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SCHOOL and CONFERENCE on COMPLEX SYSTEMS and NONEXTENSIVE STATISTICAL MECHANICS

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Dynamics and Statistical Mechanics of (Hamiltonian) many body systems with long-range interactions

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CACTUS
Chaos And Complexity Theoretical University Study
Group
Catania

Group web page: ***www.ct.infn.it/cactus***

Plan of the lectures

1st lecture

Part 1

- Dynamics and Thermodynamics of the **Hamiltonian Mean Field (HMF) model**: a simple model of fully-coupled particles on the unitary ring or classical spins, whose behavior seems to be paradigmatic for a large class of nonextensive systems: **Equilibrium case**

Part 2

- Dynamical anomalies in the **out-equilibrium regime** and the role of initial conditions:
  - Quasi-Stationary States & Negative specific heat
  - Chaos suppression & slow dynamics
  - Anomalous diffusion & Lèvy walks
  - Non Gaussian velocity PDF, power law relaxation, aging

2nd lecture

- Generalized HMF model: the α-XY model
- Links to **Tsallis generalized statistics**
- Importance of initial conditions
- Interpretation of the QSS regime in terms of a **glassy phase**

Part 3

- **Non Hamiltonian systems**:
- a) The Kuramoto model
- b) Coupled map lattices with noise
- c) Soc models for earthquakes dynamics
- Conclusions

4 Motivation

Why one should study long-range interacting systems...

•Long-range interactions are important for phase transitions in finite size systems, for example: fragmenting nuclei and atomic clusters

•For understanding how one can treat statistically self-gravitating objects and plasmas.

•Long-range correlations are also frequently observed in out-of-equilibrium and complex systems

In general long-range interactions pose fundamental problems to standard statistical mechanics, so one needs new statistical tools to treat them properly

5 The HMF model

The Hamiltonian Mean Field (HMF) model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\vartheta_i - \vartheta_j)]$$

Antoni and Ruffo PRE 52 (1995) 2361

- The system has an infinite range force
- It is a useful paradigmatic model to study Hamiltonian long-range interacting (nonextensive) systems as for example astrophysical systems, but also fragmenting nuclei and atomic clusters

Phase transition at equilibrium

The model can be seen as N classical interacting spins or particles moving on the unit circle. One can define the total magnetization **M** as an **order parameter**

$$\vec{M} = \frac{1}{N}\sum_{i=1}^{N}\vec{m}_i$$

where the single spin is

$$\vec{m}_i = (\cos\vartheta_i, \sin\vartheta_i)$$

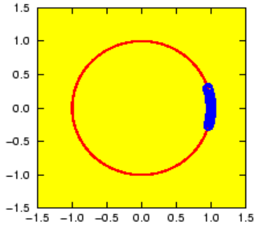
The model shows a ***second-order phase transition***, passing from a clustered phase to a homogeneous one as a function of energy

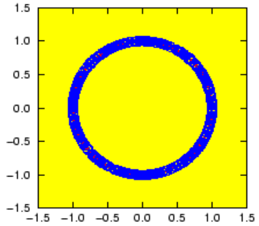
M=1

clustered phase for $U<U_c$

M=0

homogeneous phase for $U>U_c$





7 Equilibrium solution

By using the saddle point method, one gets for the free energy F

$$-\beta F = \frac{1}{2}\log\left(\frac{2\pi}{\beta}\right) - \frac{\beta}{2} + \max_y\left(\frac{-y^2}{2\beta} + \log\left(2\pi I_0(y)\right)\right)$$

where $\beta = \frac{1}{k_B T}$ and $k_B = 1$ is the Boltzmann constant

Then one gets the consistency equation $\frac{y}{\beta} - \frac{I_1}{I_0} = 0 \quad (1)$,

where $M = \frac{I_1}{I_0}$ and I_i is the modified Bessel function of *ith* order.

Solving eq.(1) one gets the **exact canonical equilibrium expression**

$$U = \frac{E}{N} = \frac{\partial(\beta F)}{\partial \beta} = \frac{1}{2\beta} + \frac{1}{2}\left(1 - M^2\right)$$

Caloric curve

8 Critical behavior of the model

The model has a second order phase transition.

The critical point is at $U_c = \frac{3}{4}$ and $T_c = \frac{1}{2}$

Close to the critical point one gets for $\beta \to \beta_c^+$

$$M \approx \frac{4}{\beta}\sqrt{\frac{1}{2} - \frac{1}{\beta}} \qquad U \approx \frac{1}{2\beta}\left[1 - \frac{8(\beta-2)}{\beta}\right] + \frac{1}{2}$$

Hence M vanishes with the classical critical mean field exponent 1/2

On the other hand, the specific heat $C_V = \frac{\partial(U)}{\partial T}$ is

$C_V(T_c) = \frac{5}{2}$ and $C_V = \frac{1}{2}$ for $T > T_c$

Close to the critical point $C_V \approx (T_c - T)^{-\alpha}$ with $\alpha = 0$

9 Dynamics vs Thermodynamics

- The model can be studied also dynamically by means of microcanonical numerical simulations.
- One can compare dynamical aspects with thermodynamical ones.
- One can study finite size effects.
- One can also study how the system relax to equilibrium

10 Dynamics

The equations of motion are:

$$\begin{cases} \dfrac{\partial \theta_i}{\partial t} = p_i \\ \dfrac{\partial p_i}{\partial t} = -M_x \sin\theta_i + M_y \cos\theta_i \end{cases}$$

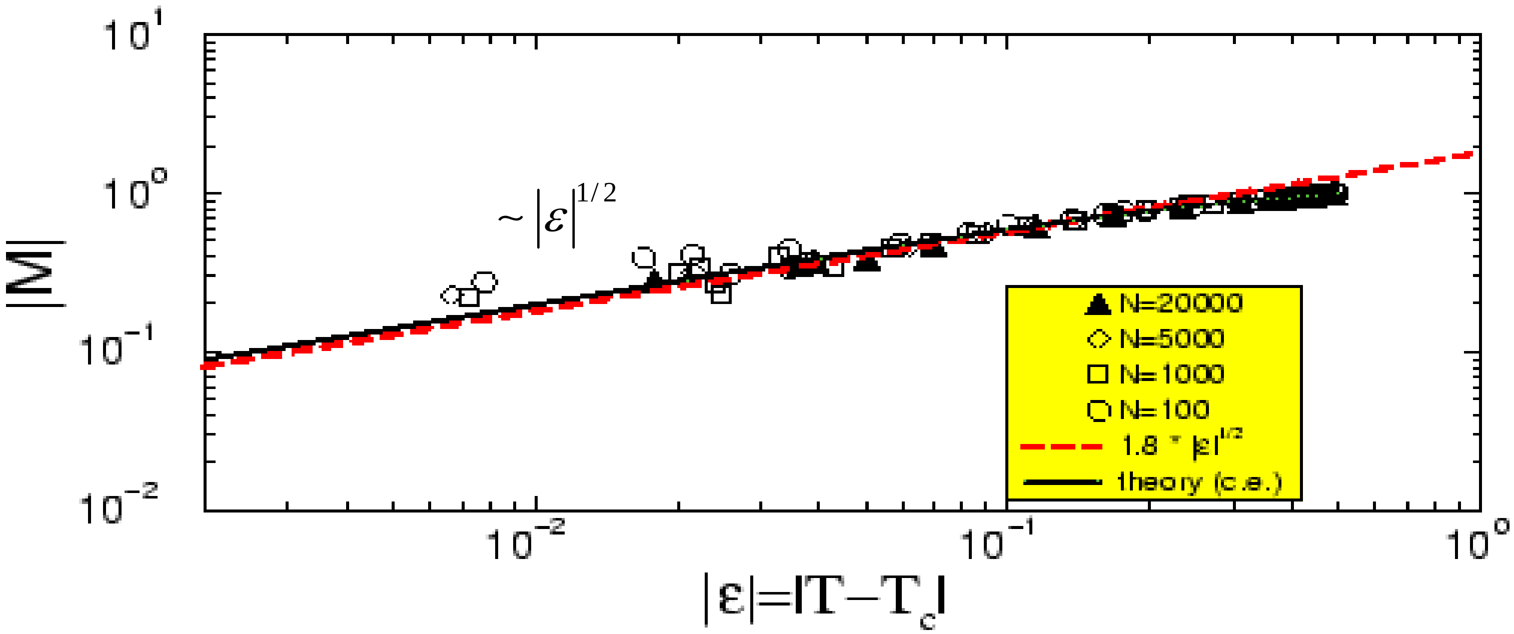
The potential is connected to the magnetization **M** as

$$V = \frac{N}{2}\left[1 - \left(M_x^2 + M_y^2\right)\right] = \frac{N}{2}\left(1 - M^2\right)$$

The equations are solved numerically by using a fourth order simplectic algorithm (Yoshida , Physica A **150** (1990) 262). Energy is conserved with an error smaller than $\frac{\Delta E}{E} = 10^{-5}$ for a number of time steps $\approx 10^6$

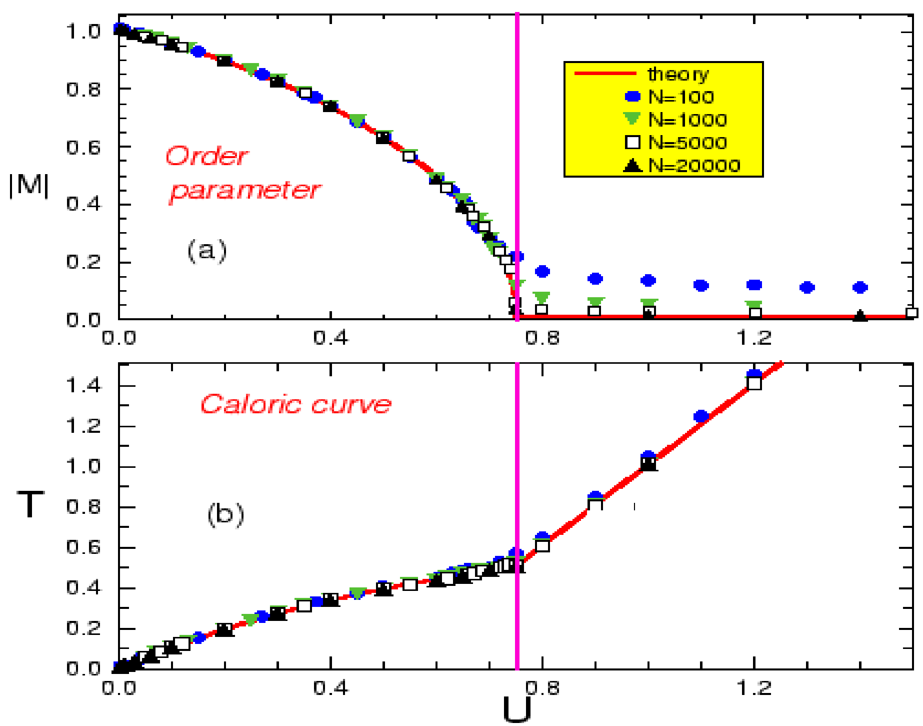
11 Critical behavior of the model for finite sizes

Microcanonical simulations follow the canonical prediction, even for N=100



12 Equilibrium

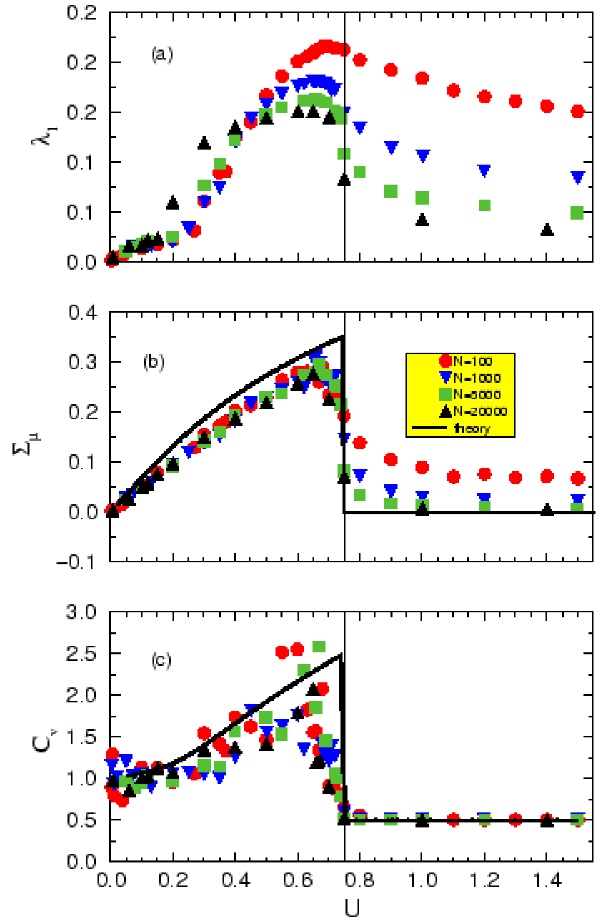
Good agreement between exact canonical solution and numerical microcanonical simulations at equilibrium for various sizes N of the system



Latora, Rapisarda and Ruffo PRL **80** (1998) 692,
Physica D **131** (1999) 38

13 Dynamics at Equilibrium

One finds a maximum of the largest Lyapunov exponent (LLE) in connection to the critical point, where both the fluctuations in kinetic energy and temperature and the specific heat present a peak



Latora, Rapisarda and Ruffo
Physica D **131** (1999) 38

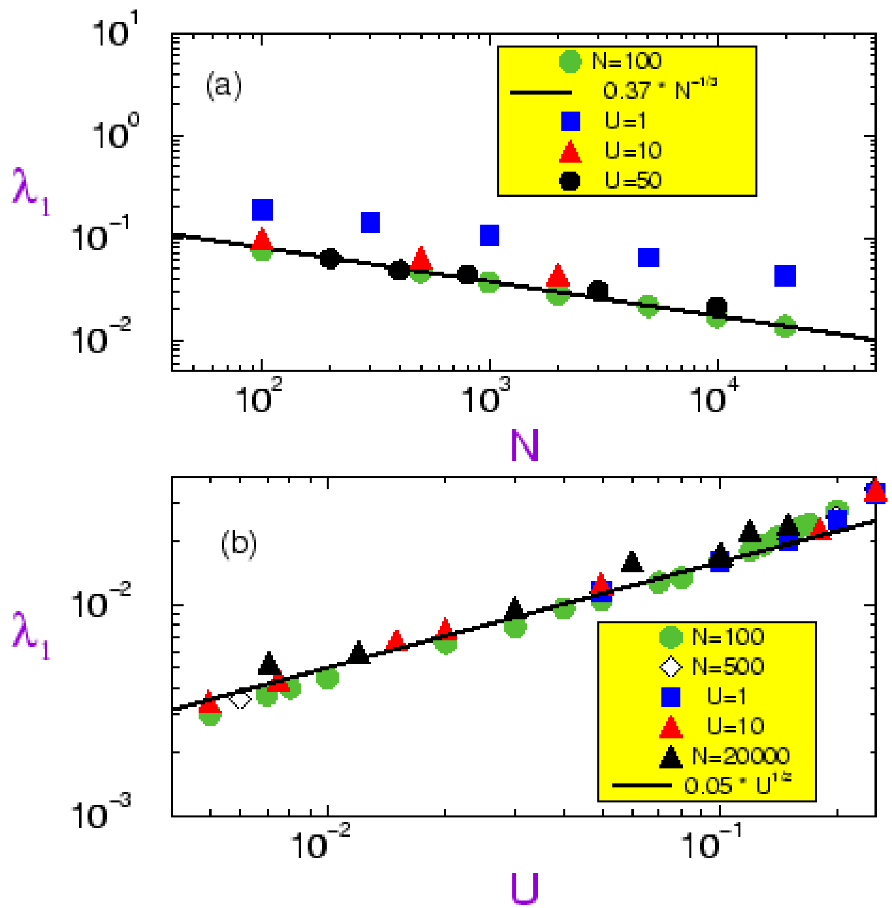
14 Dynamics at Equilibrium: scaling of the LLE

$$\lambda_1 \propto N^{-1/3}$$

for $U > U_c$

$$\lambda_1 \propto U^{1/2}$$

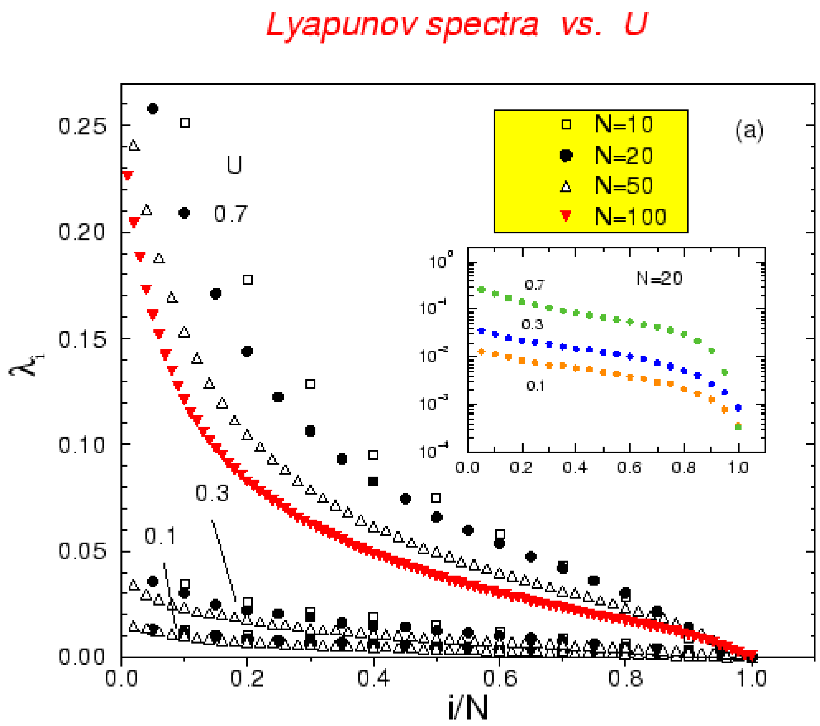
for $U \ll U_c$



15 Lyapunov spectra at Equilibrium

In Hamiltonian systems with N degrees of freedom

$$\lambda_i = -\lambda_{2N-i+1}$$



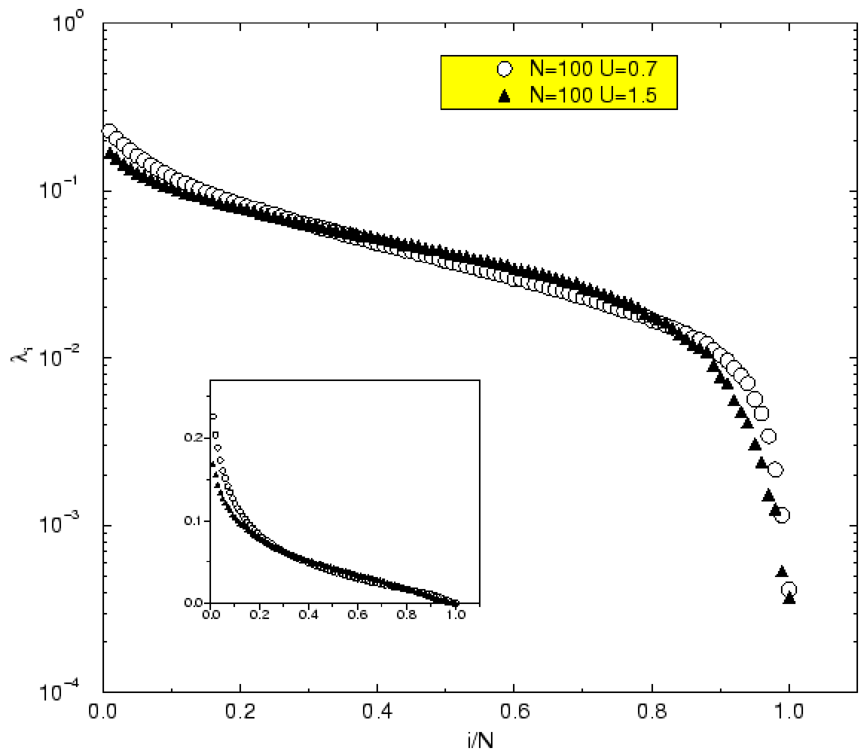
(positive part only)

At low energy only a few degrees of freedom are active.

