SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

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Dynamics and Statistical Mechanics of
(Hamiltonian) many body systems
with long-range interactions

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Group web page: [www.ct.infn.it/cactus](http://www.ct.infn.it/cactus)
Plan of the lectures

Part 1

- **Dynamics and Thermodynamics** of the Hamiltonian Mean Field (HMF) model: a simple model of fully-coupled particles on the unitary ring or classical spins, whose behavior seems to be paradigmatic for a large class of nonextensive systems: **Equilibrium case**

Part 2

- **Dynamical anomalies in the out-equilibrium regime** and the role of initial conditions:
  - Quasi-Stationary States & Negative specific heat
  - Chaos suppression & slow dynamics
  - Anomalous diffusion & Lévy walks
  - Non Gaussian velocity PDF, power law relaxation, aging
- Generalized HMF model: the $\alpha$-XY model
- Links to **Tsallis generalized statistics**
- Importance of initial conditions
- Interpretation of the QSS regime in terms of a **glassy phase**

Part 3

- **Non Hamiltonian systems**:
  - a) The Kuramoto model
  - b) Coupled map lattices with noise
  - c) Soc models for earthquakes dynamics
- Conclusions
4 Motivation

Why one should study long-range interacting systems...

- Long-range interactions are important for phase transitions in finite size systems, for example: fragmenting nuclei and atomic clusters.

- For understanding how one can treat statistically self-gravitating objects and plasmas.

- Long-range correlations are also frequently observed in out-of-equilibrium and complex systems.

In general long-range interactions pose fundamental problems to standard statistical mechanics, so one needs new statistical tools to treat them properly.
5 The HMF model

The Hamiltonian Mean Field (HMF) model

\begin{equation}
H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\vartheta_i - \vartheta_j)]
\end{equation}

Antoni and Ruffo PRE 52 (1995) 2361

• The system has an infinite range force

• It is a useful paradigmatic model to study Hamiltonian long-range interacting (nonextensive) systems as for example astrophysical systems, but also fragmenting nuclei and atomic clusters
Phase transition at equilibrium

The model can be seen as $N$ classical interacting spins or particles moving on the unit circle. One can define the total magnetization $M$ as an order parameter where the single spin is $\vec{m}_i = (\cos \vartheta_i, \sin \vartheta_i)$.

The model shows a second-order phase transition, passing from a clustered phase to a homogeneous one as a function of energy.

- $M=1$: clustered phase for $U<U_c$
- $M=0$: homogeneous phase for $U>U_c$
Equilibrium solution

By using the saddle point method, one gets for the free energy $F$

$$- \beta F = \frac{1}{2} \log \left( \frac{2\pi}{\beta} \right) - \frac{\beta}{2} + \max_y \left( -\frac{y^2}{2\beta} + \log \left( 2\pi I_0(y) \right) \right)$$

where $\beta = \frac{1}{k_B T}$ and $k_B = 1$ is the Boltzmann constant.

Then one gets the consistency equation

$$\frac{y}{\beta} - \frac{I_1}{I_0} = 0 \quad (1)$$

where $M = \frac{I_1}{I_0}$ and $I_i$ is the modified Bessel function of $i$th order.

Solving eq. (1) one gets the exact canonical equilibrium expression

$$U = \frac{E}{N} = \frac{\partial (\beta F)}{\partial \beta} = \frac{1}{2\beta} + \frac{1}{2} \left( 1 - M^2 \right)$$

**Caloric curve**
Critical behavior of the model

The model has a second order phase transition. The critical point is at

\[ U_c = \frac{3}{4} \quad \text{and} \quad T_c = \frac{1}{2} \]

Close to the critical point one gets for \( \beta \to \beta^+ \)

\[ M \approx \frac{4}{\sqrt{2}} \frac{1}{\beta} \left( 1 - \frac{1}{\beta} \right) \]

\[ U \approx \frac{1}{2\beta} \left[ 1 - \frac{8(\beta - 2)}{\beta} \right] + \frac{1}{2} \]

Hence \( M \) vanishes with the classical critical mean field exponent \( 1/2 \)

On the other hand, the specific heat

\[ C_V = \frac{\partial U}{\partial T} \]

\[ C_V(T_c) = \frac{5}{2} \quad \text{and} \quad C_V = \frac{1}{2} \quad \text{for} \quad T > T_c \]

Close to the critical point

\[ C_V \approx (T_c - T)^\alpha \quad \text{with} \quad \alpha = 0 \]
The model can be studied also dynamically by means of microcanonical numerical simulations.

One can compare dynamical aspects with thermodynamical ones.

One can study finite size effects.

One can also study how the system relax to equilibrium.
10 Dynamics

The equations of motion are:

\[
\begin{align*}
\frac{\partial \theta_i}{\partial t} &= p_i \\
\frac{\partial p_i}{\partial t} &= -M_x \sin \theta_i + M_y \cos \theta_i
\end{align*}
\]

The potential is connected to the magnetization \( \mathbf{M} \) as

\[
V = \frac{N}{2} \left[ 1 - \left( M_x^2 + M_y^2 \right) \right] = \frac{N}{2} \left( 1 - M^2 \right)
\]

The equations are solved numerically by using a fourth order simplectic algorithm (Yoshida, Physica A 150 (1990) 262). Energy is conserved with an error smaller than \( \frac{\Delta E}{E} = 10^{-5} \) for a number of time steps \( \approx 10^6 \).
Critical behavior of the model for finite sizes

Microcanonical simulations follow the canonical prediction, even for \( N=100 \)

\[ |\varepsilon| \sim |T-T_c|^{1/2} \]
Good agreement between exact canonical solution and numerical microcanonical simulations at equilibrium for various sizes $N$ of the system.
One finds a maximum of the largest Lyapunov exponent (LLE) in connection to the critical point, where both the fluctuations in kinetic energy and temperature and the specific heat present a peak.

