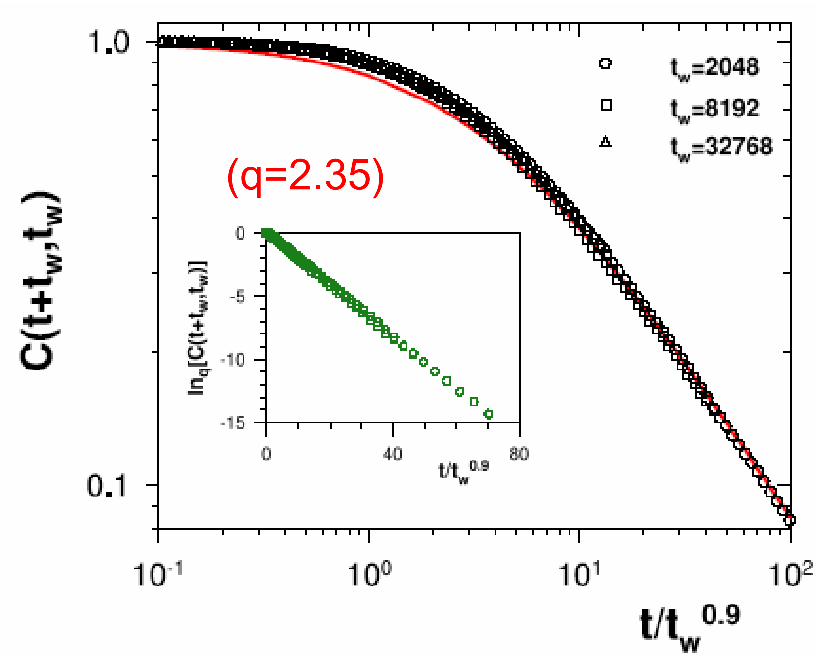
K.E. Daniels, C. Beck and E. Bodenschatz, Physica D **193**, 208 (2004)

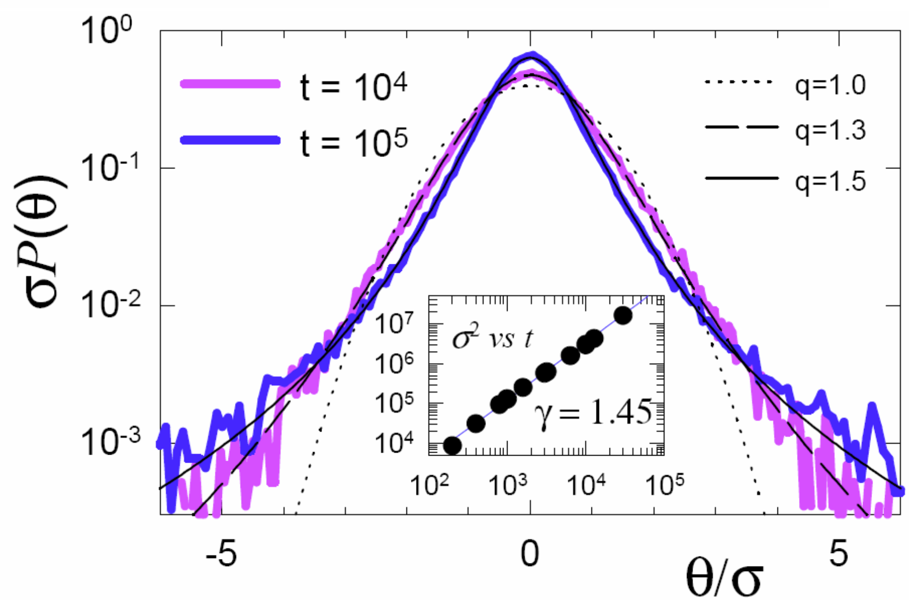
XY FERROMAGNET WITH LONG-RANGE INTERACTIONS:



A. Rapisarda and A. Pluchino, Europhys News **36**, 202 (2005)
(European Physical Society)







$\alpha = 0$ *model* :

$q_{rel} \simeq 2.35$?

$q_{stat} \simeq 1.5$?

F.A. Tamarit and C. Anteneodo
Europhysics News **36** (6), 194 (2005) [European Physical Society]

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

Theoretical predictions by E. Lutz, Phys Rev A **67**, 051402(R) (2003):

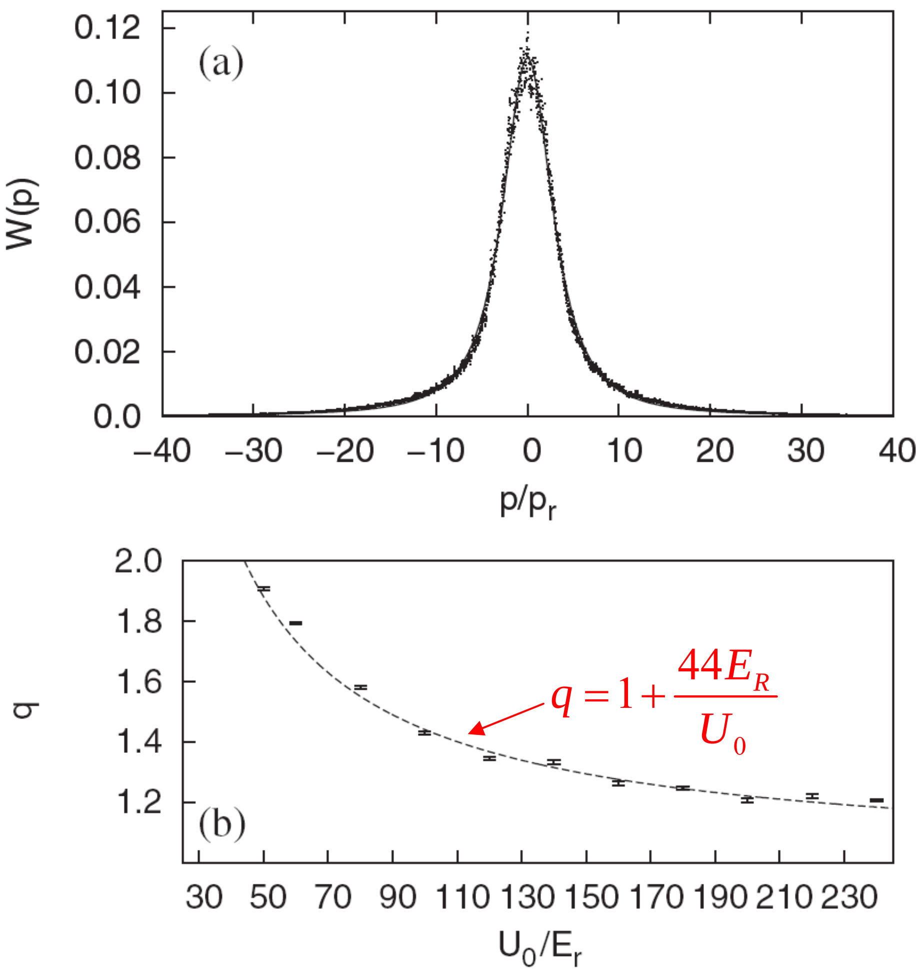
(i) The distribution of atomic velocities is a *q*-Gaussian;

(ii) $q = 1 + \frac{44 E_R}{U_0}$ where $E_R \equiv$ recoil energy

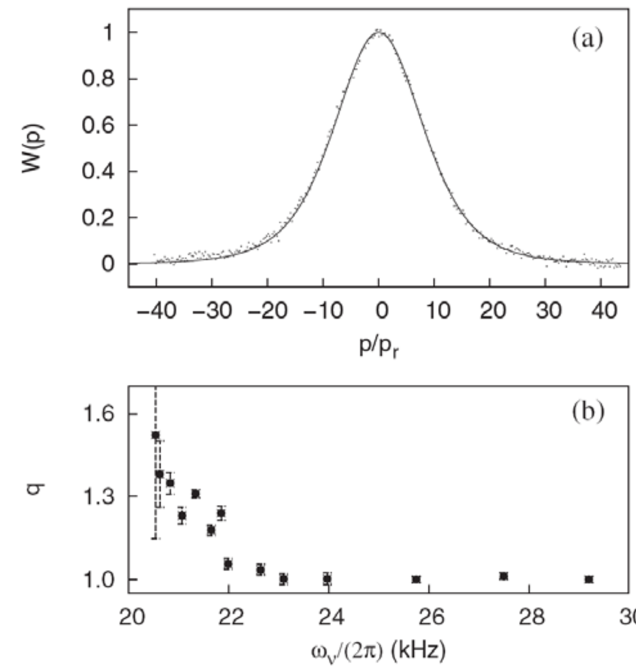
$U_0 \equiv$ potential depth

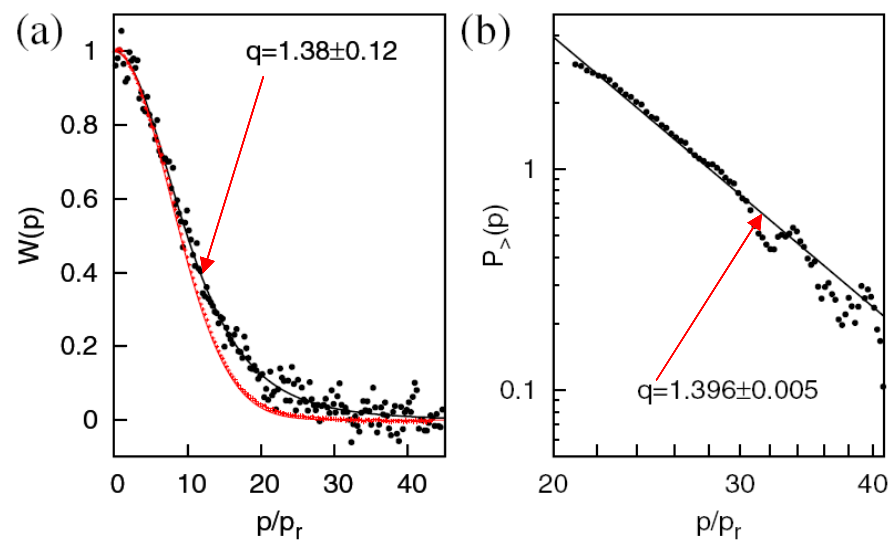
Experimental and computational verifications

by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)



(Computational verification: quantum Monte Carlo simulations)





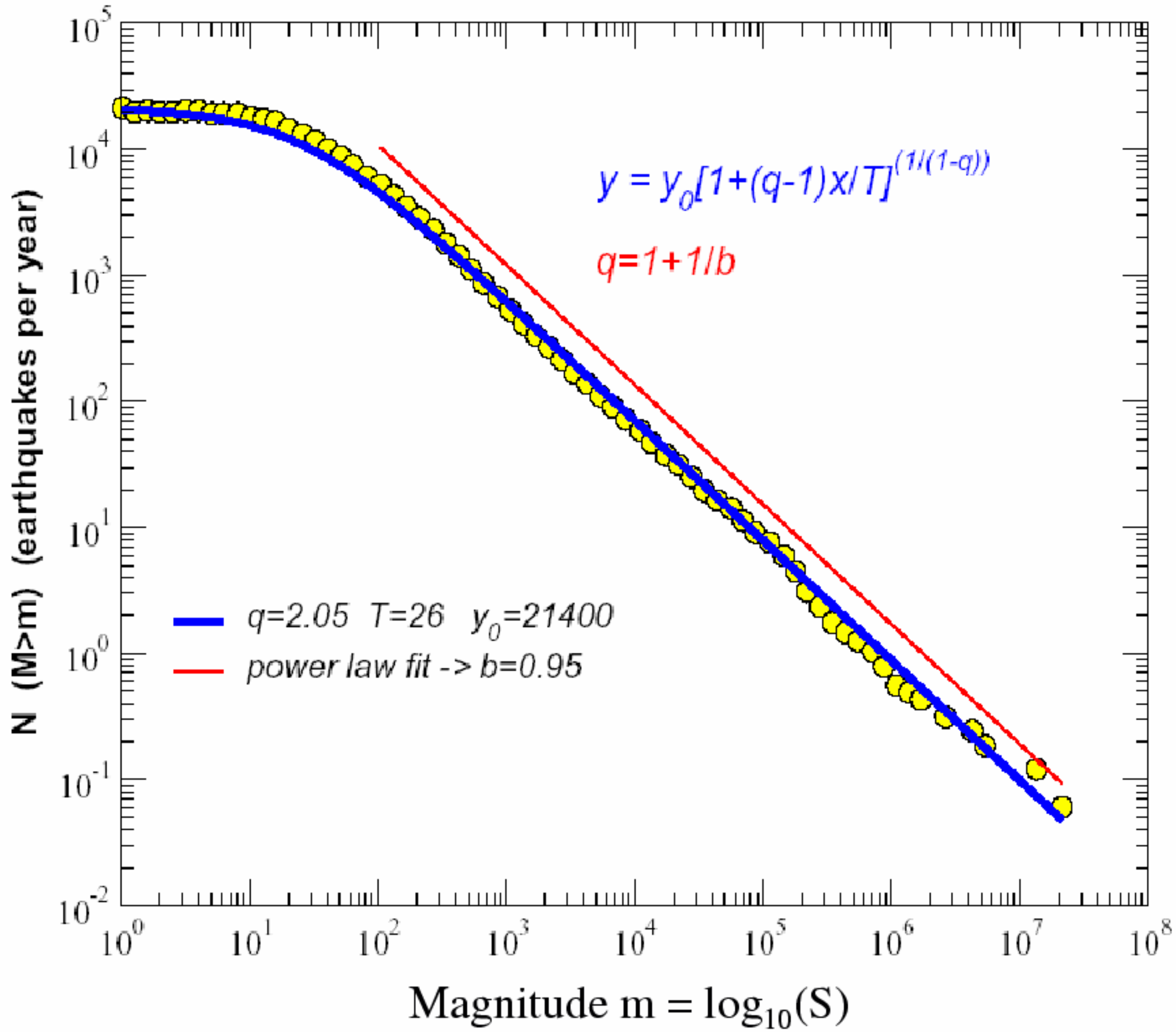
(Experimental verification)

EARTHQUAKES

Earthquakes

Data from

P.Bak, K. Christensen, L. Danon and T. Scanlon,
Phys Rev Lett **88**, 178501 (2002)



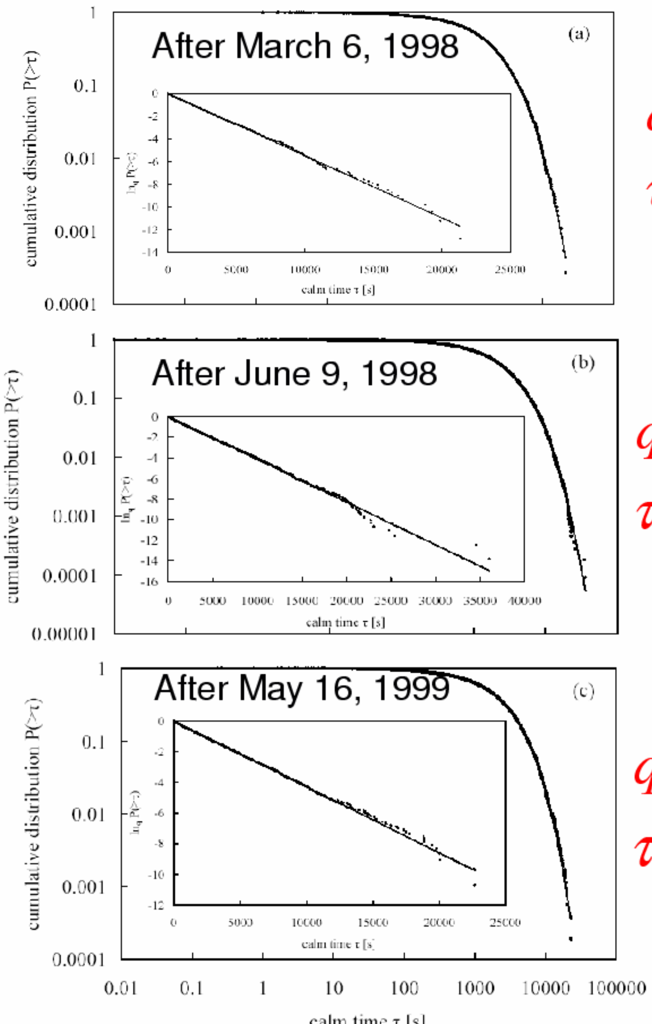
TIME INTERVALS BETWEEN EARTHQUAKES

Southern California data [S. Abe and N. Suzuki (2004)]