16 Lyapunov spectra at Equilibrium

No significant change in the shape of the spectra across the critical point is observed

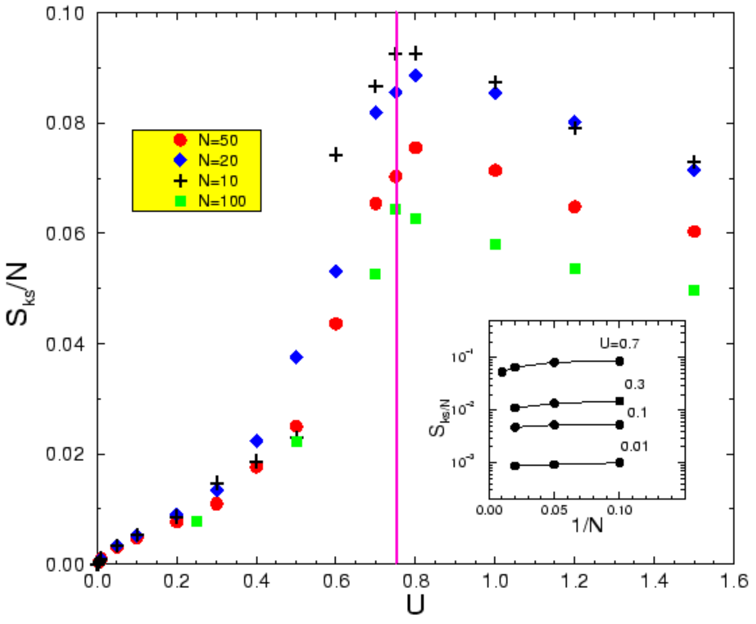
Lyapunov spectra below and above the critical point



17 Kolmogorov Sinai entropy

$$S_{KS} = \sum_{i=1}^{N} \lambda_i \qquad \text{with} \qquad \lambda_i > 0$$

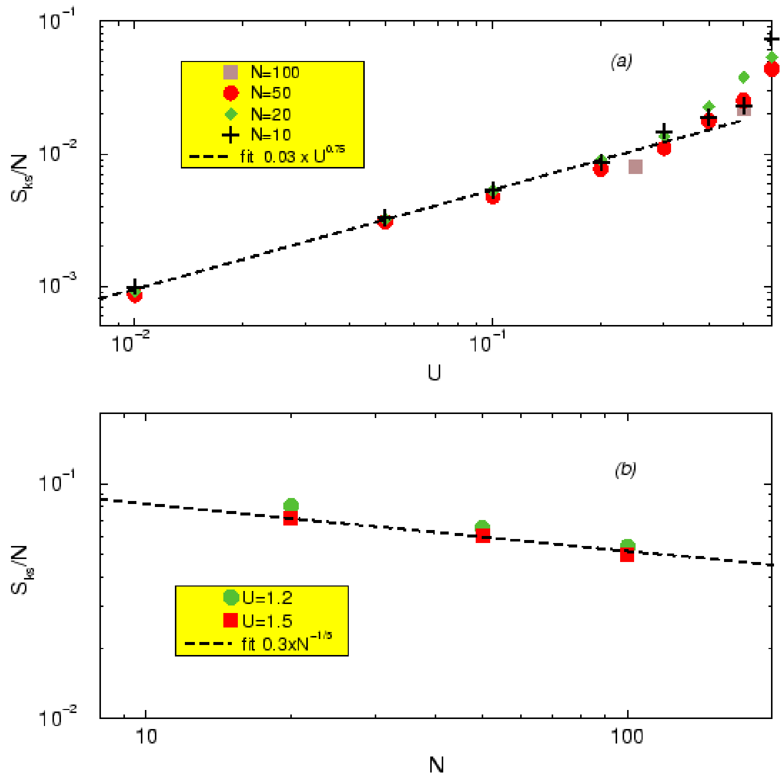
A peak close to the critical point is found also for S_{KS}



18 Scaling laws fot the Kolmogorov Sinai entropy

$$S_{KS} \propto U^{3/4} \qquad U << U_c$$

$$S_{KS} \propto N^{-1/5} \qquad U > U_c$$



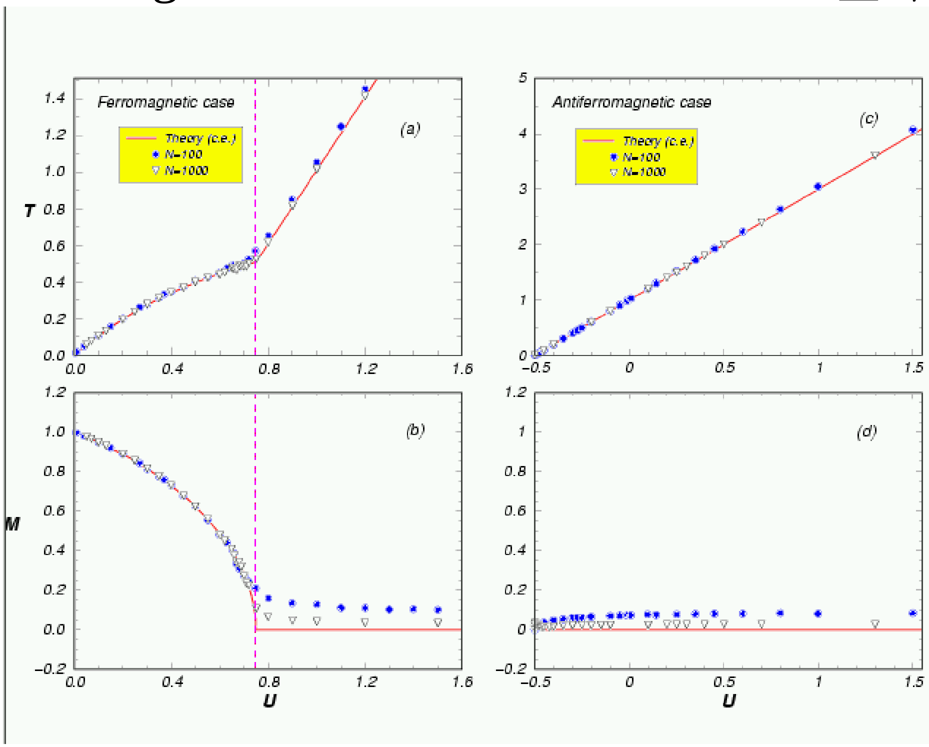
19

Antiferromagnetic behavior of HMF

The HMF model can have also an antiferromagnetic behavior if one considers

$$H = K - V$$

The general canonical solution for $\pm V$ is

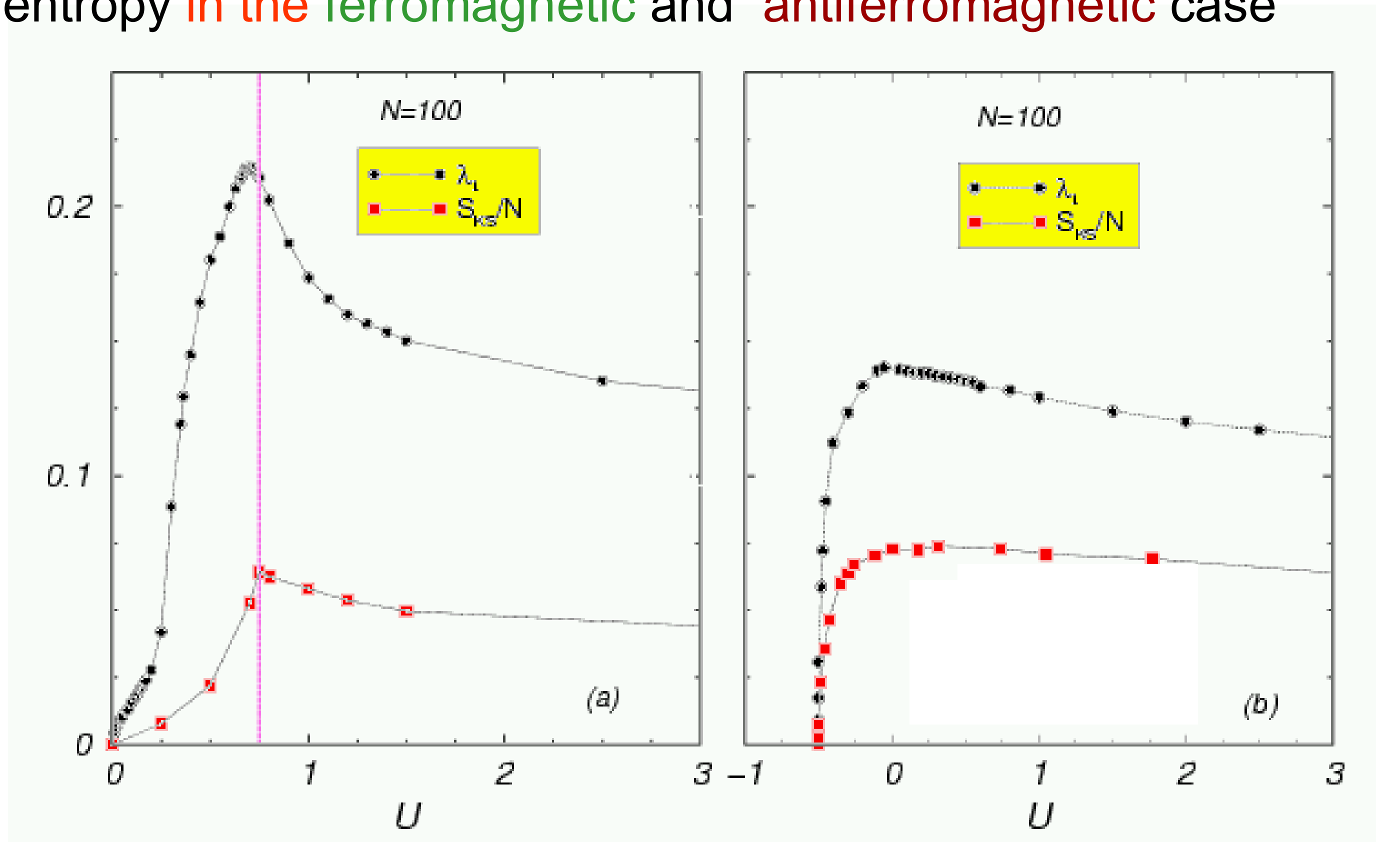


$$U = \frac{1}{2\beta} + \frac{\varepsilon}{2}\left(1 - M^2\right)$$

with $\varepsilon = \pm 1$

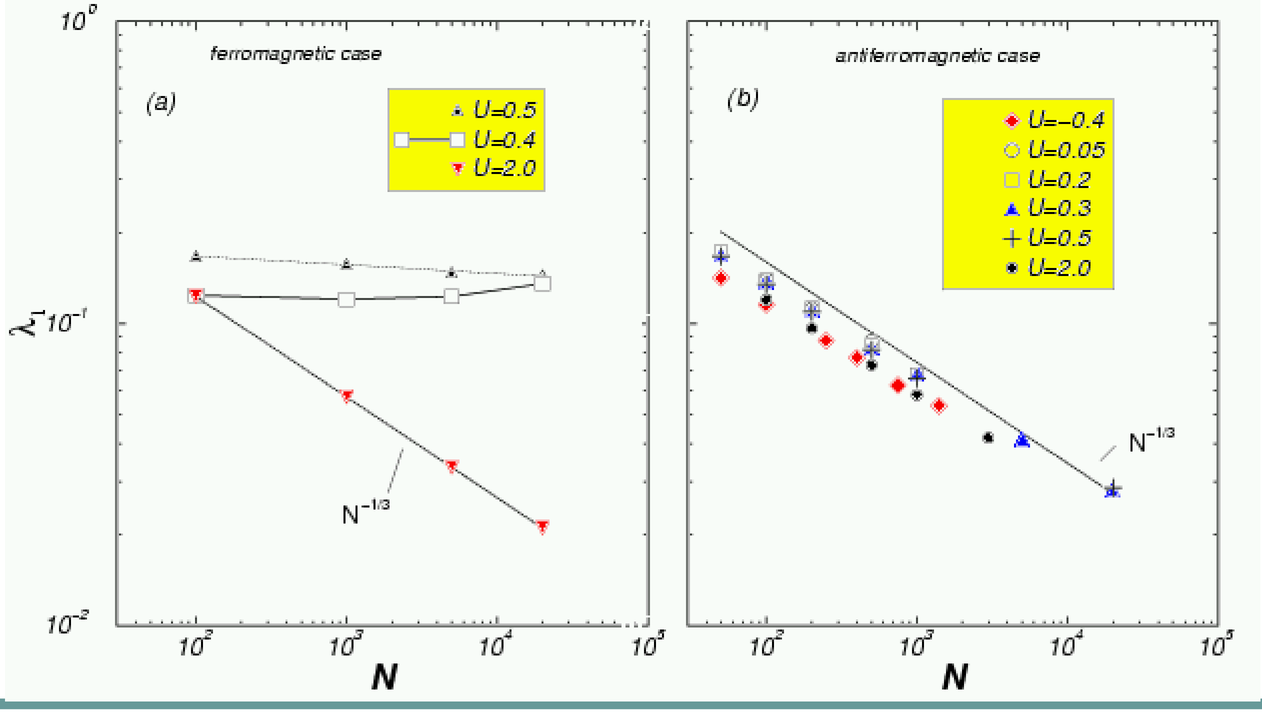
20 Dynamics at Equilibrium

One has a different behavior of the Largest Lyapunov exponent and the KS entropy in the ferromagnetic and antiferromagnetic case



21 LLE in the thermodynamical limit

In the thermodynamic limit, the LLE goes to zero for the whole energy range in the antiferromagnetic case, while it remains finite, for energies smaller than the critical one (Uc=0.75), in the ferromagnetic one. In the latter case it goes to zero for overcritical energies as $\lambda_1 \propto N^{-1/3}$

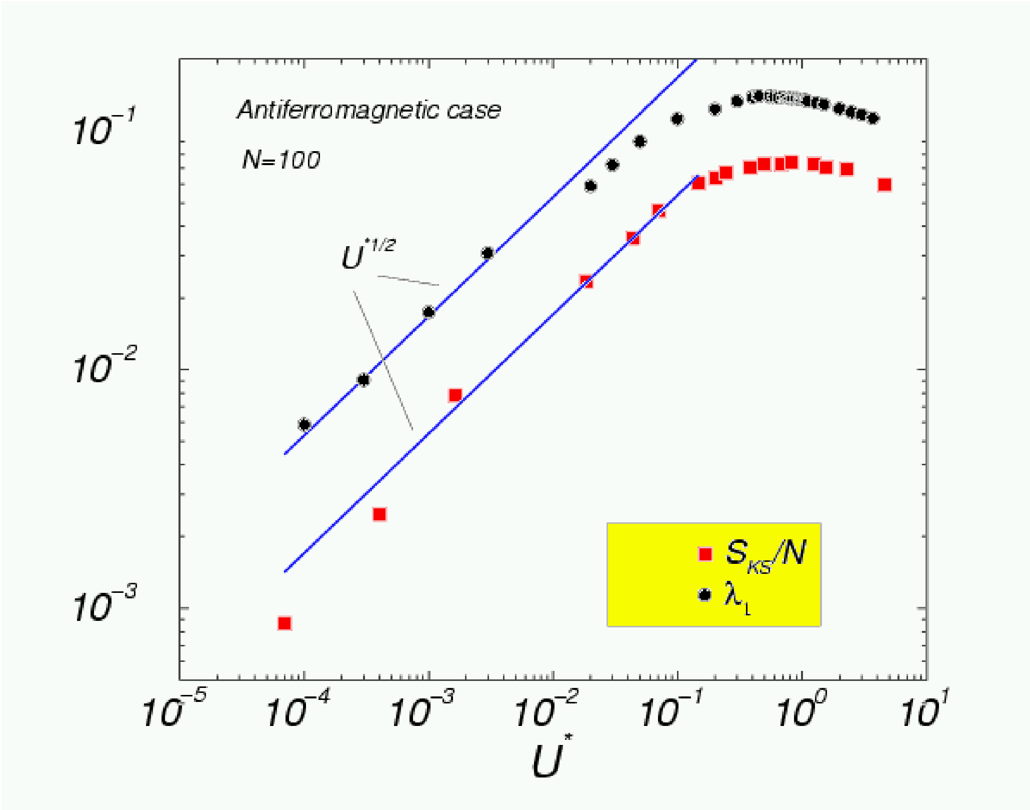


22 LLE in the antiferromagnetic case

Both the LLE and the KS entropy go to zero as in the antiferromagnetic case

$$\lambda_1 \propto \sqrt{U^*}$$

$$with \quad U^* = U + \frac{1}{2}$$



23 Equilibrium PDFs for the HMF model

In the continuum limit, considering the one-body distribution function F, the evolution of the HMF model is described by the Vlasov equation

$$\frac{\partial F}{\partial t} + p\frac{\partial F}{\partial \vartheta} - \frac{\partial V}{\partial \vartheta}\frac{\partial F}{\partial p} = 0$$

Supposing a factorization of the distribution function

$$F = f(p)g(\vartheta, t)$$

One gets the stationary equilibrium solution

$$f = f_0 \frac{1}{\sqrt{2\pi T}} e^{-p^2/2T}$$

$$g = g_0 e^{M\cos(\vartheta - \phi)/T}$$

where $g_0 = \frac{1}{2\pi I_0(M/T)}$, ϕ is the phase of M

and I_0 is the Bessel function

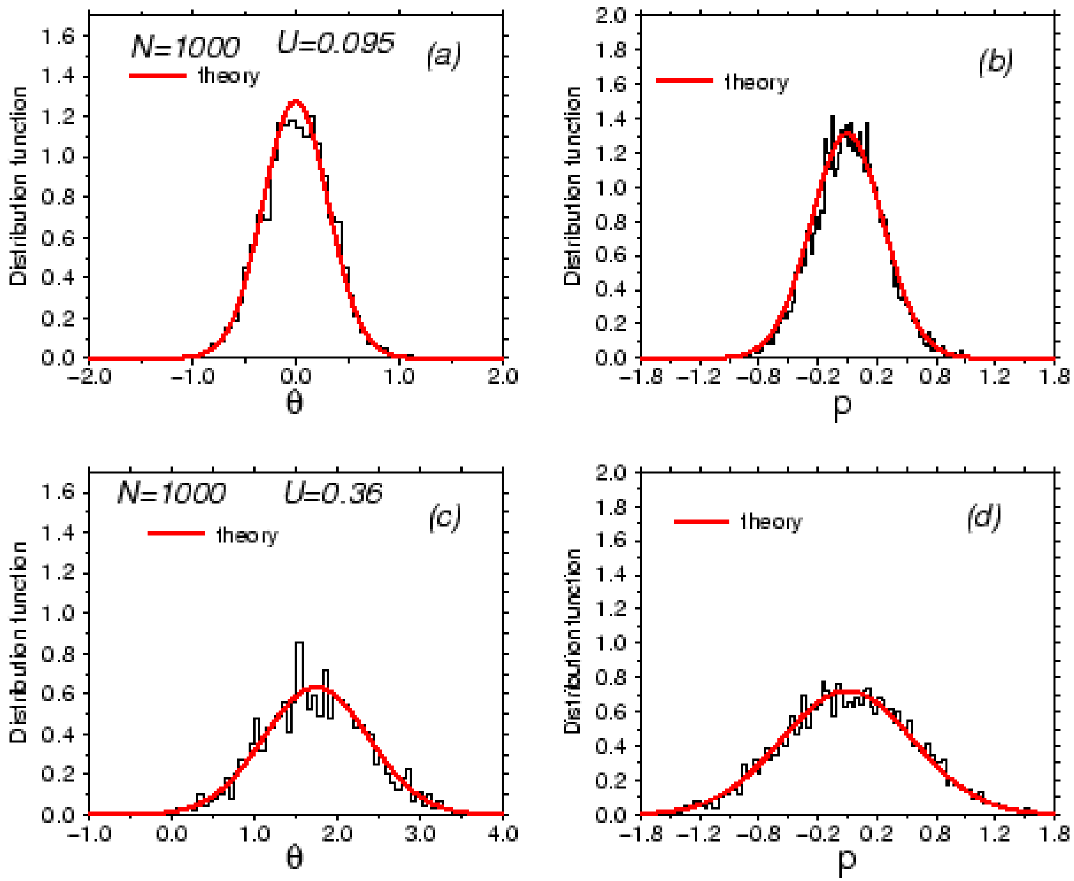
In the overcritical region $M = 0 \Rightarrow g = \frac{1}{2\pi}$

In the low energy region $I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + \dots\right] \Rightarrow g \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\vartheta^2/2\sigma^2}$

Latora, Rapisarda and Ruffo Physica D **131** (1999) 38

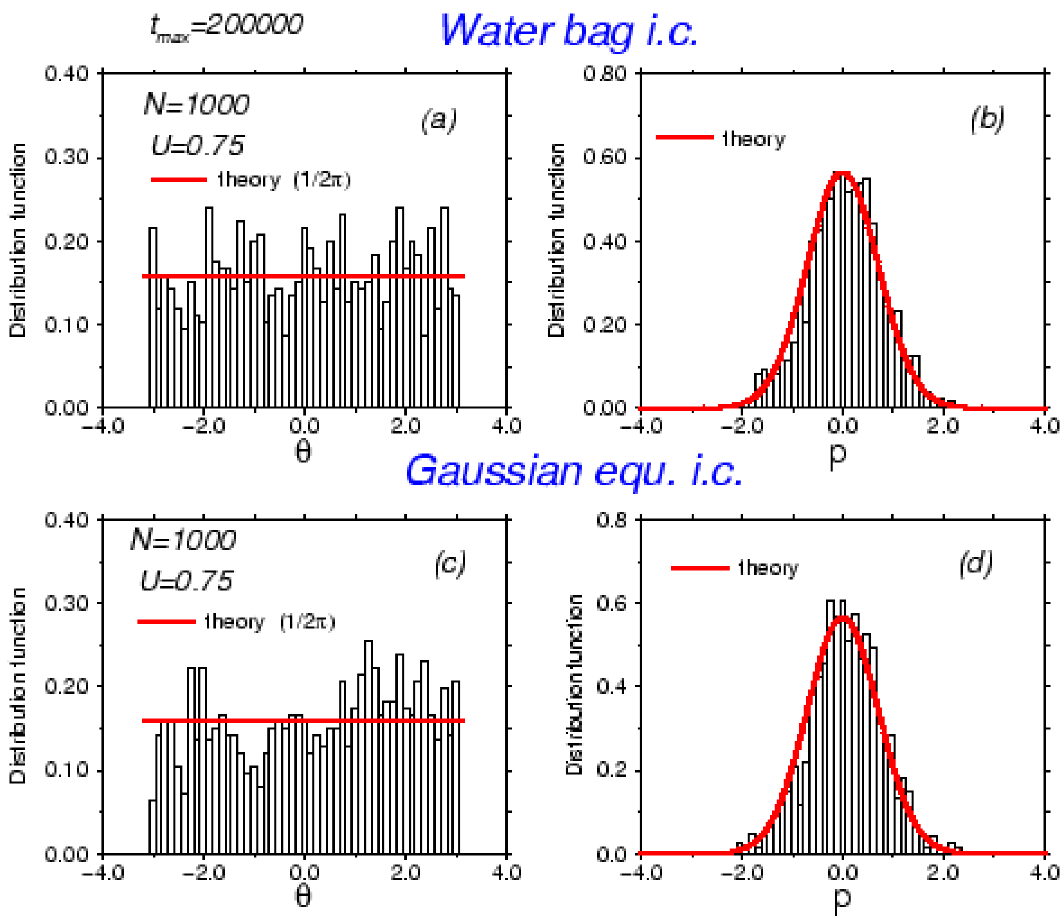
24 Comparison with numerical pdfs

At low energy



25 Comparison with numerical pdfs at equilibrium

At the critical point



Anomalous dynamics

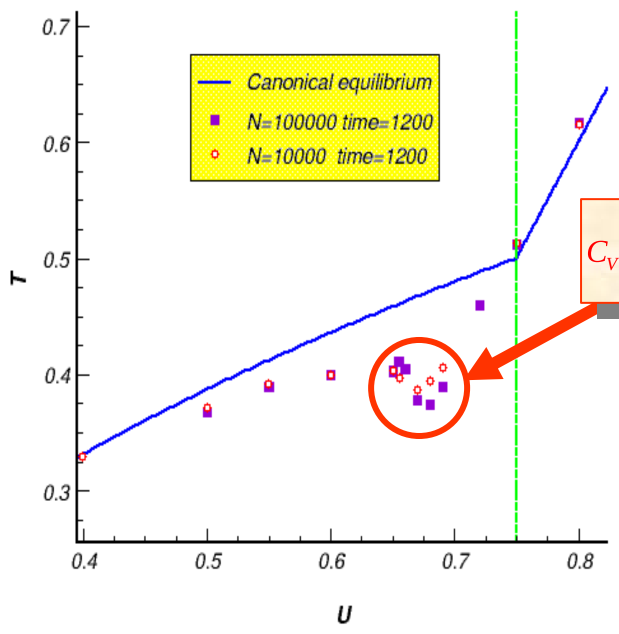
HMF model in the out-of-equilibrium case

OUT-OF-EQUILIBRIUM CASE

When the system is started with initial conditions very far from equilibrium…..