Latora, Rapisarda and Ruffo
Physica D 131 (1999) 38
\[ \lambda_1 \propto N^{-1/3} \]

for \( U > U_c \)

\[ \lambda_1 \propto U^{1/2} \]

for \( U \ll U_c \)
In Hamiltonian systems with $N$ degrees of freedom

$$\lambda_i = -\lambda_{2N-i+1}$$

At low energy only a few degrees of freedom are active.

(positive part only)
No significant change in the shape of the spectra across the critical point is observed.
Kolmogorov Sinai entropy

\[ S_{KS} = \sum_{i=1}^{N} \lambda_i \quad \text{with} \quad \lambda_i > 0 \]

A peak close to the critical point is found also for \( S_{KS} \).
Scaling laws for the Kolmogorov Sinai entropy

\[ S_{KS} \propto U^{3/4} \quad U \ll U_c \]

\[ S_{KS} \propto N^{-1/5} \quad U > U_c \]
The HMF model can have also an antiferromagnetic behavior if one considers

\[ H = K - V \]

The general canonical solution for \( \pm V \) is

\[ U = \frac{1}{2\beta} + \frac{\varepsilon}{2} \left(1 - M^2\right) \]

with \( \varepsilon = \pm 1 \)
One has a different behavior of the Largest Lyapunov exponent and the KS entropy in the ferromagnetic and antiferromagnetic case.
In the thermodynamic limit, the LLE goes to zero for the whole energy range in the antiferromagnetic case, while it remains finite, for energies smaller than the critical one ($U_c=0.75$), in the ferromagnetic one. In the latter case it goes to zero for overcritical energies as $\lambda_1 \propto N^{-1/3}$.
Both the LLE and the KS entropy go to zero as in the antiferromagnetic case.

\[ \lambda_1 \propto \sqrt{U^*} \]

with \( U^* = U + \frac{1}{2} \)
In the continuum limit, considering the one-body distribution function $F$, the evolution of the HMF model is described by the Vlasov equation

$$\frac{\partial F}{\partial t} + p \frac{\partial F}{\partial \vartheta} - \frac{\partial V}{\partial \vartheta} \frac{\partial F}{\partial p} = 0$$

Supposing a factorization of the distribution function $F = f(p)g(\vartheta, t)$, one gets the stationary equilibrium solution

$$f = f_0 \frac{1}{\sqrt{2\pi T}} e^{-\frac{p^2}{2T}}$$

$$g = g_0 e^{\frac{M \cos(\vartheta - \phi)}{T}}$$

where $g_0 = \frac{1}{2\pi I_0(M/T)}$, $\phi$ is the phase of $M$, and $I_0$ is the Bessel function.

In the overcritical region $M = 0 \Rightarrow g = \frac{1}{2\pi}$

In the low energy region $I_0(z) \approx \frac{e^z}{\sqrt{2\pi}} \left[1 + \frac{1}{8z} + \ldots\right] \Rightarrow g \approx \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{\vartheta^2}{2\sigma^2}}$

Latora, Rapisarda and Ruffo Physica D 131 (1999) 38
24 Comparison with numerical pdfs

At low energy
Comparison with numerical pdfs at equilibrium

At the critical point

Water bag i.c.

Gaussian equ. i.c.
HMF model in the out-of-equilibrium case
When the system is started with initial conditions very far from equilibrium.....

...... we observe many dynamical anomalies, in particular in an energy range below the critical point.
In a region before the critical point the specific heat becomes negative: 
the temperature decreases, by increasing the energy density.

This phenomenon has been observed in multifragmentation nuclear reactions and atomic clusters, but also in self-gravitating stellar objects, i.e. for nonextensive systems.

See for example:

- Lynden-Bell, Physica A 263 (1999) 293
29 Quasi Stationary States

The system lives for a very long time in a *metastable quasi stationary state* (QSS), whose temperature defined as

$$T = \frac{2\langle K \rangle}{N}$$

is smaller than the equilibrium one.

- The larger $N$, the longer the QSS lifetime.
- The temperature $T$ tends to $0.38$ for $N \rightarrow \infty$.
QSS regime is reached almost immediately

Time evolution from WBIC --> QSS

$U=0.69 \quad WBIC$

$2\langle K \rangle/N$

$T_{\text{cut}}=0.476$

- $N=500$
- $N=1000$
- $N=5000$
- $N=10000$

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QSS do not depend on the accuracy of the integration
32 QSS lifetime and Temperature

Notice that being

\[ M^2 = T + 1 - 2U = T - 0.38 \]

one has

\[ M^2 \propto N^{-1/3} \]
The majority of the results refer to the case $U=0.69$, where anomalies are most evident.
In the QSS regime the force $F_i$ on the $i$th spin goes to zero with the size $N$ being

$$F_i = -M_x \sin \theta_i + M_y \cos \theta_i$$

Thus the QSS dynamics is slower and slower by increasing $N$. For $N \rightarrow \infty$ the system remains frozen in the QSS regime.
In the QSS regime the largest Lyapunov exponent tends to zero as the size of the system tends to infinity.

\[ \lambda_{QSS} \sim N^{-1/9} \]

This scaling can be obtained considering that

\[ \lambda \propto M^{2/3} \propto \left( N^{-1/3} \right)^{1/3} = N^{-1/9} \]

Latora, Rapisarda, Tsallis Physica A 305 (2002) 129
Simulations show that, going towards the thermodynamic limit, it is very crucial the time order of the size limit and time limit to infinity.

