The Abdus Salam
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SMR.1763- 19

SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

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Superstatistics: Applications in turbulence and particle physics

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Superstatistics

— applications in turbulence and particle physics

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in these lectures (ICTP 2006)

- what is superstatistics?
- why is it of interest?
- what is its relation to nonextensive stat. mech.?
- physical applications
  - ⊕ hydrodynamic turbulence (Eulerian & Lagrangian)
  - ⊕ high energy physics (e^+e^- annihilation, cosmic rays)
  - ⊕ pattern forming systems (defect turbulence)
  - ⊕ general time series

C.B., E.G.D. Cohen, Physica A 322, 267 (2003)

Foundations of (ordinary) statistical mechanics:

entropy $S = -\sum_i p_i \ln p_i$

prob. of microstates

extremize subject to constraints

$$\sum p_i = 1$$

$$\sum p_i E_i = U$$

energies of microstates

result: $p_i = \frac{1}{Z} e^{-\beta E_i}$ (canonical distribution)

$Z = \sum_i e^{-\beta E_i}$ (partition function)

Q: Why this particular function $S = -\sum p_i \ln p_i$?

A1: Because it works! (physics correctly described)

A2: Because it satisfies certain nice axioms of information measures

(see, e.g., C. Beck, F. Schlögl, Thermodynamics of Chaotic Systems, Cambridge University Press 1993)

but: generalized Khinchin axioms (Abe, 2000)

↪ more general information measures possible e.g. Tsallis entropies

Khinchin axioms

(desirable properties of an information measure)

i) $S = S(p_1, p_2, \dots, p_W)$

(function of probabilities only)

ii) $p_i = \frac{1}{W} \Rightarrow S = \max$

(maximum for equal probability distr.)

iii) $S(p_1, \dots, p_W, 0) = S(p_1, \dots, p_W)$

(no change by event with prob. zero)

iv) $S(\underbrace{I + II}_{\text{composed system}}) = S(I) + S(II \mid I)$

conditional entropy

i) – iv) $\Rightarrow$ $S = -\sum p_i \ln p_i$ uniquely

But: If you allow a slightly more general form of iv)

iv*) $S(I + II) = S(I) + S(II \mid I) + (1-q)\, S(I) \cdot S(II \mid I)$

then you end up uniquely with

$$S = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right)$$

Tsallis entropies (Abe, PLA 2000)

In particular, for independent subsystems I and II

$\boxed{\mathrm{I} \mid \mathrm{II}}$

$$S_q(\mathrm{I}+\mathrm{II}) = S_q(\mathrm{I}) + S_q(\mathrm{II}) + (1-q)\, S_q(\mathrm{I})\, S_q(\mathrm{II})$$

entropy is not extensive any more
(for $q \neq 1$)

but for specially <u>correlated</u> subsystems S_q is additive again!
Tsallis, Gell-Mann, Sato cond-mat/0502274

Can now do generalized version of stat. mech. by extremizing Tsallis entropies S_q subject to constraints

'nonextensive stat. mech.'

$$\sum_i p_i = 1$$

$$\sum_i p_i E_i = U_q$$

$$\left(\text{or} \quad \sum P_i E_i = U_q \quad \text{with} \quad P_i = \frac{p_i^q}{\sum p_i^q} \right)$$

escort distributions

$$\rightsquigarrow \quad p_i = \frac{1}{Z_q}\left(1-\beta(1-q)E_i\right)^{\frac{1}{1-q}}$$

(generalized canonical distribution)

$$Z_q = \sum_i \left(1-(1-q)\beta E_i\right)^{\frac{1}{1-q}}$$

(partition function)

Entire formalism of thermodynamics has q-generalization / q-invariance

helpful tool:
define

$$e_q^x := \left(1 + (1-q)\,x\right)^{\frac{1}{1-q}} \qquad \rightarrow e^x \quad (q \to 1)$$

q-exponential

$$\ln_q x := \frac{x^{1-q} - 1}{1-q} \qquad \rightarrow \ln x \quad (q \to 1)$$

q-logarithm

$$e_q^{\ln_q x} = x \qquad \forall q$$

canonical distributions become

$$p(E) \sim e_q^{-\beta E} = \left(1 - \beta(1-q)E\right)^{\frac{1}{1-q}}$$

generalized Boltzmann factor

can also derive

$$F_q = U_q - T S_q = -\frac{1}{\beta} \ln_q Z_q$$

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q}$$

$$\frac{\partial^2 S}{\partial U_q^2} = -\frac{1}{T^2}\,\frac{1}{C_q} \qquad \text{etc....}$$

When } could this be physically relevant?
Why }

Basic idea:

If system (for whatever reason) Cannot extremize Shannon entropy it then chooses to extremize some other ~~the second best~~ information measures. suitable to describe the problem. These are e.g. the Tsallis entropies.

- convex
- Lesche-stable
- pres. Legendre transform struct.

(+ possibly more ...)

reason could be

— system non-mixing
:

— long range interaction

— complicated multifractal phase space structure

— complicated networks

— external energy input (nonequilibrium system with stationary state)

— scale invariance

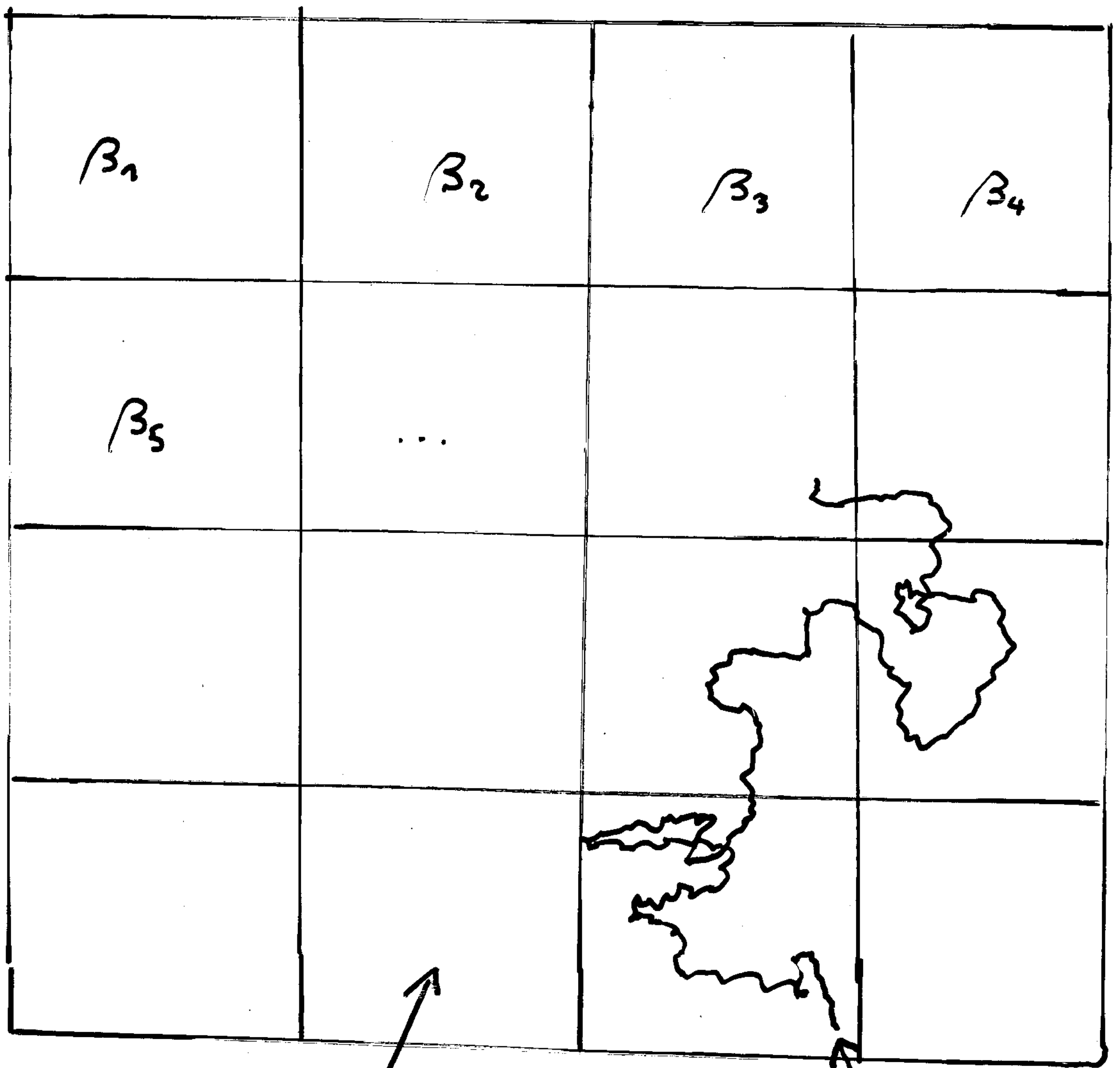
— fluctuations of temp. or energy dissipation rate

— strongly inelastic collisions

Non equilibrium system

with fluctuations of (e.g.) inverse temperature β on long time scale

(can also be pressure, chemical potential, energy dissipation rate, ...)



local equilibrium

$$p(E) \sim e^{-\beta_i E}$$

in each cell

test particle

simplest model

$E = \frac{1}{2} u^2$

model: choose a random configuration $\{\beta_j\}$

(β distributed according to density $f(\beta)$)

then choose next random config., and so on.