…… we observe many dynamical anomalies, in particular in an energy range below the critical point.

28 Negative specific heat



$$C_V = \left(\frac{\partial T}{\partial U}\right)_V^{-1} < 0$$

In a region before the critical point the specific heat becomes negative:

the temperature decreases, by increasing the energy density.

This phenomenon has been observed in multifragmentation nuclear reactions and atomic clusters, but also in self-gravitating stellar objects, i.e. for nonextensive systems.

See for example:

▪Thirring, Zeit. Physik 235 (1970) 339

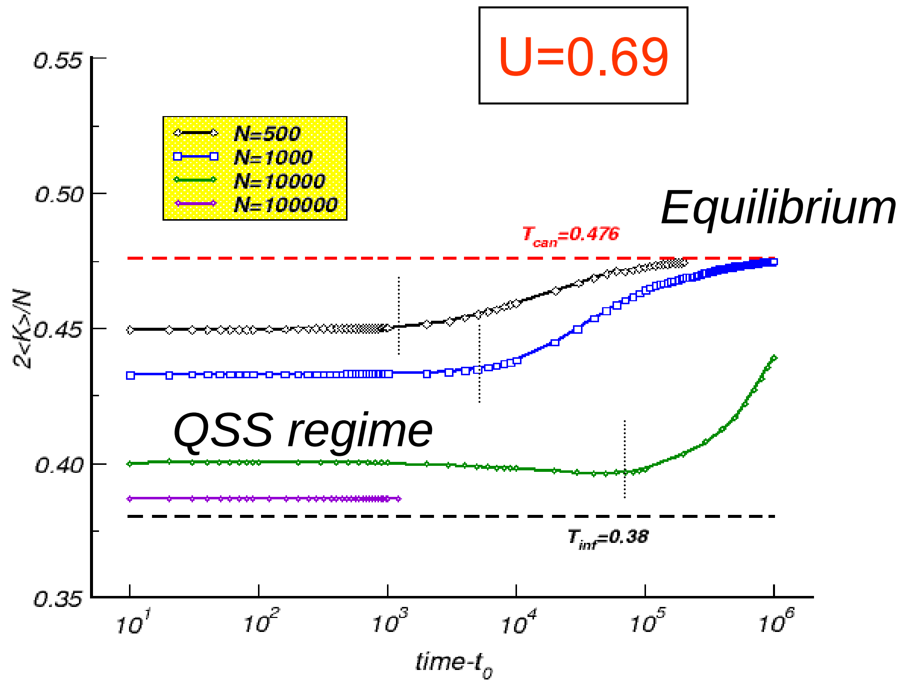
▪Lynden-Bell, Physica A 263 (1999) 293

▪D.H.E.Gross, *Microcanonical Thermodynamics: Phase transitions in Small systems*, World Scientific (2001).

▪M. D'Agostino et al, Phys. Lett. B 473 (2000) 279

▪Schmidt et al, Phys. Rev. Lett. 86 (2001) 1191

29 Quasi Stationary States



The system lives for a very long time in a ***metastable quasi stationary state*** (QSS), whose temperature defined as

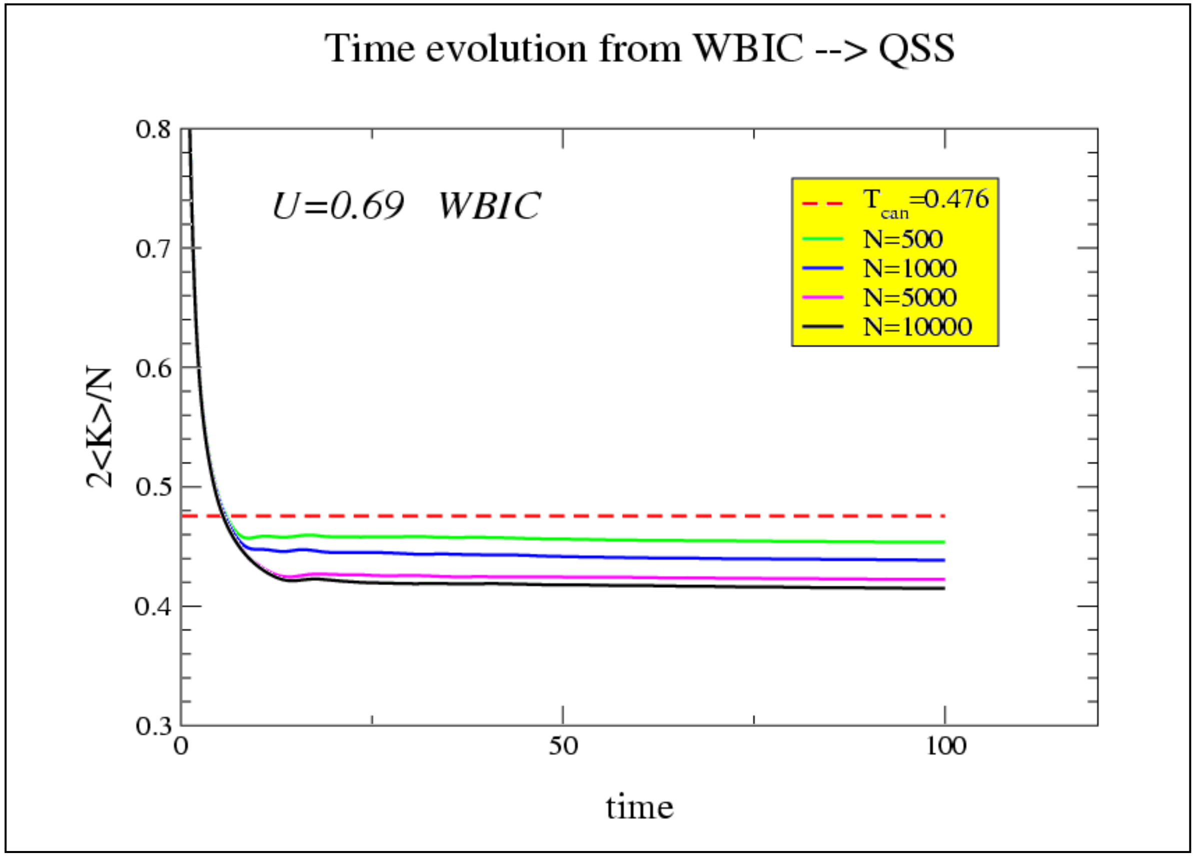
$$T = \frac{2\langle K \rangle}{N}$$

is smaller than the equilibrium one.

- The larger N, the longer the QSS lifetime.
- The temperature T tends to 0.38 for $N \rightarrow \infty$

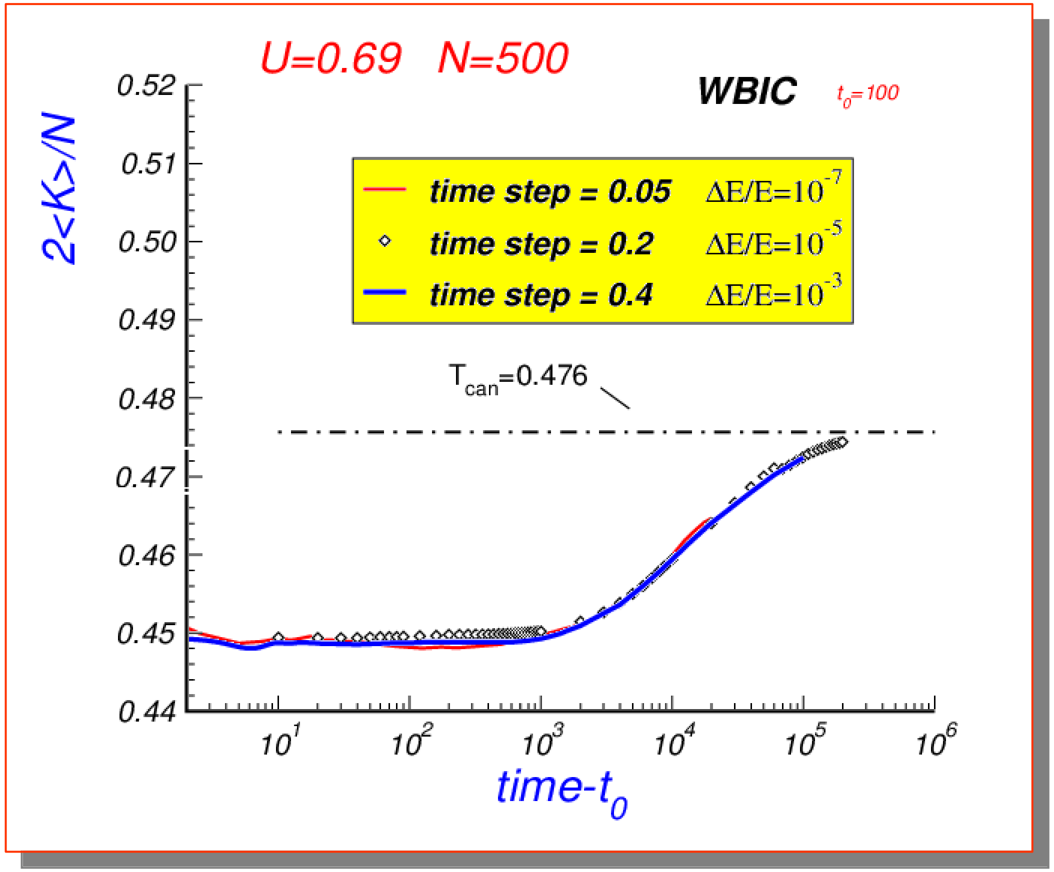
QSS: initial evolution

QSS regime is reached almost immediately

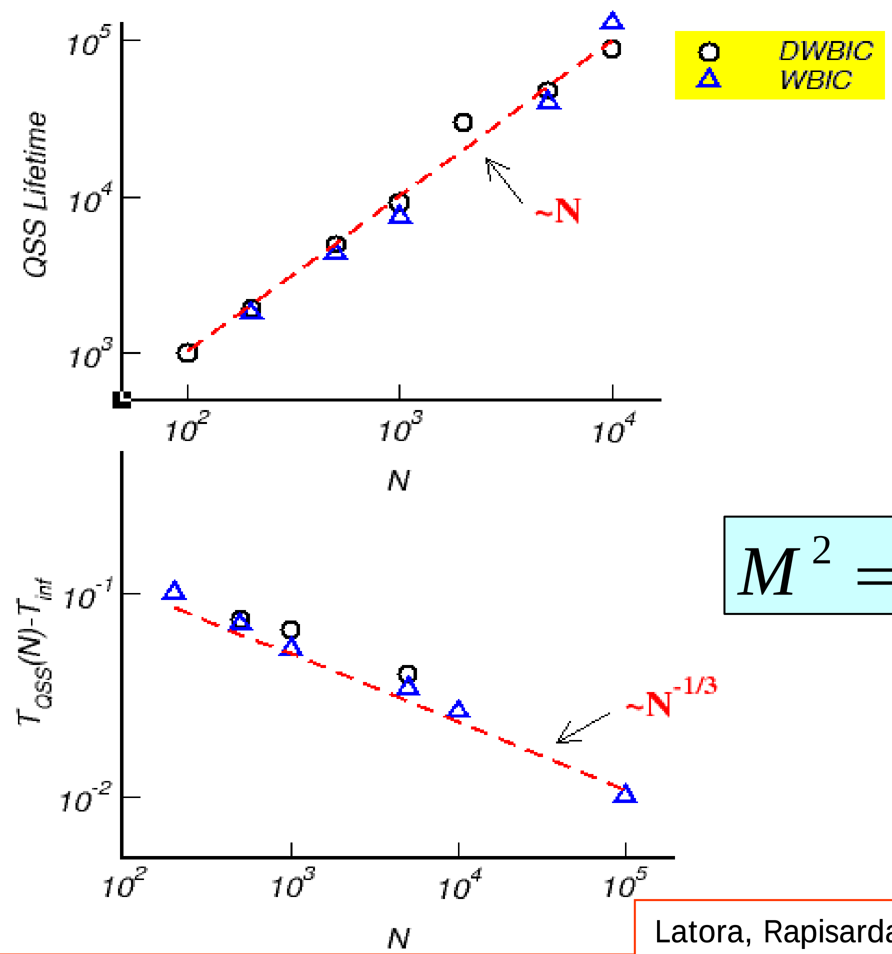


31 QSS and numerical accuracy

QSS ***do not depend*** on the accuracy of the integration



32 QSS lifetime and Temperature



The QSS lifetime diverges with the size N

The QSS temperature goes to T_{inf}=0.38 (for U=0.69) as $N \to \infty$

Notice that being

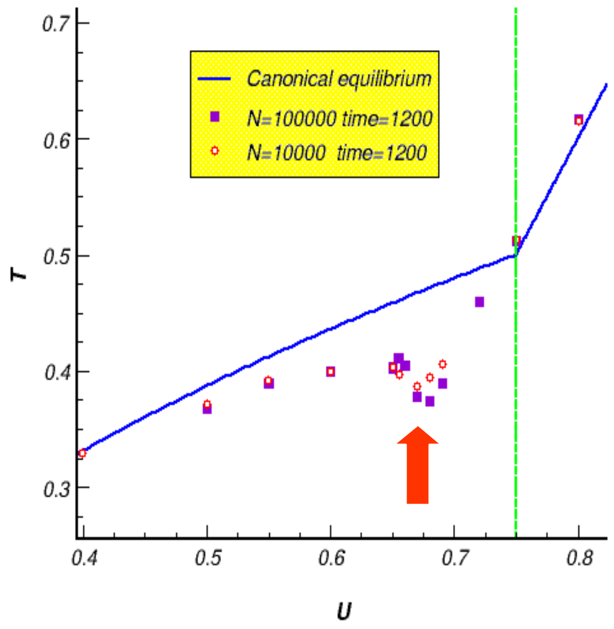
$$M^2 = T + 1 - 2U = T - 0.38$$

one has

$$M^2 \propto N^{-1/3}$$

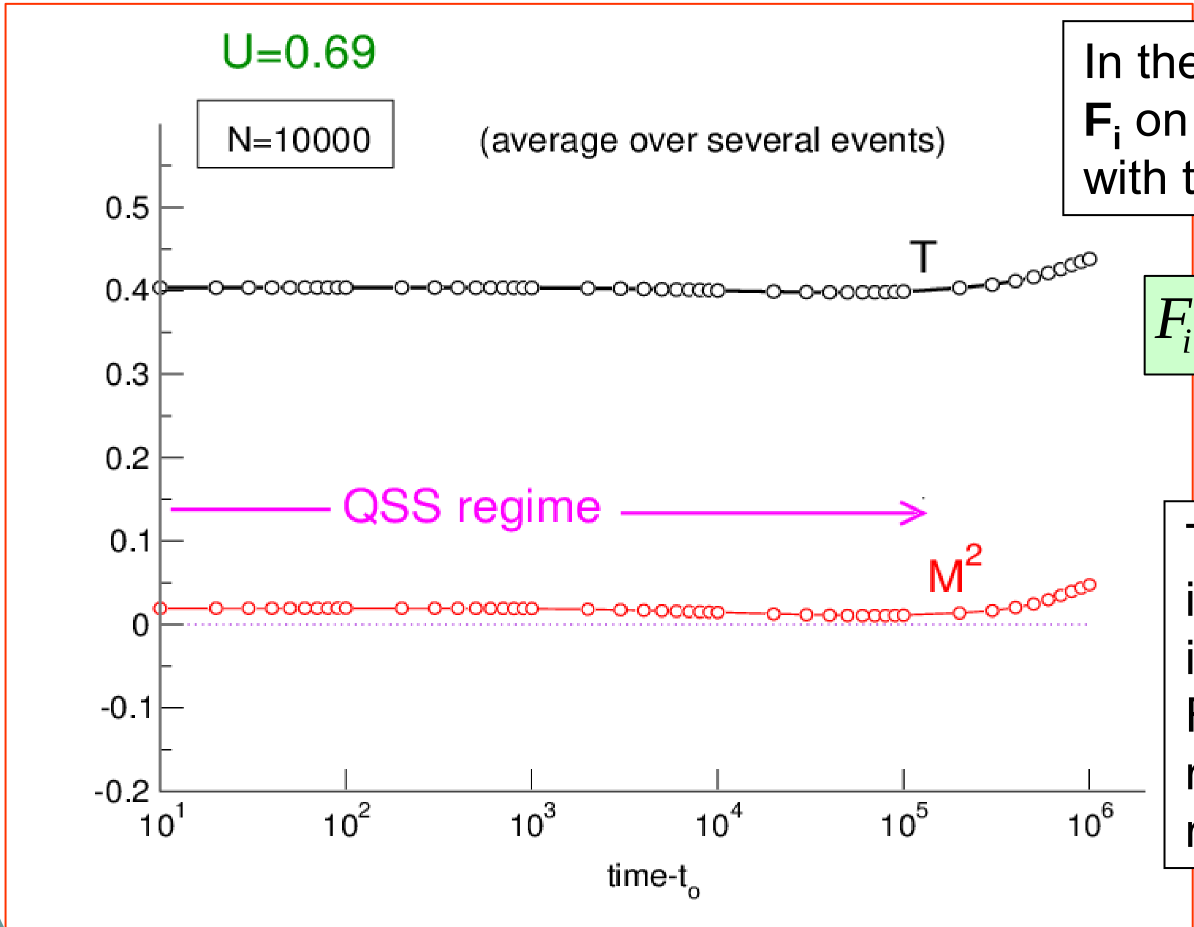
Latora, Rapisarda and Tsallis PRE 64 (2001) 056134

15 The energy density U=0.69



The majority of the results refer to the case U=0.69, where anomalies are most evident

Force in the QSS regime

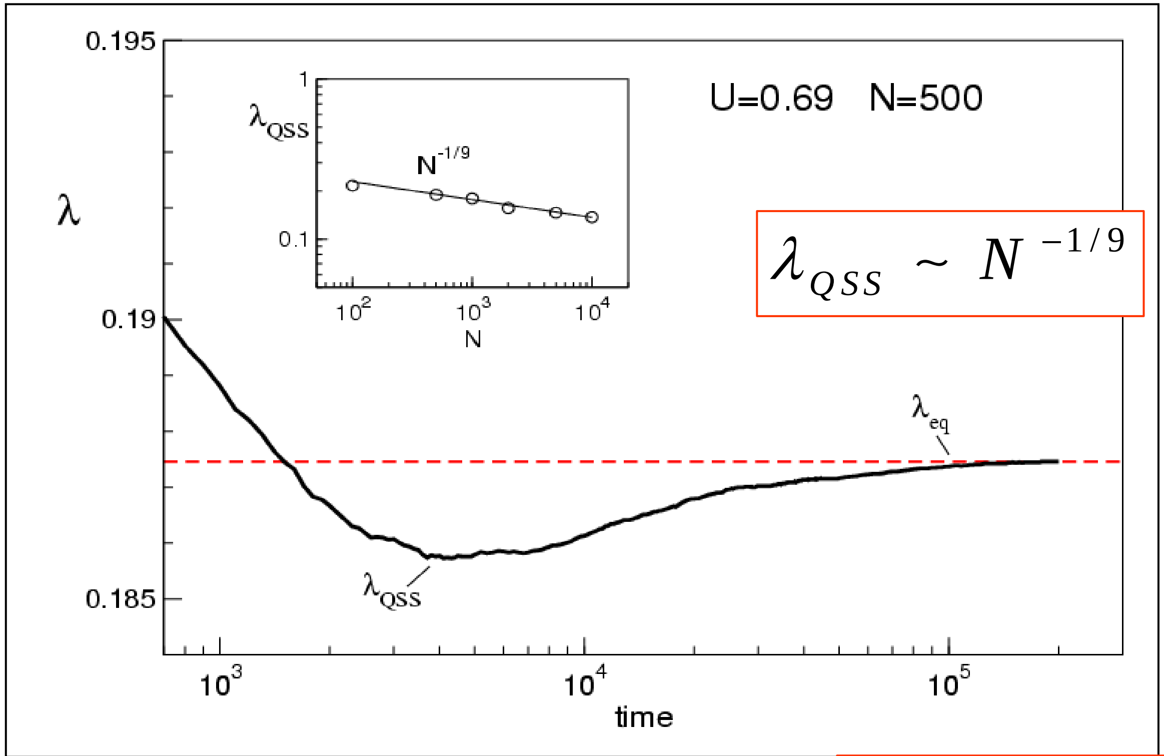


In the QSS regime **the force F_i** on the ith spin **goes to zero** with the size N being

$$F_i = -M_x \sin\theta_i + M_y \cos\theta_i$$

Thus the QSS dynamics is slower and slower by increasing N.
For $N \to \infty$ the system remains frozen in the QSS regime

35 Vanishing Lyapunov exponents



$$\lambda_{QSS} \sim N^{-1/9}$$

In the QSS regime the largest Lyapunov exponent tends to zero as the size of the system tends to infinity

This scaling can be obtained considering that

$$\lambda \propto M^{2/3} \propto \left(N^{-1/3}\right)^{1/3} = N^{-1/9}$$

Latora, Rapisarda, Tsallis Physica A 305 (2002) 129

36 Importance of the order in the limits

Simulations show that, going towards the thermodynamic limit, it is very crucial the time order of the size limit and time limit to infinity