Calm periods (stationary states) between major earthquakes,

i.e., excluding the Omori-regime periods (nonstationary states)

$$P(t > \tau) = \frac{1}{[1 + (q-1)\tau / \tau_0]^{1/(q-1)}}$$



$q = 1.10$

$\tau_0 = 1830$ s

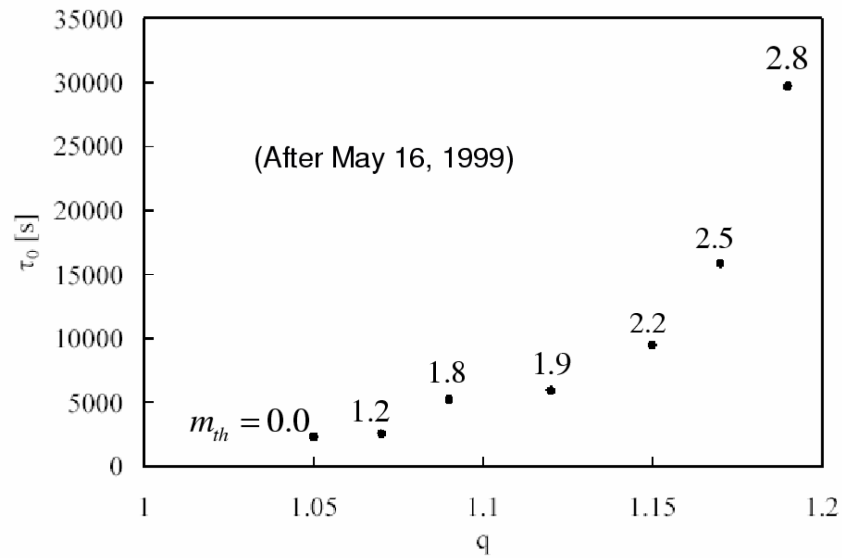
$q = 1.08$

$\tau_0 = 2410$ s

$q = 1.05$

$\tau_0 = 2330$ s

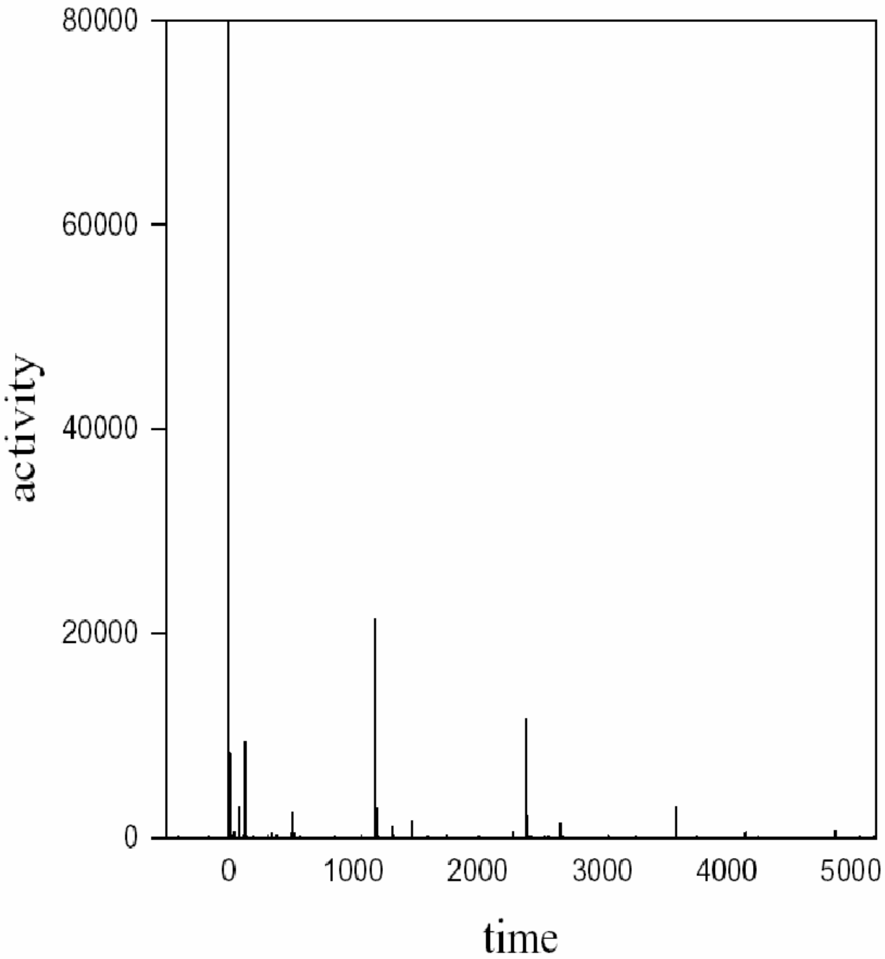
Influence of the threshold m_{th}

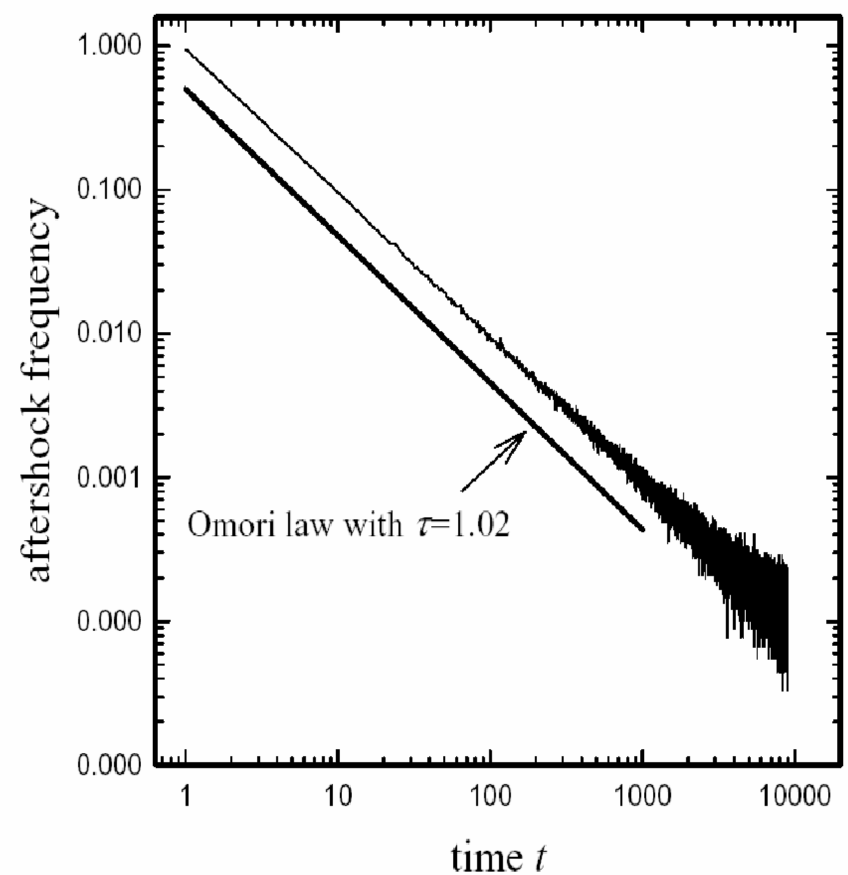


AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:
M.E.J. Newman, Proc. R. Soc. London B **263,** 1605 (1996);
M.E.J. Newman and K. Sneppen, Phys. Rev. E **54,** 6226 (199

Aging:
U. Tirnakli and S. Abe, Phys. Rev. E **70**, 056120 (2004)

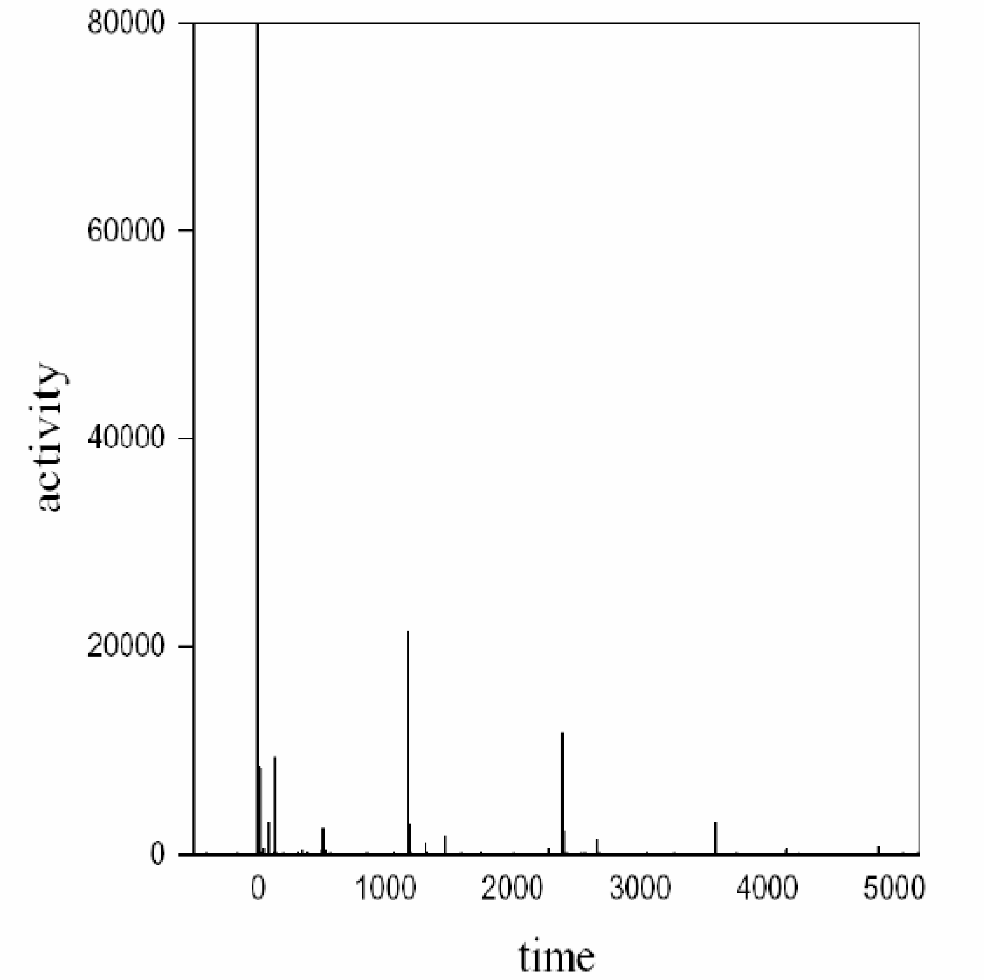


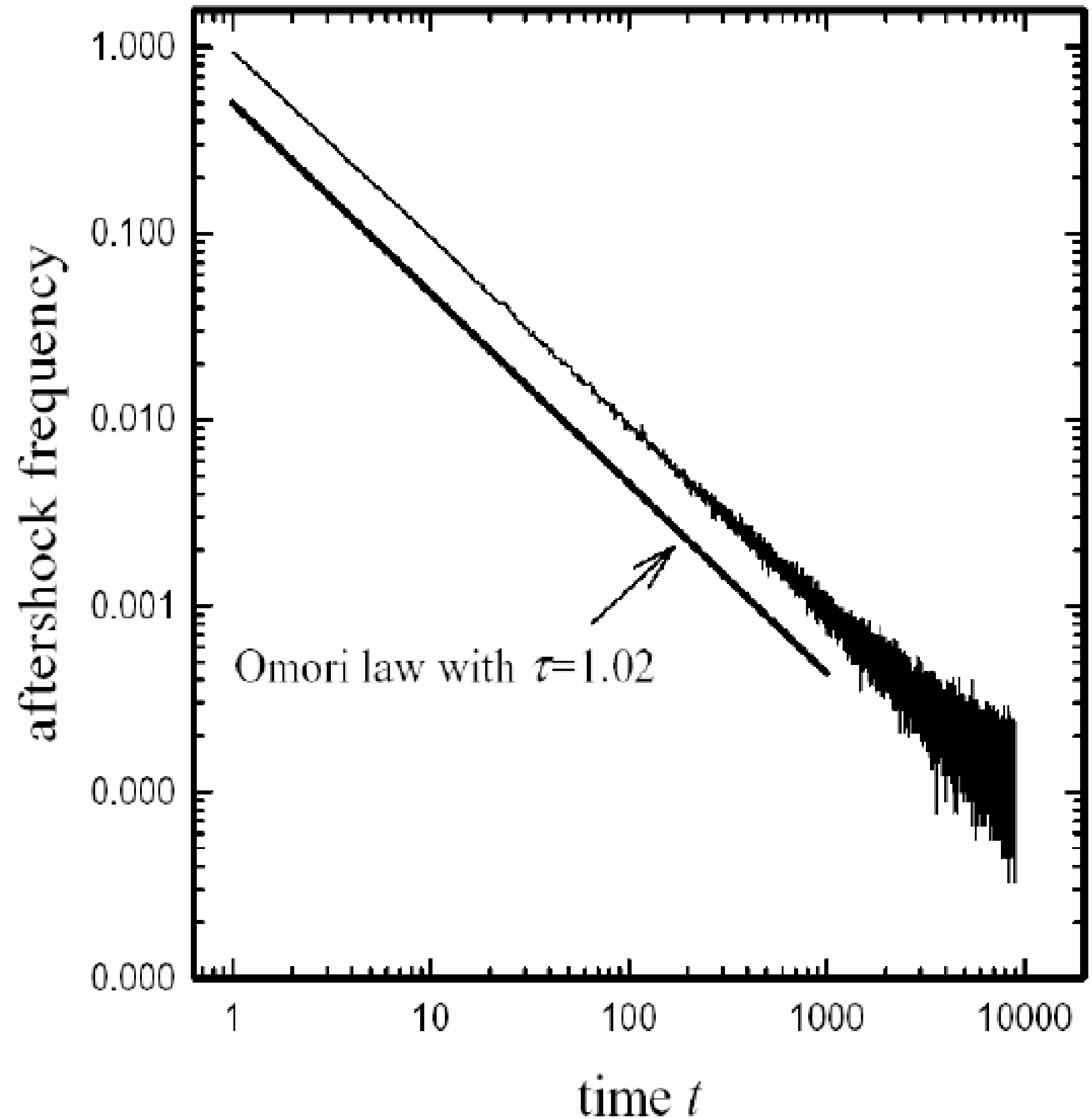


AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:
M.E.J. Newman, Proc. R. Soc. London B **263,** 1605 (1996);
M.E.J. Newman and K. Sneppen, Phys. Rev. E **54,** 6226 (1996).

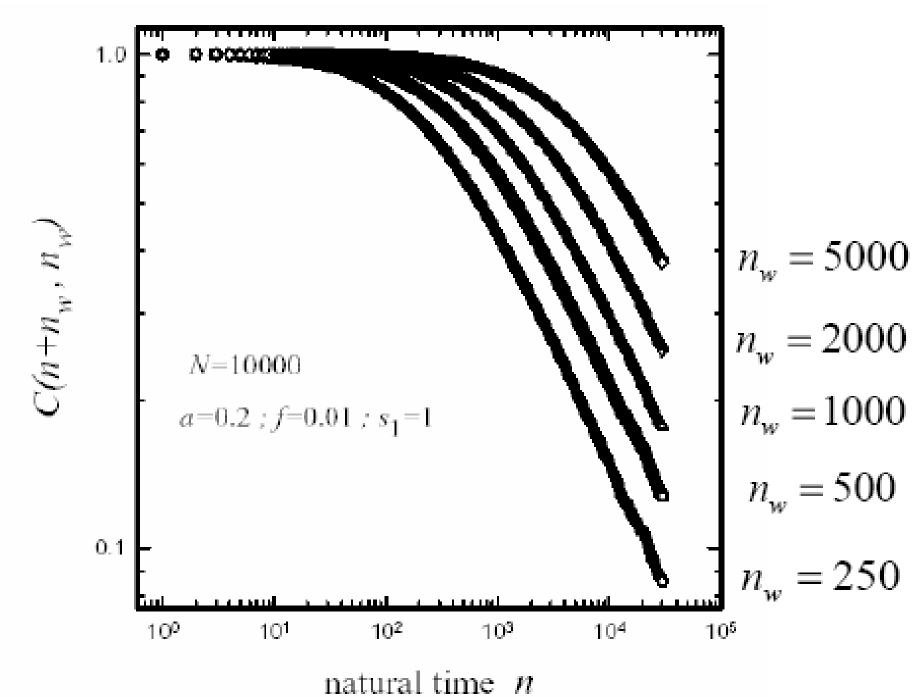
Aging:
U. Tirnakli and S. Abe, Phys. Rev. E **70**, 056120 (2004)



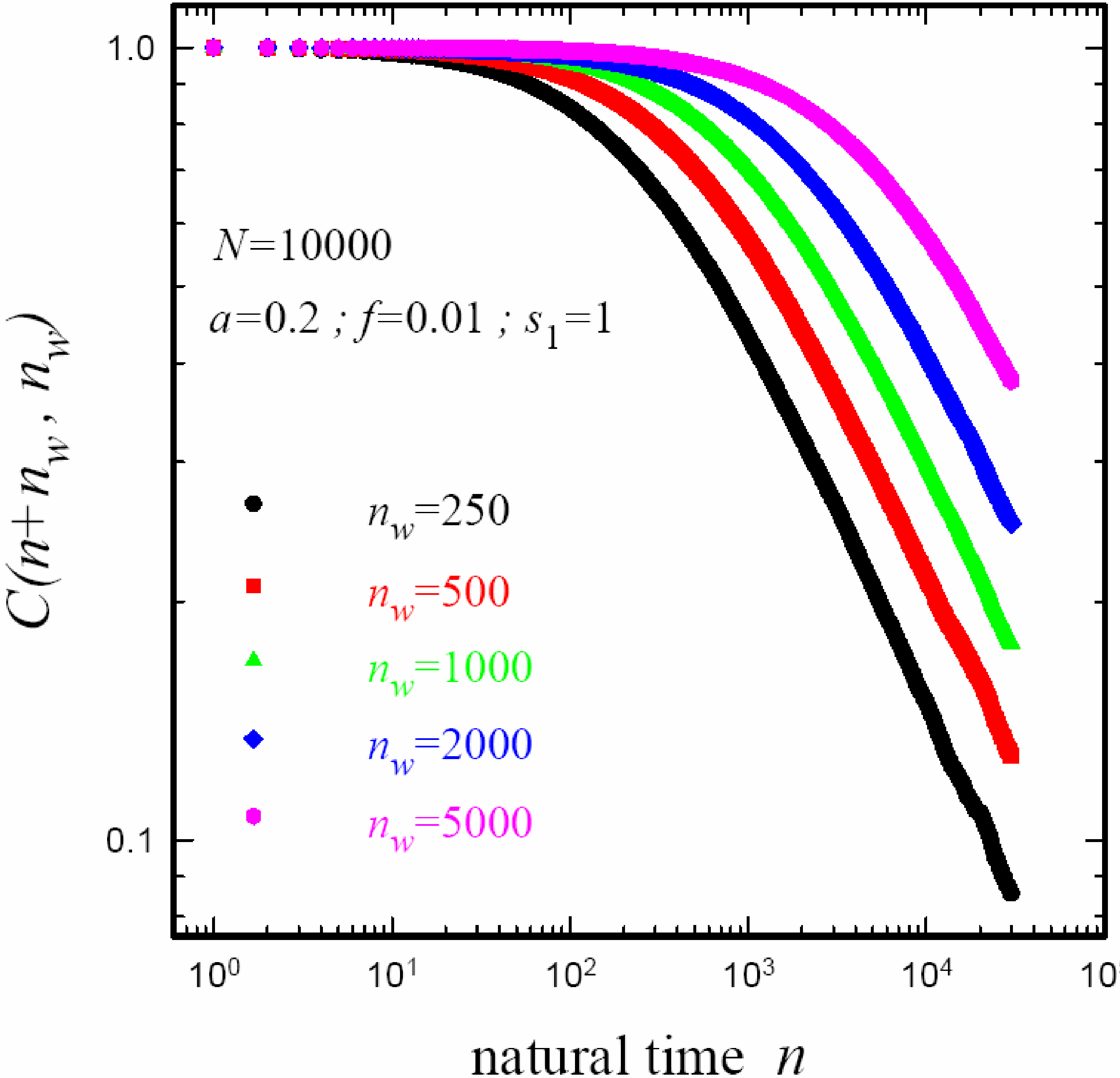


1.000
0.100
0.010
0.001
0.000
0.000
aftershock frequency
Omori law with τ=1.02
1
10
100
1000
10000
time t

$$C(n+n_w, n_w) \equiv \frac{\langle t_{n+n_w} t_{n_w} \rangle - \langle t_{n+n_w} \rangle \langle t_{n_w} \rangle}{(\sigma^2{}_{n+n_w} \sigma^2{}_{n_w})}$$

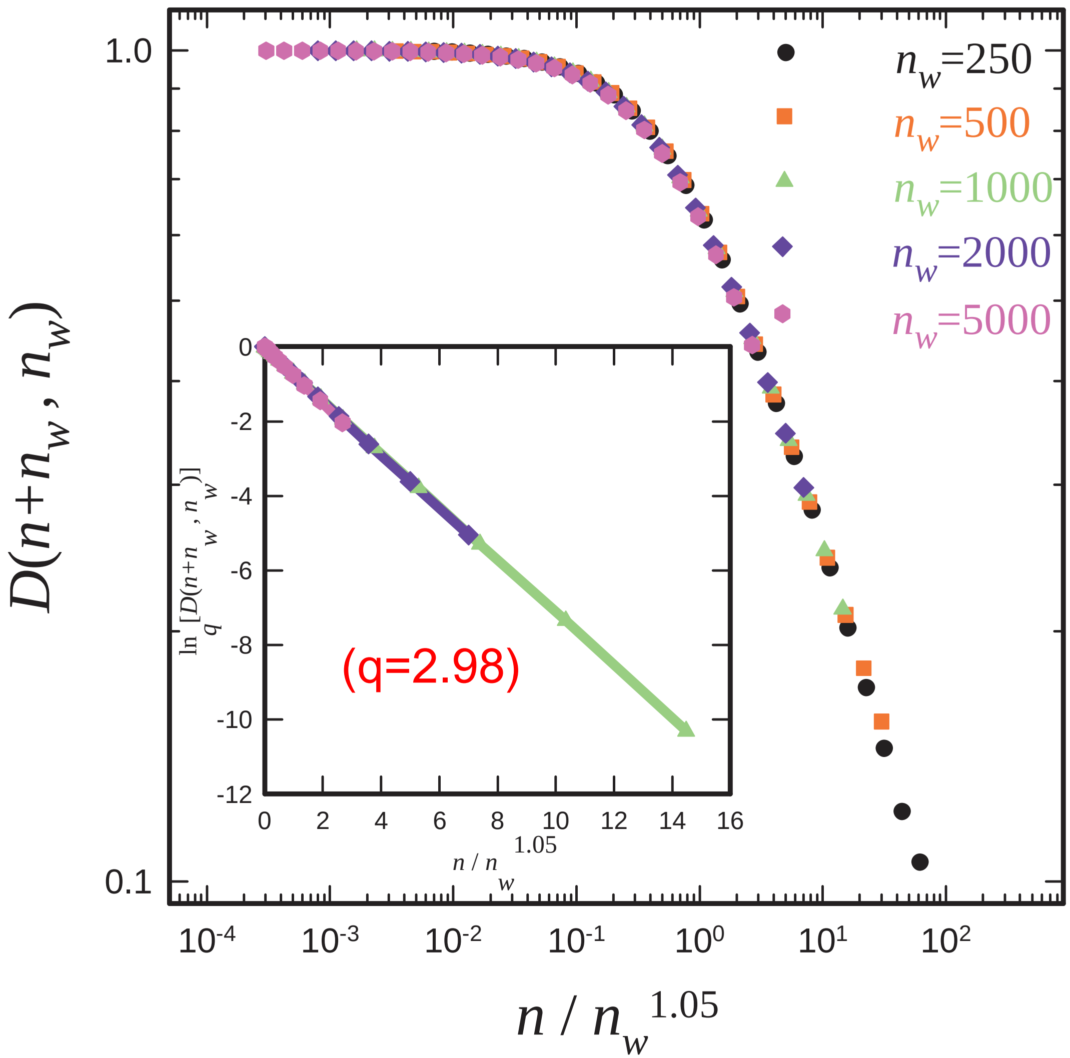


"Natural time" suggested in
P.A. Varotsos, N.V. Sarlis and E.S. Skordas,
Phys Rev E **66**, 011902 (2002); **67**, 021109; **68**, 031106 (2003).