Dynamical foundation of nonextensive stat. mech. for systems with fluctuating temperature or energy dissipation rate.

Brownian particle (Ornstein-Uhlenbeck process)

$$\dot{u} = -\gamma u + \sigma L(t)$$

$\uparrow$ Gaussian white noise

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{1}{2}\beta u^2\right\}$$

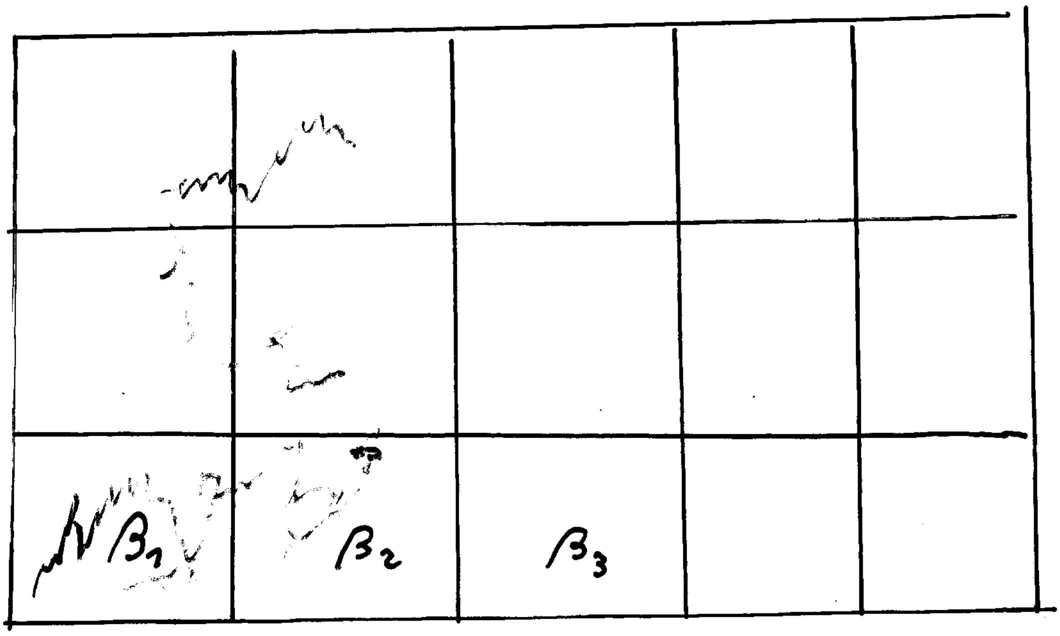
$\beta := \frac{\gamma}{2\sigma^2}$ inverse temperature

Assume γ and/or σ fluctuate on large time scale s.t. $\beta = \frac{\gamma}{2\sigma^2}$ is χ^2 distributed with degree n

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n}{2}\frac{\beta}{\beta_0}\right\}$$

$\uparrow$ prob. density

e.g. $\beta = \sum_{i=1}^{n} X_i^2 \leftarrow$ Gaussian (av. 0)



conditional prob.

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{1}{2}\beta u^2\right\}$$

joint prob.

$$p(u,\beta) = p(u|\beta)\cdot f(\beta)$$

marginal prob.

$$p(u) = \int_0^\infty p(u|\beta)\, f(\beta)\, d\beta$$

$$= \frac{1}{Z_q} \frac{1}{\left(1+\frac{1}{2}\tilde{\beta}(q-1)u^2\right)^{1/q-1}} \qquad \left(E=\frac{1}{2}u^2\right)$$

where

$$q = 1+\frac{2}{n+1}$$

$$\tilde{\beta} = \frac{2}{3-q}\beta_0$$

C. B., PRL $\underline{87}$, 180601 (2001)

$$\beta_0 := \int f(\beta)\cdot\beta\, d\beta = \text{average of } \beta$$

Simple dynamical model
where Tsallis statistics can be proved rigorously.

Various generalizations possible.

e.g. $\dot{u} = -\gamma F(u) + \sigma L(t)$

$$F(u) = -\frac{\partial}{\partial u} V(u) \qquad V(u) \sim |u|^{2\alpha}$$

$$\Rightarrow q = 1 + \frac{2\alpha}{\alpha n+1}$$

Fluctuations of β and Tsallis statistics

integral representation of Γ function:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

substitute

$$t = \beta\left(E(u) + \frac{1}{(q-1)\beta_0}\right)$$

$$z = \frac{1}{q-1}$$

$$\Downarrow$$

any Hamiltonian
or effect. energy!

$$\left(1 + (q-1)\beta_0 E(u)\right)^{-\frac{1}{q-1}} = \int_0^\infty e^{-\beta E(u)} f(\beta)\, d\beta$$

'generalized' Boltzmann factor — ordinary Boltzmann factor

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left\{\frac{1}{(q-1)\beta_0}\right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{1}{q-1}\frac{\beta}{\beta_0}\right\}$$

χ^2 distribution
(occurs in many circumstances)

$$\langle\beta\rangle = \int_0^\infty \beta f(\beta)\, d\beta = \beta_0$$

$$q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2}$$

e.g. $\beta = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Gaussian
av. 0

$$n = \frac{2}{q-1}$$

Wilk et. al.
PRL 2000

C.B.
PRL 87, 180601 (2001)

More generally one can consider generalized Boltzmann factors.

S. Abe
J. Phys. A 36 8733 (2003)
A. Souza, C. Tsallis
PLA 2003
↓
effectively maximizes more general information measures

$$B(E) = \int_0^\infty e^{-\beta E} f(\beta)\, d\beta$$

In general $f(\beta)$: "Superstatistics"
(C.B., E.G.D. Cohen, Physica 322A, 267 (2003))

$f(\beta) = \chi^2 \Rightarrow$ Tsallis

$f(\beta) = \frac{1}{b}$ for $\beta \in [a, a+b]$
(uniform distribution)

$$\Rightarrow B(E) = \frac{1}{bE}\left(e^{-(\beta_0 - \frac{1}{2}b)E} - e^{-(\beta_0 + \frac{1}{2}b)E}\right)$$

$$= e^{-\beta_0 E}\left(1 + \frac{1}{24} b^2 E^2 + \frac{1}{1920} b^4 E^4 + \dots\right)$$

- $f(\beta) =$ log-normal

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{-\frac{\left(\log \frac{\beta}{m}\right)^2}{2s^2}\right\}$$

$w := e^{s^2}$

$$\Rightarrow B(E) = e^{-\beta_0 E}\left(1 + \frac{1}{2} m^2 w(w-1)E^2 + \frac{1}{6} m^3 w^{\frac{3}{2}}(w^3 - 3w + 2)E^3 + \dots\right)$$

- $f(\beta) =$ F-distribution

$$f(\beta) \sim \frac{\beta^{\frac{v}{2}-1}}{(1 + c\cdot\beta)^{\frac{v+w}{2}}} \Rightarrow B(E) = \dots$$

Main result: (σE small)

For small enough variance of the fluctuations of β all superstatistics behave in a universal way

can prove

$$B(E) = e^{-\beta_0 E}\left(1 + \frac{1}{2}\sigma^2 E^2 + g(q)\beta_0^3 E^3 + \dots\right)$$

variance of distribution $f(\beta)$

$\sigma^2 = \langle\beta^2\rangle - \langle\beta\rangle^2$

$\beta_0 = \langle\beta\rangle$

can define

$$q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2}$$

for <u>any</u> superstatistics

next-order term:

$$g(q) = \begin{cases} 0 & \text{(uniform)} \\ \frac{1}{3}(q-1)^2 & (\chi^2) \\ \frac{1}{6}(q^3 - 3q + 2) & \text{(log-normal)} \\ \frac{1}{3}\frac{(q-1)(5q-6)}{3-q} & (F \text{ with } v = 4) \end{cases}$$

(non-universal)

Asymptotics of superstatistics (σE large)

What is the large E behaviour of general superstatistics?

$$B(E) = \int_0^\infty f(\beta) e^{-\beta E} d\beta$$
$$= \int_0^\infty e^{-\beta E + \ln f(\beta)} d\beta$$

saddle point approximation:

maximize

'free energy'

$$\text{'entropy'} \to \Phi(\beta, E) := -\beta E + \ln f(\beta)$$

maximum attained at β_E s.t.

$$E = (\ln f(\beta))' = \frac{f'(\beta)}{f(\beta)}$$

large E: relevant is behaviour of $f(\beta)$ for $\beta \to 0$

$$B(E) \sim \frac{f(\beta_E) e^{-\beta_E E}}{\sqrt{-(\ln f(\beta_E))''}}$$

H. Touchette & C. B., Phys. Rev. E **71**, 016131 (2005)

Legendre transform formalism for asymptotics of superstatistics

Example 1

power law behaviour

$$f(\beta) \sim \beta^{\gamma} \qquad (\gamma > 0) \qquad \beta \to 0$$

e.g. χ^2 superstatistics $\quad f(\beta) \sim \beta^{\frac{n}{2}-1} e^{-\frac{n}{2}\frac{\beta}{\beta_0}}$

or F " $\quad f(\beta) \sim \dfrac{\beta^{\frac{v}{2}-1}}{\left(1+\frac{vb}{w}\right)^{\frac{v+w}{2}}}$

$$-\beta_E E + \ln f(\beta_E) \sim -\gamma \ln E$$

$$(\ln f(\beta_E))'' \sim -E^2$$

$$\Rightarrow \quad B(E) \sim E^{-\gamma-1} \qquad E \to \infty$$

power-law Boltzmann factors

'universal' relation

$$\gamma + 1 = \frac{1}{q-1}$$

↑ entropic index of nonextensive stat. mech.

power law in β implies power law in E, no matter what the rest of the distribution looks like.