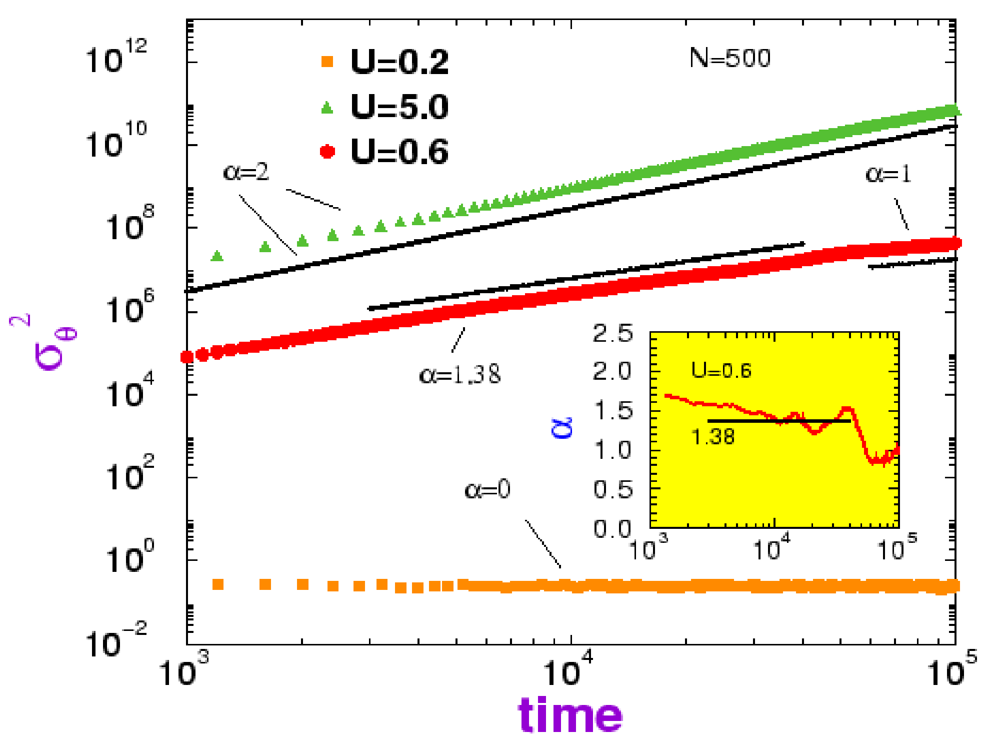
In general, the two limits do not commute:

$$N \to \infty \; t \to \infty \quad \neq \quad t \to \infty \; N \to \infty$$

Boltzmann-Gibbs equilibrium

QSS

Anomalous diffusion in the QSS regime



Latora, Rapisarda and Ruffo, PRL 83 (1999) 2104

In general one has for the mean square displacement

$$\sigma^2(t) \propto t^\alpha$$

$\alpha = 1$ Normal diffusion

$\alpha \neq 1$ Anomalous diffusion

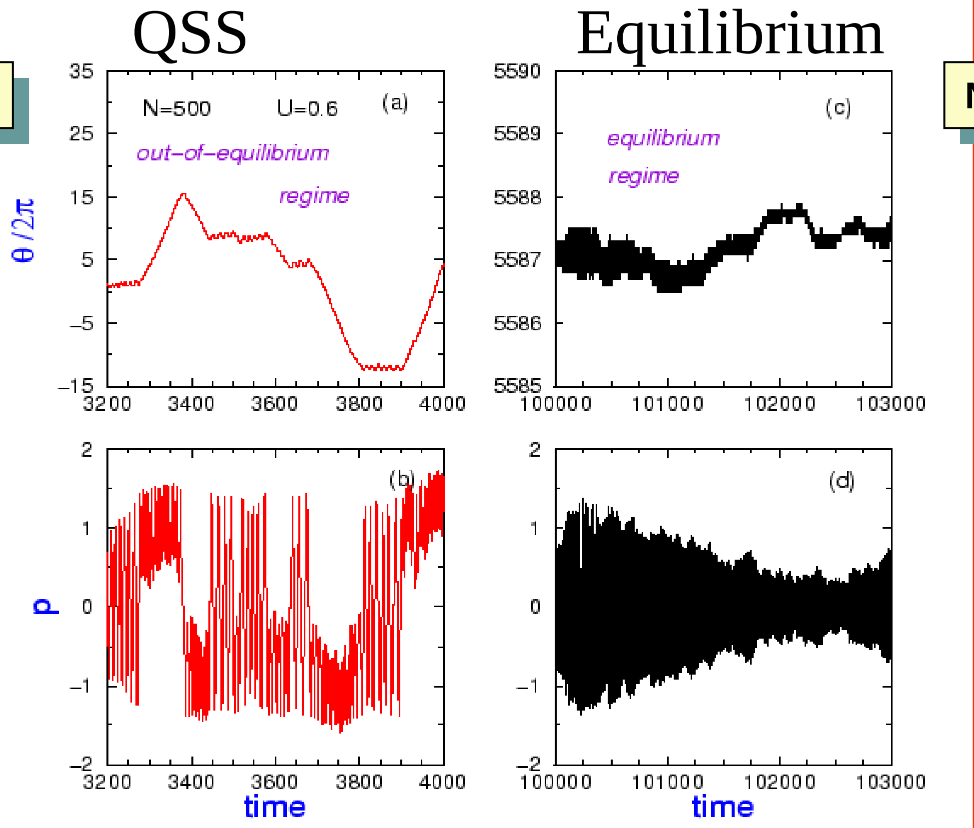
In our case we get superdiffusion with an exponent **α=1.38** in correspondence of the QSS regime.

38 Anomalous diffusion: typical dynamics

Phase space dynamics of a typical particle (spin)

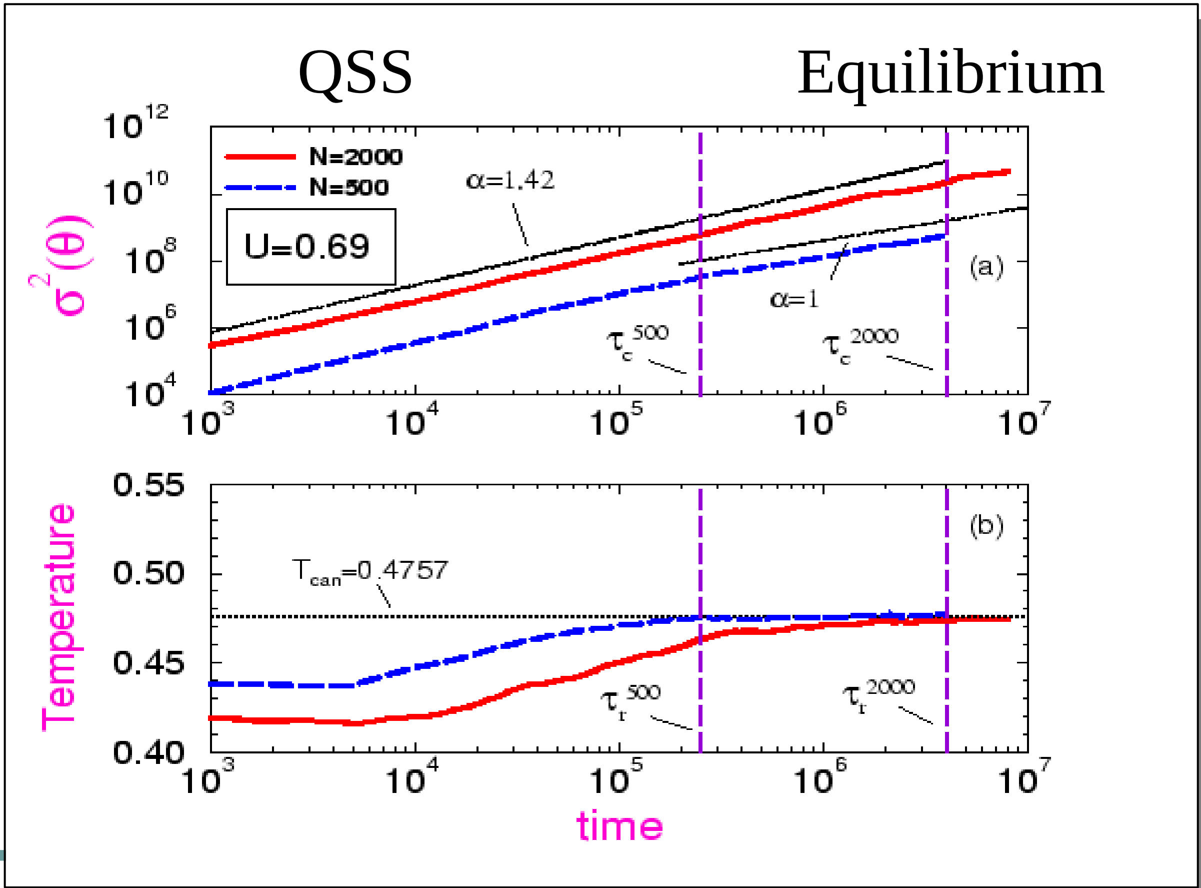
Lèvy walks

Normal random walks



39 Anomalous diffusion

The cross-over times**,** from anomalous to normal diffusion, ***coincide with the relaxation times***



Lévy walks: walking and trapping time PDFs

For a one dimensional system which shows sticking and flying particles with constant velocity, one finds:

$$P_{walk}(t) \propto t^{-\mu}$$

$$P_{trap}(t) \propto t^{-\nu}$$

with

$$\alpha = 2 + \nu - \mu$$

when

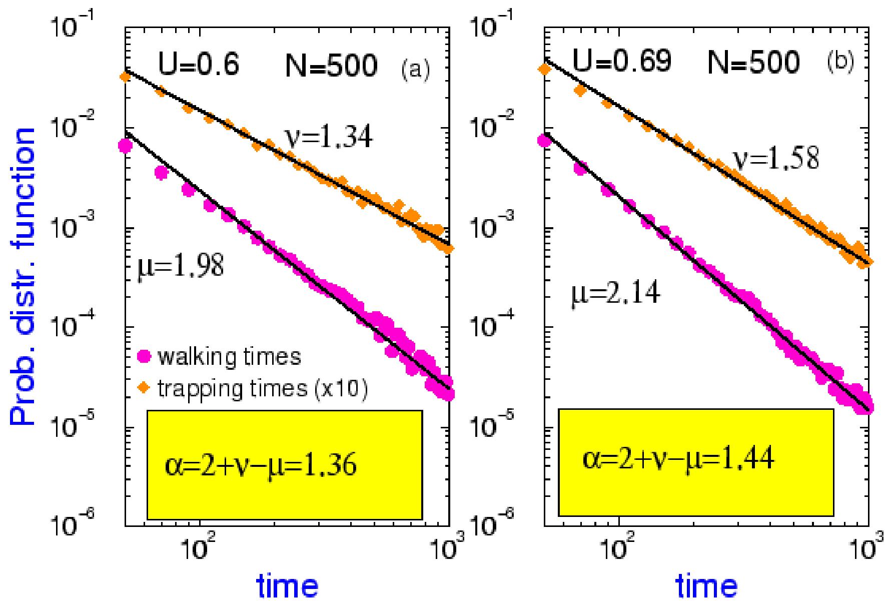
$$2 < \mu < 3$$

$$\nu < 2$$

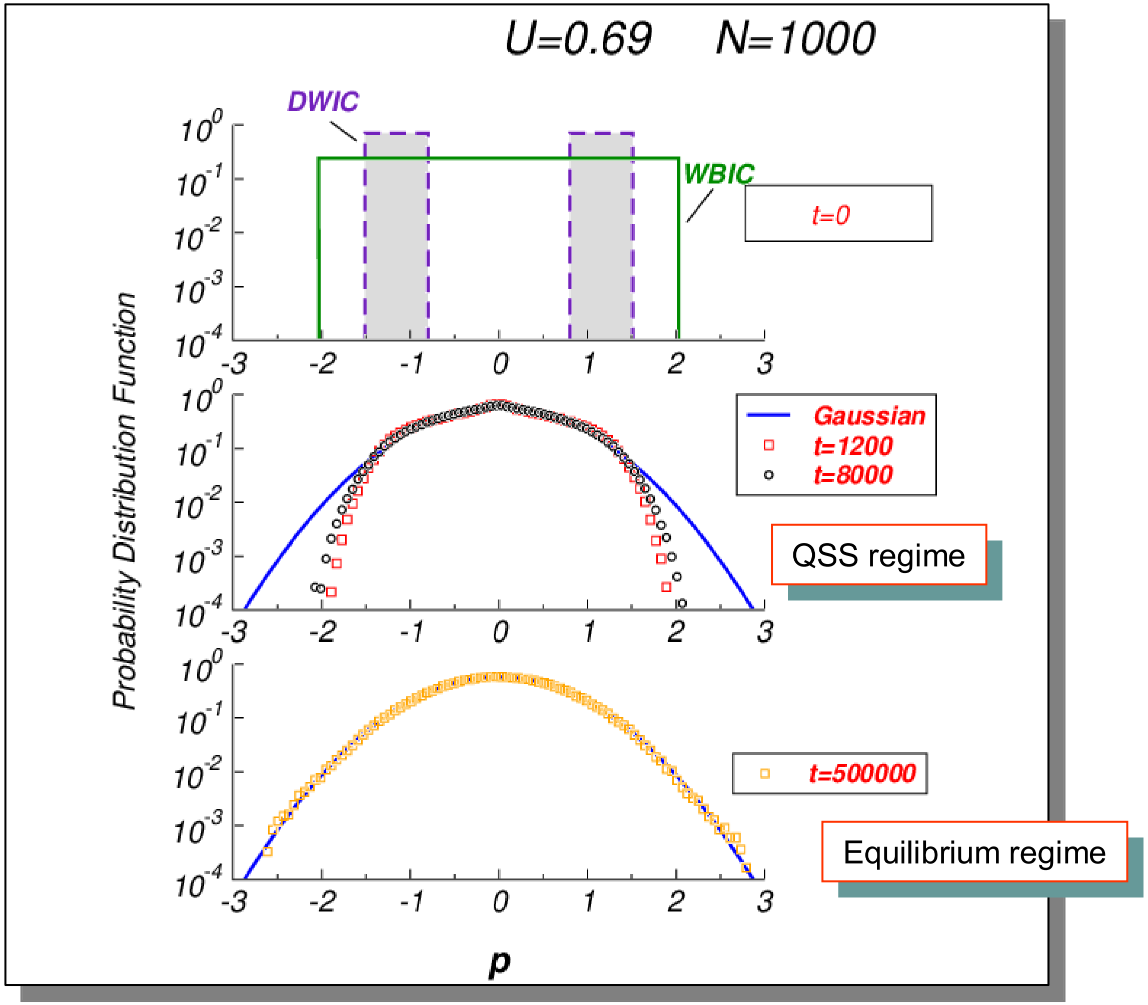
See Klafter and Zumofen

PRE 49 (1994) 4873

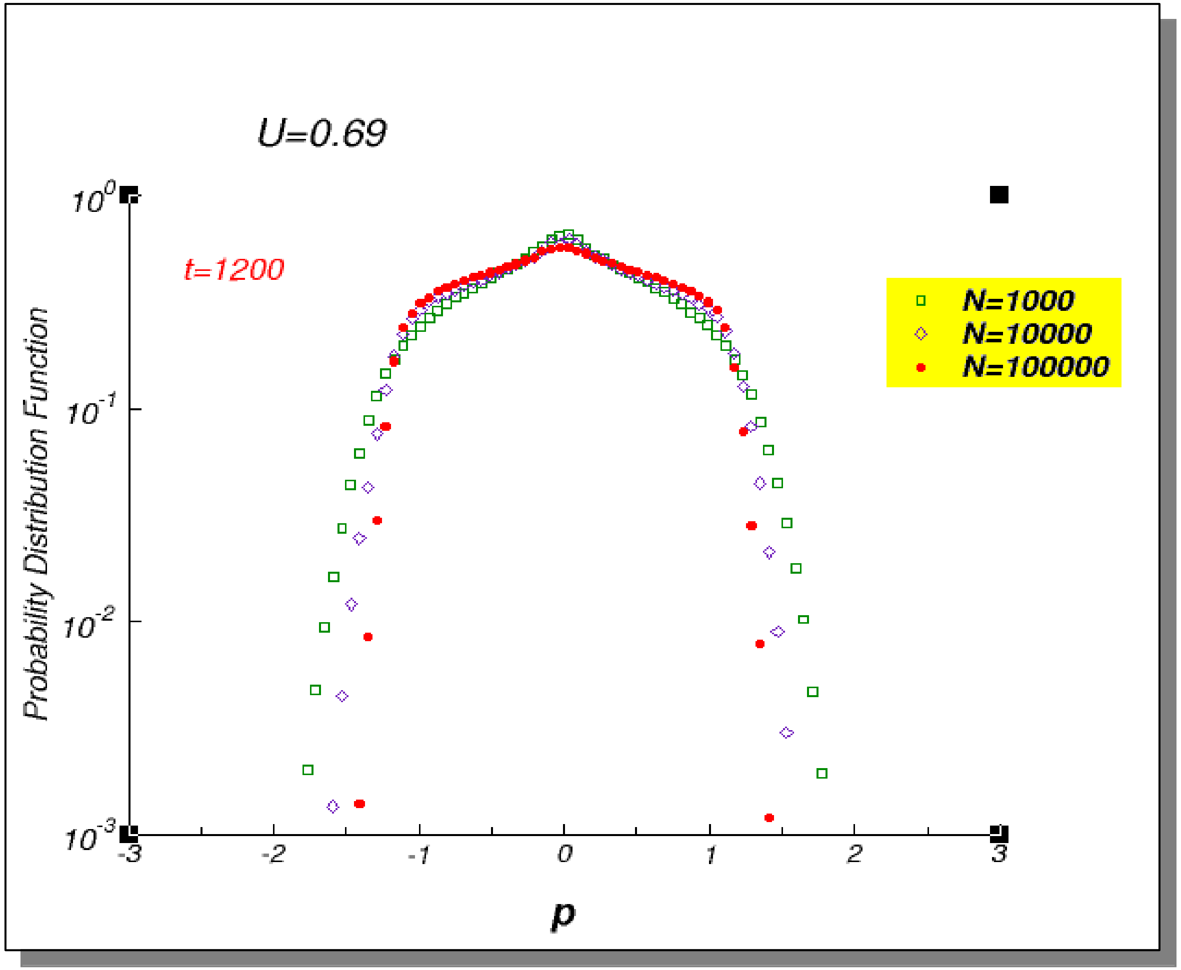
In our case we find



Time evolution of velocity PDFs



42 Non-Gaussian velocity PDF



Velocity PDFs in the QSS regime for different sizes N of the system

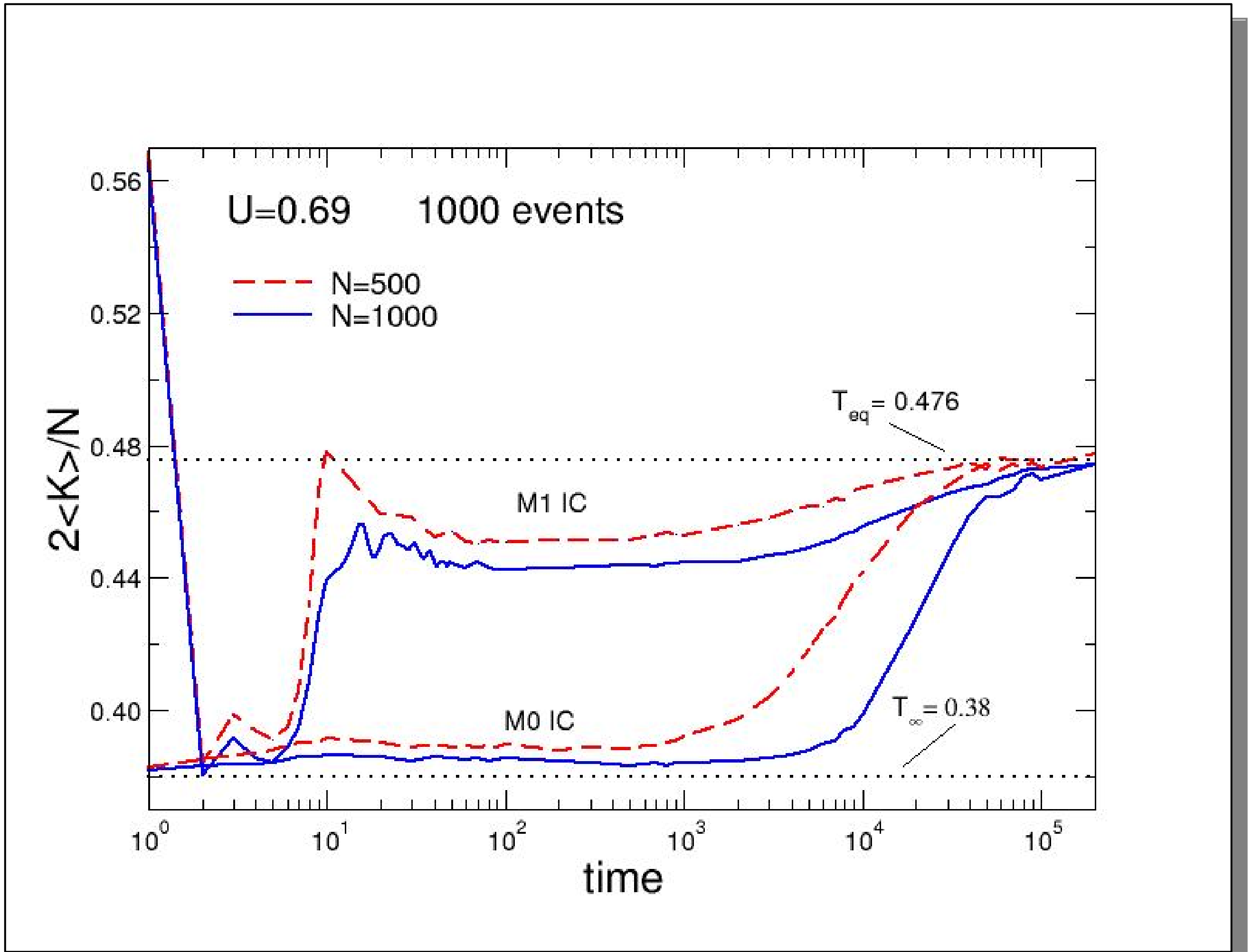
43 Role of initial conditions for QSS

We have recently studied the nature of the anomalous QSS regime by starting from different initial magnetizations with $0 \leq M \leq 1$

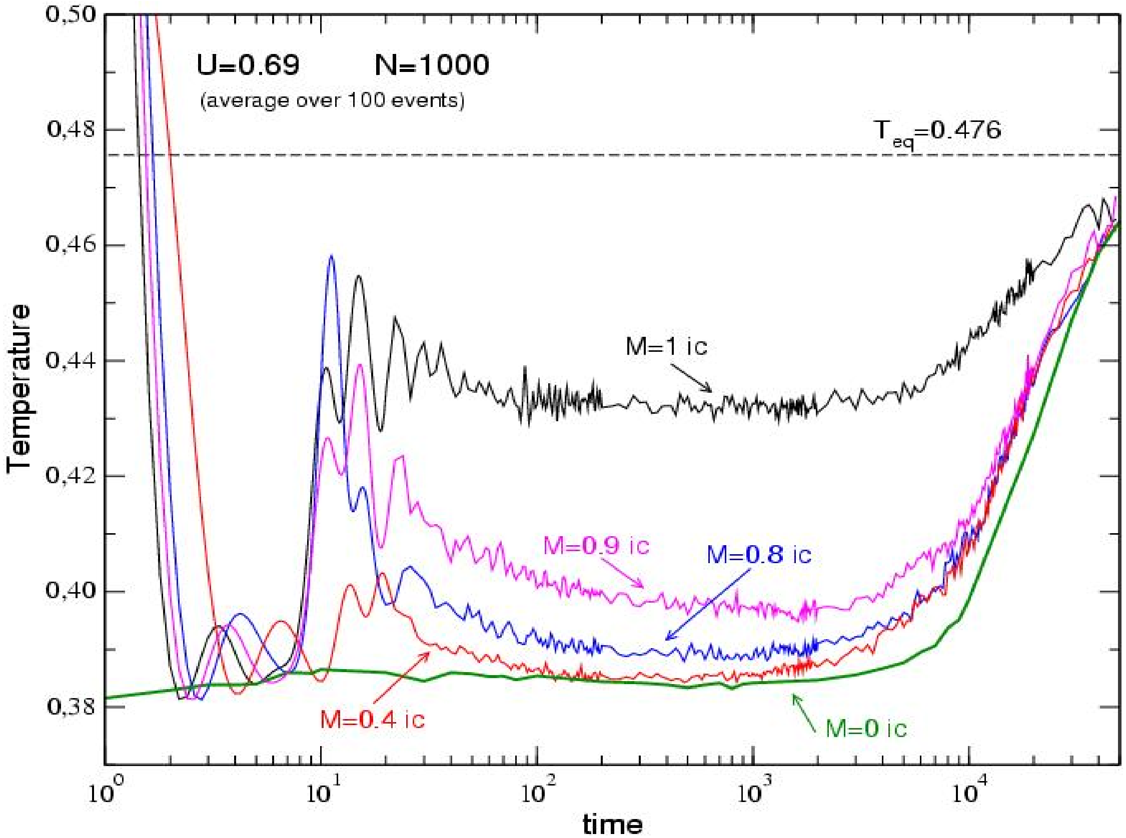
(considering in all cases a uniform distribution in momenta: *water bag*).