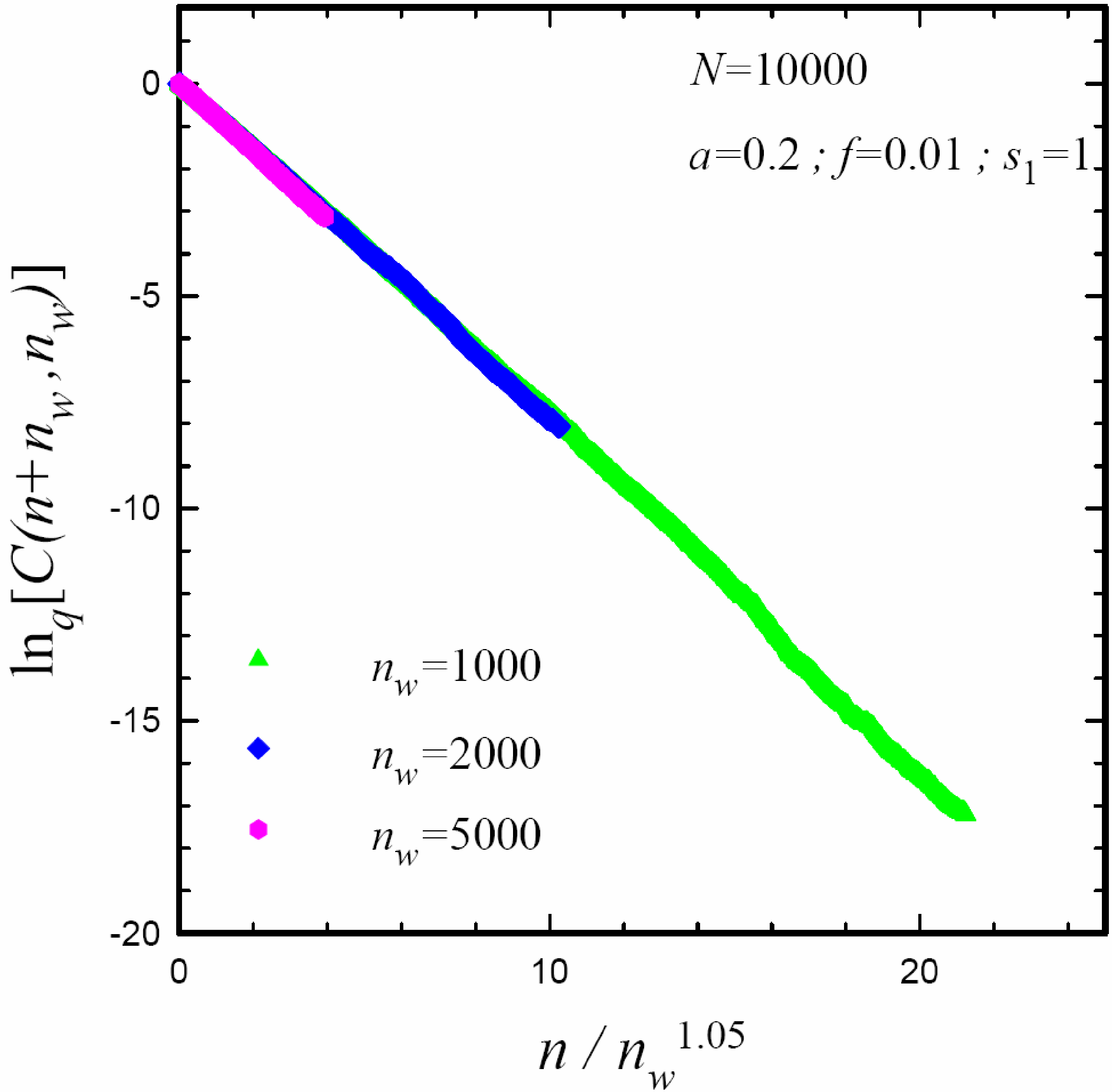


1.0
0.1
N=10000
a=0.2 ; f=0.01 ; s_1=1
$C(n+n_w, n_w)$
n_w=250
n_w=500
n_w=1000
n_w=2000
n_w=5000
10^0
10^1
10^2
10^3
10^4
10^5
natural time n

MODEL FOR EARTHQUAKES (OMORI REGIME):



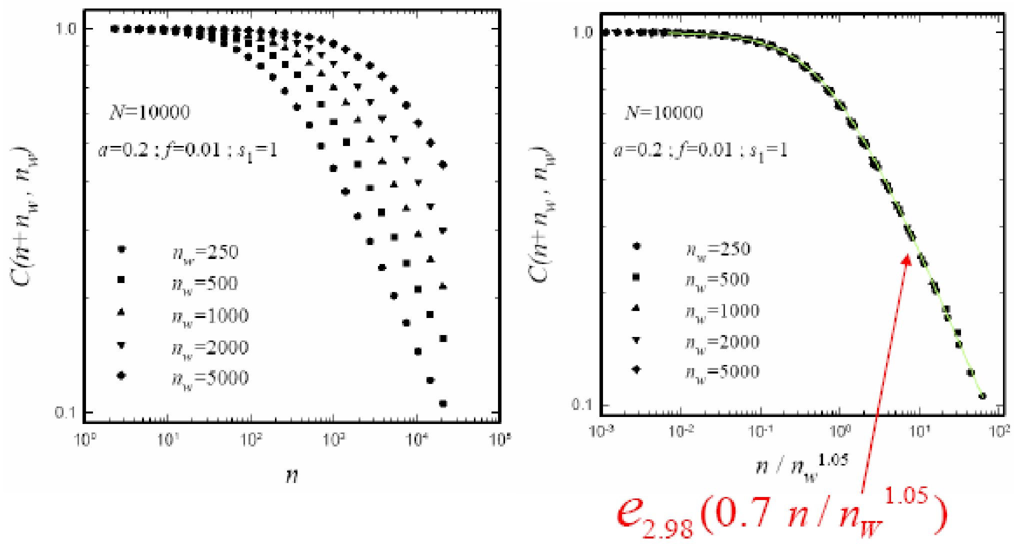
S. Abe, U. Tirnakli and P.A. Varotsos
Europhysics News **36** (6), 206 (2005) [European Physical Society]



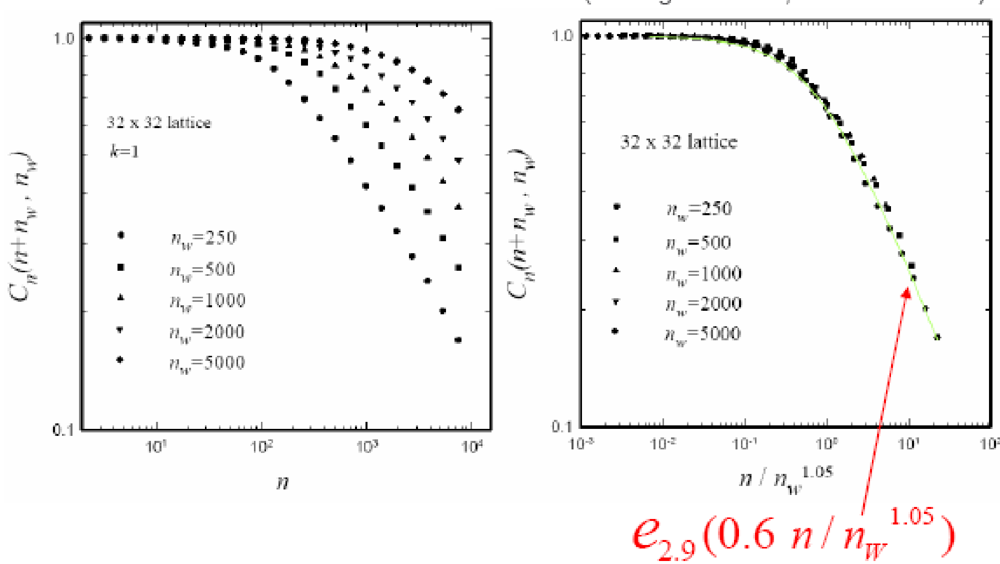
N=10000
a=0.2 ; f=0.01 ; s1=1
$\ln_q[C(n+n_w, n_w)]$
n_w=1000
n_w=2000
n_w=5000
$n / n_w^{1.05}$
0
-5
-10
-15
-20
0
10
20

EARTHQUAKES:

NEWMAN MODEL (average over 100,000 realizations)



OLAMI-FEDER-CHRISTENSEN MODEL (average over 20,000 realizations)

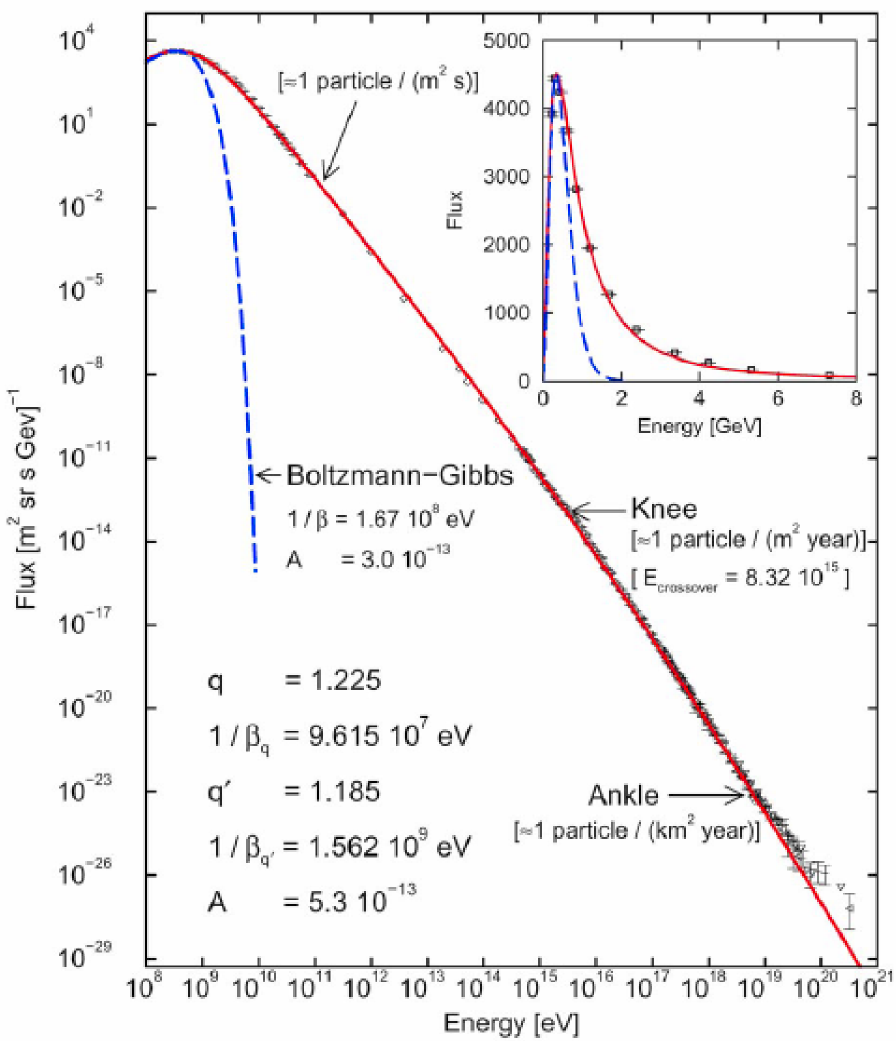


U. Tirnakli, in *Complexity, Metastability and Nonextensivity*,
eds. C. Beck, G. Benedek, A. Rapisarda and C. T. (World Scientific, Singapore, 2005), page 350

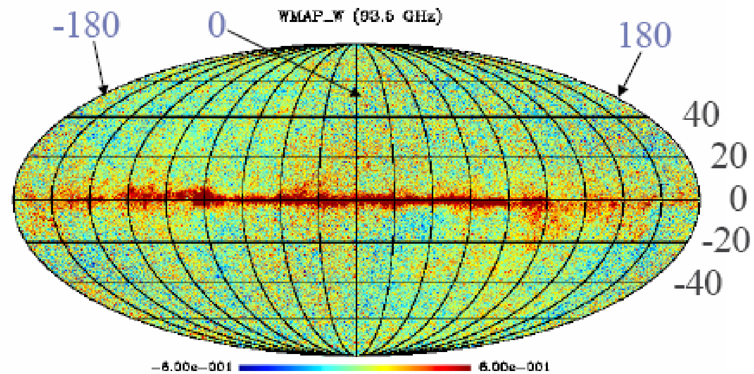
ASTROPHYSICS

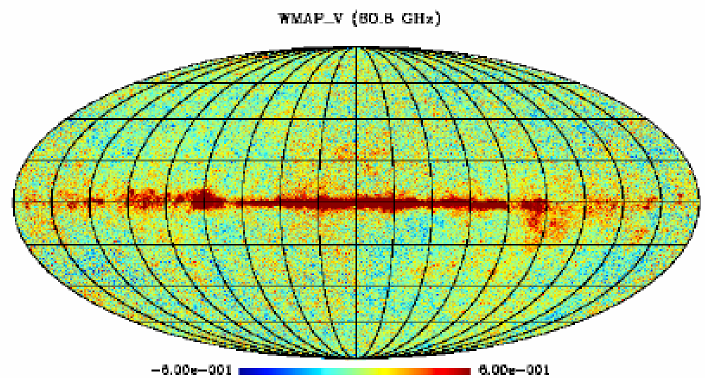
FLUXES OF COSMIC RAYS

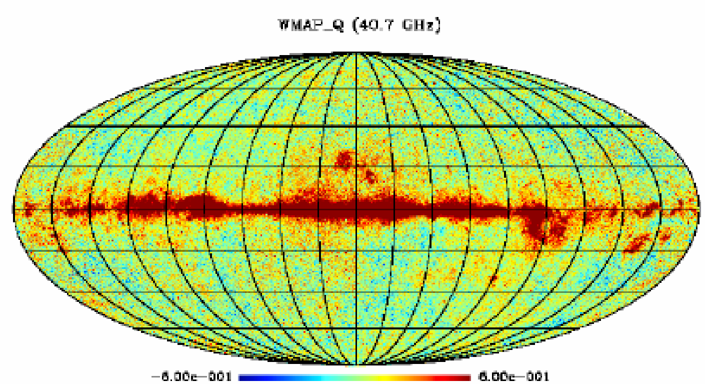
Tsallis., Anjos and Borges, Phys. Lett. A **310**, 372 (2003)