Example 2

$$f(\beta) \sim e^{-c\beta^\delta} \qquad (c>0\,,\ \delta<0) \quad \beta \to 0$$

$$\beta_E = \left(\frac{E}{c|\delta|}\right)^{\frac{1}{\delta-1}}$$

$$(\ln f(\beta_E))'' \sim -\ E^{\frac{\delta-2}{\delta-1}}$$

$$-\beta_E E + \ln f(\beta_E) \sim E^{\frac{\delta}{\delta-1}}$$

$$\Rightarrow\ B(E) \sim E^{\frac{2-\delta}{2\delta-2}}\ e^{E^{\frac{\delta}{\delta-1}}} \qquad E \to \infty$$

stretched exponentials

important special case: $\delta = -1$

$$\Rightarrow\ B(E) \sim E^{-\frac{3}{4}}\ e^{\sqrt{E}}$$

If $E = \frac{1}{2}u^2$ is Kinetic energy we have exponential tails in $|u|$

Application to turbulent flows

(Eulerian high Re turbulence)

Experimentally measured:

velocity difference in the flow

$$u(t) = v_r(\vec{x}+\vec{r},t) - v_r(\vec{x},t)$$

component in $\vec{r}$-direction

of two points separated by distance $r = |\vec{r}|$

Simple dynamical model:

$$\dot{u} = -\gamma s |u|^{2\alpha-1} + \sigma L_\tau(t)$$

$s = \text{sign}(u)$

rapidly fluctuating 'chaotic' noise

$\beta := \frac{\gamma}{2\sigma^2}$ fluctuating as well (superstatistics)

basic idea:

velocity differences relax and are driven rapidly by fluctuating forces

Dynamical model generating Tsallis statistics of observed type

$$p(u) \sim \left(1 + \langle\beta\rangle (q-1) E(u)\right)^{-\frac{1}{q-1}}$$

$$E(u) = \frac{1}{2}|u|^{2\alpha} - c\sqrt{\gamma\tau}\, \mathrm{sign}(u)\left(|u|^{\alpha} - \frac{1}{3}|u|^{3\alpha}\right) + \ldots$$

$p(u) = \int d\beta\, f(\beta)\, p(u,\beta)$ marginal distr.

$$\dot{u} = -\gamma\, \mathrm{sign}(u)\, |u|^{2\alpha-1} + \sigma L_\tau(t)$$

$$\beta := \frac{\gamma}{2\sigma^2} \neq \text{const}$$

χ^2 distributed $(T \gg \tau)$

$$f(\beta) \sim \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{\beta}{(q-1)\beta_0}\right\} \quad \text{prob. density}$$

$$\beta_0 = \int_0^\infty \beta f(\beta)\, d\beta = \langle\beta\rangle$$

$$q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2} \qquad L_\tau(t) = (\gamma\tau)^{\frac{1}{2}} \sum_j^{\lfloor \frac{t}{\tau} \rfloor} \underset{\text{chaotic map } T}{x_j}\, \delta(t - j\tau)$$

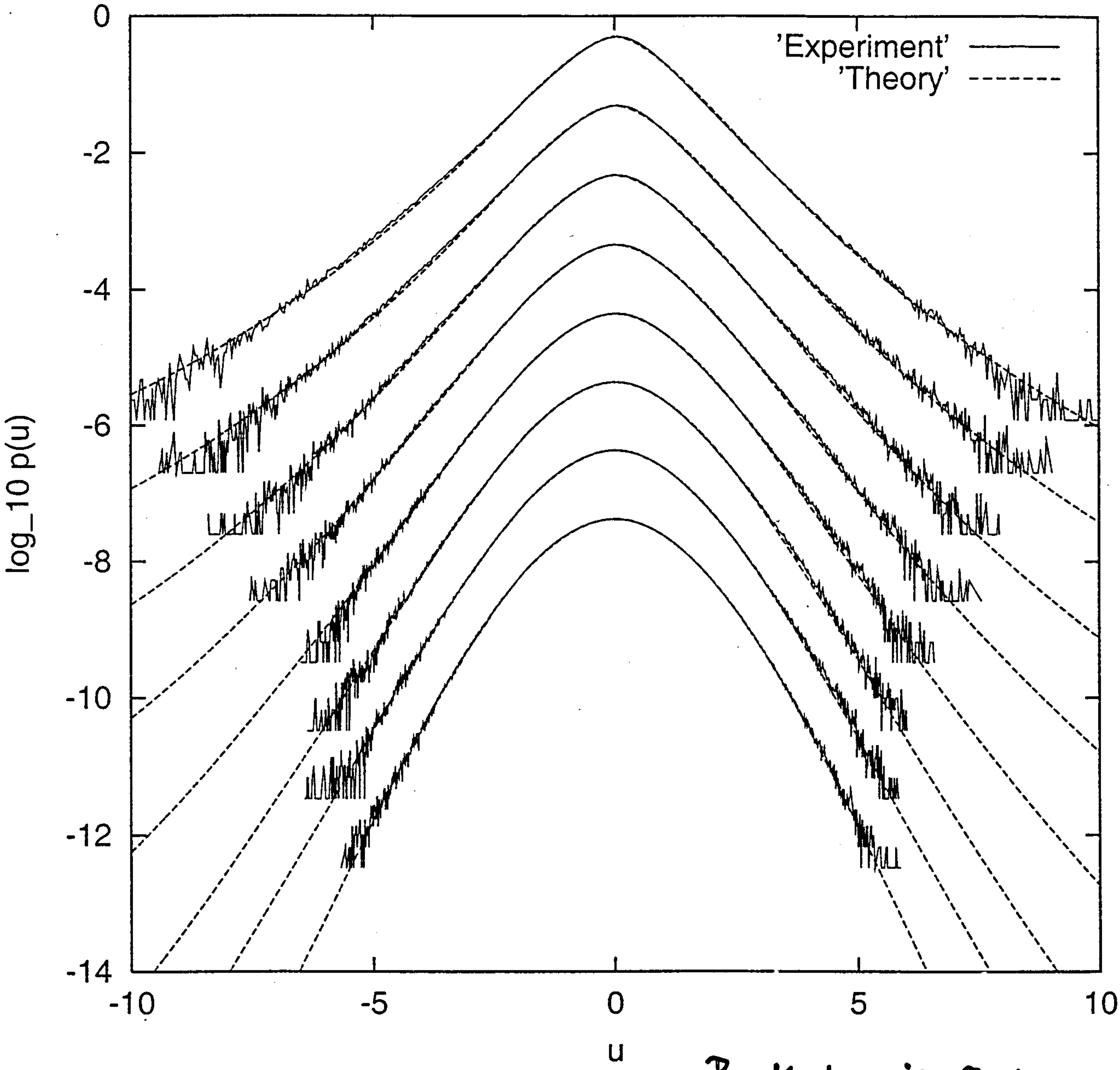
generalization of ordinary Brownian motion in various ways

ordinary	generalized
u: velocity	u: velocity difference
$\beta = \frac{\gamma}{2\sigma^2} = \text{const}$ (inverse temperature)	$\beta = \frac{\gamma}{2\sigma^2} = (\hat{\varepsilon}\, T_\eta)^\alpha$ fluctuating (energy diss. rate)
Ornstein-Uhlenbeck proc. $\alpha = 1$	turbulent process $\alpha \neq 1$ allowed
$L(t)$ Gaussian white noise	$L_\tau(t)$ chaotic noise
Gaussian stat. density extremizes Shannon entropy	Tsallis distribution extremizes Tsallis entropy
ordinary stat. mech.	nonextensive stat. mech.