Dynamical anomalies depend on the initial conditions. The most interesting are those observed for initial magnetization $M \sim 1$

QSS for different initial conditions: M1 vs. M0

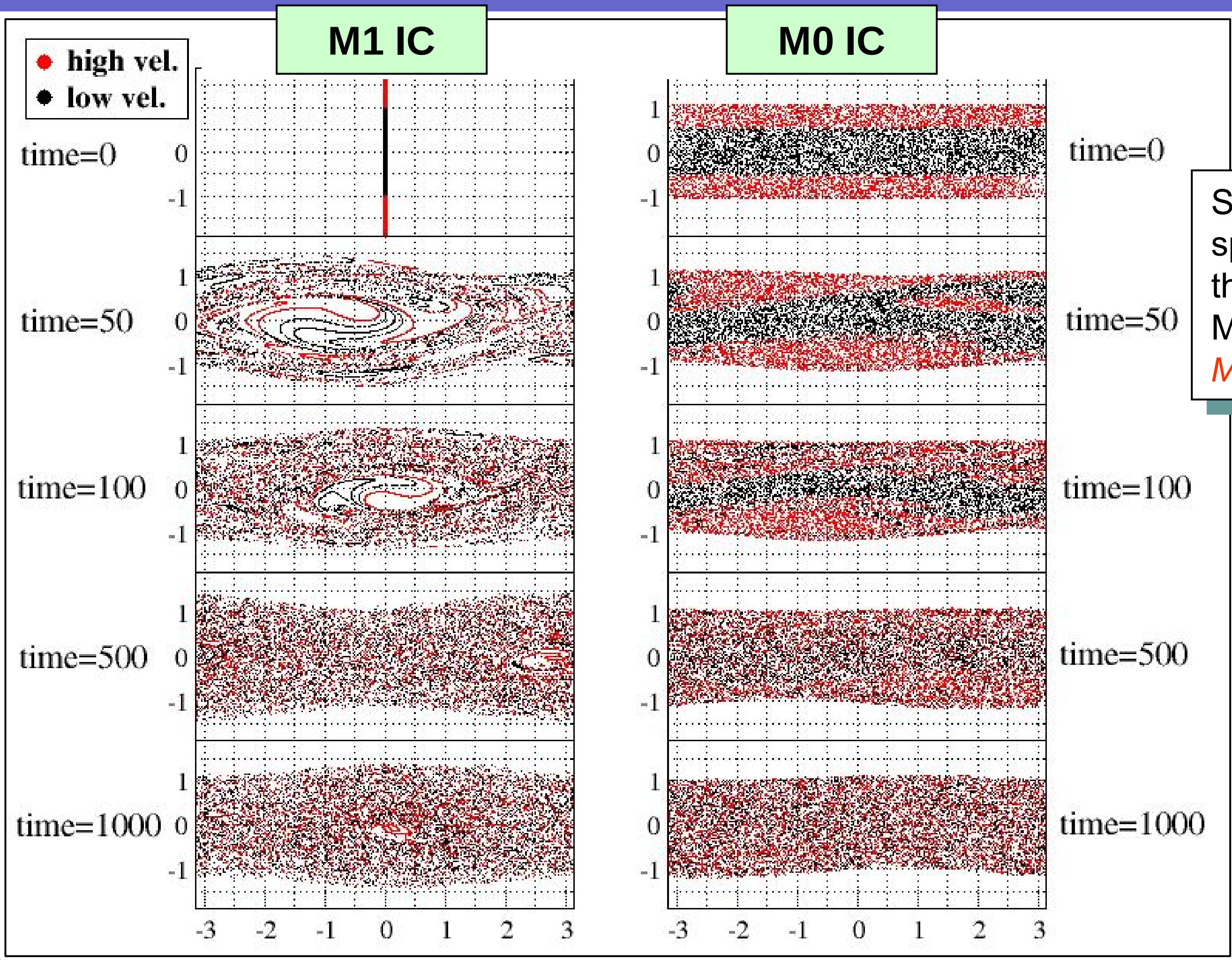


Dependence of QSS on the initial magnetization



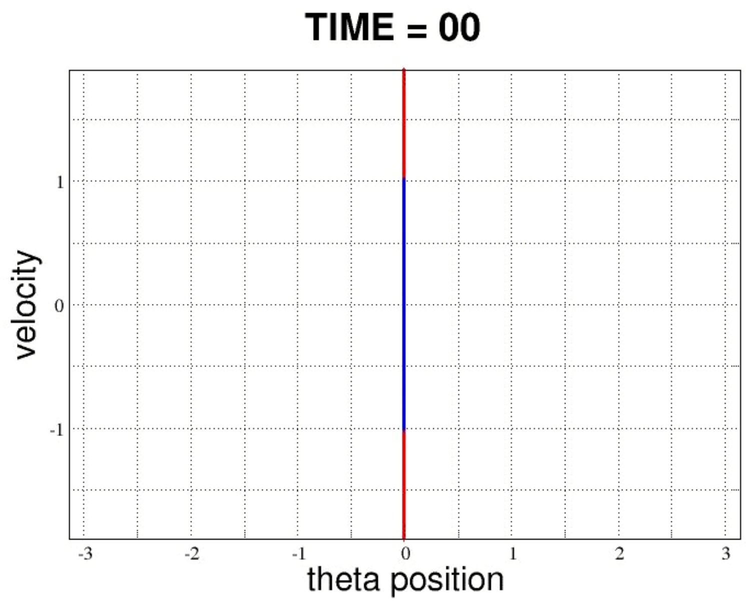
Pluchino, Latora, Rapisarda Physica A **338 (2004) 60**

46 Correlations in phase space for different IC

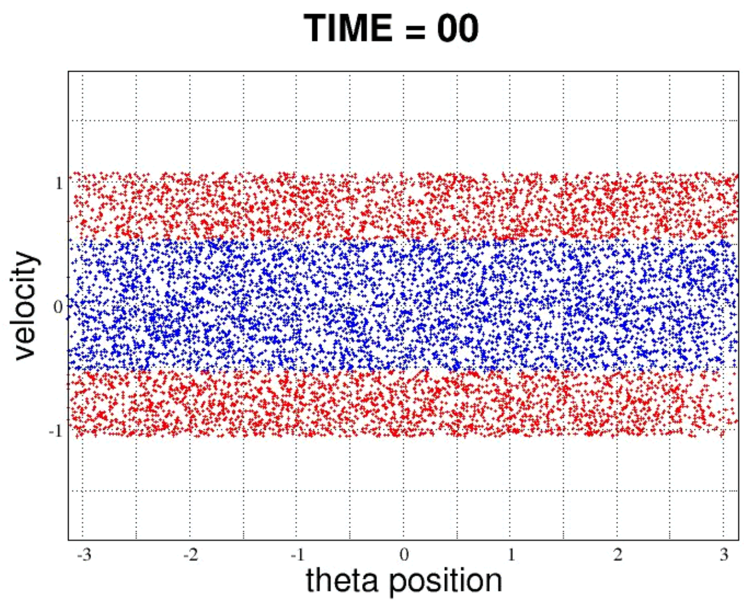


Structures in phase-space appear, in the QSS regime for M1IC, *but not for M0IC*

47 Dynamical evolution for different initial conditions

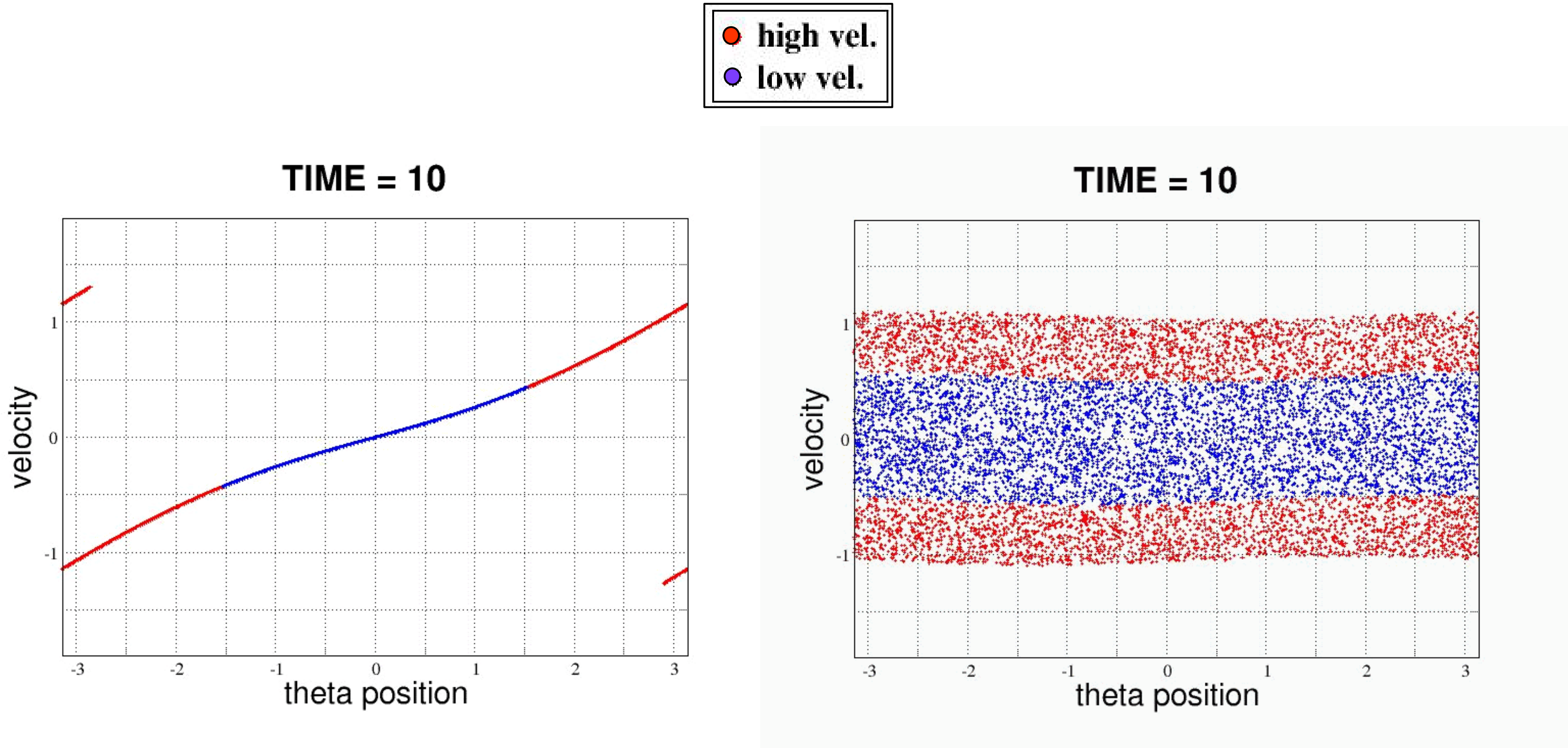


M1 IC



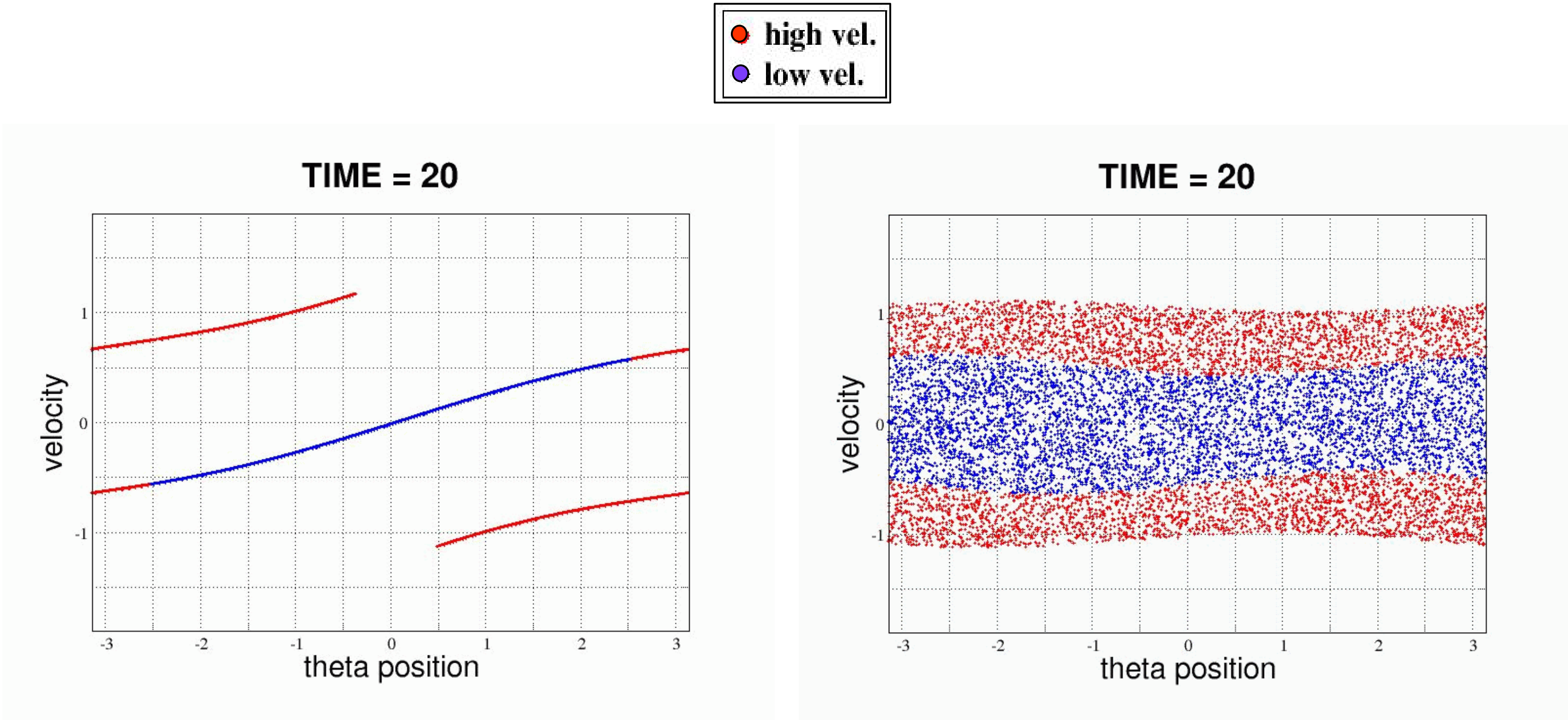
M0 IC

Dynamical evolution for different initial conditions



M1 IC **M0 IC**

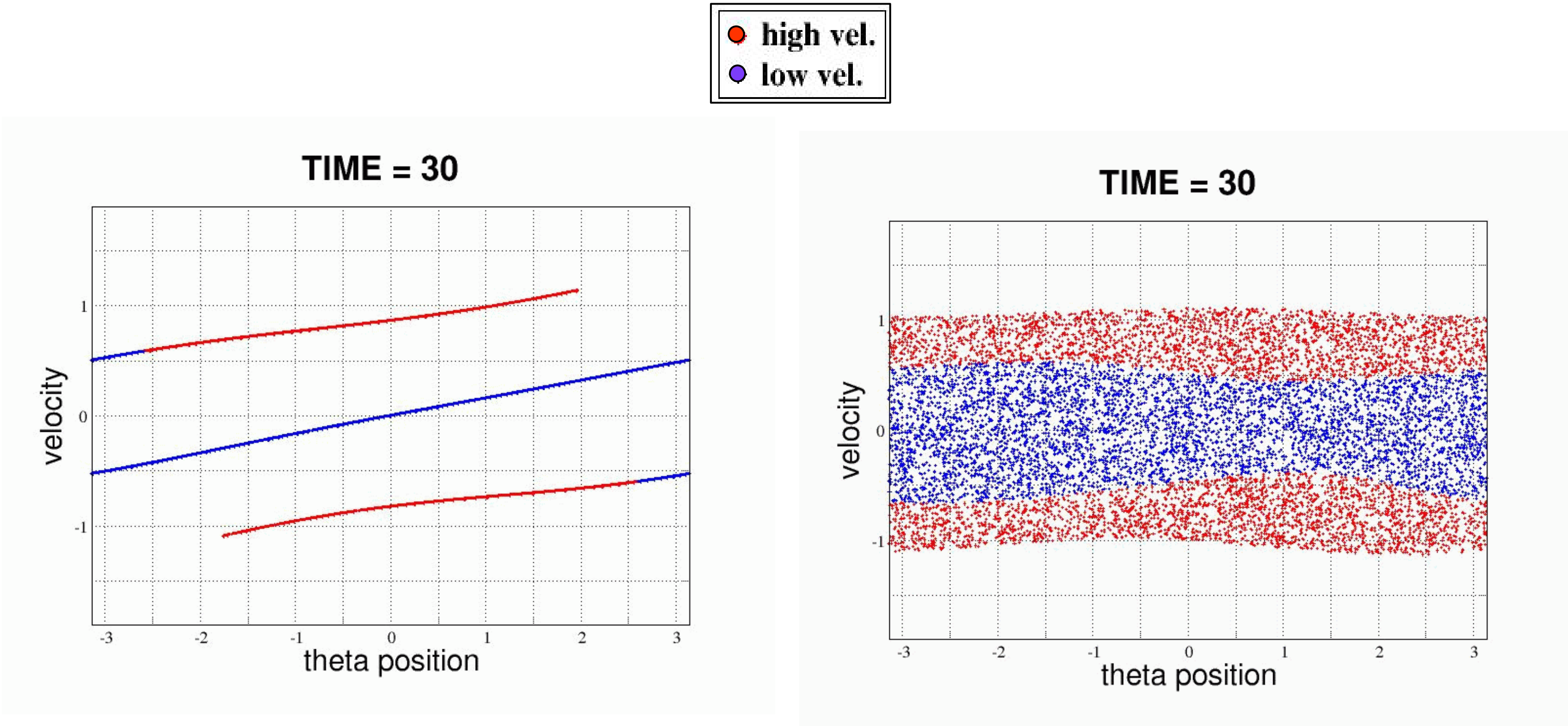
Dynamical evolution for different initial conditions