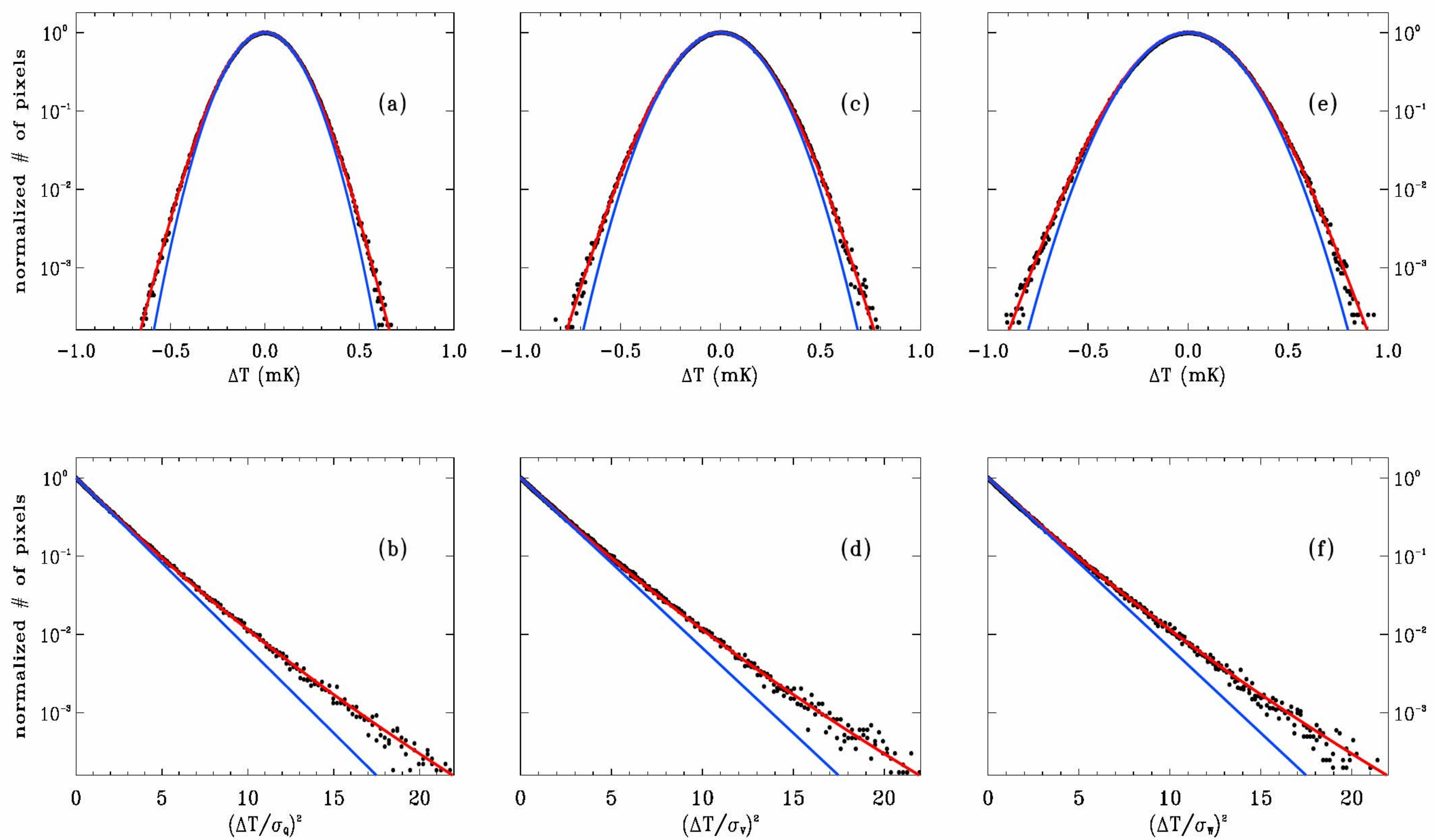
COSMIC MICROWAVE BACKGROUND RADIATION: TEMPERATURE FLUCTUATIONS





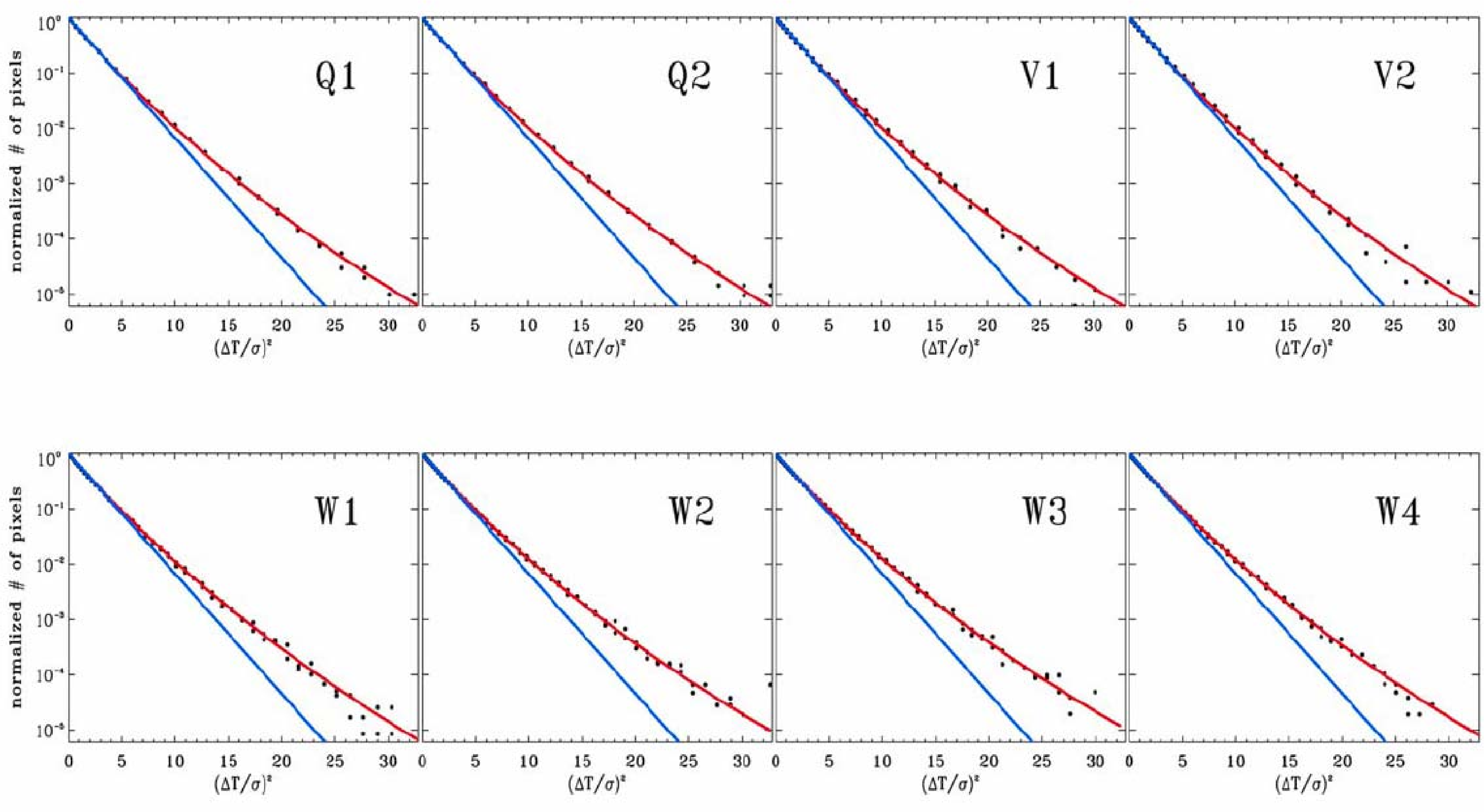


(Band Q: 22.8 GHz) (Band V: 60.8 GHz) (Band W: 93.5 GHz)



(Data after using Kp0 mask) $q = 1.045 \pm 0.005$ (99 % confidence level)

A. Bernui, C. T. and T. Villela, Phys Lett **356**, 426 (2006)



$$q = 1.045 \pm 0.005 \quad (99\ \% \ \text{confidence level})$$

A. Bernui, C. T. and T. Villela, Phys Lett **356**, 426 (2006)

ECONOMICS

$$dv = -\gamma \left(v - \frac{\alpha + 1}{\beta} \right) dt + \sqrt{2\, v\, \frac{\gamma}{\beta}}\, dW_t,$$

$$P(v) = \frac{1}{Z} \left(\frac{v}{\theta} \right)^{\alpha} \exp_q \left(-\frac{v}{\theta} \right) \qquad e_q^x \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}} \qquad (e_1^x \equiv e^x)$$

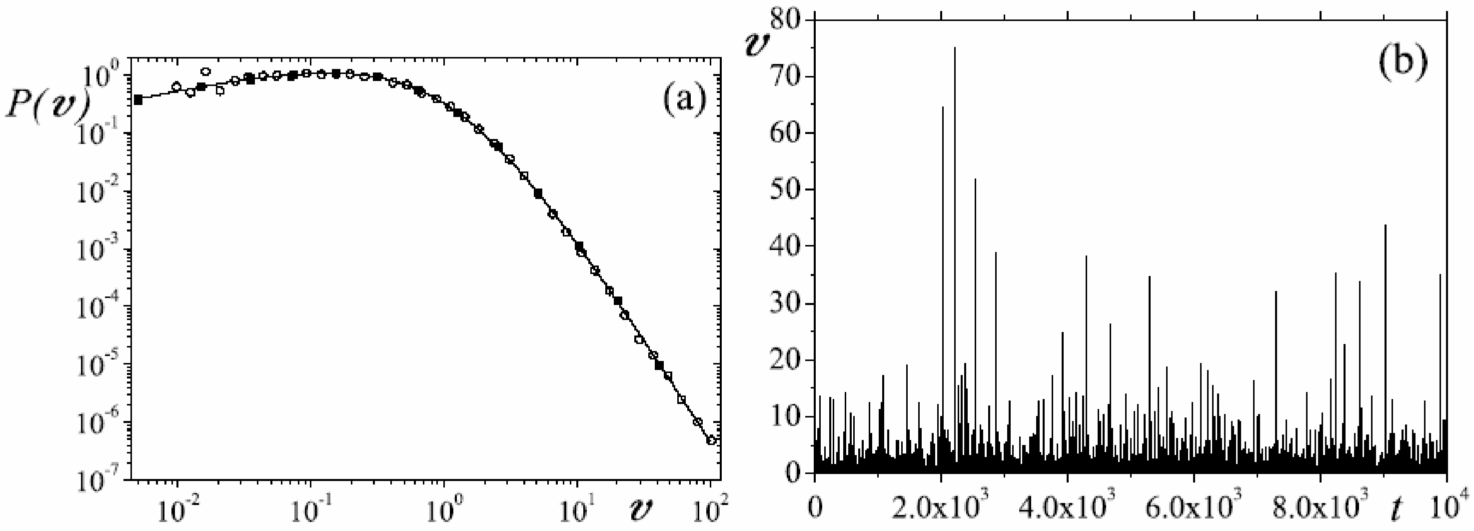
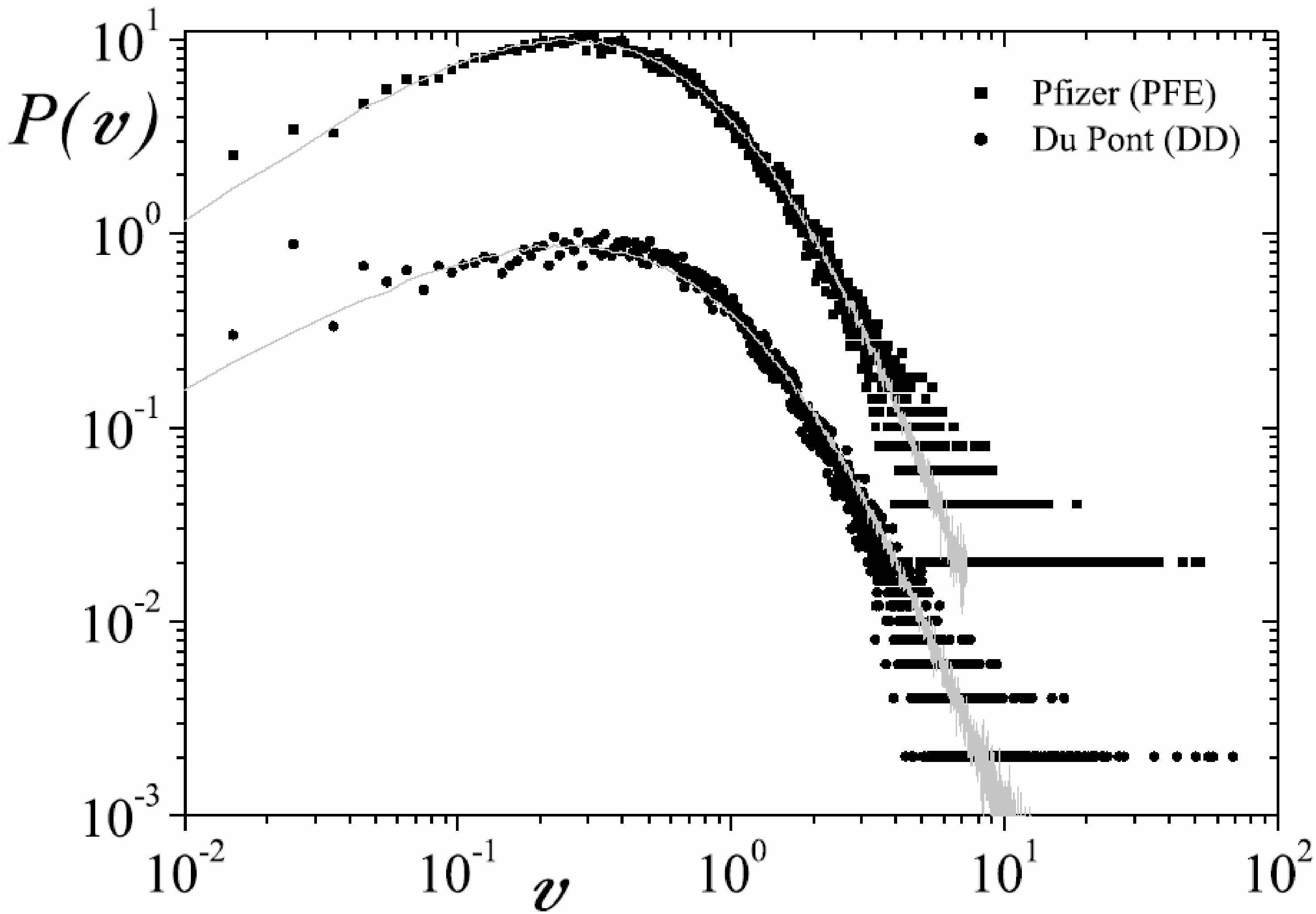


Figure 6. In panel (a) open symbols represents the PDF for the ten-high 1 minute traded volume stocks in NYSE exchange; solid symbols represent the PDF obtained for the numerical realization depicted in panel (b) and line the theoretical PDF Eq. (28). Parameters are $q = 1.17$, $\alpha = 1.79$, $\lambda = 1.42$ and $\delta = 3.09$.

STOCK VOLUMES:

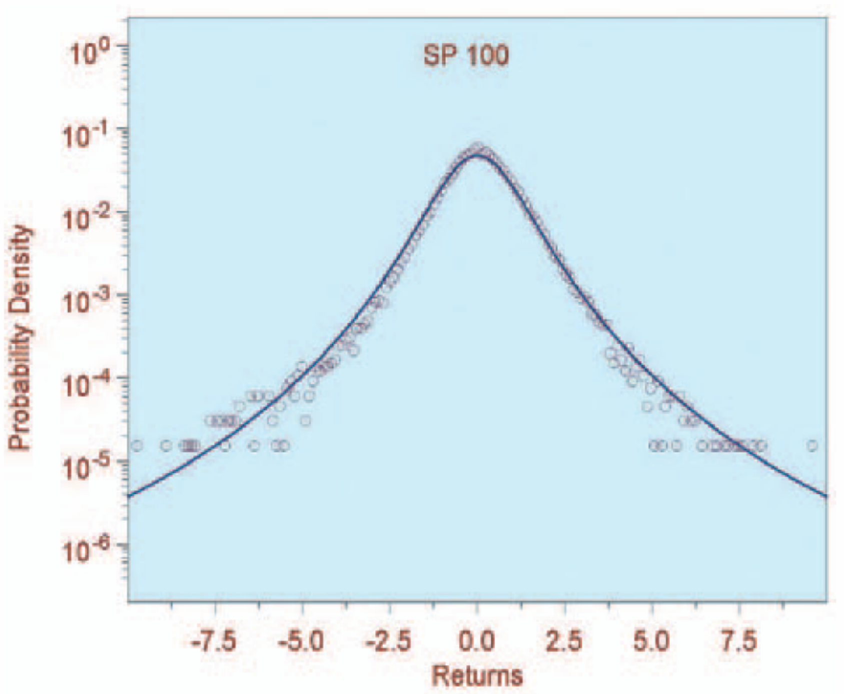


$$p\left(u\right)=\frac{1}{Z}\left(\frac{v}{\theta}\right)^{-\alpha-2}\exp_q\left[-\frac{\theta}{v}\right]$$

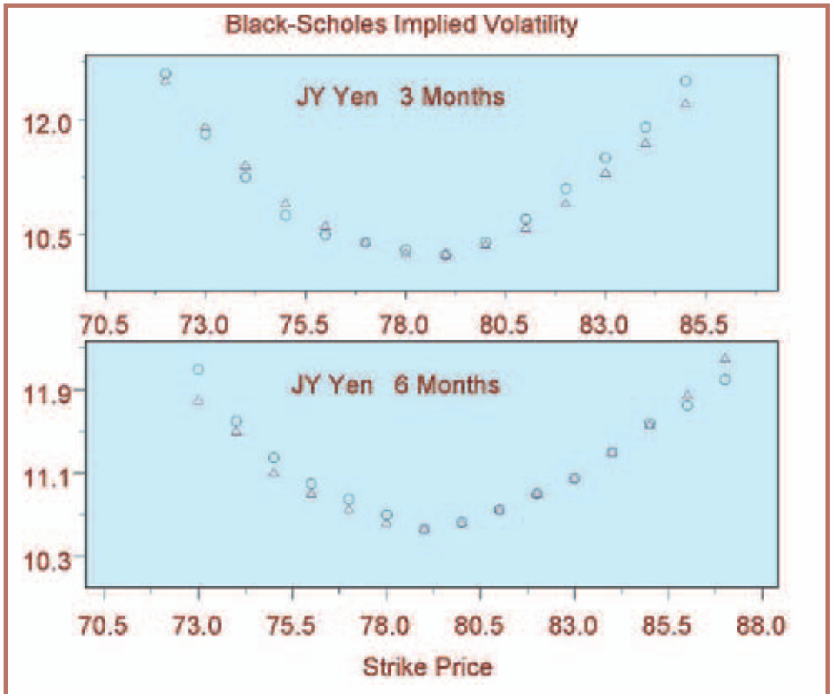
J de Souza, SD Queiros and LG Moyano, physics/0510112 (2005)

q-GENERALIZED BLACK-SCHOLES EQUATION:

L Borland, Phys Rev Lett **89**, 098701 (2002), and Quantitative Finance **2**, 415 (2002)
L Borland and J-P Bouchaud, cond-mat/0403022 (2004)
L Borland, Europhys News **36**, 228 (2005)
See also H Sakaguchi, J Phys Soc Jpn **70**, 3247 (2001)
C Anteneodo and CT, J Math Phys **44**, 5194 (2003)



▲ Fig.2: The empirical distribution of daily returns from the stocks comprising the SP 100 (red) is fit very well by a q-Gaussian with $q = 1.4$ (blue).