In general, the two limits do not commute:

\[ N \rightarrow \infty \quad t \rightarrow \infty \quad \neq \quad t \rightarrow \infty \quad N \rightarrow \infty \]

Boltzmann-Gibbs equilibrium

QSS
In general one has for the mean square displacement

$$\sigma^2(t) \propto t^\alpha$$

\[\alpha = 1\] Normal diffusion

\[\alpha \neq 1\] Anomalous diffusion

In our case we get superdiffusion with an exponent $\alpha = 1.38$ in correspondence of the QSS regime.
38 Anomalous diffusion: typical dynamics

Phase space dynamics of a typical particle (spin)

Lévy walks

Normal random walks

QSS

Equilibrium

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The cross-over times, from anomalous to normal diffusion, coincide with the relaxation times.
For a one dimensional system which shows sticking and flying particles with constant velocity, one finds:

\[
P_{\text{walk}}(t) \propto t^{-\mu}
\]

\[
P_{\text{trap}}(t) \propto t^{-\nu}
\]

with

\[\alpha = 2 + \nu - \mu\]

when

\[2 < \mu < 3\]

\[\nu < 2\]

See Klafter and Zumofen

PRE 49 (1994) 4873
41 Time evolution of velocity PDFs

$U=0.69 \quad N=1000$

Equilibrium regime

QSS regime

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Velocity PDFs in the QSS regime for different sizes $N$ of the system.
We have recently studied the nature of the anomalous QSS regime by starting from different initial magnetizations with \(0 \leq M \leq 1\) (considering in all cases a uniform distribution in momenta: *water bag*).

**Dynamical anomalies** depend on the initial conditions. The most interesting are those observed for initial magnetization \(M \sim 1\).
QSS for different initial conditions: M1 vs. M0

U=0.69, 1000 events

\[ \frac{2\langle K \rangle}{N} \]

- \( N=500 \)
- \( N=1000 \)

\( T_{eq} = 0.476 \)

\( T_e = 0.38 \)
Dependence of QSS on the initial magnetization

Structures in phase-space appear, in the QSS regime for M1IC, but not for M0IC.
Dynamical evolution for different initial conditions

TIME = 00

TIME = 00

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Dynamical evolution for different initial conditions

TIME = 10

velocity

M1 IC

TIME = 10

velocity

M0 IC

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Dynamical evolution for different initial conditions

**TIME = 20**

- **M1 IC**
  - Red dots: high vel.
  - Blue dots: low vel.

- **M0 IC**
  - Red dots: high vel.
  - Blue dots: low vel.
Dynamical evolution for different initial conditions

TIME = 30

M1 IC

TIME = 30

M0 IC

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Dynamical evolution for different initial conditions

**M1 IC**

**M0 IC**

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Dynamical evolution for different initial conditions

TIME = 50

M1 IC

TIME = 50

M0 IC
Dynamical evolution for different initial conditions

TIME = 100

M1 IC

TIME = 100

M0 IC

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Dynamical evolution for different initial conditions

TIME = 150

velocity

theta position

M1 IC

TIME = 150

velocity

theta position

M0 IC

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Dynamical evolution for different initial conditions

TIME = 200

M1 IC

M0 IC
Dynamical evolution for different initial conditions

TIME = 250

M1 IC

TIME = 250

M0 IC

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Dynamical evolution for different initial conditions

TIME = 300

velocity

theta position

M1 IC

TIME = 300

velocity

theta position

M0 IC

high vel.
low vel.
Dynamical evolution for different initial conditions

TIME = 400

M1 IC

TIME = 400

M0 IC

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Dynamical evolution for different initial conditions

TIME = 500

velocity

theta position

TIME = 500

velocity

theta position

M1 IC

M0 IC

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Dynamical evolution for different initial conditions

TIME = 600

velocity

theta position

M1 IC

TIME = 600

velocity

theta position

M0 IC

high vel.
low vel.

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Dynamical evolution for different initial conditions

TIME = 700

velocity vs. theta position

TIME = 700

velocity vs. theta position

M1 IC

M0 IC
Dynamical evolution for different initial conditions

TIME = 800

M1 IC

TIME = 800

M0 IC

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Dynamical evolution for different initial conditions

TIME = 900

-3 -2 -1 0 1 2 3
theta position

velocity

TIME = 900

-3 -2 -1 0 1 2 3
theta position

velocity

M1 IC

M0 IC
Dynamical evolution for different initial conditions

TIME = 1000

velocity

theta position

M1 IC

TIME = 1000

velocity

theta position

M0 IC

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Dependence of μ-space structures on the i.c.
Also the entropy and the free energy show a different dynamical evolution for the two initial conditions M1IC and M0IC.
One can study the process of relaxation to equilibrium by means of velocity correlation functions,

\[ C(t,0) = \frac{\langle P(t)P(0) \rangle - \langle P(t) \times P(0) \rangle}{\sigma(t)\sigma(0)} \]

where \( P(t) = (p_1, p_2, \ldots, p_N) \) is the velocity vector, the brackets indicate an average over the events, and \( \sigma(t) \) is the standard deviation at time \( t \).