Turbulent
Couette-Taylor flow (Lewis & Swinney)
(Eulerian turbulence)

$Re = 540\,000$

Fig. 1a



Beck, Lewis, Swinney
Phys. Rev. E (2001)
63E, 035303 (R)

from top to bottom:

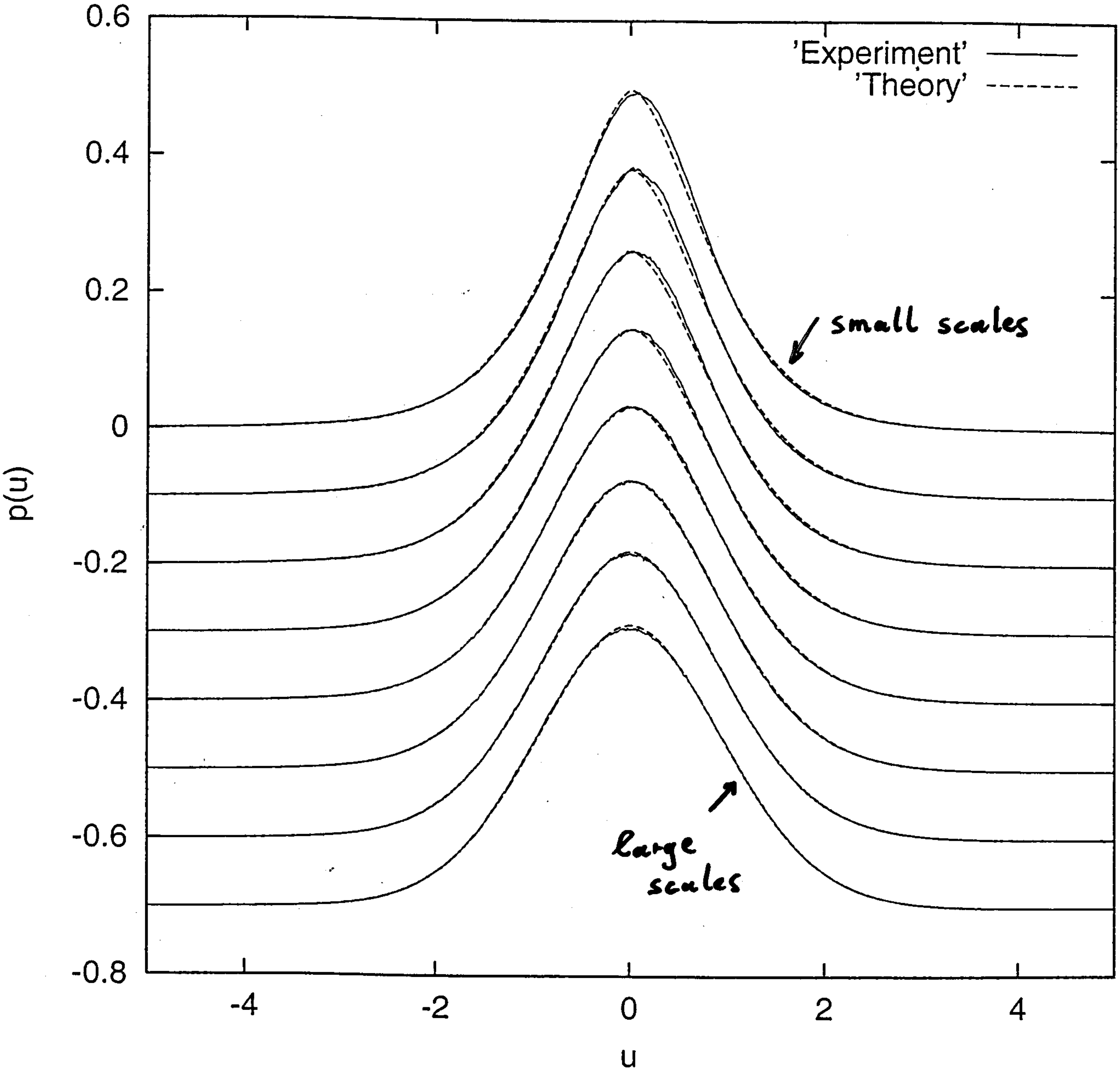
$\frac{r}{\eta} = 11.6,\ 23.1,\ 46.2,\ 92.5,\ 208,\ 399,\ 827,\ 14450$

$q = 1.168,\ 1.150,\ 1.124,\ 1.105,\ 1.084,\ 1.065,\ 1.055,\ 1.038$

$\alpha = 2 - q$

(shift by -1 unit for better visibility)

Fig. 1b



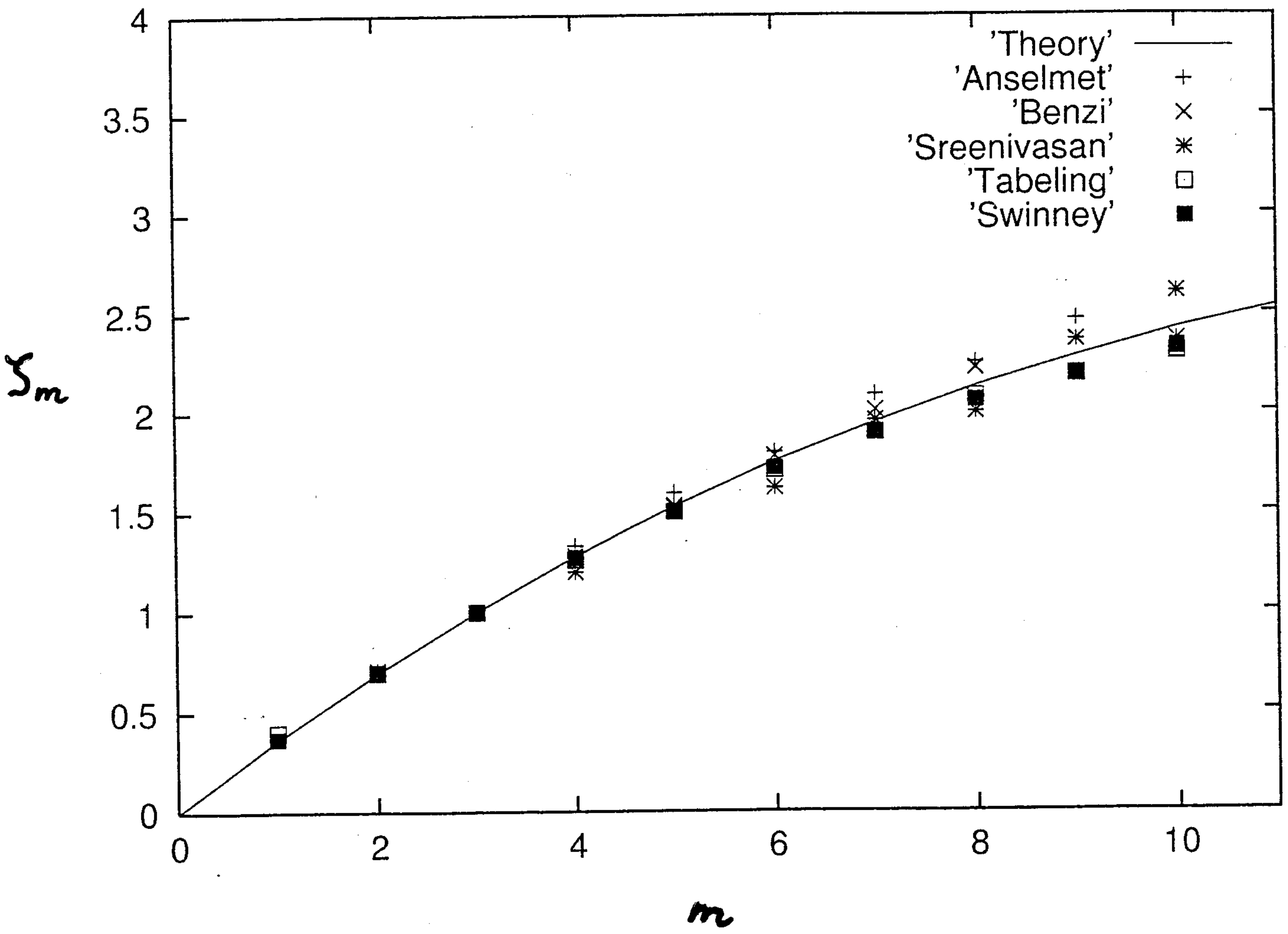
(shift by -0.1 units for better visibility)