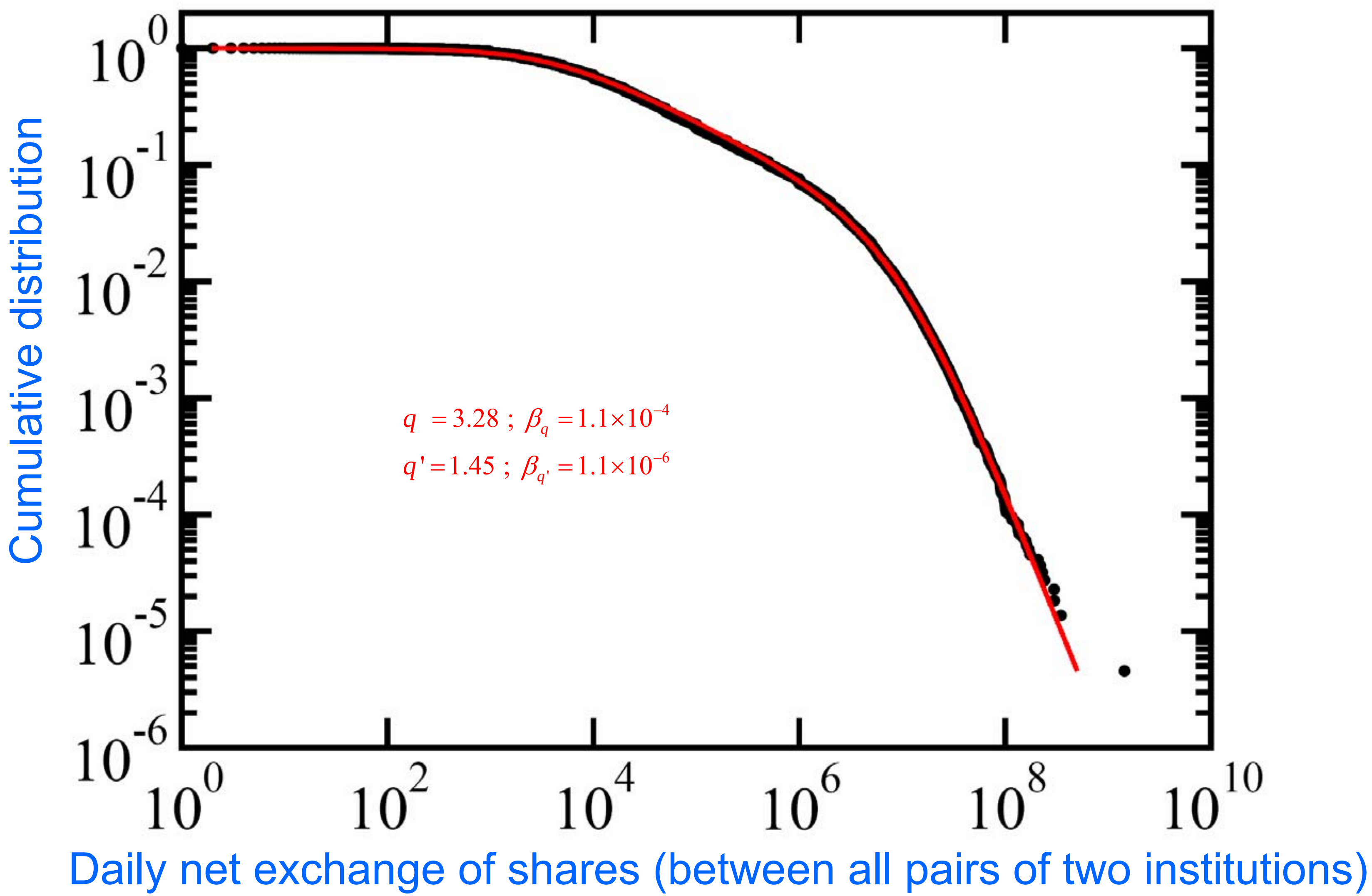
▲ Fig.3: Theoretical implied Black-Scholes volatilities from the $q = 1.4$ model (triangles) match empirical ones (circles) very well, across all strikes and for different times to expiration.

[*REMARK: Student t−distributions are the particular case of q−Gaussians when* $q = \frac{n+3}{n+1}$ *with n integer*]

LONDON STOCK EXCHANGE (Block market):

Data: I.I. Zovko; **Fitting**: E.P. Borges (2005)

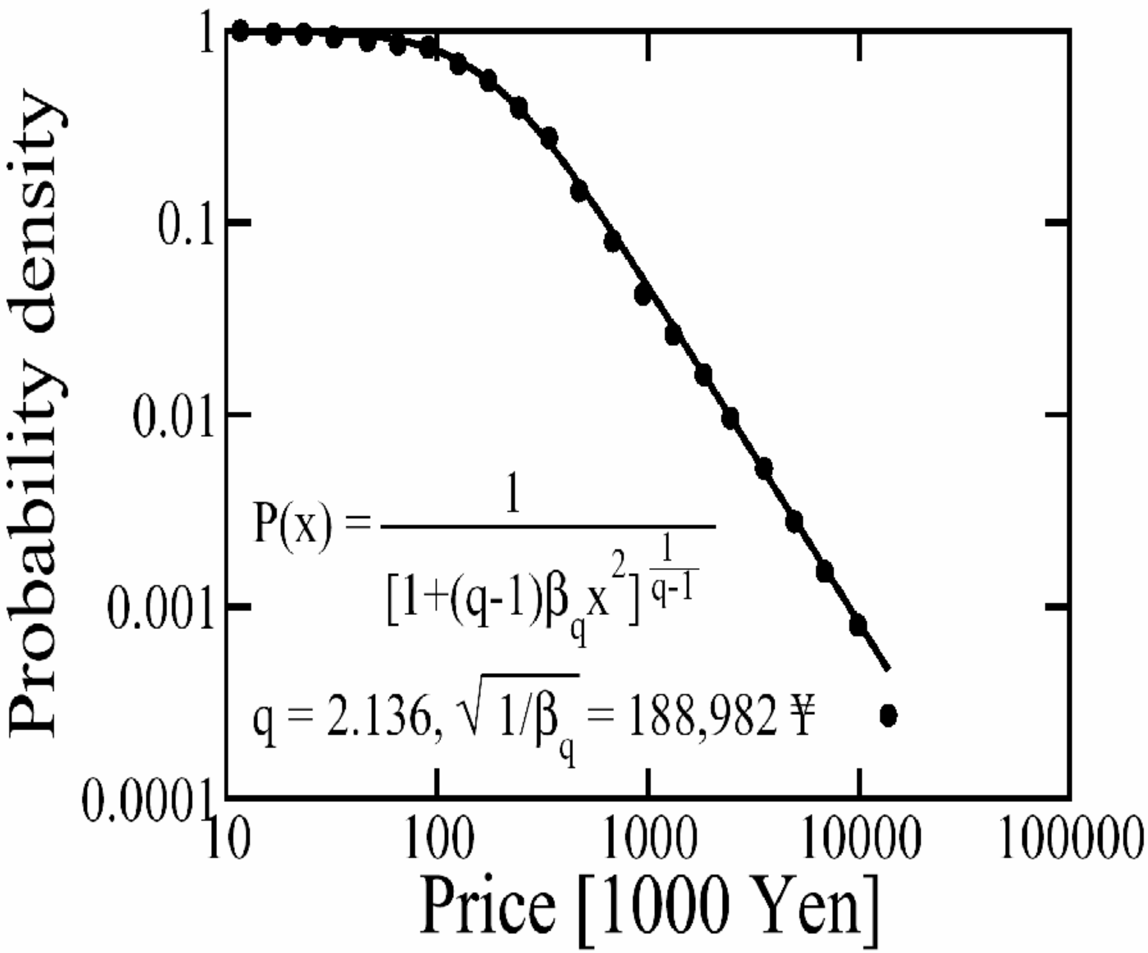
VODAPHONE stocks (31 May 2000 to 31 December 2002)



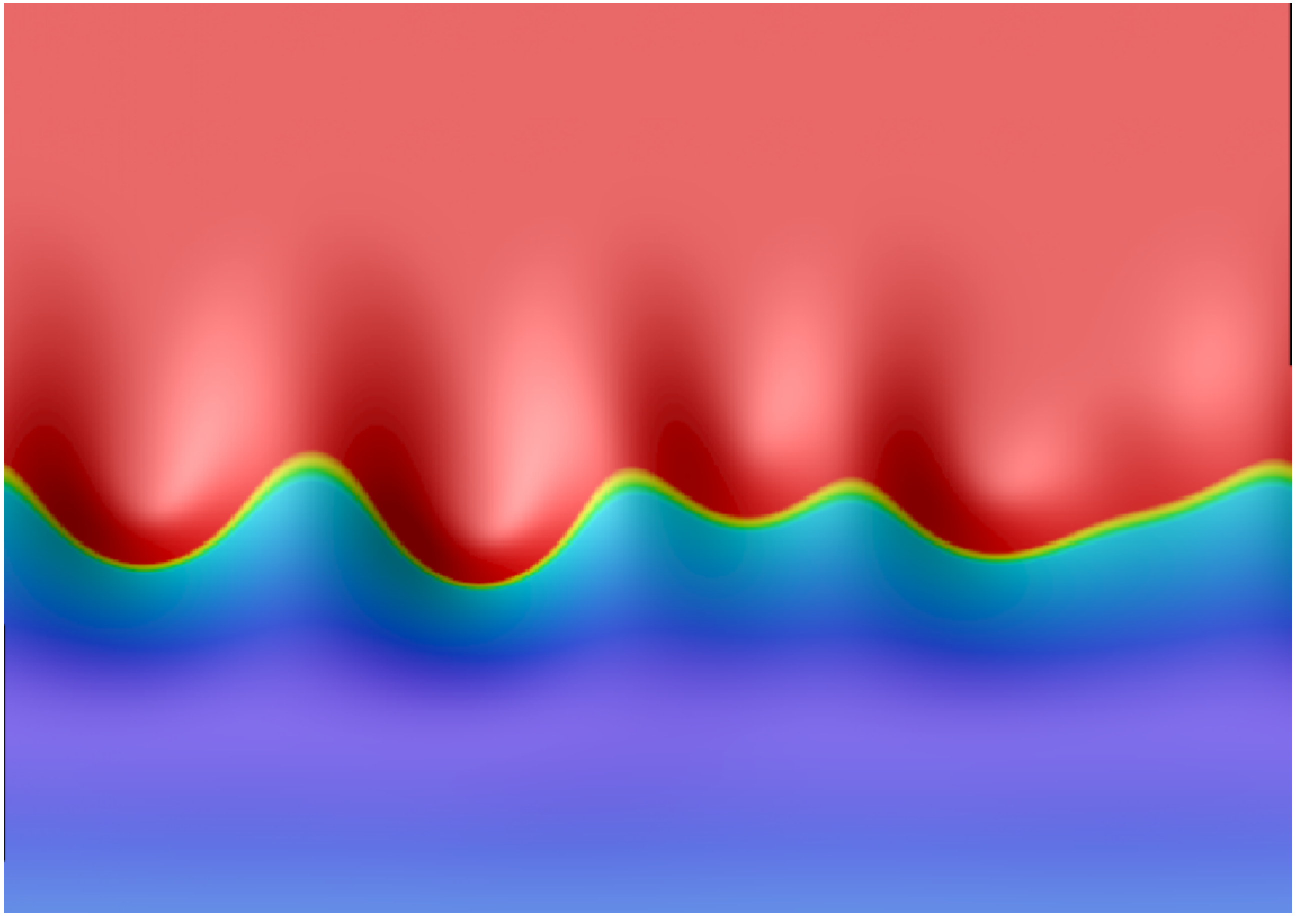
LAND PRICES IN JAPAN

(cumulative distribution)

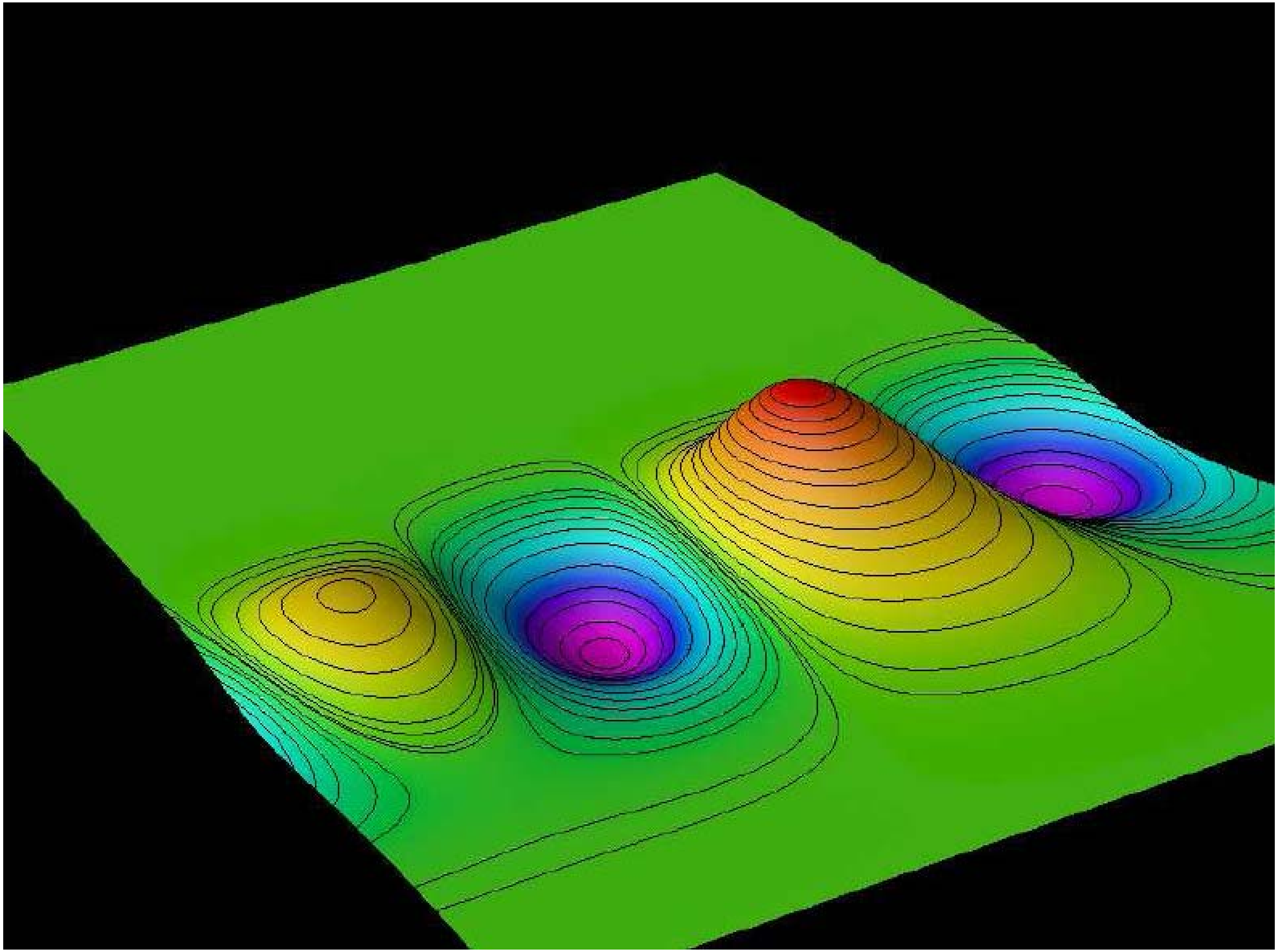
Data: T. Kaizoji, Physica A **326**, 256 (2003)
Curve: E.P. Borges (2003)



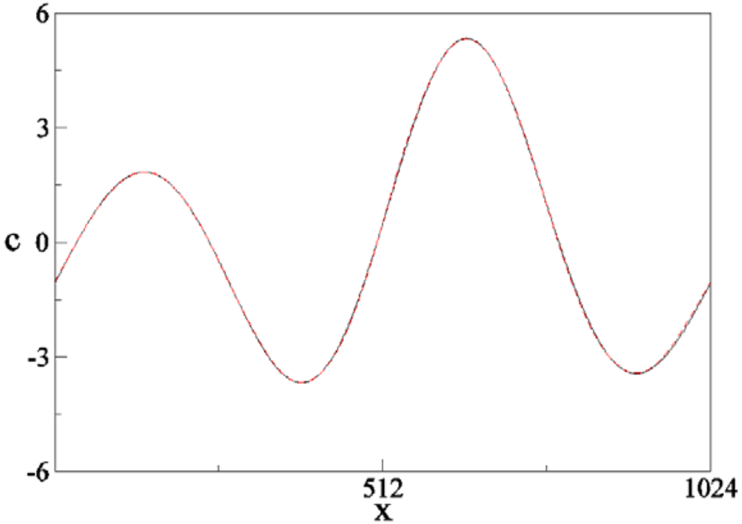
FINGERING

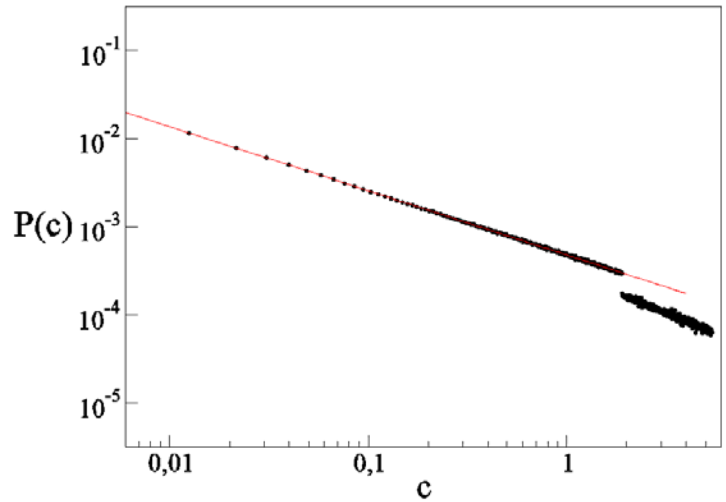


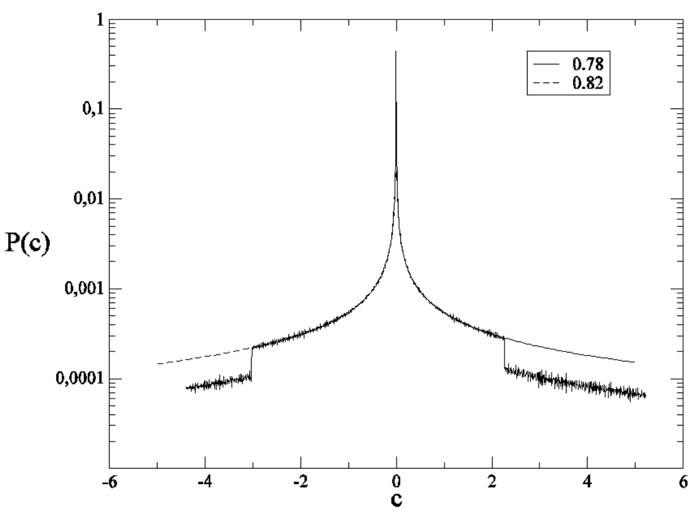
P. Grosfils and J.P. Boon, 2005



P. Grosfils and J.P. Boon, 2005







P. Grosfils and J.P. Boon, 2005

GENERALIZED SIMULATED ANNEALING AND RELATED ALGORITHMS

HYBRID LEARNING OF NEURAL NETWORKS

A.D. Anastasiadis and G.D. Magoulas, Physica A **344**, 372 (2004)

(Hybrid Learning Scheme = HLS; q > 1)