Correlations show a power law decay for M1IC and an exponential decay for M0IC.
For M1 IC, the system, in going towards equilibrium, shows strong memory effects and aging, i.e. the correlation functions depend on $t$ and on the waiting time $t_w$

$$C(t+t_w,t_w) = \frac{\langle P(t+t_w)P(t_w) \rangle - \langle P(t+t_w) \rangle \langle P(t_w) \rangle}{\sigma(t+t_w)\sigma(t_w)}$$

Montemurro, Tamarit and Anteneodo PRE 67 (2003) 031106
Dynamical frustration

M=1 initial conditions:
- Competition between clusters of rotating particles in the QSS regime
- Each cluster tries to capture the maximum number of particles in order to reach the final equilibrium configuration:

This competition leads to a \textit{dynamical frustration} in the QSS regime!

M=0 initial conditions:
No competition and no frustration are present

At equilibrium only one big rotating cluster is present for both the IC
"Critical" cluster size distribution

The cumulative cluster size distribution is a power law for the QSS regime, which is robust for the entire plateau.
Cluster size distribution vs initial conditions

(a) $N=10000, U=0.69$

Sums over 20 realizations

Cumulative Number of Clusters vs Clusters Size

(time = 200)

Cumulative Number of Clusters $\propto N^{-1.6}$

(b) Different initial conditions $M$:

- $M=1$ ic
- $M=0.98$ ic
- $M=0.95$ ic
- $M=0.9$ ic
- $M=0.8$ ic
- $M=0.4$ ic
- $M=0$ ic

(time = 500)

Cumulative Number of Clusters $\propto N^{-1.6}$
Cluster formation vs initial conditions

M1 ic

M0 ic

$t=200$
Cluster formation vs initial conditions

M095 ic

M08 ic
2nd lecture
The HMF model has been also generalized to study the dynamic and thermodynamic behavior as a function of the range of the interaction

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i \neq j}^{N} \frac{1 - \cos(\theta_i - \theta_j)}{r_{ij}^\alpha} \]

- Anteneodo and Tsallis, PRL 80 (1998) 5313
- Campa, Giansanti and Moroni, PRE 62 (2000) 303
- Tamarit and Anteneodo, PRL 84 (2000) 208

For $\alpha \leq d$ this generalized model reduces to HMF.

For $\alpha \to \infty$ one has interaction only among nearest neighbour spins.
\( \alpha \)-XY model and nonextensive effects

Anomalies depend in a crucial way on the range of the interaction

The lifetime \( \tau \) of the QSS does not diverge for all \( \alpha \ldots \)

see A. Campa et al. Physica A 305 (2002) 137

Decreasing the range of the interaction, i.e. diminishing nonextensivity \( \alpha > 0 \)
this anomalous behaviour disappears:

- Relaxation is very fast \( (\tau \propto e^{-\alpha}) \)
- No negative specific heat is observed
Another example which support the long-range nature of the interaction as the origin of negative specific heat is the 2-d Lennard-Jones gas with attractive potential

\[ V \propto r^{-\alpha} \]


Again decreasing the range of the interaction, i.e. diminishing non extensivity, this anomalous specific heat disappears. In correspondence they also find non-Boltzmann velocity distributions

Further similar examples have been found in self-gravitating systems, see Sota et al, PRE 64 (2001) 056133
More precisely the potential energy is

\[
V = C_\alpha \sum_{i < j}^N \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^\alpha \right]
\]

\[
C_\alpha = \varepsilon \left( \frac{b}{\alpha^\alpha} \right)^{\frac{1}{12-\alpha}} / N (12 - \alpha)
\]

\(\sigma\) is the diameter of the particles, \(b\) is a constant and \(\varepsilon\) is the energy scale.

For \(\alpha=6\) one gets the usual Lennard-Jones
The anomalous QSS regime is the effect of non-extensivity or, in other words, of the long-range character of the interaction.

These anomalies seem to be connected to Tsallis thermostatistics.

Very recently a link with a glassy phase has also been found.
In the last decade a lot of effort has been devoted to understand if thermodynamics can be generalized to nonequilibrium complex systems.

In particular one of these attempts is that one started by Constantino Tsallis with his seminal paper *J. Stat. Phys. 52 (1988) 479*.

For recent reviews see for example:

For a regularly updated bibliography: [http://tsallis.cat.cbpf.br/biblio.htm](http://tsallis.cat.cbpf.br/biblio.htm)
Tsallis conjecture for nonextensive systems

C. Tsallis, Braz. Jour. of Phys. 29 (1999) 1

Figure 4: Central conjecture of the present work, assuming a Hamiltonian system which includes two-body (attractive) interactions which, at long distances, decay as $r^{-3}$. The crossover at $t = \tau$ is expected to be slower than indicated in the figure (for space reasons).
82 Generalized velocity pdfs

\[ U = 0.69 \]

Scaling to the theoretical PDF

\[ 1 - (1 - q) \left( \frac{p^2}{2T} \right)^{\frac{1}{1-q}} \]

\( t = 1200 \)

\( N = 1000 \)