Scaling exponents ζ_m

$$\langle |u|^m \rangle \sim r^{\zeta_m}$$

is obtained using Nonextensive Stat. Mech. $(\alpha = 2-q)$

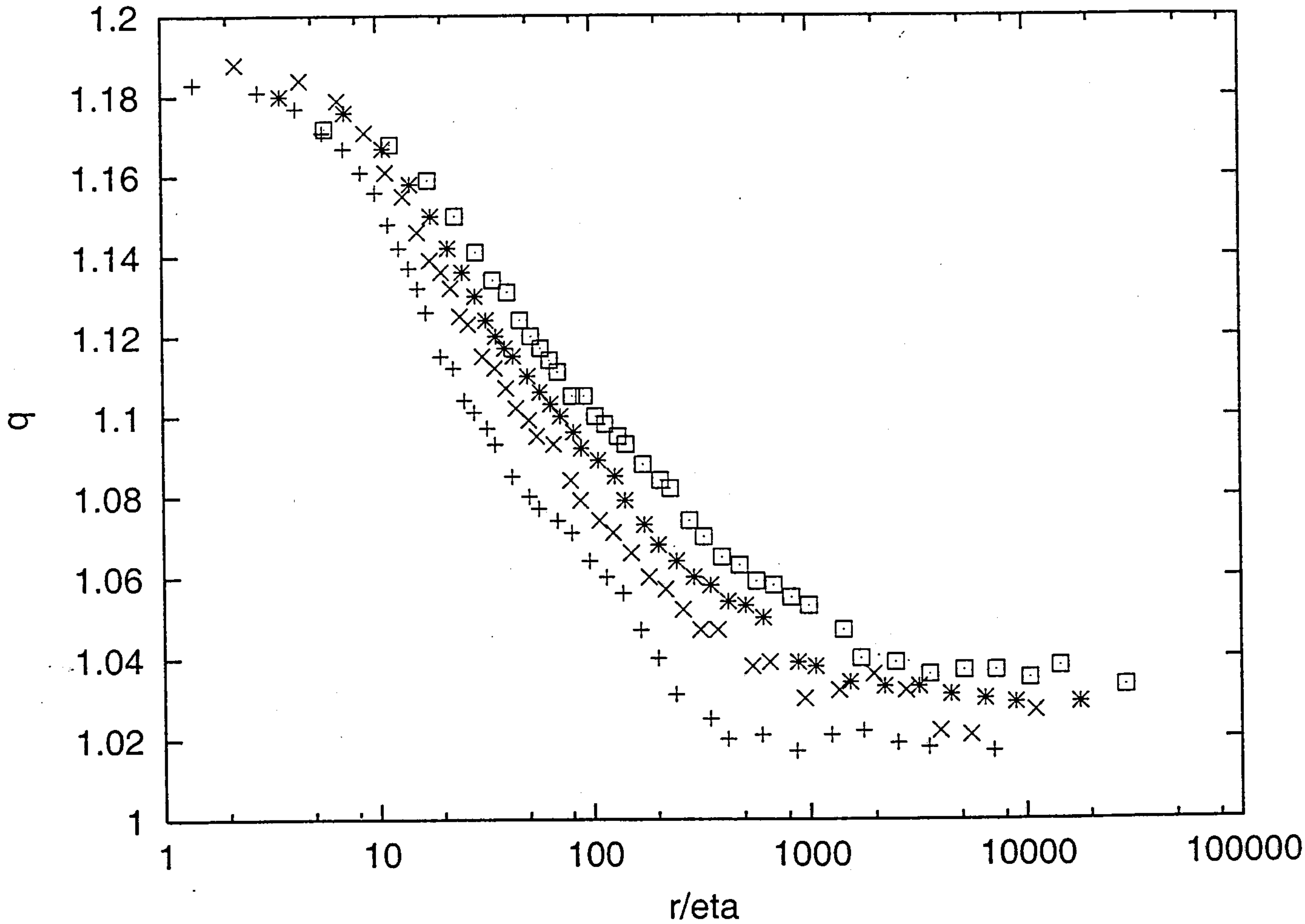
+ ESS (extended self similarity)



C. Beck, Physica 295 A, 195 (2001)

Experimentally measured curves $q(r, Re)$ in Lewis-Swinney experiment

Fig. 3



$Re = 69\,000$ +
$Re = 133\,000$ ×
$Re = 266\,000$ ∗
$Re = 540\,000$ □

Why does entropic index q change with scale r?

recall $\boxed{\mathrm{I} \mid \mathrm{II}}$

$$S_q(\mathrm{I}+\mathrm{II}) = S_q(\mathrm{I}) + S_q(\mathrm{II}) + (1-q)\, S_q(\mathrm{I})\, S_q(\mathrm{II})$$

'pseudo-additivity'

System could compensate correction term $(1-q)\, S_q(\mathrm{I})\, S_q(\mathrm{II})$ by changing $q \to q'$ on larger scale of system $\mathrm{I}+\mathrm{II}$

i.e.

$$S_{q'}(\mathrm{I}+\mathrm{II}) = S_q(\mathrm{I}) + S_q(\mathrm{II})$$

'quasi-additivity'

C.B., Europhys. Lett. 57, 329 (2002)

Quasi-additivity implies a power law

if $|q-1|$ is small

consistent with Lewis-Swinney data

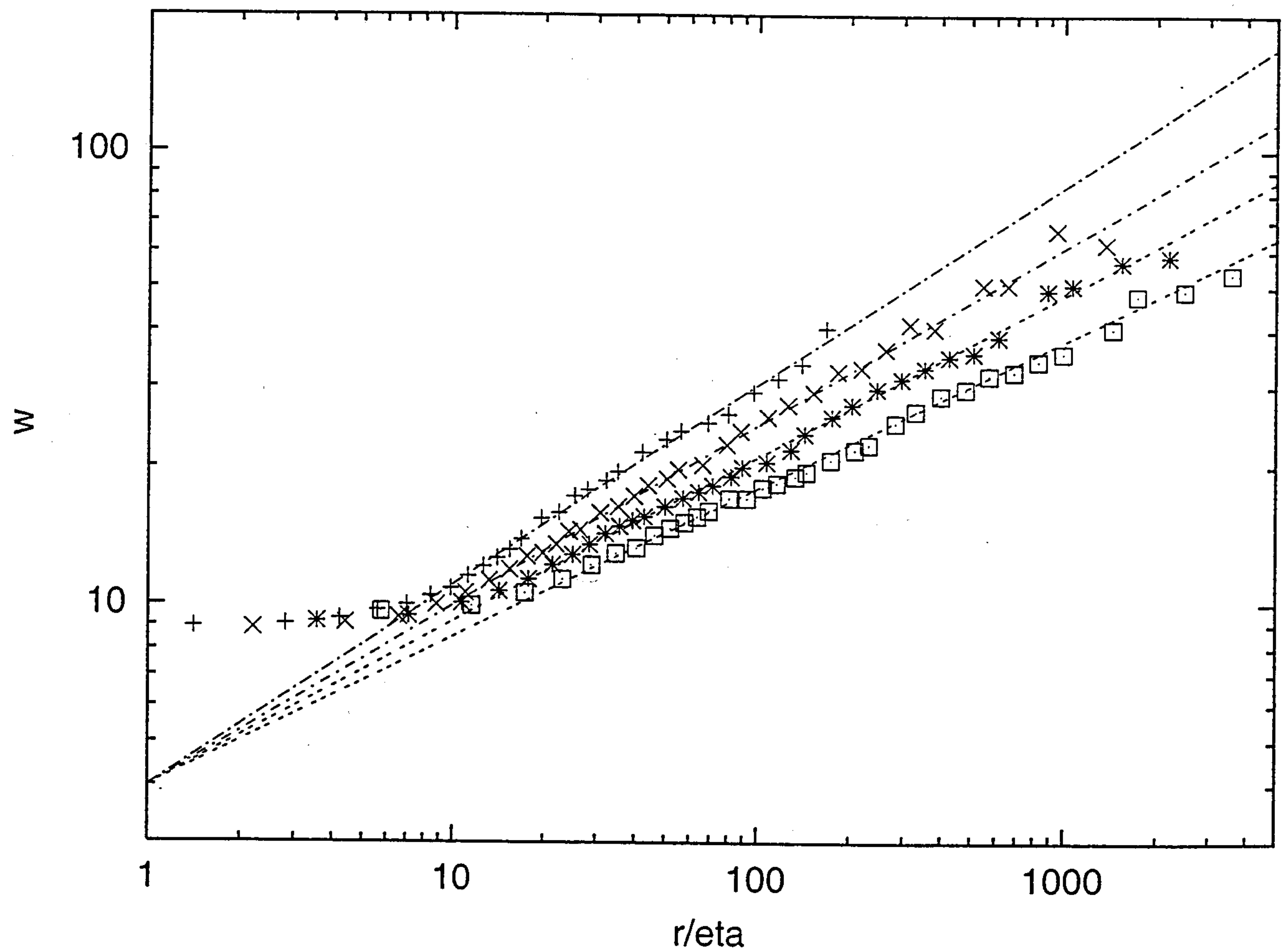
for large u $\qquad p(u) \sim |u|^{-w}$

$$w = \frac{2\alpha}{q-1} = \frac{4-2q}{q-1}$$

'physical' meaning of w:

only moments $\langle |u|^m \rangle$ with $m < w-1$ exist

Fig. 4



experimentally observed: $\quad w(r) = 4 \left(\frac{r}{\eta}\right)^{\delta}$

$\delta = 0.440,\ 0.395,\ 0.360,\ 0.326 \quad (Re = 69K,\ 133K,\ 266K,\ 540K)$

Patterns in turbulent flows for yet another experiment

(rotating annulus)