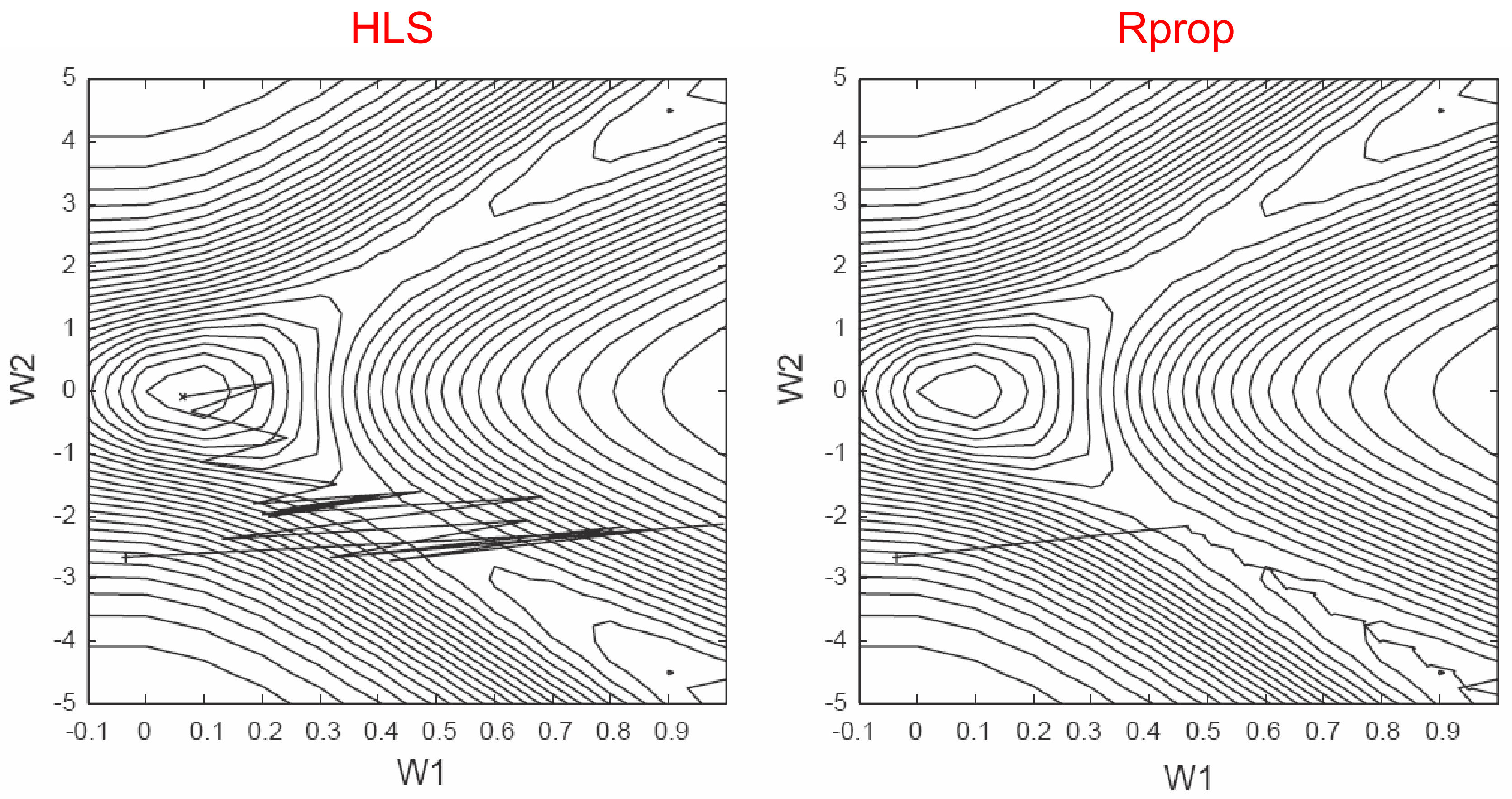


Fig. 1. The weights trajectory of the Hybrid Learning Scheme converges to the global minimum (left), whilst the trajectory of Rprop to a local minimizer (right).

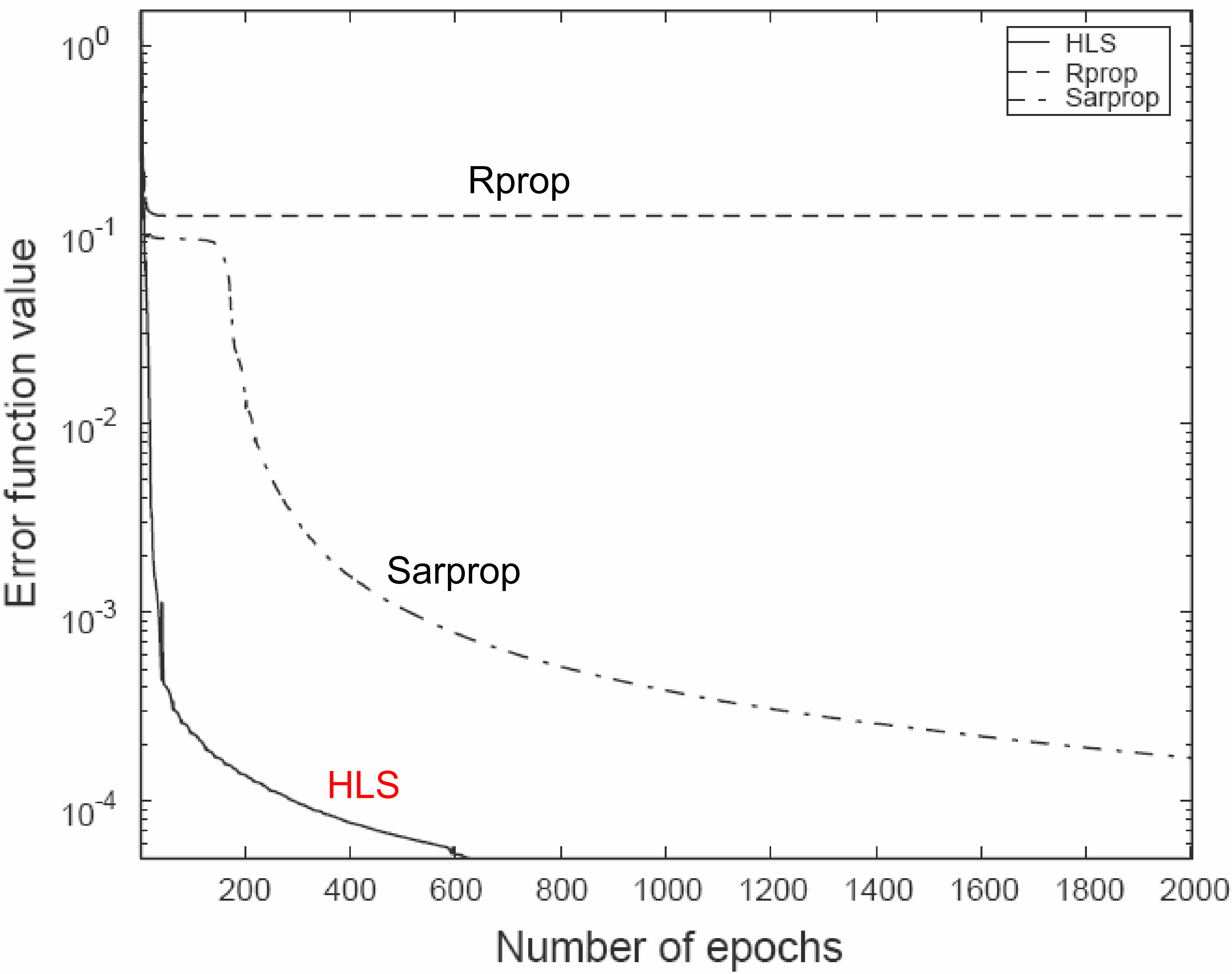


Fig. 2. Typical learning error curve for the Parity-3 problem.

Algorithm	Iris			Cancer		
	IT	*GEN.*	*CONV.*	*IT*	*GEN.*	*CONV.*
Rprop	1400 (+)	98.4 (+)	96	279 (+)	97.2 (−)	94
SARprop	1430 (+)	98.9 (+)	96	282 (+)	97.6 (−)	87
HLS	1377	99.5	100	157	97.5	100

Algorithm	Diabetes			Thyroid		
	IT	*GEN.*	*CONV.*	*IT*	*GEN.*	*CONV.*
Rprop	357 (+)	75.9 (+)	96	793 (+)	98.0 (−)	78
SARprop	325 (+)	75.8 (+)	96	736 (+)	98.1 (−)	92
HLS	223	76.1	100	460	98.2	100

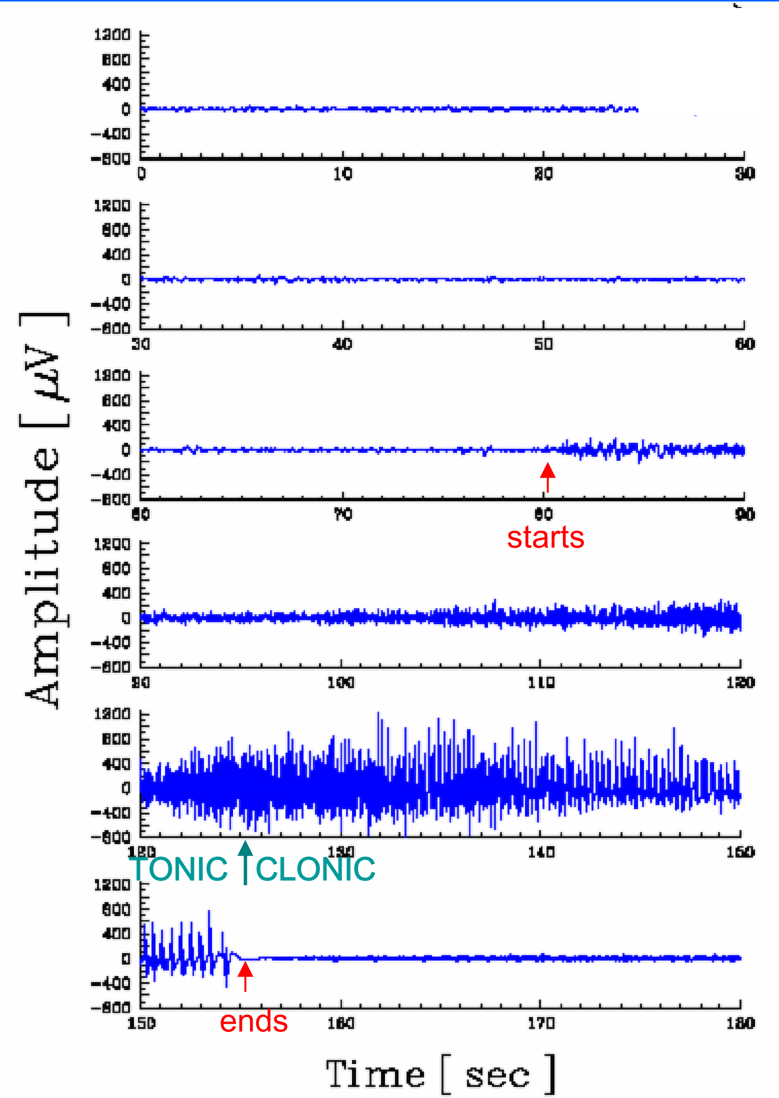
Algorithm	XOR			Parity 4		
	IT	*GEN.*	*CONV.*	*IT*	*GEN.*	*CONV.*
Rprop	1110 (+)	100 (−)	23	1360 (+)	100 (−)	42
SARprop	168 (+)	100 (−)	98	1378 (+)	100 (−)	48
HLS	49	100	100	1270	100	100

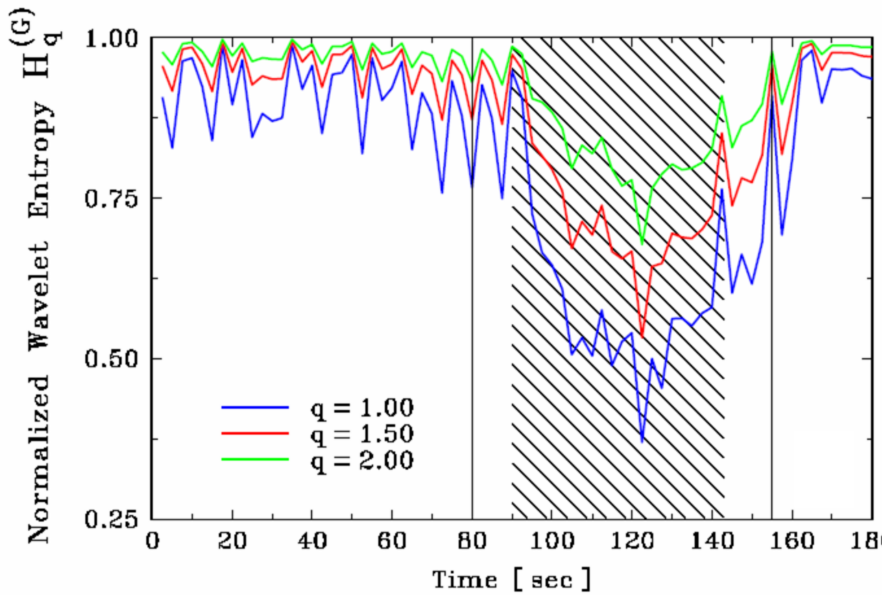
Algorithm	Parity 3			Parity 5		
	IT	*GEN.*	*CONV.*	*IT*	*GEN.*	*CONV.*
Rprop	1105 (+)	100 (−)	22	416 (+)	100 (−)	67
SARprop	882 (+)	100 (−)	78	394 (+)	100 (−)	95
HLS	640	100	100	20	100	100

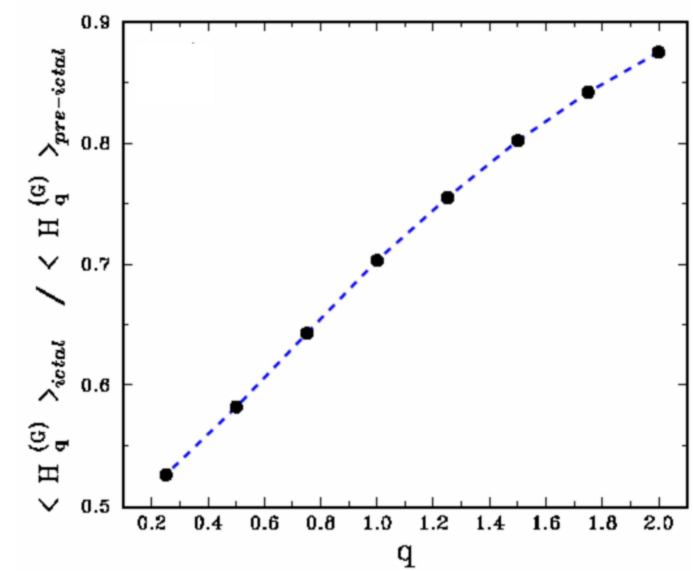
Anastasiadis and Magoulas (2004)

ELECTROENCEPHALOGRAMS (tonic-clonic transition in epilepsy):

(INCLUDING MUSCULAR ACTIVITY CONTRIBUTION)







A. Plastino and O.A. Rosso
Europhysics News **36** (6), 224 (2005) [European Physical Society]

IMAGE THRESHOLDING:

M.P. de Albuquerque, I.A. Esquef, A.R.G. Mello and M.P. de Albuquerque
Pattern Recognition Letters **25**, 1059 (2004)

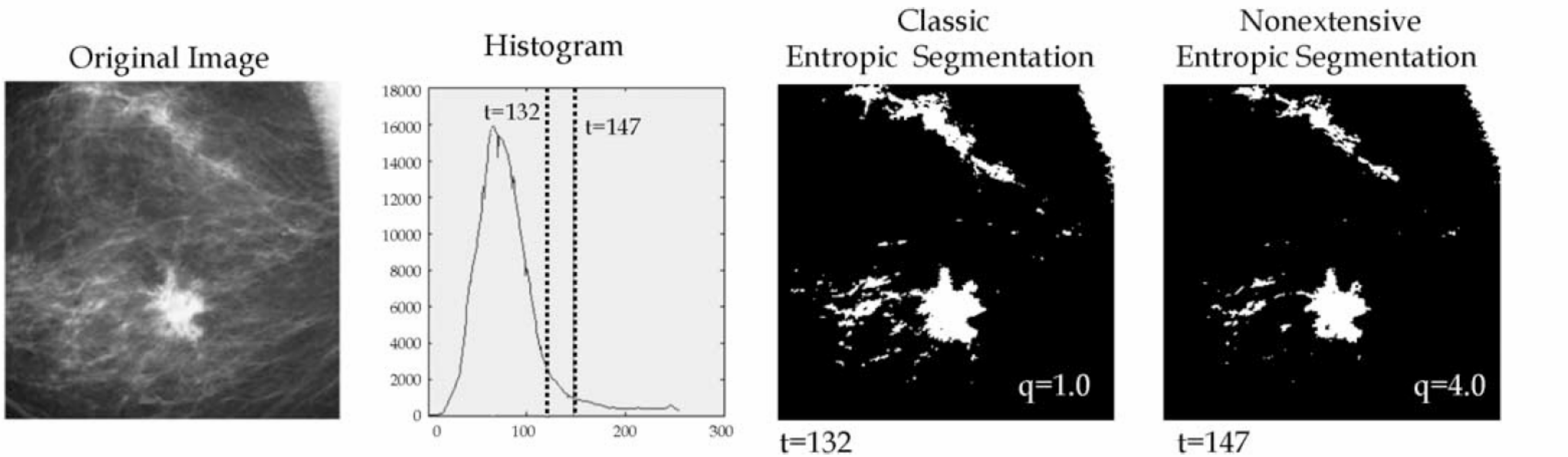


Fig. 3. Example of entropic segmentation for mammography image with an inhomogeneous spatial noise. Two image segmentation results are presented for $q = 1.0$ (classic entropic segmentation) and $q = 4.0$.

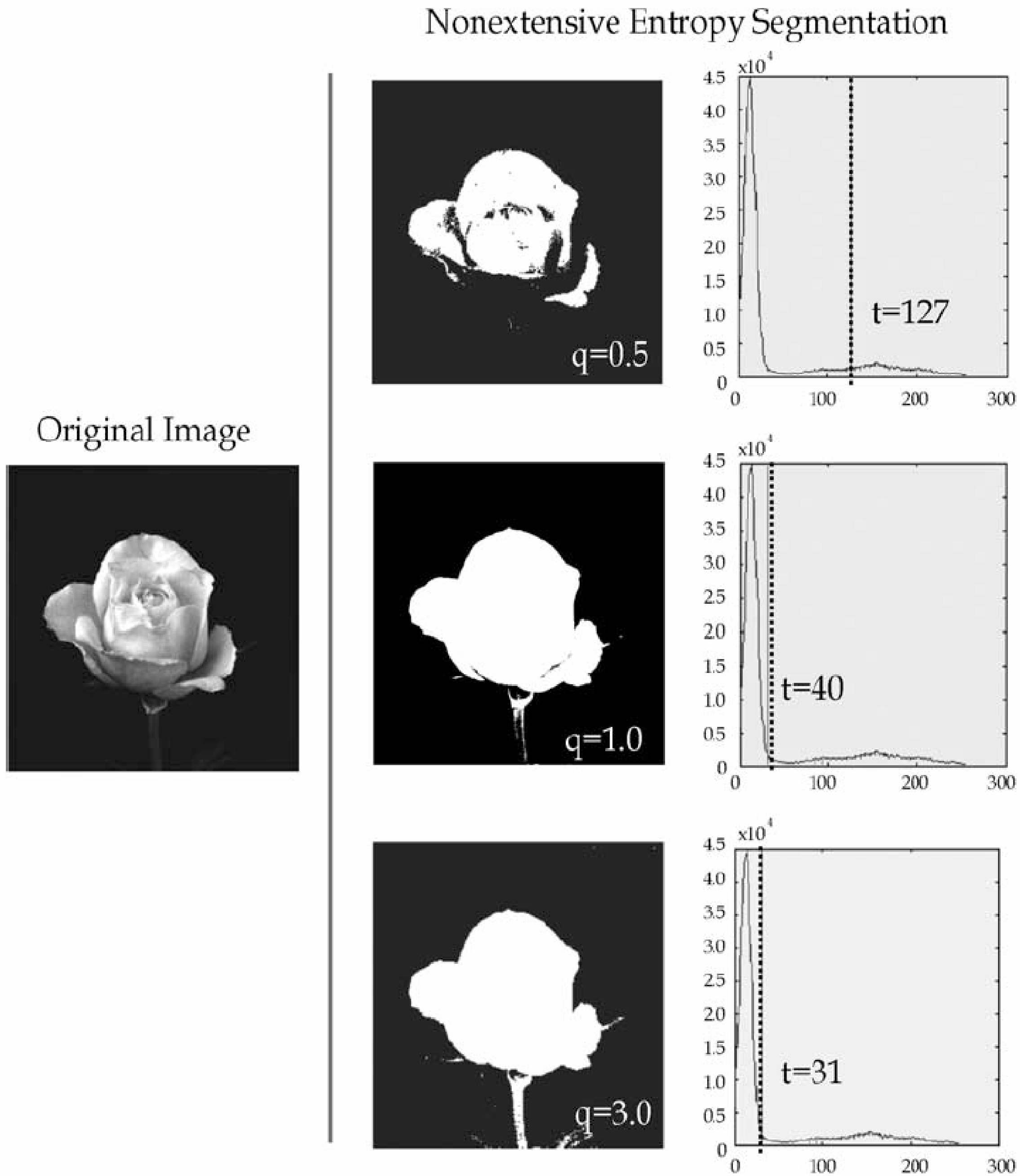


Fig. 4. Influence of parameter q in natural images: $q = 0.5$, $q = 1.0$ (classical entropic segmentation) and $q = 3.0$.