\( N = 10000 \)

\( N = 100000 \)

Tsallis \( q = 7 \) \( T = 0.38 \) with cutoff

\( \text{Gaussian} \)
The decay of the velocity correlation function can be reproduced very well by means of the generalized q-exponential

\[ Ae_q(x) = A[1 + (1 - q)x]^{1/q} \]

In our case \( x = -t/\tau \). Within a generalized Fokker-Plank equation which generates Tsallis q-exponential pdfs [1], one can extract the following relation between the exponent of the anomalous diffusion and \( q \)

\[ \gamma = \frac{2}{3 - q} \]

In our case \( \gamma = 1.38 - 1.4 \), thus we expect \( q = 1.55 - 1.6 \), which is confirmed by the fit in the figure for M1IC. On the other hand, for M0IC the decay is almost exponential.
q-exponential decay also for different i.c.
Also for the aging behavior, the power law decay of the correlation functions, after a proper rescaling, can be reproduced with a q-exponential function. In this case we get $q=1.65$. 

85 q-exponential decay for aging
64 q vs initial Magnetization

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Anomalous diffusion for different i.c.
Correlation decay vs range $\alpha$

(N=1000, $E=0.69$, transiente=100, 30 Ev., M1 i.c.)
Summary of most recent results regarding HMF model and q-statistics

Anomalous diffusion vs q-exponential decay

Rapisarda and Pluchino,
Europhysics News 36 (2005) 202

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90 Most recent results

Some unpublished results on recent criticism
Recent criticism

A few comments on recent criticism regarding anomalous diffusion in the HMF model and q-statistics
It seems that velocity pdfs can be described using Lynden-Bell entropy within a Vlasov approach. There are however several discrepancies due to dynamical effects and initial conditions.

Antoniazzi et al cond-mat/0601518

FIG. 2: Velocity distribution functions. Symbols refer to numerical simulations, while dashed solid lines stand for the theoretical profile. Panels a), b) and c) present the three cases $M_0 = 0.3$, $M_0 = 0.5$ and $M_0 = 0.7$ in lin-log scale, while panel d) shows the case $M_0 = 0.3$ in lin-lin scale. The numerical curves are computed from one single realization with $N = 10^7$ at time $t = 100$. Here $e = H/N = 0.69$. 

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It seems that velocity pdfs can be described using Lynden-Bell entropy within a Vlasov approach. There are however several discrepancies… due to dynamical effects and initial conditions.

Antoniazzi et al cond-mat/0601518
Latora, Rapisarda Tsallis
PRE 2001

\[ U = 0.69 \]

\[ P(p) \propto \left[ 1 - \left( \frac{1-q}{2T} \right) p^2 \right]^{1-q} \]
FIG. 3: The exponent $\gamma = \frac{d \log(\sigma^2)}{d \log(t)}$ is plotted as a function of the rescaled time $\tau = t/N$. Starting from the initial ballistic value 2, it converges to the normal diffusion exponent 1. Simulations refer to $M_0 = 0.3$, and $M_0 = 0.9$. Here $N = 10^5$ and $e = H/N = 0.69$. 

Antoniazzi et al
cond-mat/0601518
On the diffusive anomalies in a long-range Hamiltonian system

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We scrutinize the anomalies in diffusion observed in an extended long-range system of classical rotors, the HMF model. Under suitable preparation, the system falls into long-lived quasi-stationary states presenting super-diffusion of rotor phases. We investigate the diffusive motion of phases by monitoring the evolution of their probability density function for large system sizes. These densities are shown to be of the $q$-Gaussian form, $P(r) \propto (1 + (q-1)(x/R)^2)^{1/(q-1)}$, with parameter $q$ increasing with time before reaching a steady value $q \approx 3/2$. From this perspective, we also discuss the relaxation to equilibrium and show that diffusive motion in quasi-stationary trajectories strongly depends on system size.

PACS numbers: 05.20.-y, 05.60.Cd, 05.90.+m

I. INTRODUCTION

Systems with long-range interactions constitute a very appealing subject of research as they display a variety of dynamic and thermodynamic features very different from those of short-range systems treated in the textbooks (see [1] for a review on the subject). Moreover, in recent years, the study of long-range models have raised at energies close below $\varepsilon_c[5]$. In a QS state, the temperature (twice the specific mean kinetic energy) is almost constant in time and lower than the canonical value to which it eventually relaxes. However, the duration of QS states increases with the system size $N$, indicating that these states are indeed relevant in the $(N \to \infty)$ thermodynamical limit (TL).

Several other peculiar features have been found for
FIG. 1: Histograms of rotor phases at different instants of the dynamics (symbols). Simulations for $N = 1000$ were performed starting from regular water-bag initial conditions at $\varepsilon = 0.69$ (conditions leading to Q$S$ states). Countings were accumulated over 100 realizations, at times $t_k = 2^k$, with $k = 6, 8, \ldots, 14$, growing in the direction of the arrow up to $t = 16384$. Solid lines correspond to $q$-Gaussian fittings. Histograms were shifted for visualization. Inset: log-log representation of the fitted data.
Anomalous diffusion is a finite size effect?