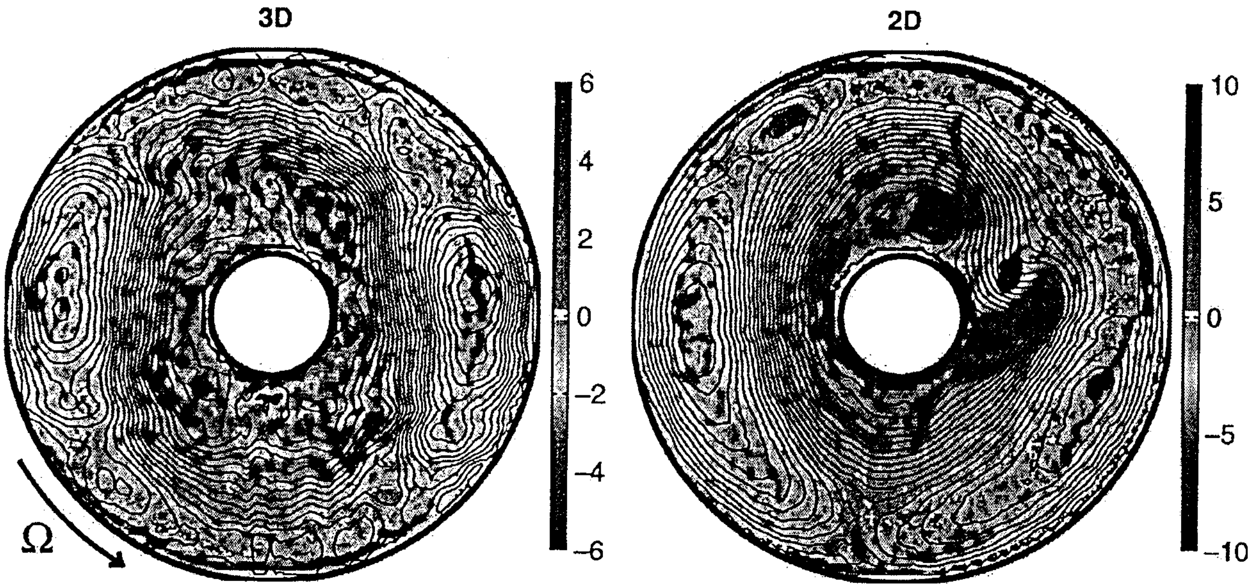


Fig. 1. Vorticity and streamfunction maps for the 3D and 2D flows, at $\Omega = 1.57$ and 11.0 rad/s, respectively. The cyclonic (red cente anti-cyclonic (blue center) vortices are advected clockwise by the mean anti-cyclonic jet, as the tank rotates counter-clockwise. The sp of the streamline contours is $12\,\mathrm{cm}^2/\mathrm{s}$ for the 3D case and $30\,\mathrm{cm}^2/\mathrm{s}$ for the 2D case, and the color bars show the vorticity values (

from:

C.N. Baroud, H.L. Swinney, Physica 184 D, 21 (2003)

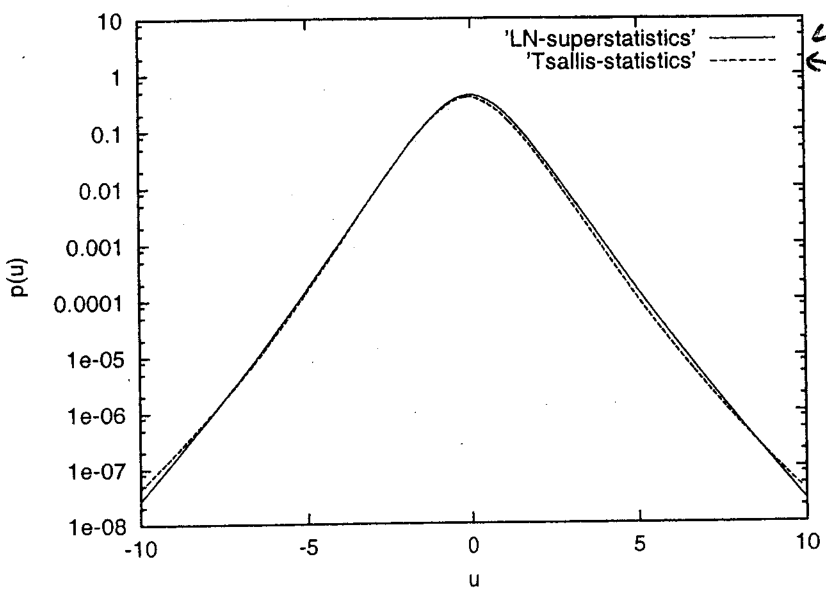
Probability densities of velocity differences well approximated by 'canonical' distributions of generalized stat. mech

$q \approx 1.3$ (2D)

q scale dep. (3D)

$\alpha = 2 - q$ not satisfied for this experiment!

Various superstatistics yield similar results if q-1 is small (as proved)



α = 1

α = 2-q

Big advantage of log-normal superstatistics: linear damping force (α=1) sufficient to fit data.

Fig. 3 Comparison between log-normal superstatistics as given by eq. (15) with $s^2 = 0.28$ and Tsallis statistics as given by eq. (13) with $q = 1.11$ and $\alpha = 2-q$. For the range of values accessible in the experiment, $|u| < 8$, there is no visible difference between the two curves.

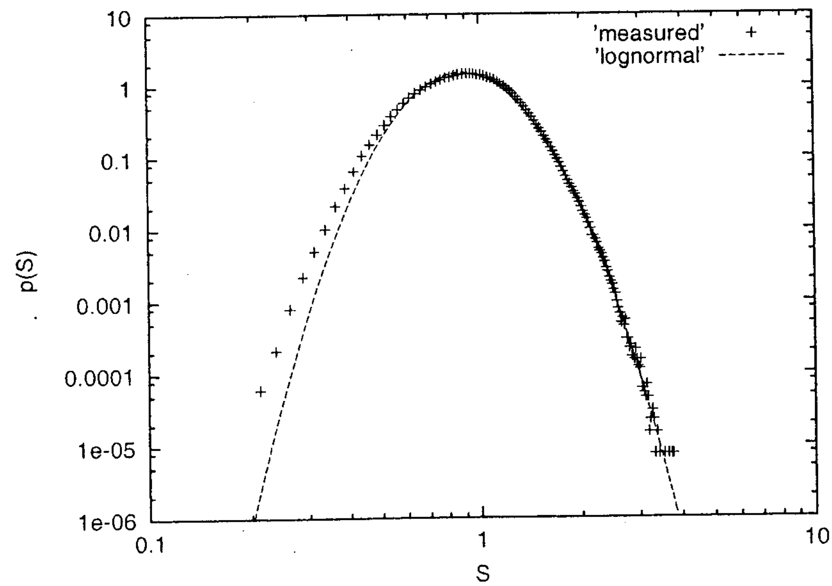
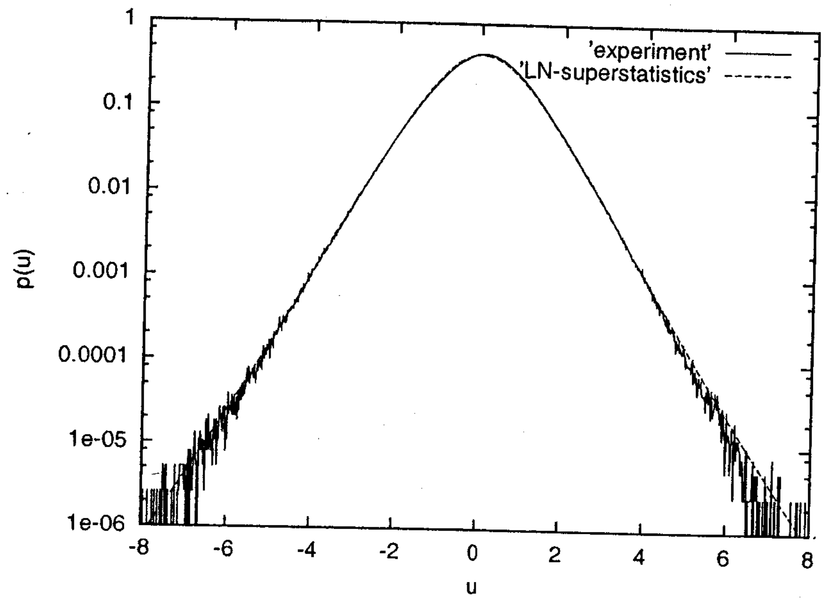


Fig. 4 Swinney's measurements of the shear stress distribution at the outer cylinder of the Taylor-Couette experiment, and comparison with a log-normal distribution.

Swinney et al — Taylor-Couette Flow
data from — C.B., G. Lewis, H. Swinney, PRE (2001)



C.B.
Physica D (2004)

Fig. 1 Histogram of velocity differences u as measured in Swinney's experiment and the log-normal superstatistics prediction eq. (15) with $s^2 = 0.28$.