IMAGE EDGE DETECTION

[A. Ben Hanza, J. Electronic Imaging **15**, 013011 (2006)]

Original image

q = 1.5

Canny edge detector

q = 1 (Jensen-Shannon)

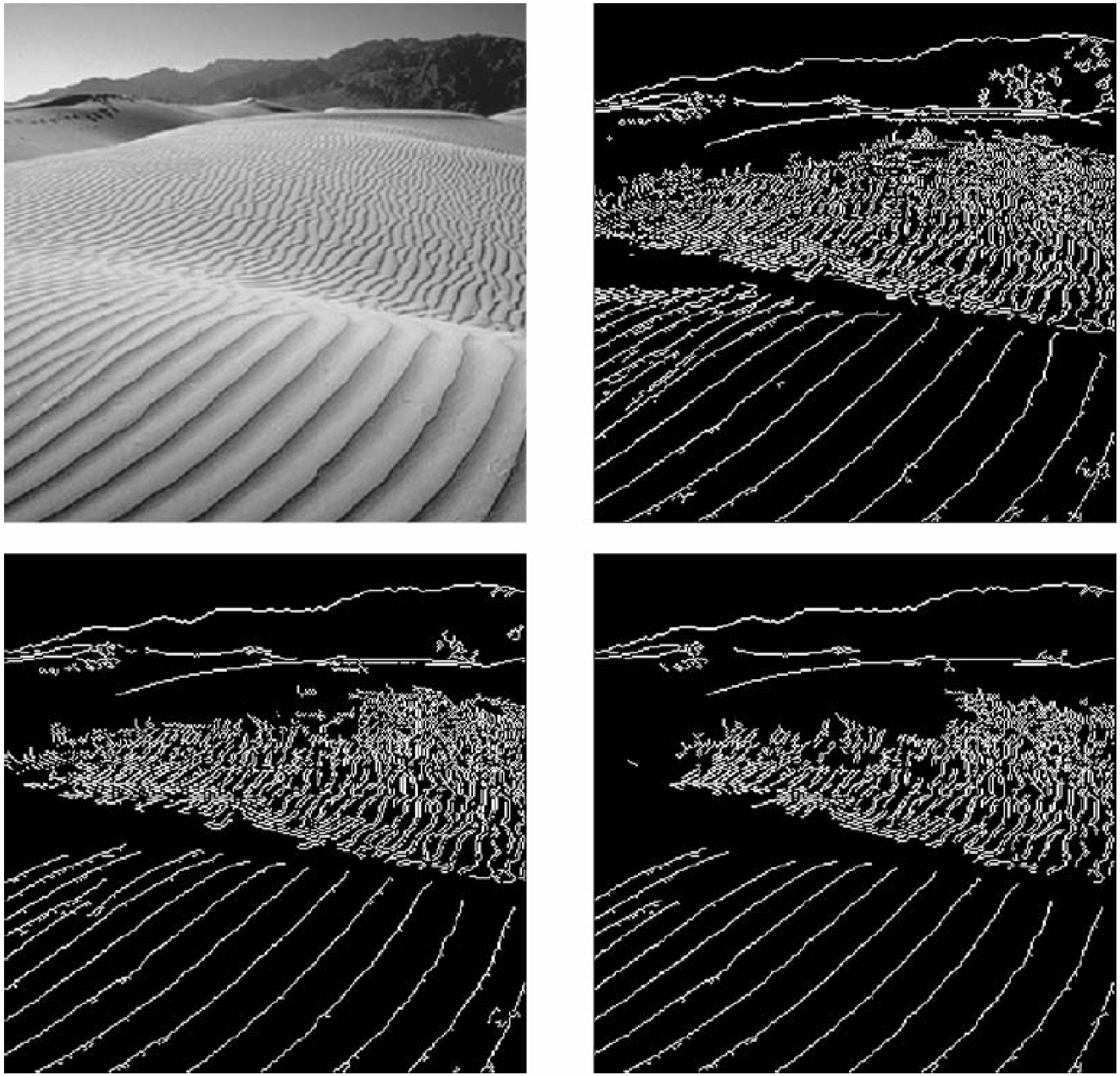
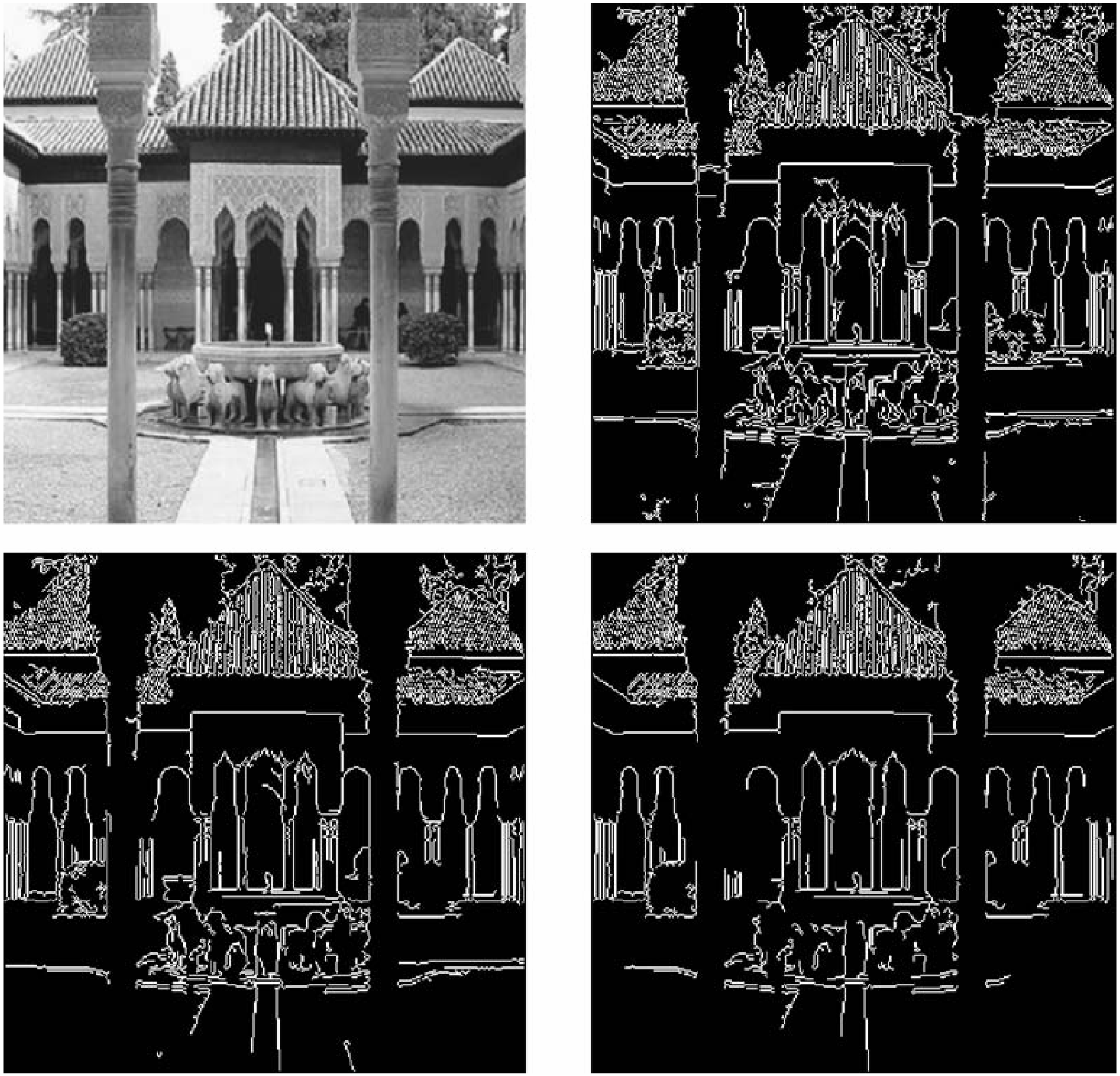


IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging **15**, 013011 (2006)]

Original image

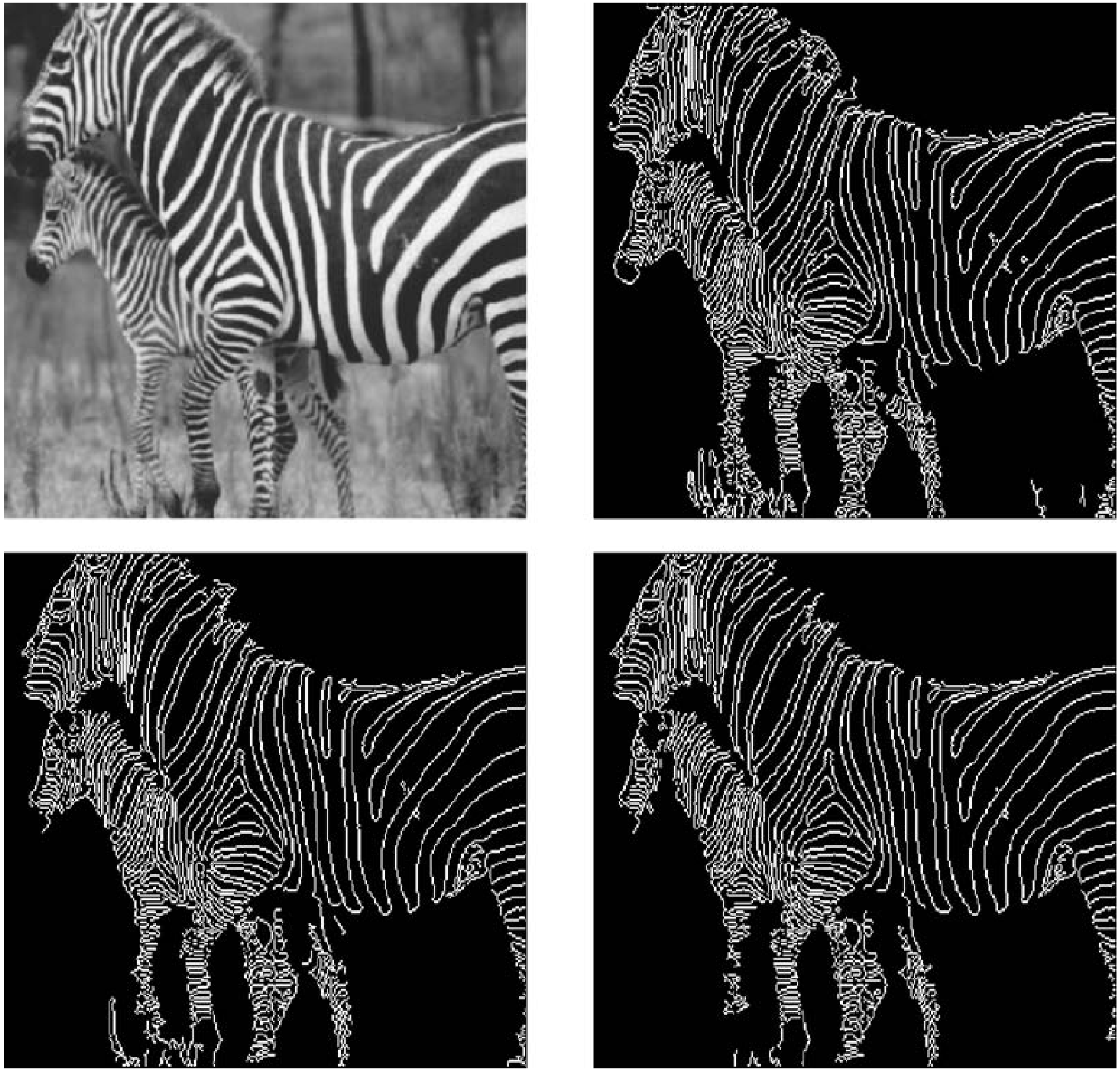
q = 1.5



Canny edge detector

q = 1 (Jensen-Shannon)

IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging **15**, 013011 (2006)]



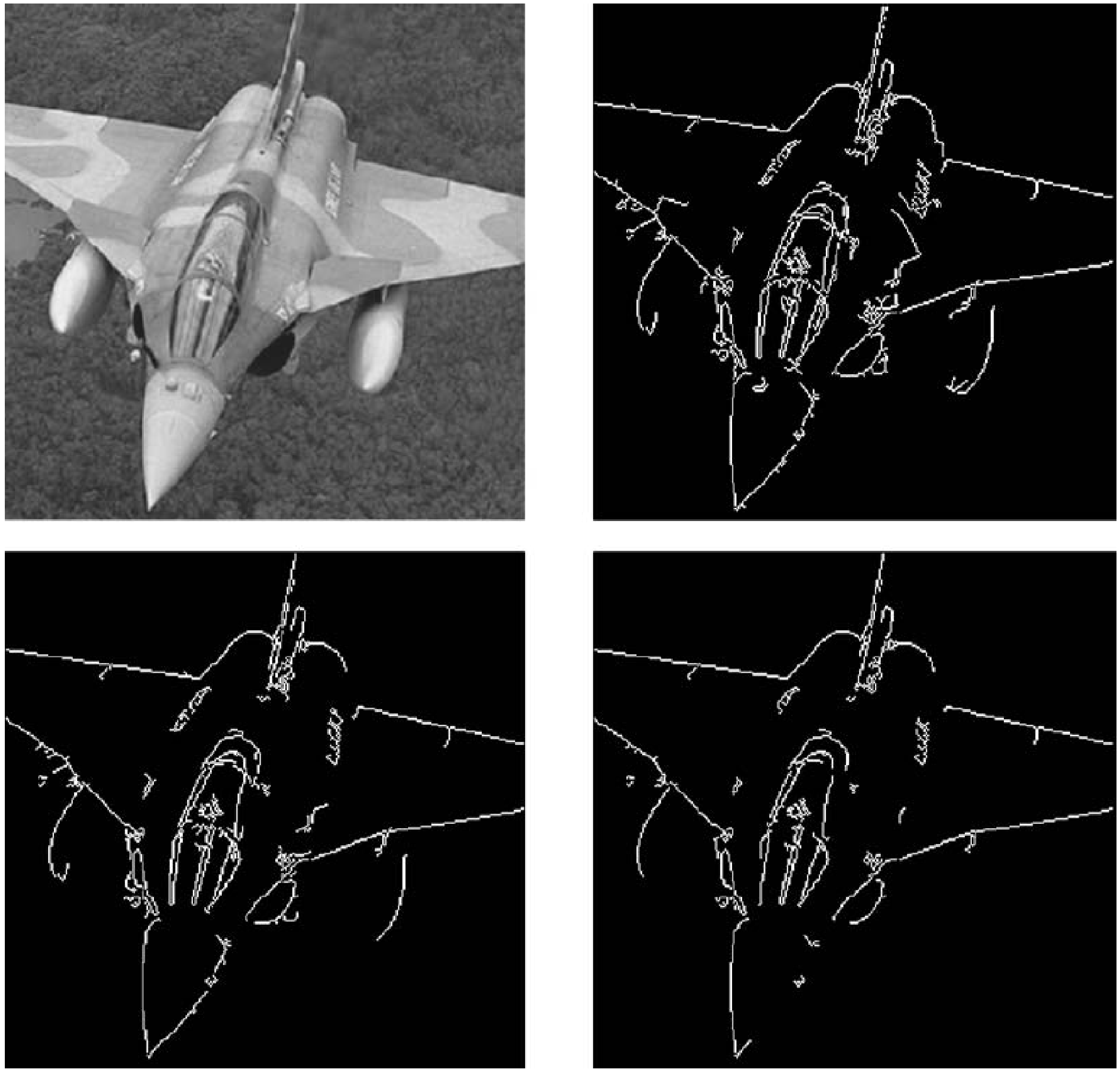
Original image

q = 1.5

Canny edge detector

q = 1 (Jensen-Shannon)

IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging **15**, 013011 (2006)]



Original image

q = 1.5

Canny edge detector

q = 1 (Jensen-Shannon)

q-GENERALIZED SIMULATED ANNEALING (GSA):

C.T. and D.A. Stariolo, Notas de Fisica / CBPF (1994); Physica A **233**, 395 (1996)