Moyano and Anteneodo
cond-mat/0601518

FIG. 2: Averaged time series of (a) temperature $T$, (b) deviation $\sigma$, (c) diffusion exponent $\gamma$ and (d) parameter $q$, for $\varepsilon = 0.09$ and different values of $N$ ($N = 500 \times 2^k$, with $k = 0, \ldots, 9$). Bold lines correspond to $N = 500$, as reference, and $N$ increases in the direction of the arrows up to $N = 250000$. Averages were taken over $2.56 \times 10^6/N$ realizations, starting from a waterbag configuration at $t = 0$. In panel (d), the fitting error is approx. 0.05. Dotted lines are drawn as references. In (a), they correspond to temperatures at equilibrium ($T_{EQ} = 0.476$) and at QS states in the TL ($T_{EQ} = 0.38$). In (b), to ballistic motion ($\gamma = 2$) and normal diffusion ($\gamma = 1$).

FIG. 3: Averaged time series of (a) temperature $T$, (b) local exponent $\gamma$ and parameter $q$ (symbols), as a function of $t/N^{0.4}$. Data are the same presented in Fig. 2.

FIG. 4: Histograms of coarse phases at different instants of the dynamics (symbols). Simulations were performed for $N = 500$ and $\varepsilon = 0.09$, starting from an equilibrated initial condition. Counting was accumulated over 200 realizations, at times $t_k = 0.1 \times 4^k$, $k = 3, 4, \ldots, 10$, growing in the direction of the arrow up to $t = 1.05 \times 10^6$. The $q$-Gaussian function with $q = 1.5$ was plotted for comparison (solid line). Histograms were shifted for visualization.

Onò corresponds to normal diffusion, $\gamma < 1$ to sub-diffusion and super-diffusion occurs for $\gamma > 1$. The evolution of $\sigma$ is shown in Figs. 2b and 2b, for waterbag and equilibrium initial preparations, respectively. In order to detect different regimes, it is useful to obtain an instantaneous exponent $\gamma$ as a function of time by taking the logarithm in both sides of Eq. (6) and differentiating with respect to $\ln t$:
It is not completely clear if anomalous diffusion disappears at large \( N \) at the moment. However, initial conditions are extremely important even for large \( N \).

Tsallis, Rapisarda, Pluchino in preparation
Single realizations can have a very different dynamics. So a good statistics is needed even for large N.

Tsallis, Rapisarda, Pluchino
in preparation

A. RAPISARDA  ICTP 2006
Tsallis, Rapisarda, Pluchino
in preparation
Recently a dynamical transition as a function of the initial magnetization was claimed by Chavanis.
A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

Recently a dynamical transition as a function of the initial magnetization was claimed by Chavanis

See Chavanis
cond-mat/0604234

For an initial magnetization $M > 0.897$ (if $U = 0.69$)

The homogeneous Linden-Bell distribution becomes unstable and the system can be trapped into an incomplete mixed state, where Tsallis statistics is a possible explanation.

Fig. 10. Initial magnetization $b = M_0$ as a function of the degeneracy parameter $\mu$ for a given value of the energy. There exists a critical magnetization, corresponding to $\mu_{\text{crit}}(\epsilon)$, above which the homogeneous Lynden-Bell distribution is unstable.
A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

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Antoniazzi et al cond-mat/0601518
A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics.

Several numerical simulations confirm this transition, but further investigation is needed.

\[ U = 0.69 \quad N = 100000 \]

![Graph showing cumulative number of clusters vs. cluster size](image)
Collective Oscillation in a Hamiltonian System

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Oscillation of macroscopic variables is discovered in a metastable state of the Hamiltonian system of the mean-field XY model. The duration of the oscillation is divergent with the system size. This long-lasting periodic or quasiperiodic collective motion appears through Hopf bifurcation, which is a typical route in low-dimensional dissipative dynamical systems. The origin of the oscillation is explained, with a self-consistent analysis of the distribution function, as the self-organization of a self-excited swing state through the mean field. The universality of the phenomena is discussed.

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Dissipative systems often show periodic, quasiperiodic, and chaotic motion at a macroscopic level, when they are far away from equilibrium. The motion is described as low-dimensional dynamics, and its discovery has marked an epoch of nonlinear dynamics studies in physics. Recalling that the microscopic degrees of freedom involved are large, such macroscopic behavior is a result of collective motion that emerges out of high-dimensional microscopic dynamics. The collective motion, indeed, has been intensively and

Letter, the essence of this discovery is briefly reported, especially in the mean-field XY model [8].

We adopt the Hamiltonian system of the mean-field XY model, or globally coupled pendula [5,9,10],

\[ \mathcal{H} = -\sum_{i=1}^{N} \frac{p_{i}^{2}}{2} + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ 1 - \cos(\theta_{i} - \theta_{j}) \right] \]

(1)

All the N pendula interact with each other through phase
\[ f_0^0(\theta; M_0) = \frac{1}{Z_0(M_0)} \exp \left[ \frac{M_0}{T_{eq}(M_0)} \cos \theta \right] \] (5)

where \( Z_0(M_0) \) is the normalization. Next, the distribution function is

where we note that another metastable state in this model has been intensively investigated for a decade, especially by taking a rectangular (water bag) initial momentum distribution \([5,6]\). This metastable state exists only in the region just below the critical energy of the phase transition, and there \( M(t) \) and \( T(t) \) take smaller values than those in equilibrium, leading to a branch of negative specific heat. This state is regarded as a reflection of a stable stationary solution of the corresponding Vlasov equation. On the other hand, the metastable state that we have discovered takes larger values of \( M(t) \) and \( T(t) \) than those in equilibrium and exists over a much broader region than the negative specific heat branch. Thus the present metastable state is not explained by the above stationary solution of the Vlasov equation and is a novel one.