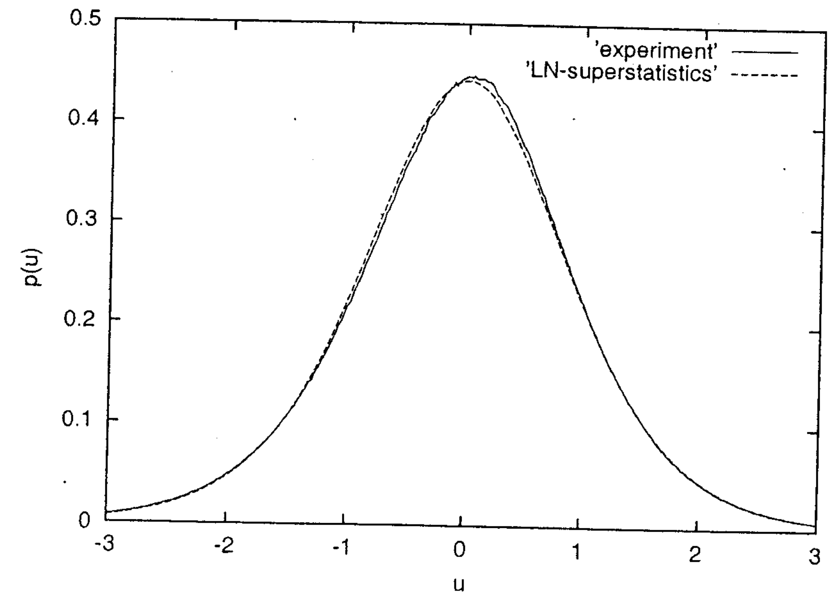


Fig. 2 Same as Fig. 1, but a linear scale is chosen. This emphasizes the vicinity of the maximum, rather than the tails.

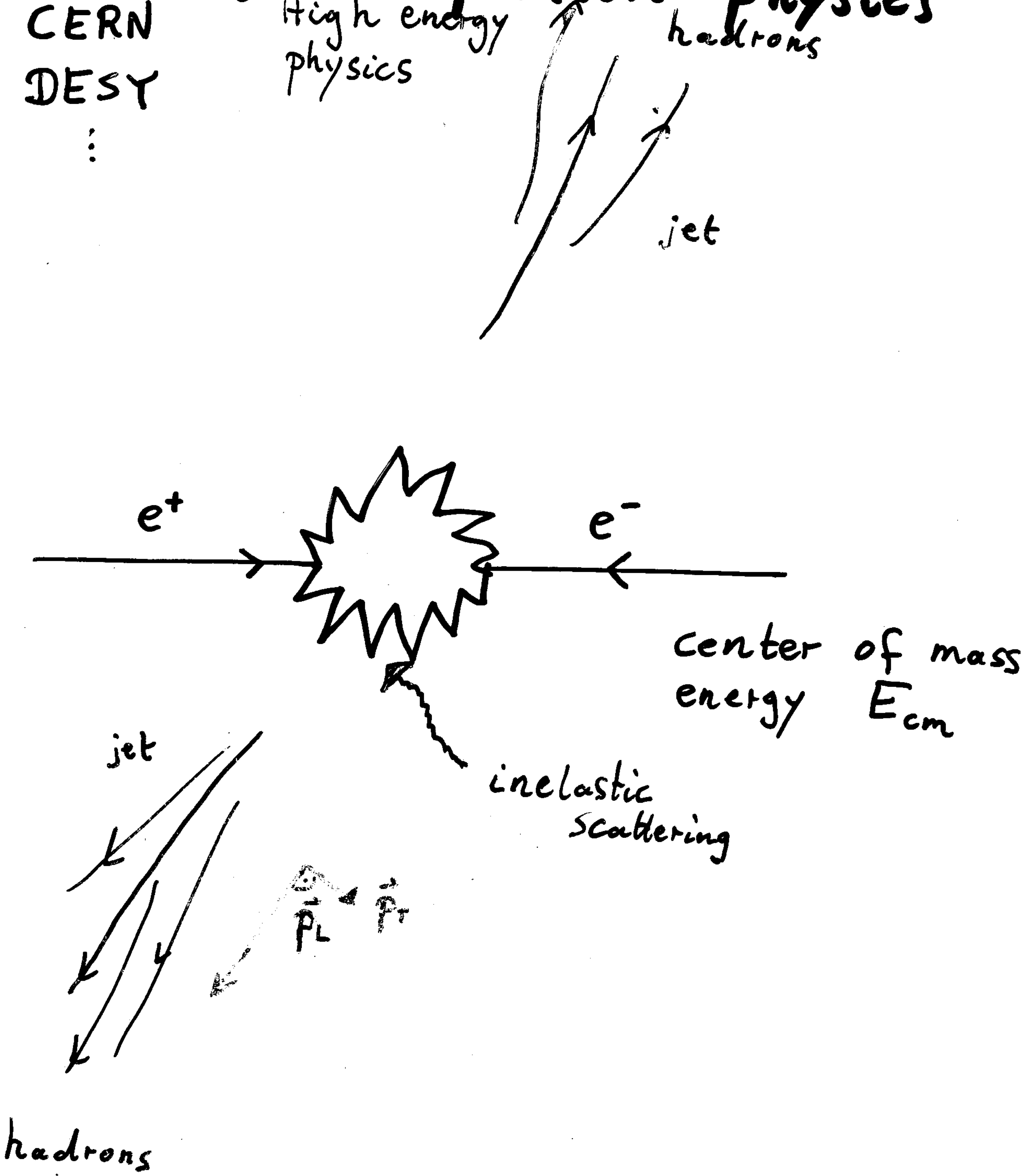
Applications in particle physics

CERN

DESY

⋮

High energy physics



differential cross section

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} \simeq \text{prob. density to observe particles with a given } p_T \times \text{number of these particles}$$

Hagedorn's theory

(R. Hagedorn, Nuovo Cim. Suppl. 3, 147 (1965))

A fireball is

a statistical equilibrium of an undetermined number of all kinds of fireballs, each of which in turn is considered to be

also of interest in superstring theory (Witten, ...)

Hagedorn phase transition

at Hagedorn temperature $T_0 \simeq 180$ MeV

like 'boiling nuclear matter'

Increasing E_{cm} to energies $\gg kT_0$ the Hagedorn temperature does not change, but all energy is put into new particle states.