$Visiting\ algorithm:$

$Boltzmann\ machine \rightarrow Gaussian$

$Generalized\ machine \rightarrow q_V - Gaussian$

$Acceptance\ algorithm:$

$Boltzmann\ machine \rightarrow Boltzmann\ weight$

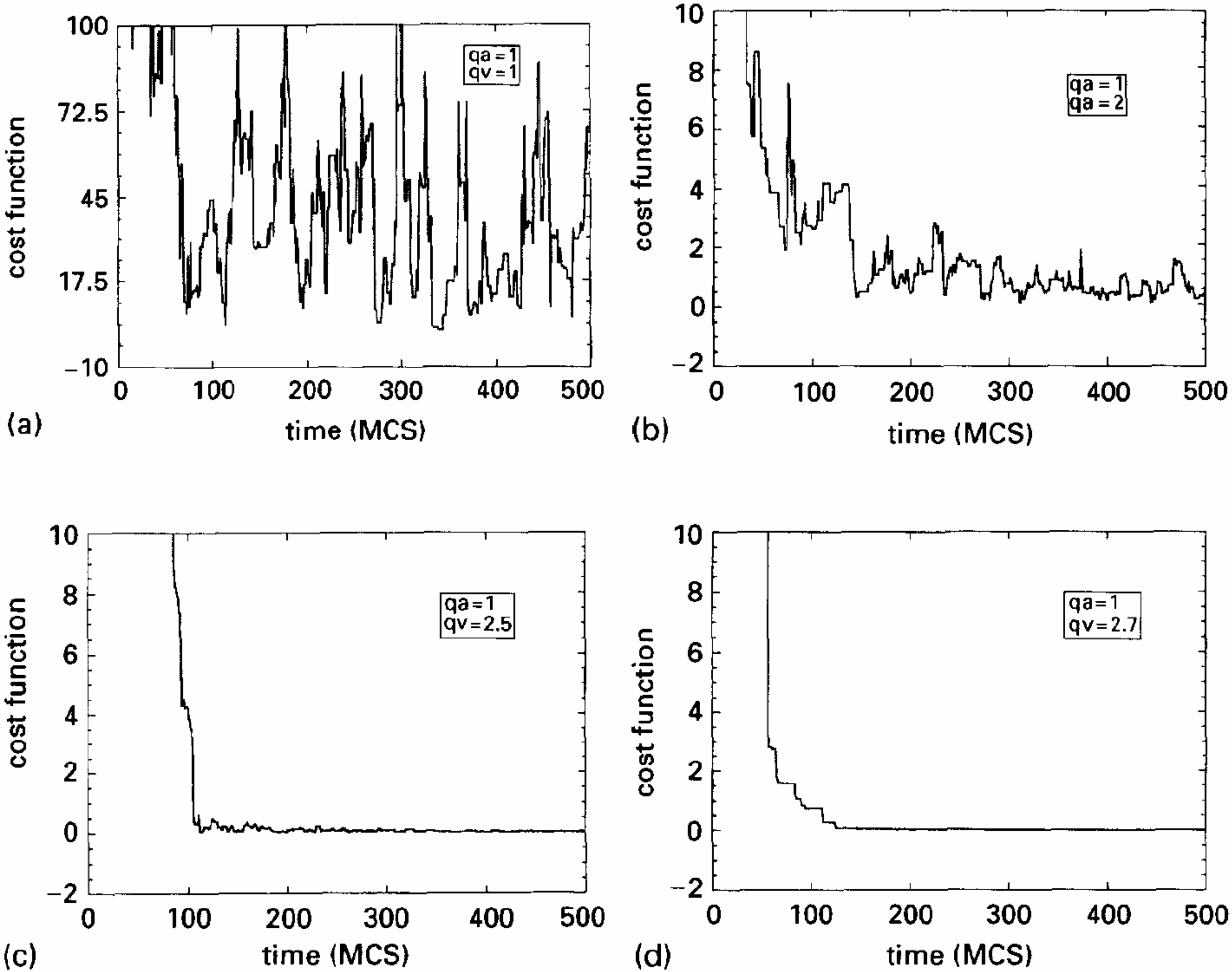
$Generalized\ machine \rightarrow q_A - exponential\ weight$

$Cooling\ algorithm:$

$$Boltzmann\ machine \rightarrow \frac{\mathrm{T(t)}}{\mathrm{T(1)}} = \frac{\ln 2}{\ln(1+t)}$$

$$Generalized\ machine \rightarrow \frac{\mathrm{T(t)}}{\mathrm{T(1)}} = \frac{2^{q_V-1}-1}{(1+t)^{q_V-1}-1}$$

$[Typical\ values:\ 1 < q_V < 3\ \ and\ \ q_A < 1]$

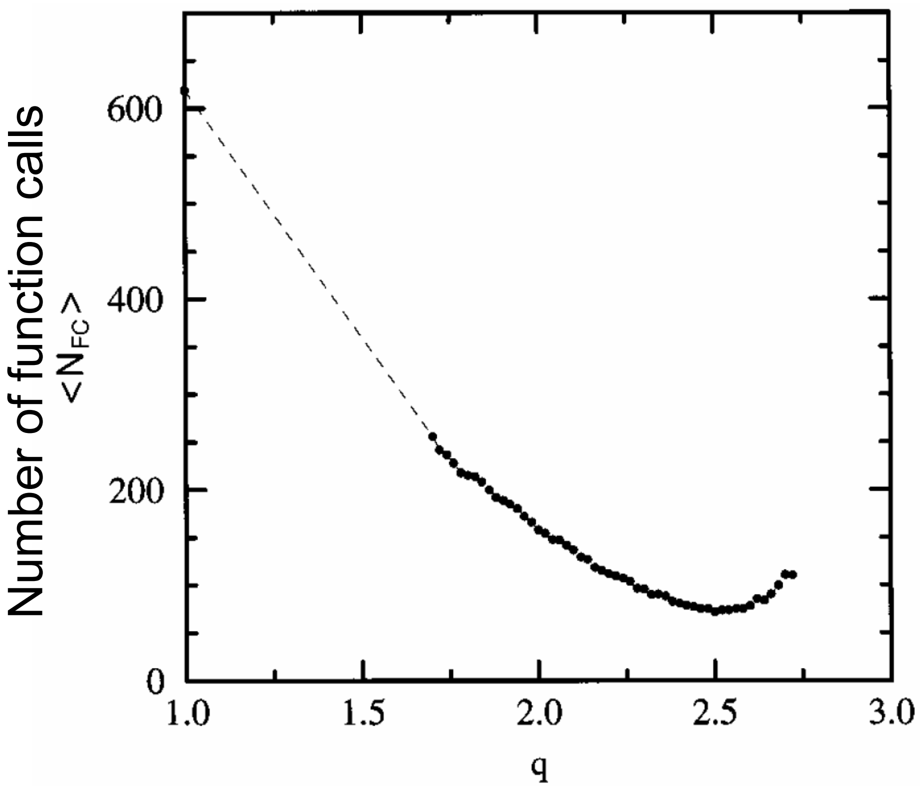


C. T. and D.A. Stariolo, Physica A **233**, 395 (1996)

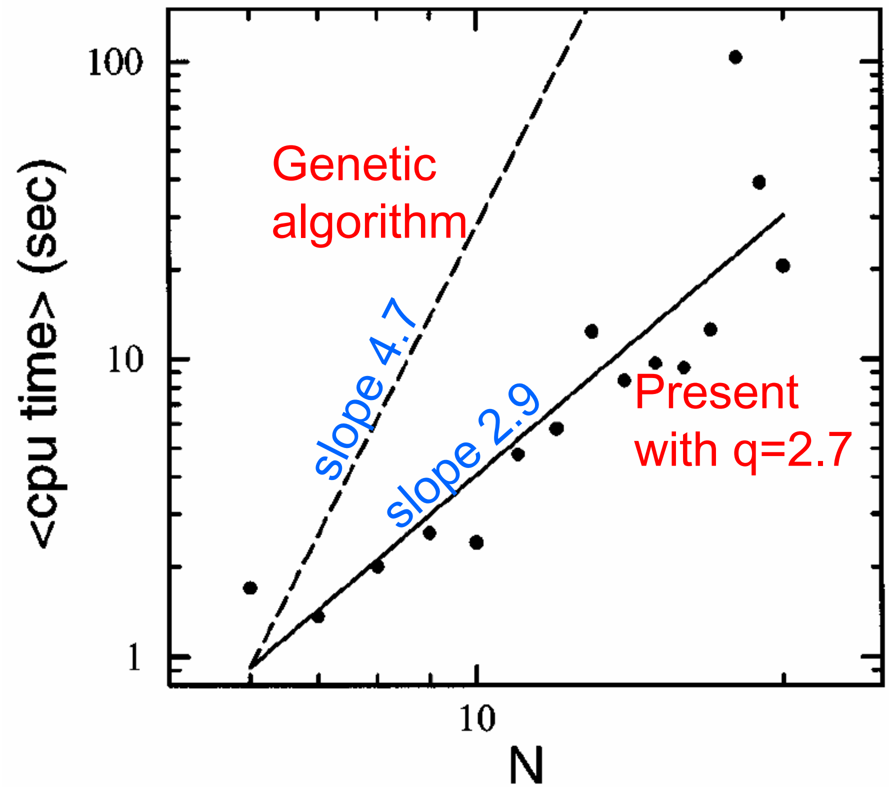
q-GENERALIZED PIVOT METHOD:

P. Serra, A.F. Stanton and S. Kais, Phys Rev E **55**, 1162 (1997)

(Branin function)



(Lennard-Jones clusters)



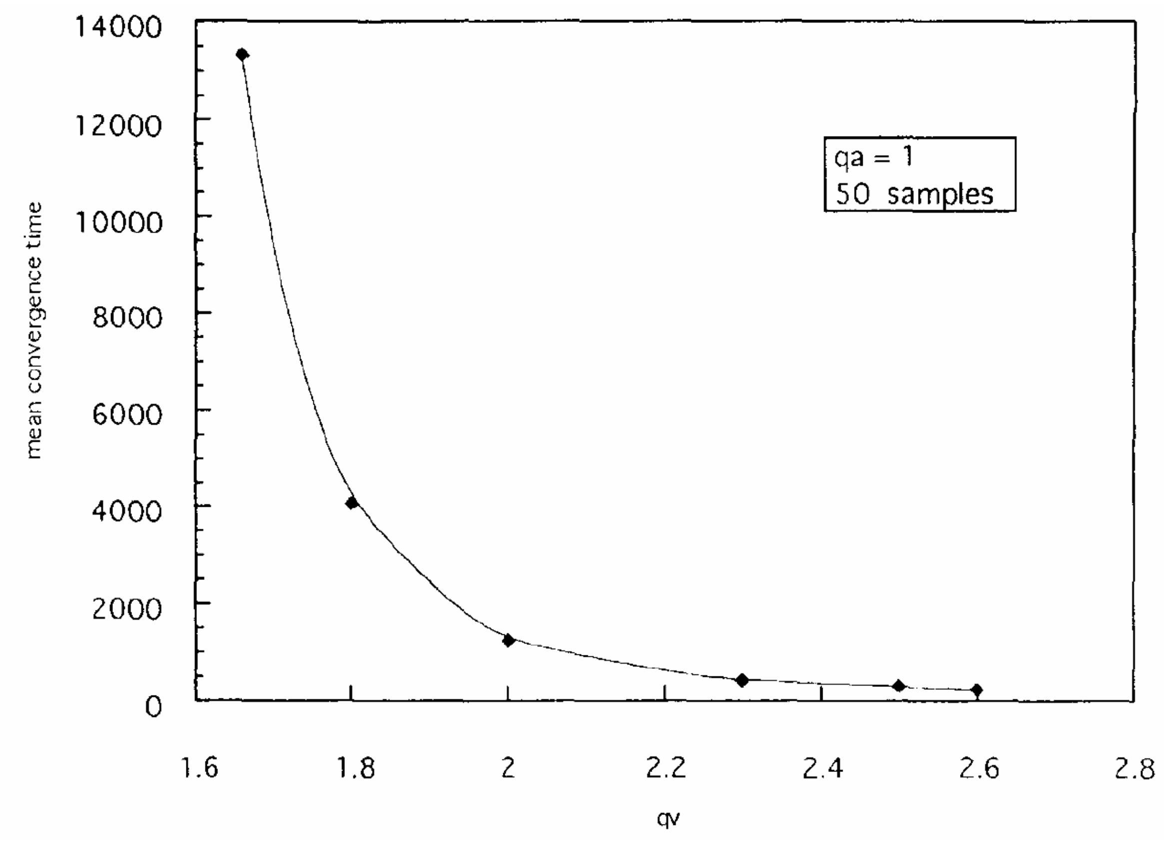
Recently: M.A. Moret, P.G. Pascutti, P.M. Bisch, M.S.P. Mundim and K.C. Mundim
Classical and quantum conformational analysis using Generalized Genetic Algorithm
Physica A (2006), in press (presumably better than both!)

q-GENERALIZED SIMULATED ANNEALING (GSA):

Illustration : $$E(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} (x_i^{\,2} - 8)^2 + 5\sum_{i=1}^{4} x_i$$

(15 *local minima and one global minimum*)

$(q_V = 1 \Rightarrow \textit{mean convergence time} \approx 50000)$



$than_q$

6 August ROUND TABLE

(Panelists: Nauenberg, Rapisarda, Robledo, Ruffo)

HMF MODEL: ABOUT THE ZEROTH PRINCIPLE OF THERMODYNAMICS

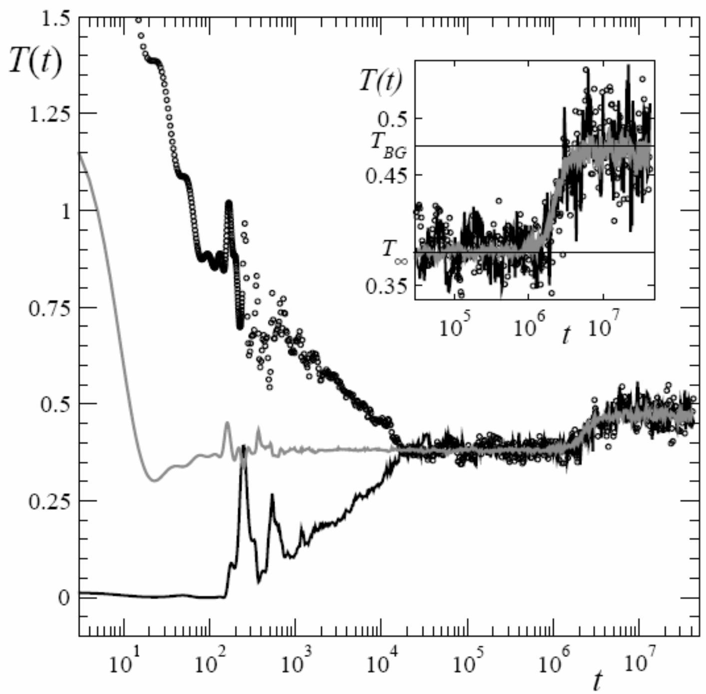


FIG. 1. Temperature evolution of an isolated N-rotor system (Eq. (1)) in grey line, and *cold* (*hot*) M-rotor subsystem in black line (circles). Inset: magnification of the crossover between T_{QSS} and T_{BG}.

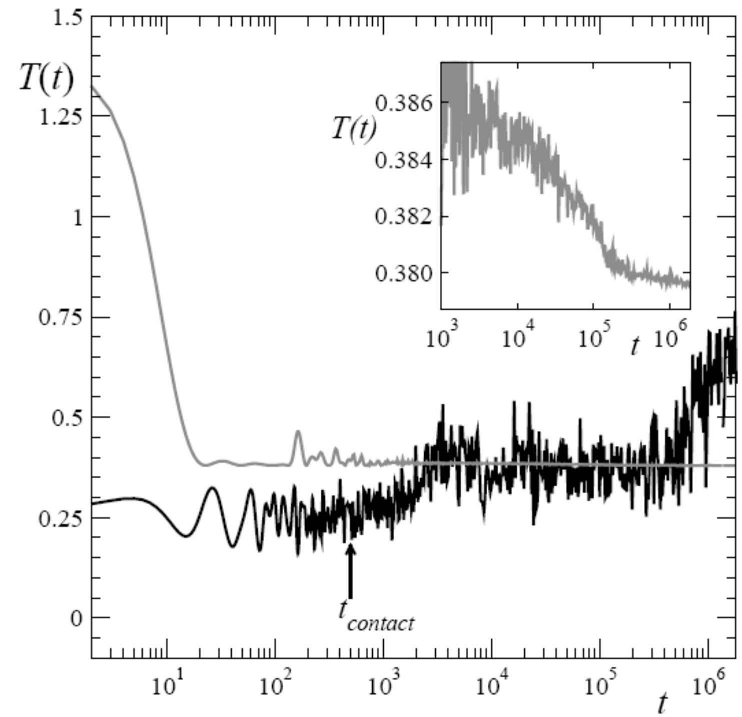


FIG. 2. Temperature evolution of an N-rotor thermostat (Eq. (1)) in grey line, and of an M-rotor thermometer (Eq. (3)) in black line. After $t_{contact}$ the systems interact through H_{int}. Inset: magnification of the thermostat temperature minimum (see text for details).

L.G. Moyano, F. Baldovin and C. T., cond-mat/0305091

Facts that make think of q-statistics for the HMF and similar models:

1) For u = 5, and d=1, 2, 3 : maximal Lyapunov exponent vanishes like 1 / N^delta with delta(alpha/d) decreasing from 1/3 to zero for alpha/d increasing from zero to 1, and remaining zero for alpha/d >1;

2) For u = 0.69, at QSS, for d = 1: maximal Lyapunov exponent vanishes like 1 / N^delta' with delta'(alpha/d) decreasing from 1/9 to zero for alpha/d increasing from zero to 1, and remaining zero for alpha/d > 1;

3) For u = 0.69, for alpha = 0 and d=1: T(t) - T(infinity) ~ exp_q (- t / tau), with q different from unity;

4) For u = 0.69 and d = 1: t_QSS ~ ln_(alpha / d) N for 0 < alpha < 1 ;

5) For u = 0.69, at QSS, for alpha = 0, and d = 1: the marginal probability of the velocity of one particle is not Maxwellian, and its central part decays like exp_q (- B p^2) for M_0 =1, with q different from unity;

6) For u = 0.69, at QSS, for alpha = 0 and d = 1: The autocorrelation function of velocities presents scalable aging and decays like exp_q [- A t / (t_W^rho)] with different from unity;

7) For u=5, alpha = 0 and d = 1: The autocorrelation function of velocities presents no aging and decays like exp_q [- A t] with q different from unity which coincides with that of point (6) ;

8) For various N, various alpha, various M_0, and d=1: gamma = 2 / (3-q)

9) For u = 0.69, N >>1, and t >>> 1, alpha = 0 and d = 1: marginal probability for the angles of one particle ~ exp_q (- C theta^2) with q different from unity;

10) For u = 0.69, alpha = 0, d=1 and finite N: the system has long memory as exhibited by the relevant influence of the initial conditions (dependence on M_0, and dependence on initial condition Catania-type or Rio de Janeiro-type);

11) For u = 0.69, alpha = 0, d=1 and finite N: the system has long memory as exhibited by the nonvanishing glassy polarization versus N along some time regime;

12) There is a manner of presenting the recent results by Baldovin and Orlandini in the canonical ensemble (instead of microcanonical) which enables them to be consistent with any value of q between unity and say 2. This is a consequence of the fact that their variation of computational total energy is only of 8 %, and of the fact that, in first order, the q-exponential function does not depend on q;

13) Werner Braun is not sure whether one can take all those derivatives in the Braun-Hepp theorem, in the case 0 < alpha / d < 1. This suggests that his intuition tells him that something quite unusual might occur in such a case.

Forse mi e scapata qualche altra raggione.

Posso qualificare con piu detagli (valori di N, valori di t, condizioni iniziali precise, algoritmo di calcolo in dinamica moleculare, etc) le condizioni sui qualli ogni una di queste raggioni e valida. Ho tutte le referenze alla tua disposizione.

Si come non abbiamo ancora una prova irrefutabile, il mio argomento e temporariamente che quello che ha il sapore di pizza, odore di pizza, e rotondo come pizza, ha pomodoro come pizza, ha muzarella come pizza, e venduto nelle pizzerie ... probabilmente e pizza!

[Fragmenti della lettera di Constantino a Stefano sul tema]