M1 IC

M0 IC

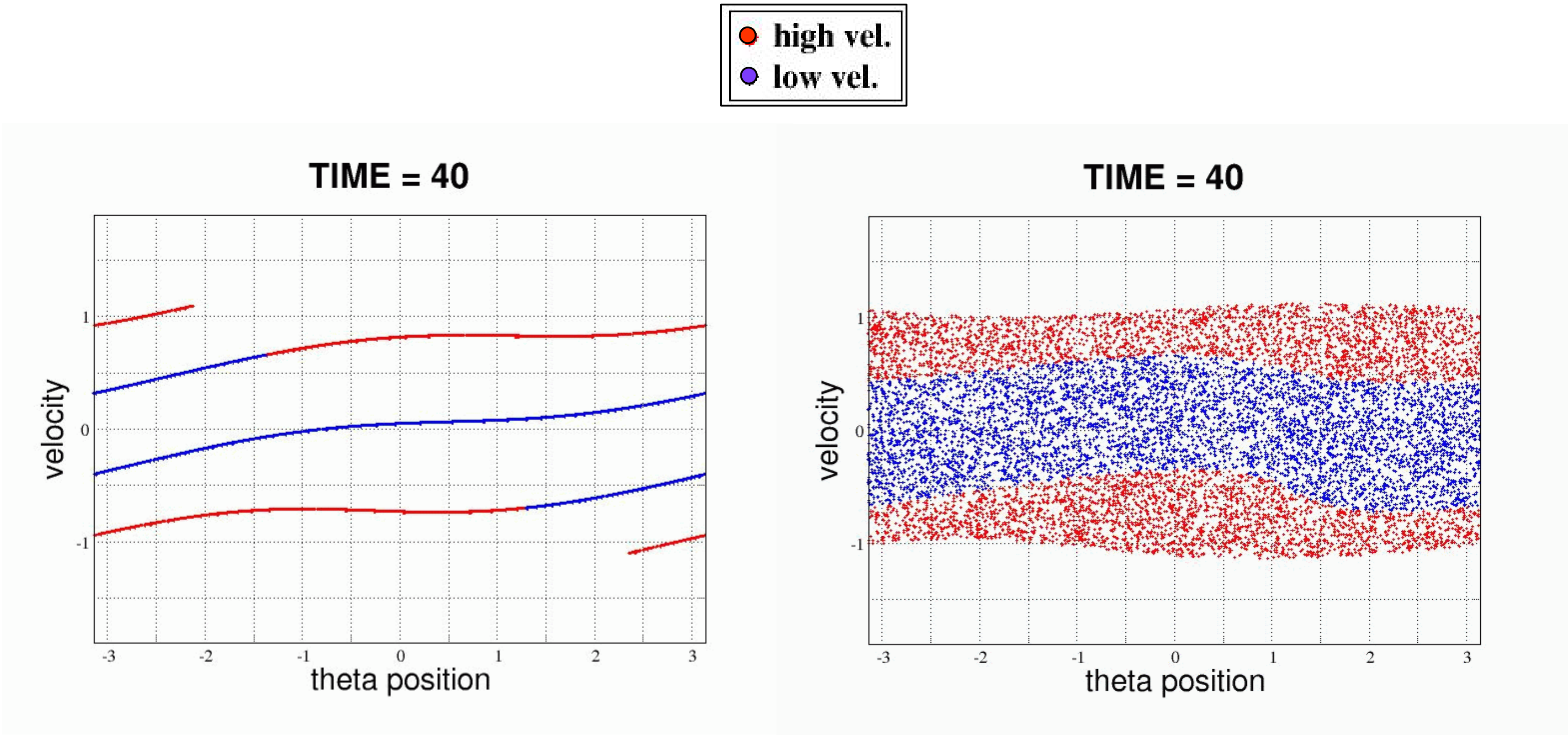
Dynamical evolution for different initial conditions



M1 IC

M0 IC

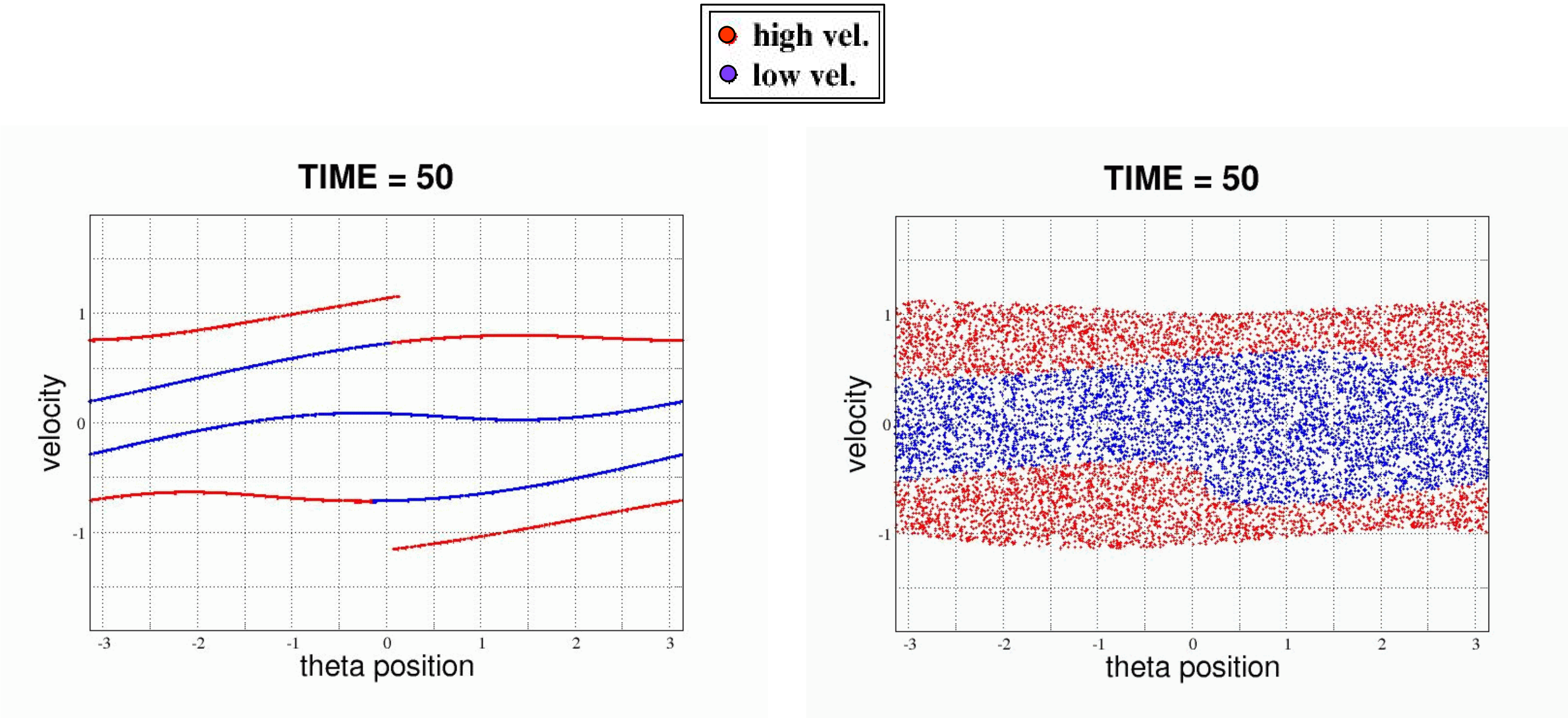
Dynamical evolution for different initial conditions



M1 IC

M0 IC

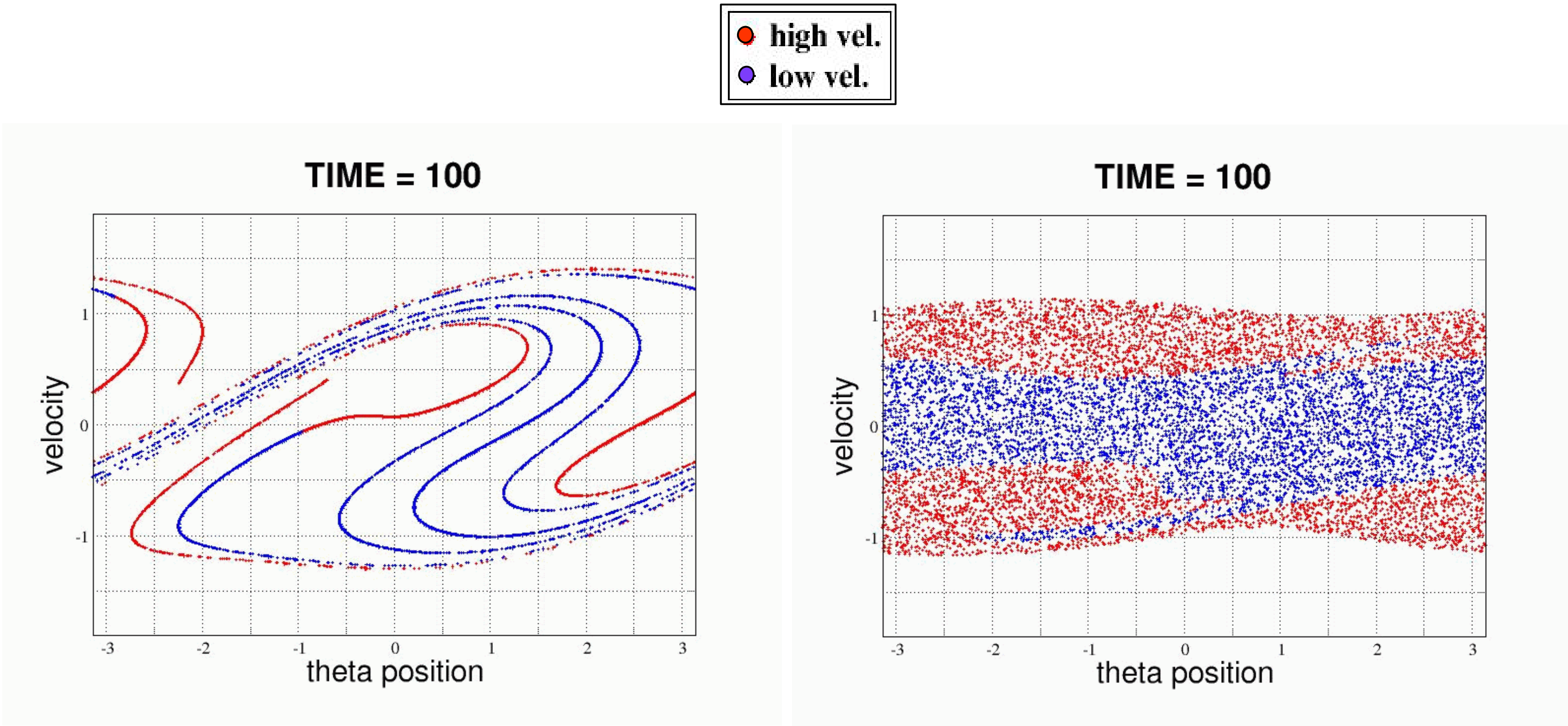
Dynamical evolution for different initial conditions



M1 IC

M0 IC

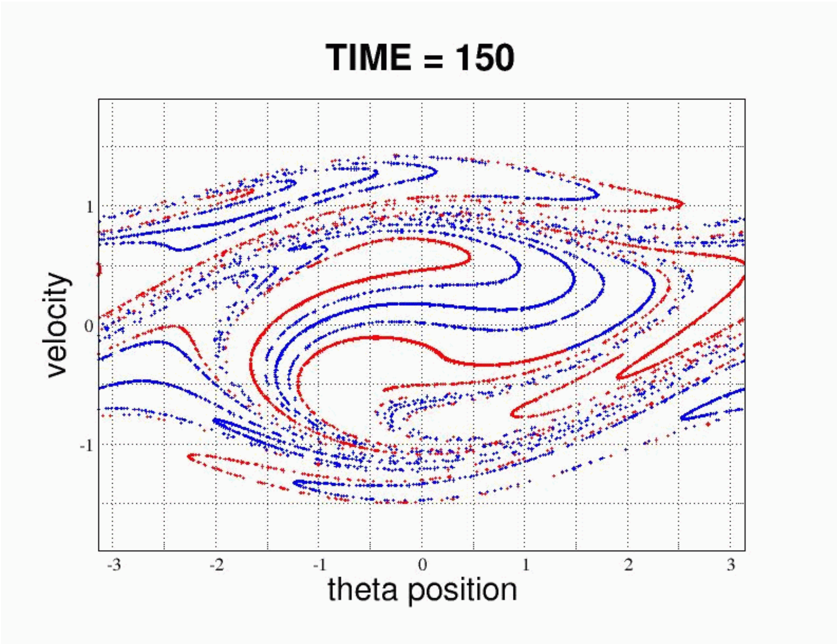
Dynamical evolution for different initial conditions

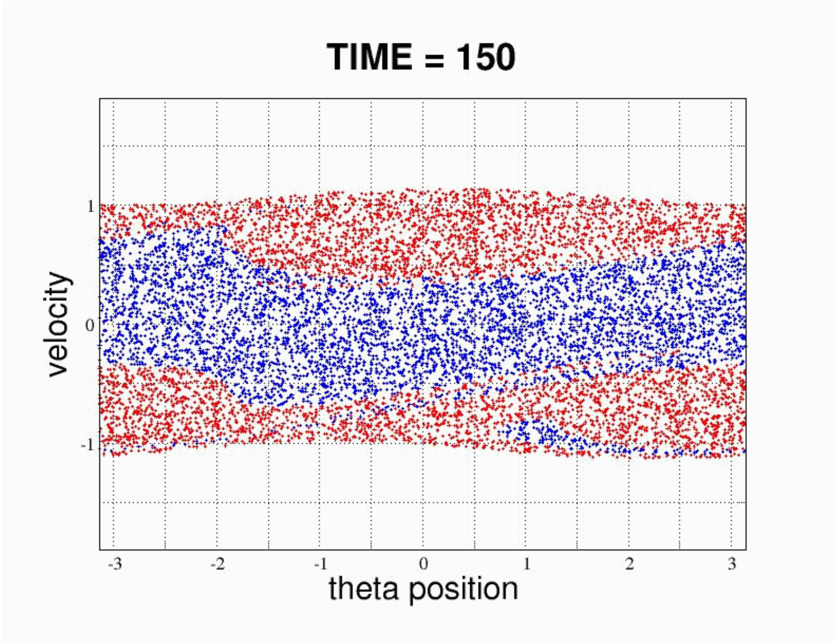


M1 IC

M0 IC

Dynamical evolution for different initial conditions

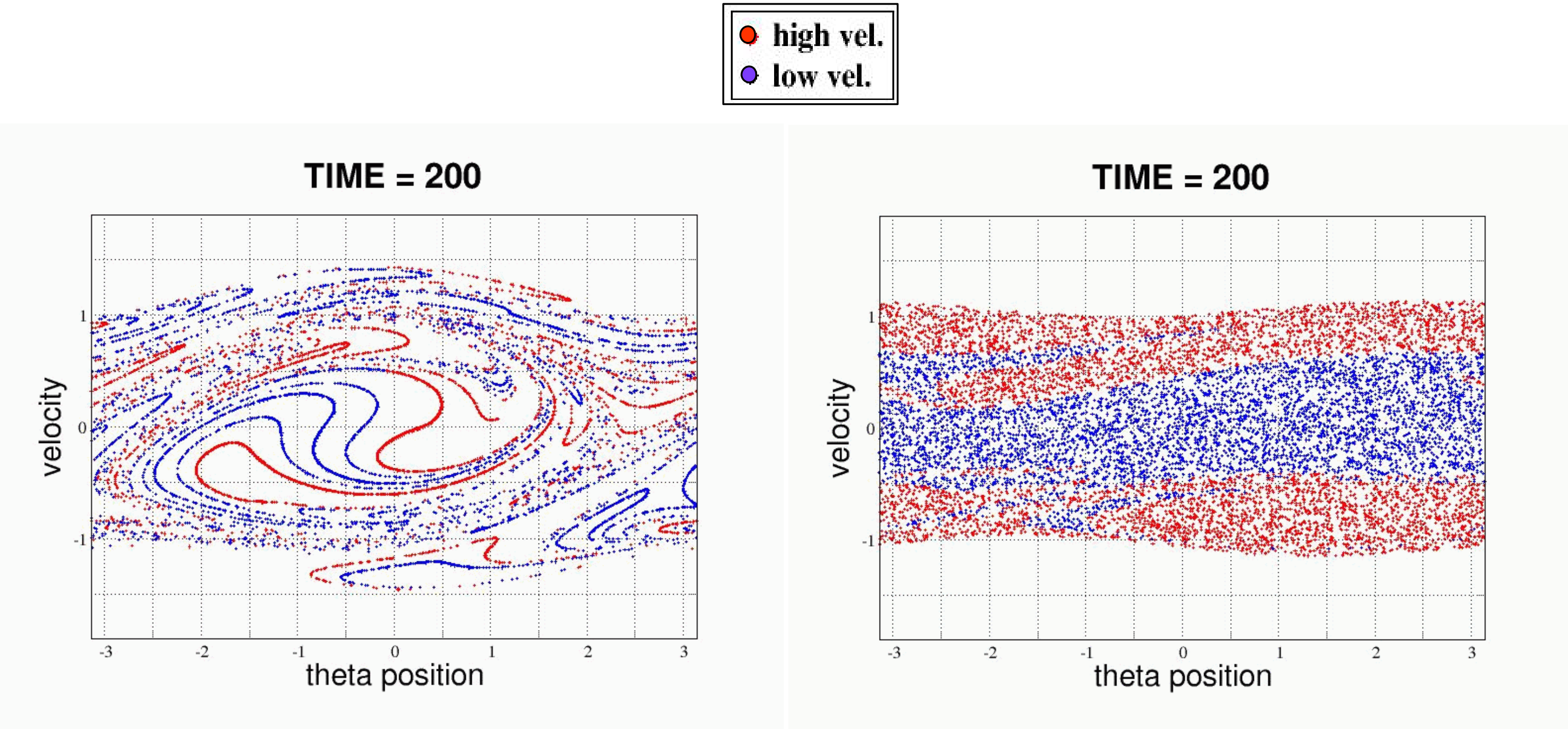




M1 IC

M0 IC

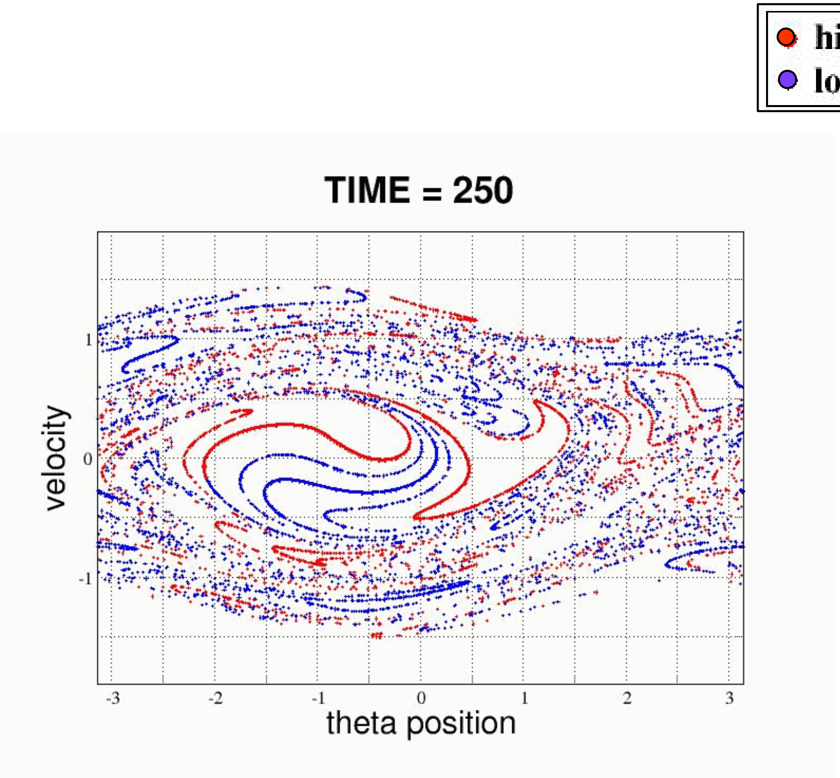
Dynamical evolution for different initial conditions