Increasing total energy, the temporal pattern of the macroscopic variable changes as stationary \( \rightarrow \) periodic \( \rightarrow \) quasiperiodic. This is regarded as a “bifurcation” of the collective motion. Here we note the similarity to the typical bifurcation route in low-dimensional dissipative dynamical systems, fixed point \( \rightarrow \) limit cycle \( \rightarrow \) torus, through Hopf bifurcations. Hence it is suggested that the present bifurcation of the collective motion is described as that of low-dimensional dynamical systems, in particular, by Hopf bifurcations.

We next investigate the bifurcation in more detail. The mean amplitude of \( M(t) \) against \( U \) in the vicinity of the

**Fig. 1** (color). A time series of \( M(t) \). The abscissa axis is a log scale. The dotted line is the equilibrium value. Inset: The duration of the plateau \( t_{\text{plateau}} \) against \( N, U = 0.69 \) [6] and \( M_0 = 1 \).

**Fig. 2** (color). (a) A time series of \( M(t) \) in the metastable state. (b) The decay rate \( \gamma \) of the amplitude of the oscillation, where \( M_{\text{amp}}(t) = M_{\text{amp}}(t_0) - \gamma \log(t/t_0) \). \( U = 0.5 \) and \( M_0 = 0.9 \).
Microscopic dynamics can be very different according to the initial conditions (as for other fully-coupled systems, for example the Kuramoto model) and different theoretical approaches can be applied.

Most probably the Vlasov equation cannot explain all the anomalies found, which on the other hand, are found also in other models with long-range interactions.

Anomalies have a clear dynamical origin.

q-statistics provides a coherent scenario and at the same time does not excludes other interpretations.
Glassy phase in HMF

Ferromagnetic Phase: $M \neq 0$

Paramagnetic Phase: $M \cong 0$

Glassy Phase: $M \cong 0$

$M = \frac{1}{N} \left| \sum_{i=1}^{N} \vec{s}_i \right|$
One can introduce the “elementary polarization”:

\[ \langle \vec{s}_i \rangle = \frac{1}{\tau} \int_1^\tau \vec{s}_i(t) \, dt \]

i.e. the temporal average, over a time interval \( \tau \), of the successive positions of each elementary spin vector.

The modulus of the “elementary polarization” has to be furtherly averaged over the quenched disorder of the \( N \) spin configurations, to finally obtain the “polarization” \( p \):

\[ p = \frac{1}{N} \sum_{i=1}^N \left| \langle \vec{s}_i \rangle \right| \]
One can consider the polarization $p$ as a new order parameter in order to measure the freezing of the spins.
In the HMF model we can use the polarization $p$ in order to characterize the QSS regime as a Glassy Phase, related with the dynamical frustration between clusters:

<table>
<thead>
<tr>
<th>PHASE</th>
<th>$p$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ferromagnetic</td>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>Paramagnetic</td>
<td>$\approx 0$</td>
<td>$\approx 0$</td>
</tr>
<tr>
<td>Glassy phase</td>
<td>$\neq 0$</td>
<td>$\approx 0$</td>
</tr>
</tbody>
</table>

In the QSS regime the magnetization goes to zero with the system size as
\[ M \propto N^{-1/6} \]
while the polarization remains constant
\[ p \sim 0.24 \]
The polarization $p$ does not depend on the integration time interval inside the QSS regime.
The polarization $p$ and the magnetization $M$ coincide at equilibrium.
Polarization is very robust as a function of $N$ and time (within the QSS regime).
103  Polarization vs initial conditions

Polarization is very sensitive to the initial conditions: it goes rapidly to zero when the initial magnetization is less than 1.
68 Monte Carlo simulations


Minimization

Standard Metropolis

Modified Monte Carlo evolution

U=0.699
N=1000

T_{mp} = 0.476

T_{zero} = 0.36

A. RAPISARDA ICTP 2006
Let us start from the following effective spin-glass Hamiltonian

\[ H = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} \vec{s}_i \cdot \vec{s}_j \]  

Using the following distribution probability for the interaction \( J_{ij} \):

\[ p(J_{ij}) = \left(\frac{1}{\sqrt{2\pi}\sigma_j}\right)^{-1} e^{-\frac{(J_{ij} - J_0)^2}{2\sigma_j^2}} \]

with average and variance

\[ \overline{J_{ij}} = J_0 = \frac{1}{N}, \quad \overline{J_{ij}^2} - \overline{J_{ij}}^2 = \sigma_j^2 = \frac{1}{N} \]

This distribution implies a random coupling among spins with a probability on average \( J_0 = 1/N \) which simulates a glassy behavior similar to what we observe in the QSS regime.

In the thermodynamic limit we have no interaction and the system remains frozen for ever.