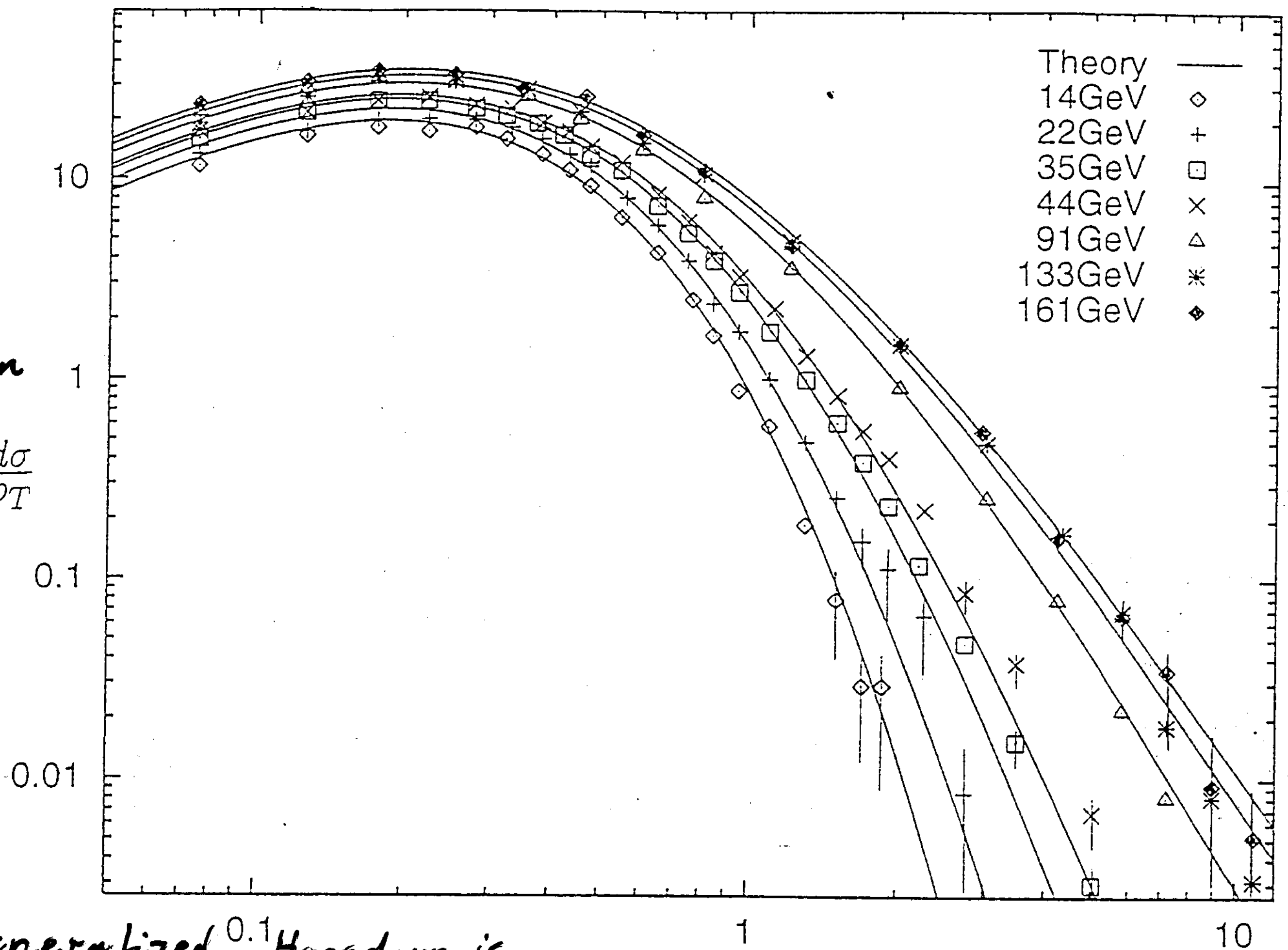
idea: q-generalize Hagedorn's theory

Effective volume
where reaction takes place
is very small !

$\Rightarrow$ Expect rather large
temperature fluctuations

If $\beta = \frac{1}{kT}$ is χ^2- distributed,
then nonextensive generalization
of Hagedorn theory makes sense

$\rightsquigarrow$ expect power law cross sections cross section



transverse momentum

<u>q-generalized Hagedorn is Another succesful application!</u>

momentum spectra of particles produced at high-energy collisions $e^+e^- \to$ hadrons at CERN (TASSO / DELPHI)

non-extensive generalization of Hagedorn theory

$$\rightsquigarrow \frac{d\sigma}{dp_T} \sim p_T^{\frac{3}{2}} \left(1 + (q-1)\beta p_T\right)^{-\frac{q}{q-1} + \frac{1}{2}}$$

inv. Hagedorn temp.

Bediaga, Curado, de Miranda, Physica <u>A 286</u>, 156 (2000)

C. B., Physica <u>286 A</u>, 164 (2000)

$$q(E_{cms}) = \frac{11 - e^{-\frac{E_{cms}}{E_0}}}{9 + e^{-\frac{E_{cms}}{E_0}}} \longrightarrow \frac{11}{9} = 1.222 \quad (E_{cms} \to \infty)$$

$E_0 \approx 1\, m$

C.B., Physica A (2004)

Model for <u>cosmic rays</u> (accelerators are supernovae!)

(based on χ^2 superstatistics)

$$p(E|\beta) = \frac{1}{Z(\beta)} E^2 e^{-\beta E}$$

$$E \approx c|\vec{p}|$$

$$Z(\beta) = \int_0^\infty E^2 e^{-\beta E} dE = \frac{2}{\beta^3}$$

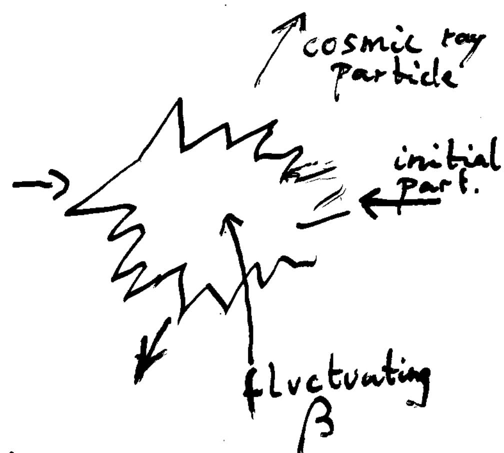
$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\}$$

$$\beta = \sum_{i=1}^n X_i^2$$

n degrees of freedom contributing to fluctuating β during production process of cosmic rays

$$p(E) = \int_0^\infty p(E|\beta) f(\beta) d\beta$$

$$= C \frac{E^2}{(1+\tilde{\beta}(q-1)E)^{\frac{1}{q-1}}}$$



where $q = 1 + \frac{2}{n+6}$

$$\tilde{\beta} = \frac{\beta_0}{4-3q}$$

inverse Hagedorn temperature

$$\frac{1}{c} E_{cms} \cdot r = \mathcal{O}(\hbar)$$

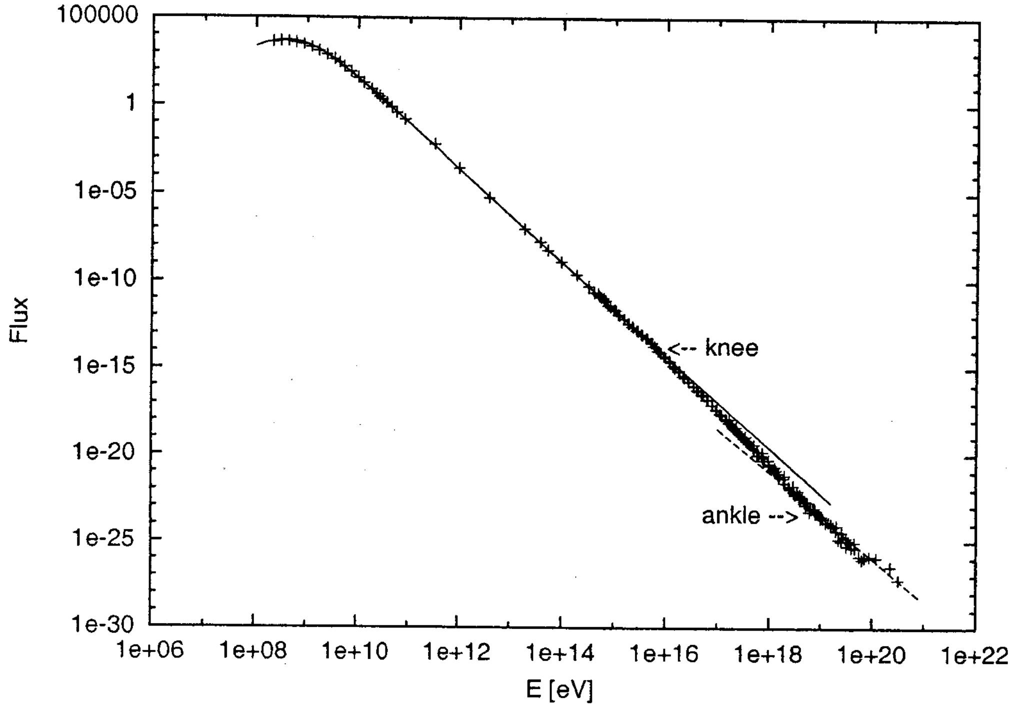
$\uparrow$ scale probed

$\Rightarrow$ Interaction volume r^3

At largest energies E_{cms}, r^3 is very small, heat can flow in 3 space directions

$$\Rightarrow n = 3 \Rightarrow q = \frac{11}{9} = 1.222$$

Measured energy spectrum of primary cosmic rays
Tsallis, Anjos, Borges, PLA 310, 372 (2003)
Kaniadakis, PRE 66, 056125 (2002)
C.B., Physica 331A, 173 (2004)



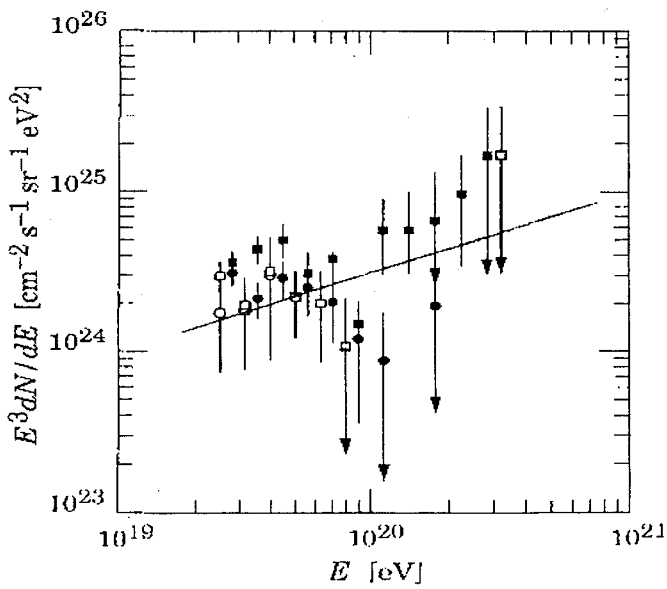


Fig. 2 Measured cosmic ray energy spectrum $E^3 \cdot dN/dE$ at largest energies (data from [19, 22, 23, 24]). The straight line is the power law prediction with exponent $\alpha = 5/2$ (corresponding to $q = 11/9$).