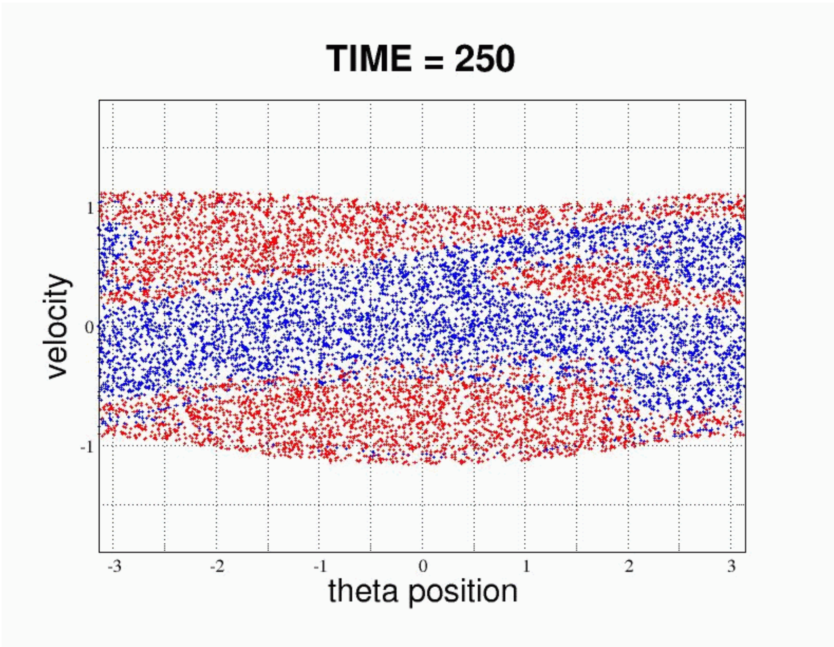


M1 IC

M0 IC

Dynamical evolution for different initial conditions

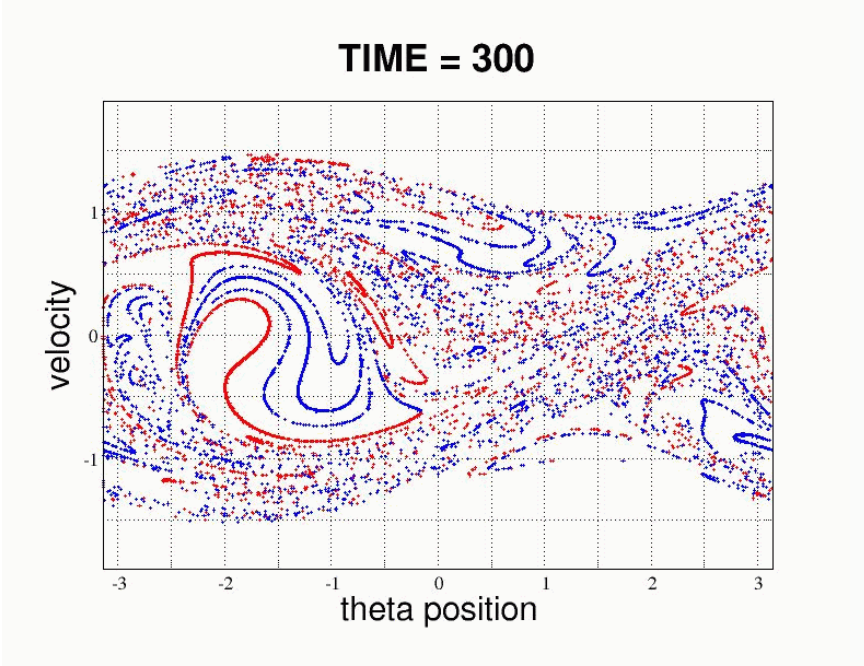


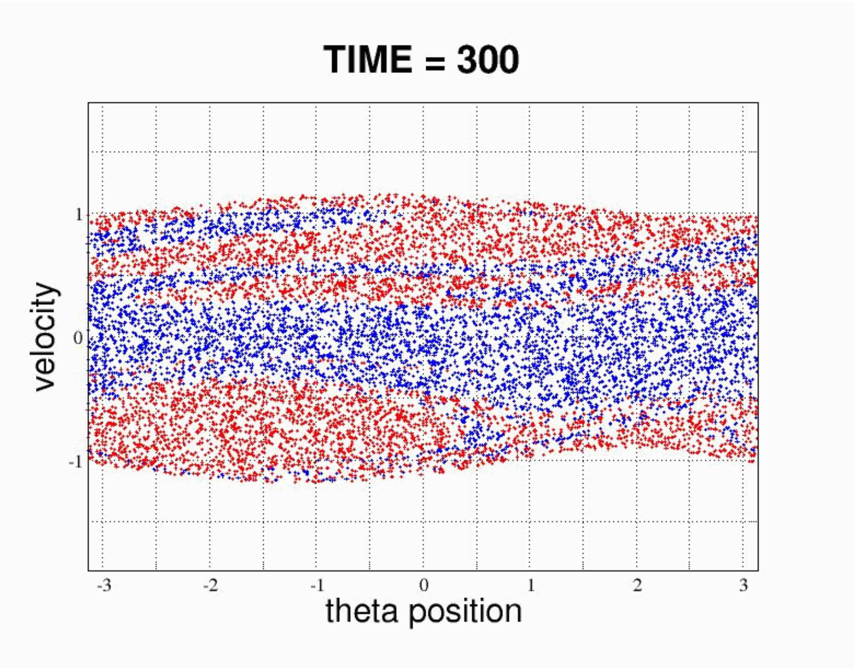


M1 IC

M0 IC

Dynamical evolution for different initial conditions

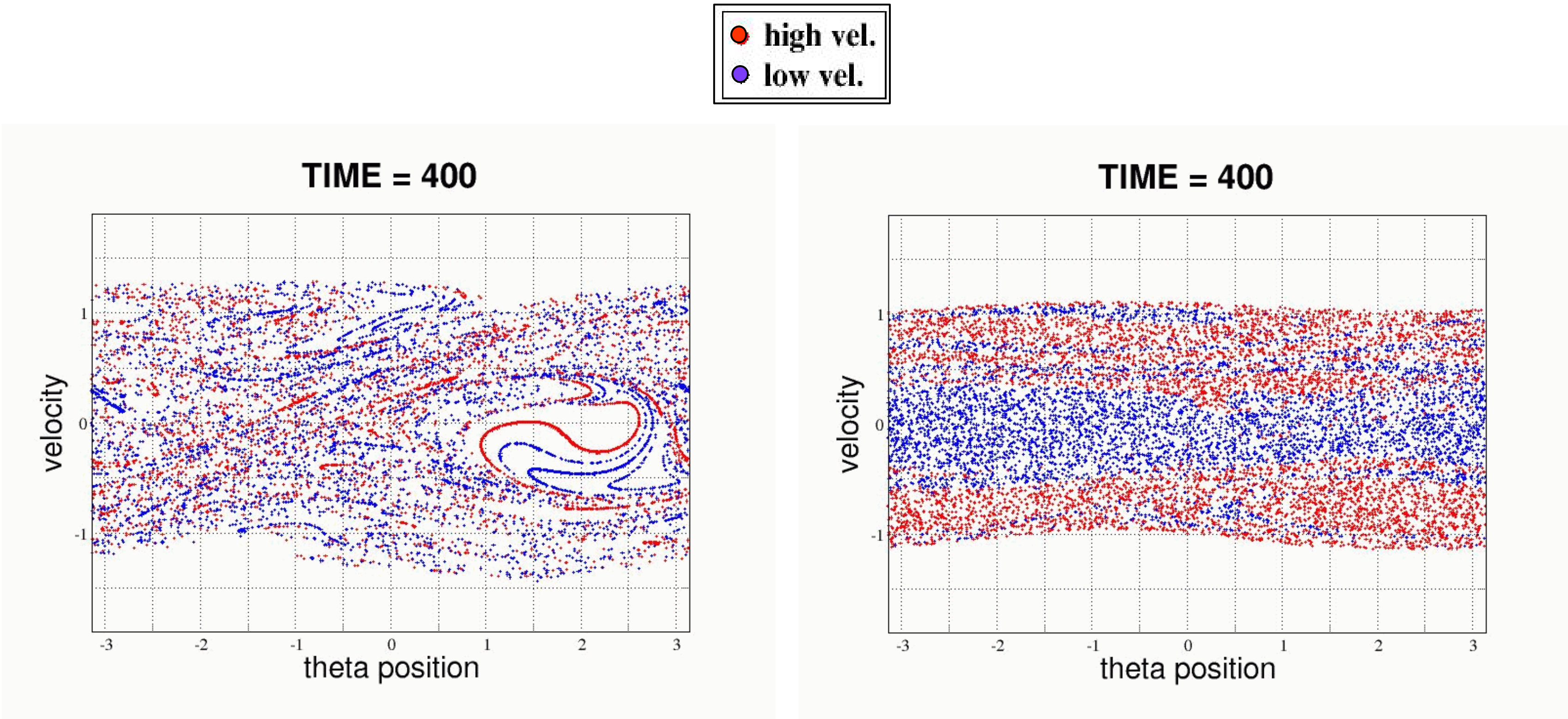




M1 IC

M0 IC

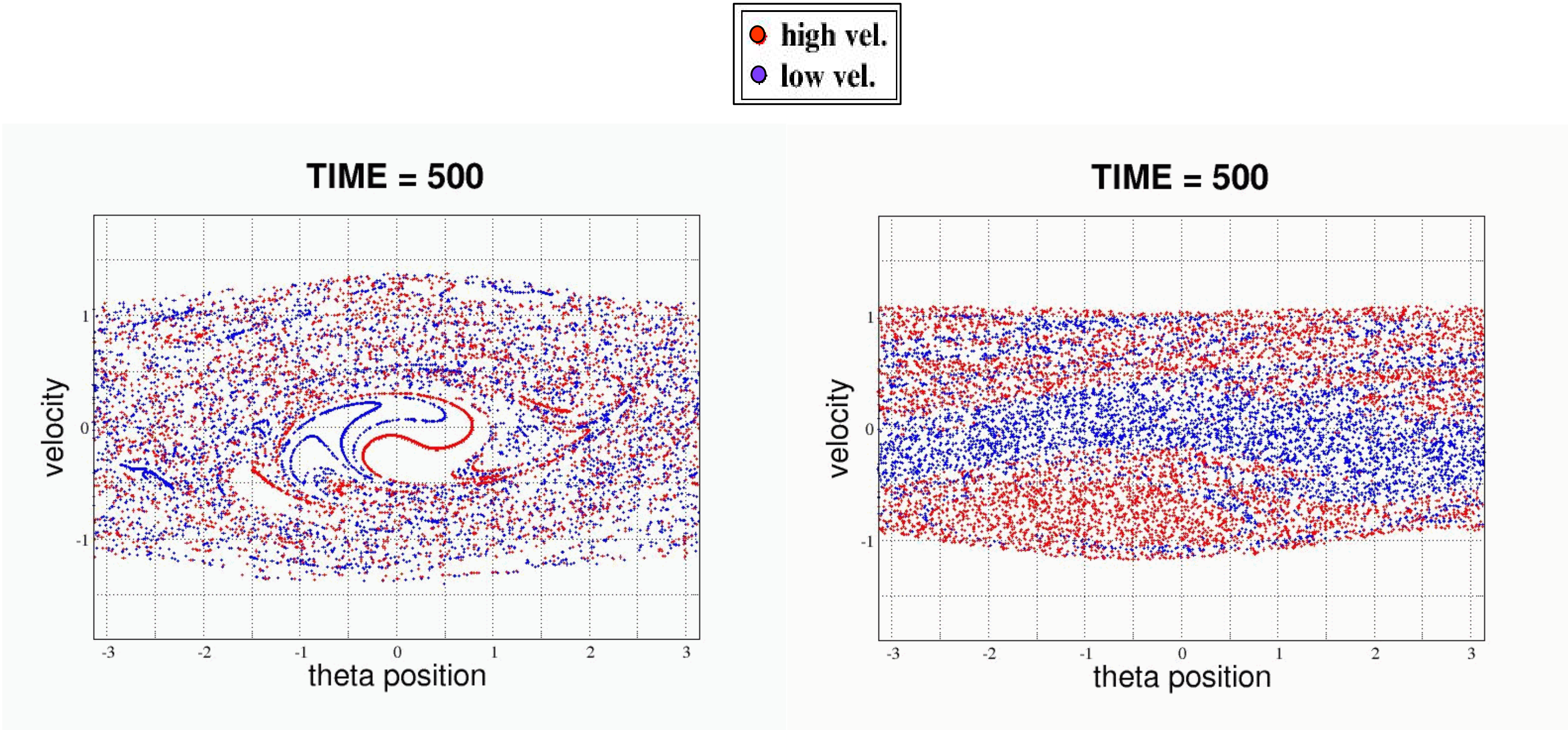
Dynamical evolution for different initial conditions



M1 IC

M0 IC

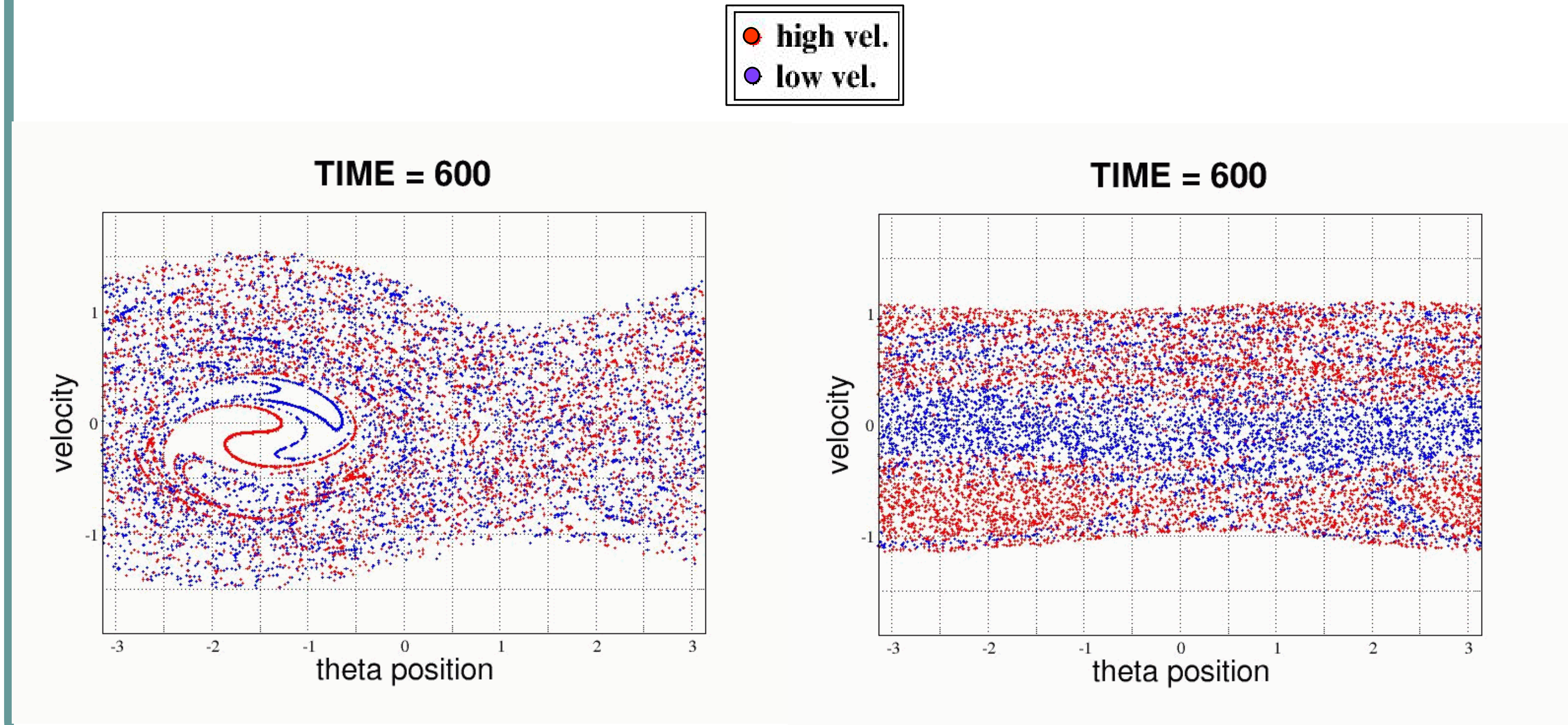
Dynamical evolution for different initial conditions



M1 IC

M0 IC

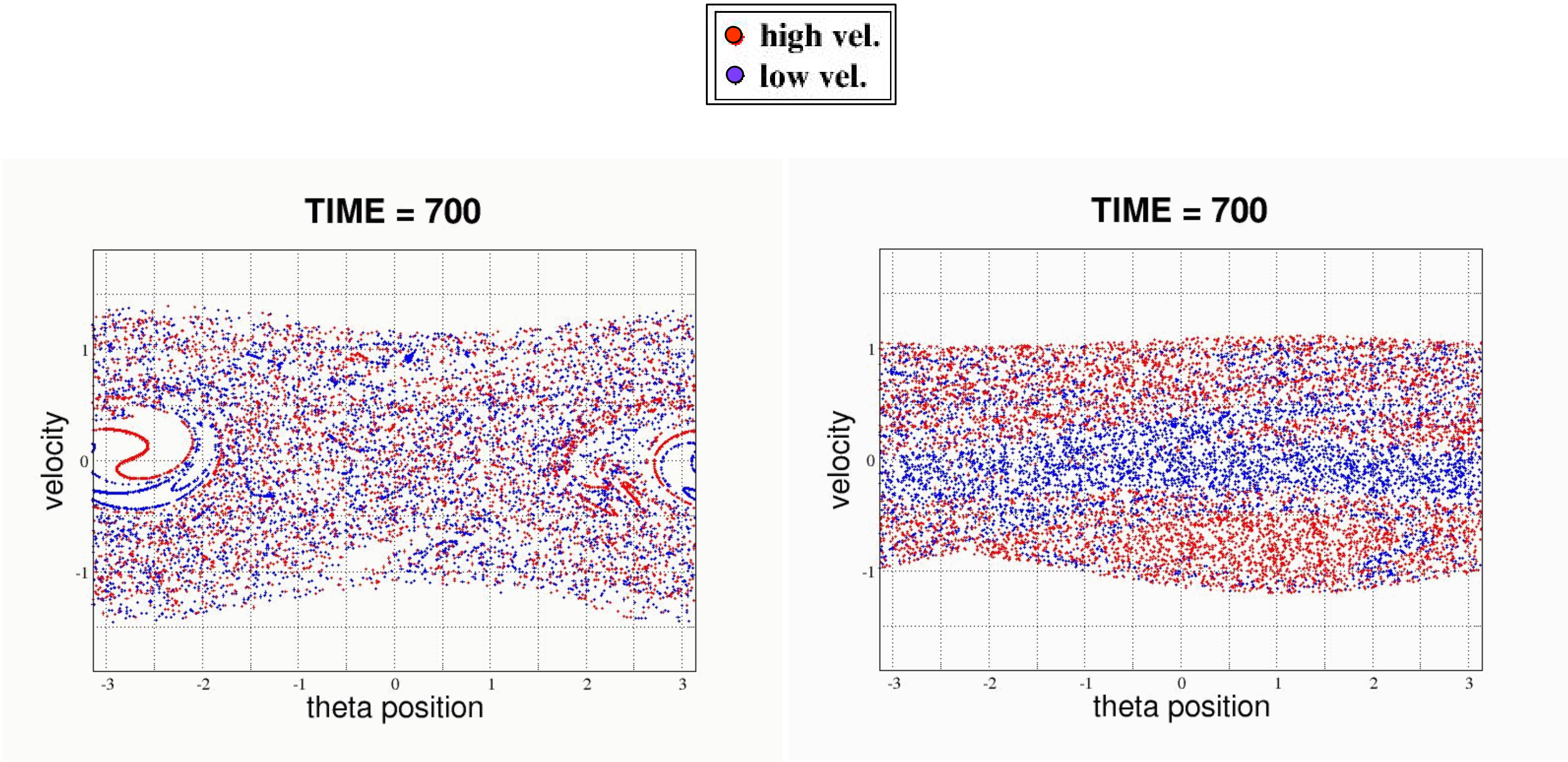
Dynamical evolution for different initial conditions



M1 IC

M0 IC

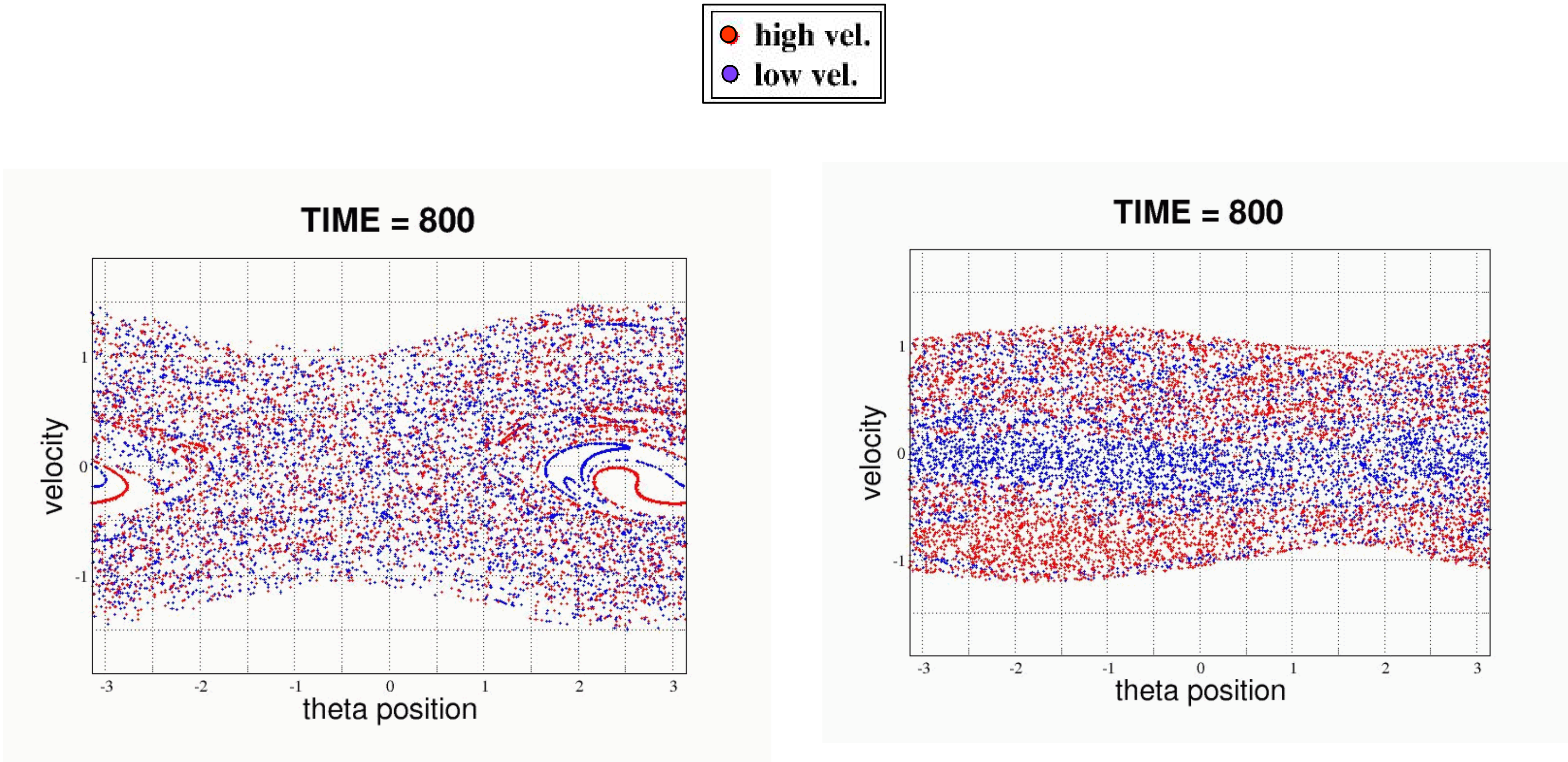
Dynamical evolution for different initial conditions



M1 IC

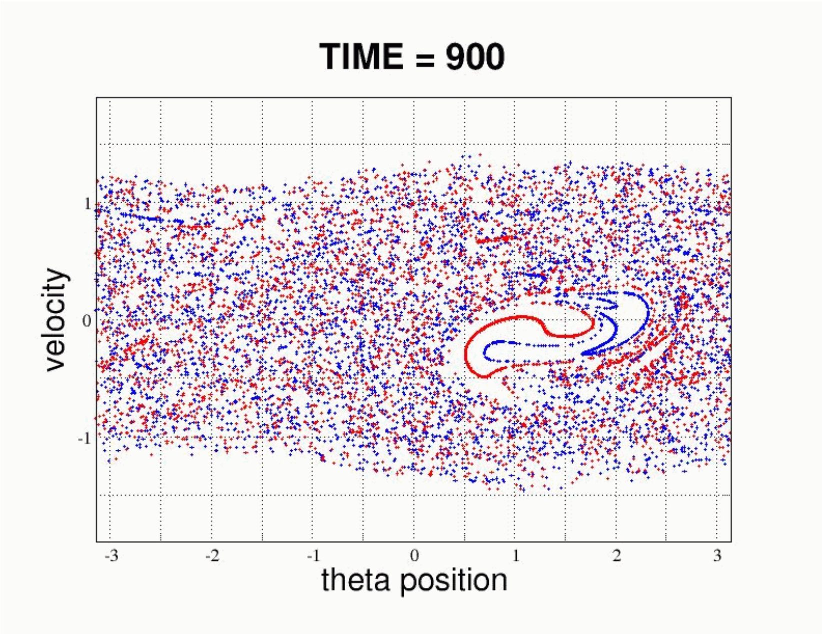
M0 IC

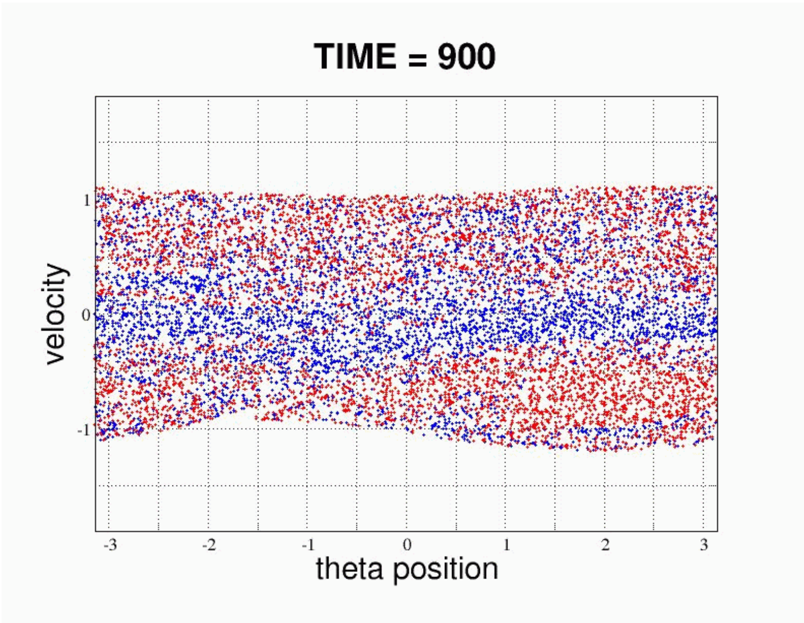
Dynamical evolution for different initial conditions