In the thermodynamic limit the Hamiltonian (1) reduces to the potential part of the HMF Hamiltonian. One can treat the kinetic part considering a heat bath with \( T = 2K/N \)

**General HMF Hamiltonian**

\[ H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} J_{ij} \vec{s}_i \cdot \vec{s}_j \]
Comparison with numerical data

By applying the replica method, after some standard calculations one can extract from the Hamiltonian (1), the following self-consistent equation for the spin-glass order parameter

\[ p_{SG} = 1 - \frac{2}{\sqrt{p}} \beta^{-1} \int_0^\infty dr \exp \left( - \frac{r^2}{2} \right) I_1 \left[ \beta r \sqrt{p/2} \right] I_0 \left[ \beta r \sqrt{p/2} \right] \]

Which can be compared with numerical molecular dynamics calculation of the polarization in the thermodynamic limit of the QSS regime considering

\[ T = T(N, \infty) = T_{QSS} \]

The comparison is good and gives further support to the connection between glassy systems and the QSS regime of the HMF model.
Schematic summary

Initial conditions $0 < M \leq 1$

*Violent relaxation*

Non-homogeneous QSS:
- Power-law decay of correlations
- Anomalous diffusion
- Fractal-like structures
- $q$-statistics
- Aging and glassy behaviour

Boltzmann-Gibbs equilibrium

Slow relaxation

Initial conditions $M = 0$

*No Violent relaxation*

Homogeneous QSS
- Vlasov stable states

Fast relaxation
Summary for the HMF model

• Summarizing, the **Hmf model** represents a **paradigmatic model** for nonextensive (long range interacting or finite ) systems, as for example self-gravitating objects, nuclear and atomic systems.

• Several **dynamical metastable anomalies** are present: negative specific heat, very slow dynamics, anomalous diffusion, power-law relaxation, fractal-like structures, vanishing of lyapunov exponents, ergodicity breaking and aging

• This **metastable anomalous behavior can become stable** if the infinite size limit is performed before the infinite time limit: **the two limits do not commute.**

• There are links to Tsallis generalized statistics.

• It is possible to treat the QSS regime in terms of a glassy phase.

• But one can find similarities with other non Hamiltonian systems....
Some references on the HMF model

✓ Latora, Rapisarda, Tsallis, Phys. Rev. E 64 (2001) 056134


For the generalized version of the HMF model see:
Non Hamiltonian systems

- The Kuramoto Model
- Coupled Logistic maps with noise
- Soc models for earthquakes dynamics
The Kuramoto model
The Kuramoto Model

(Kuramoto 1975)

Eqs. for the N coupled oscillators

\[ \dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \quad i = 1, \ldots, N \]

Order parameter

\[ r e^{i\Psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \]

mean field equation

\[ \dot{\theta}_i(t) = \omega_i + Kr \sin(\Psi - \theta_i) \]

Phase transition

![Phase transition graph](image)
Metastability in the Kuramoto Model

Also for the Kuramoto model metastability seems to diverge with the system size.
Coupled maps on a lattice
We consider a model of Coupled Logistic Maps on a lattice, i.e.

\[ f(x) = \text{sign}(x) \cdot \text{mod} \{ 2 [\alpha \cdot g_\mu(x) + (1 - \alpha) \cdot w] , 1 \} \]  

(1)

\[ x_{t+1} = f(x_t(i)) + \frac{\epsilon}{2} [ f(x_t(i+1)) - 2f(x_t(i)) + f(x_t(i-1))] \]  

(2)

\[ g_\mu = 1 - \mu x^2 \]

Here \( w \) is a white noise term controlled by the parameter \( \alpha \in [0,1] \)

See the poster by S. RIZZO
Phase diagram of CML

FRP = frozen random patterns
BD = brownian motion of defect
DT = defect turbulence
PCI = pattern competition
FDT = fully developed turbulence

We fix the control parameter at the edge of chaos

$1.8322 < \mu < 1.8323$
Dynamics of Coupled Maps on a lattice

Time evolution with no noise ($\alpha=1$)

$\mu=1.8322$

$\varepsilon=0.7$
104 Dynamics of Coupled Maps on a lattice

Time evolution with moderate noise ($\alpha=0.73$)
Dynamics of Coupled Maps on a lattice

Time evolution with strong noise ($\alpha=3$)
105 Autocorrelation for Coupled Maps on a lattice

Space autocorrelation for $\mu=1.8322$, 100 lattice sites, after 300 iterations

- $C(\delta_i, \alpha=1)$
- $C(\delta_i, \alpha=0.73)$
- $C(\delta_i, \alpha=0.30)$
- $e^{-\gamma t}$

$t=300$
One can consider the difference between two maps and study the fluctuations of this new variable. We considered for example

\[ u(t) = x(45) - x(55) \]

The time behavior of \( u(t) \) can be very intermittent and one can apply a superstatistic approach (more details in the poster by S. RIZZO).
And $w$ is a white noise term regulated by the parameter $2.7$
With no noise one gets pdfs very similar to the HMF case (black dots).

Adding some noise one gets q-gaussians which can be explained by applying the superstatistics approach proposed by Beck and Cohen.
The pdf of $\beta$, the inverse variance of $u(t)$, obeys a Lognormal distribution.

But a Gamma distribution is very close to it and represents a good approximation.
SOC model with long-range interactions for earthquakes dynamics
Conclusions

Summarizing, long-range interacting systems present several dynamical anomalies which pose severe problems to standard statistical mechanics and which can find a natural description in term of q-statistics.
“ALL THE TRUTHS PASS THROUGH THREE STAGES:
FIRST, THEY ARE CONSIDERED RIDICULOUS,
SECOND, THEY ARE VIOLENTLY ADVERSED,
THIRD, THEY ARE ACCEPTED AND CONSIDERED SELF-EVIDENT.”

A. SCHOPENHAUER