M1 IC **M0 IC**

Dynamical evolution for different initial conditions

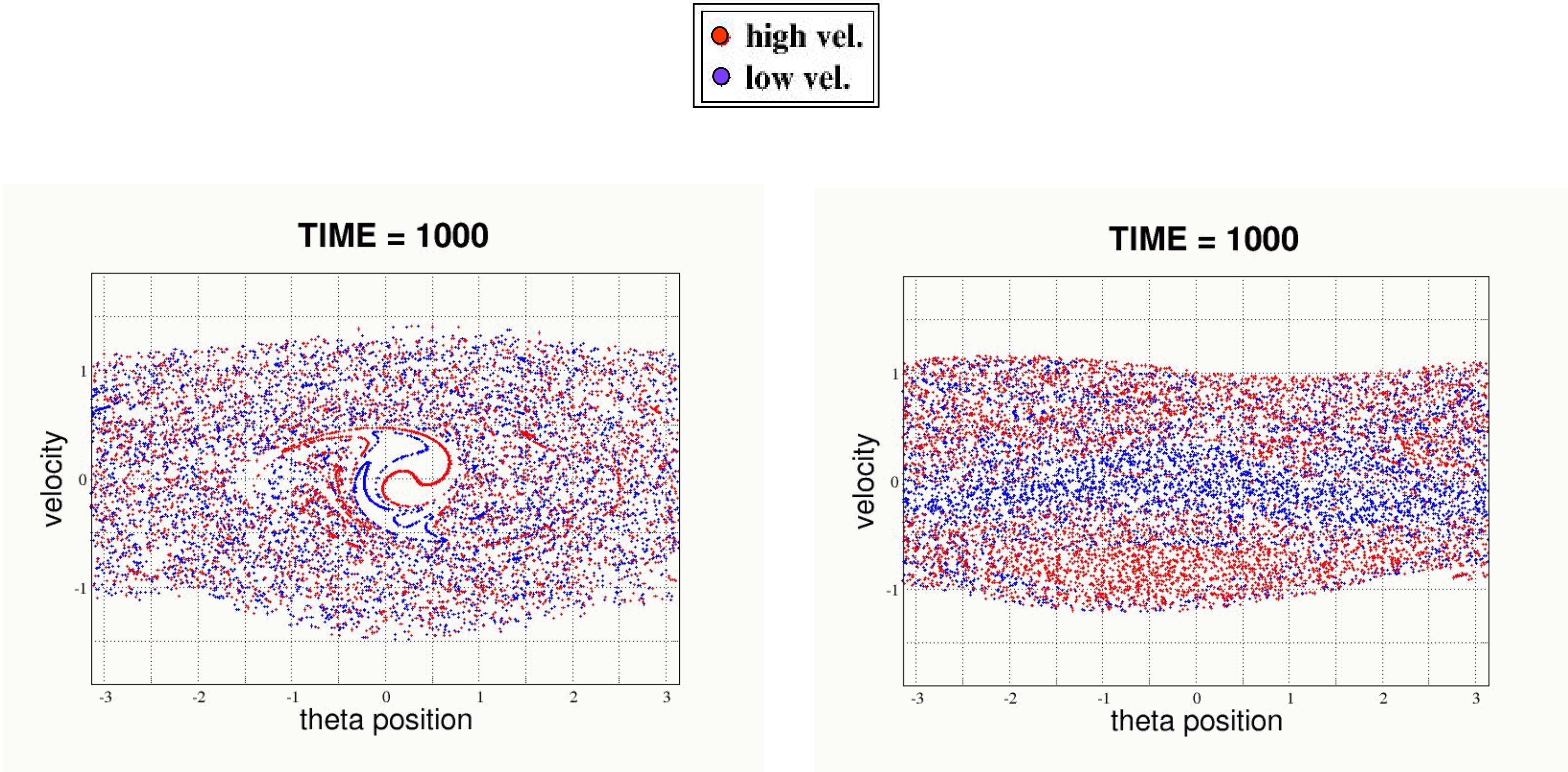




M1 IC

M0 IC

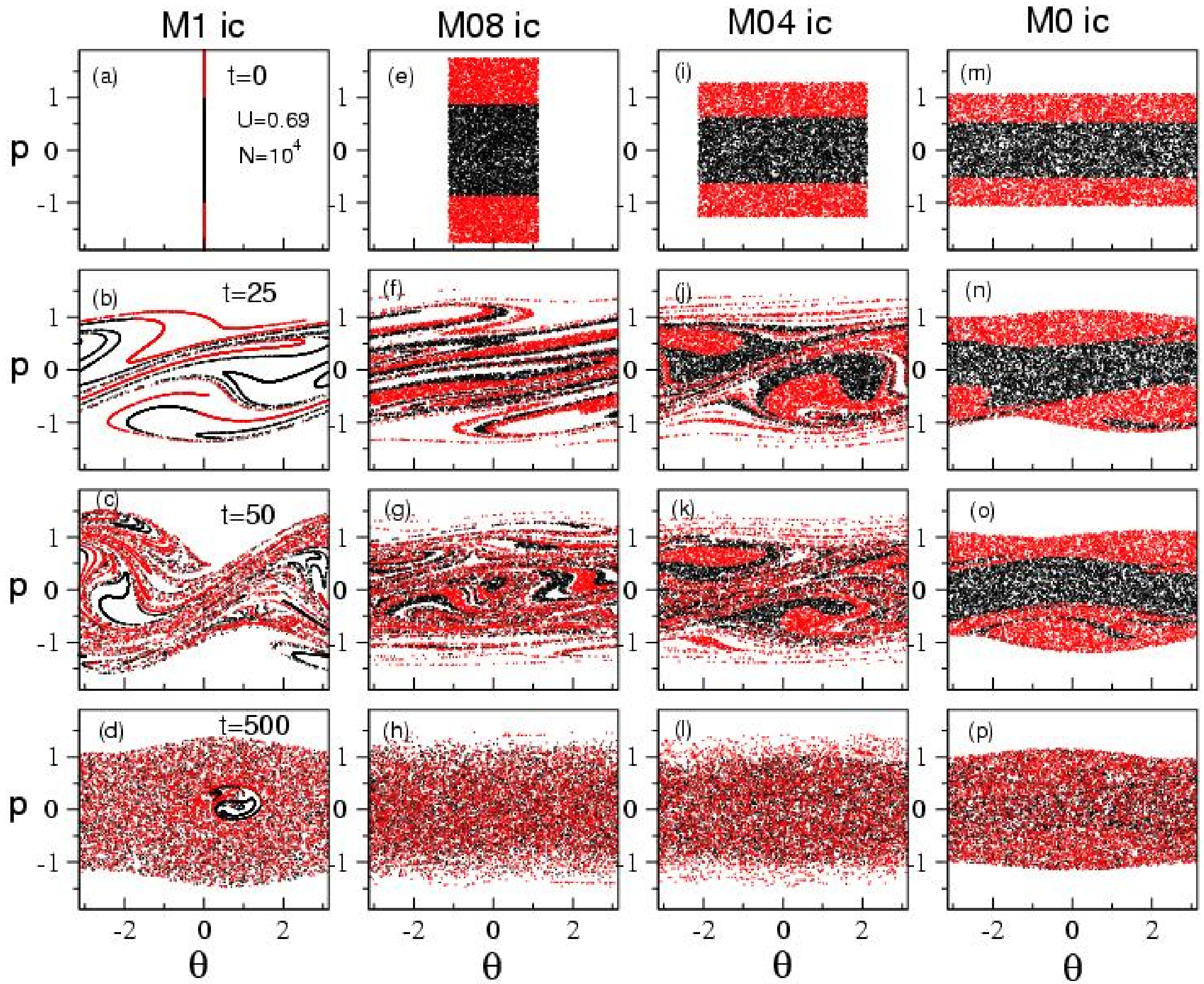
Dynamical evolution for different initial conditions



M1 IC

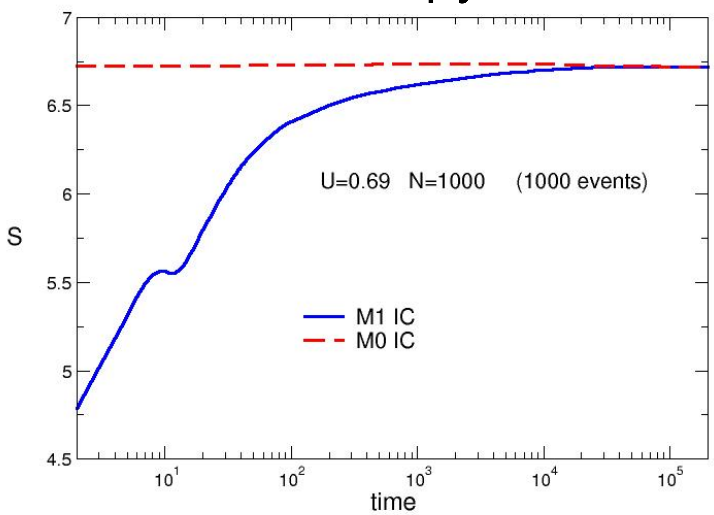
M0 IC

Dependence of μ-space structures on the i.c.

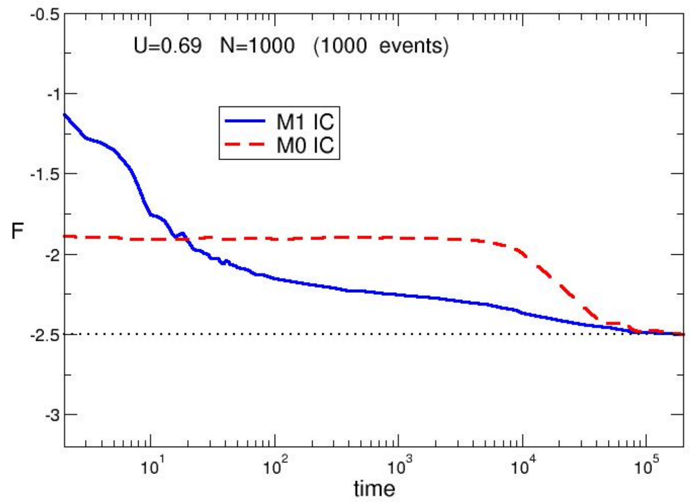


66 Entropy and free energy for QSS

Entropy

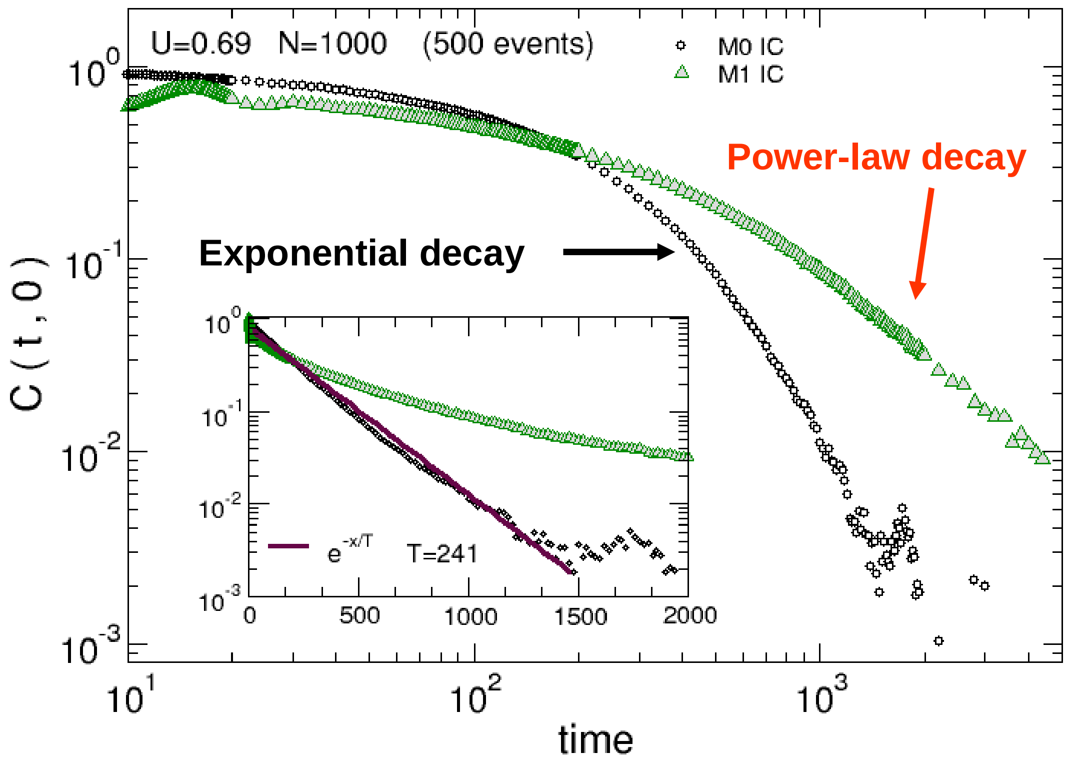


Free energy



Also the entropy and the free energy show a different dynamical evolution for the two initial conditions M1IC and M0IC

67 Relaxation to equilibrium



One can study the process of relaxation to equilibrium by means of velocity correlation functions,

$$C(t,0) = \frac{<\mathbf{P}(t)\mathbf{P}(0)> - <\mathbf{P}(t)><\mathbf{P}(0)>}{\sigma(t)\sigma(0)}$$

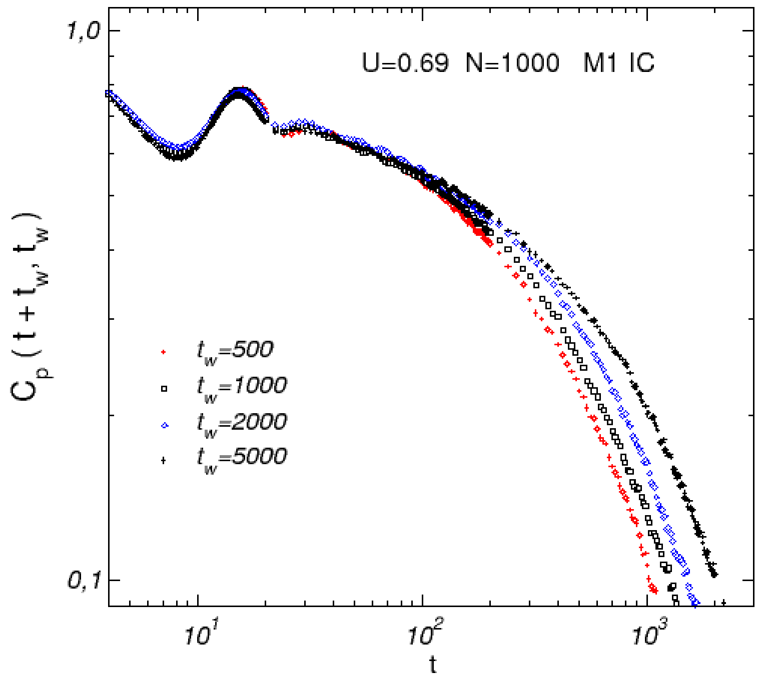
where $\mathbf{P}(t) = (p_1, p_2, \ldots p_N)$

is the velocity vector, the brackets indicate an average over the events, and $\sigma(t)$

is the standard deviation at time t.

Correlations show a power law decay for M1IC and an exponential decay for M0IC

68 Aging and strong memory effects



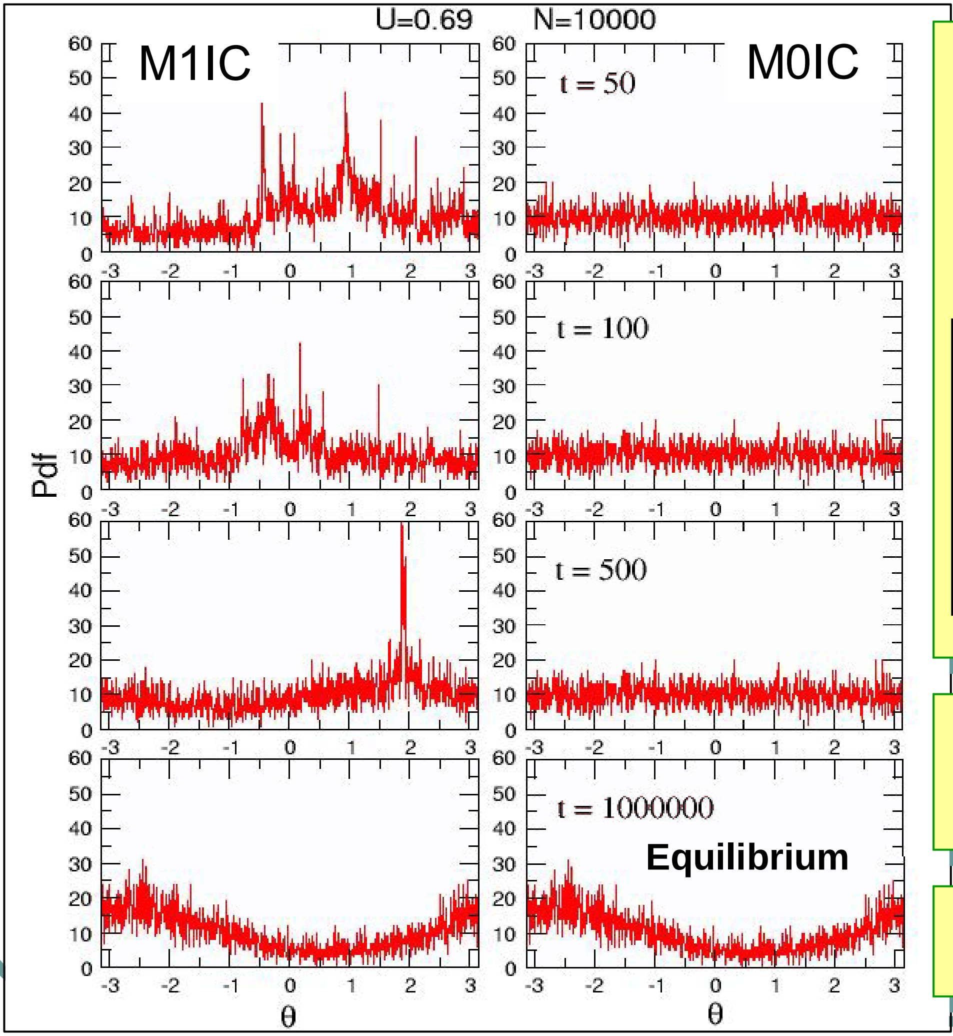
For M1 IC, the system, in going towards equilibrium, shows strong memory effects and aging, i.e. the correlation functions depend on t and on the waiting time t_w

$$C(t+t_w,t_w)=\frac{<\mathbf{P}(t+t_w)\mathbf{P}(t_w)>-<\mathbf{P}(t+t_w)><\mathbf{P}(t_w)>}{\sigma(t+t_w)\sigma(t_w)}$$

Montemurro, Tamarit and Anteneodo PRE 67 (2003) 031106

Pluchino, Latora and Rapisarda, Physica D 193 (2004) 315

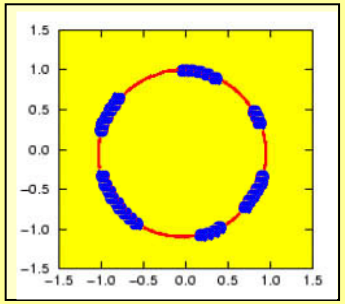
69 Dynamical frustration



M=1 initial conditions:

- **Competition between clusters of rotating particles in the QSS regime**

- **Each cluster tries to capture the maximum number of particles in order to reach the final equilibrium configuration:**



This competition leads to a *DYNAMICAL FRUSTRATION* in the QSS regime!

M=0 initial conditions:

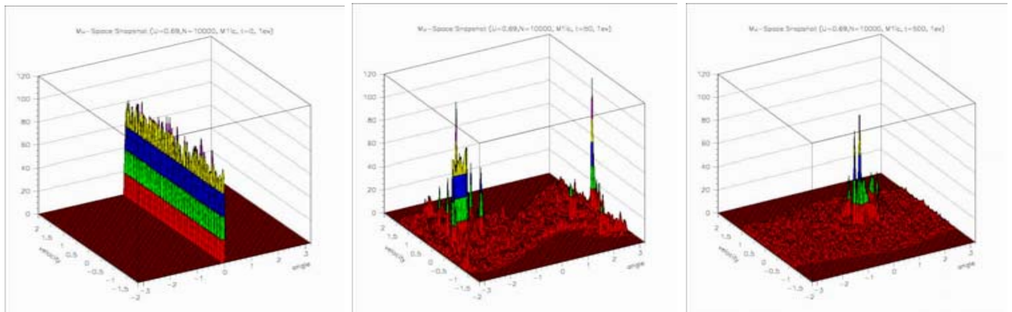
No competition and no frustration are present

At equilibrium only one big rotating cluster is present for both the IC

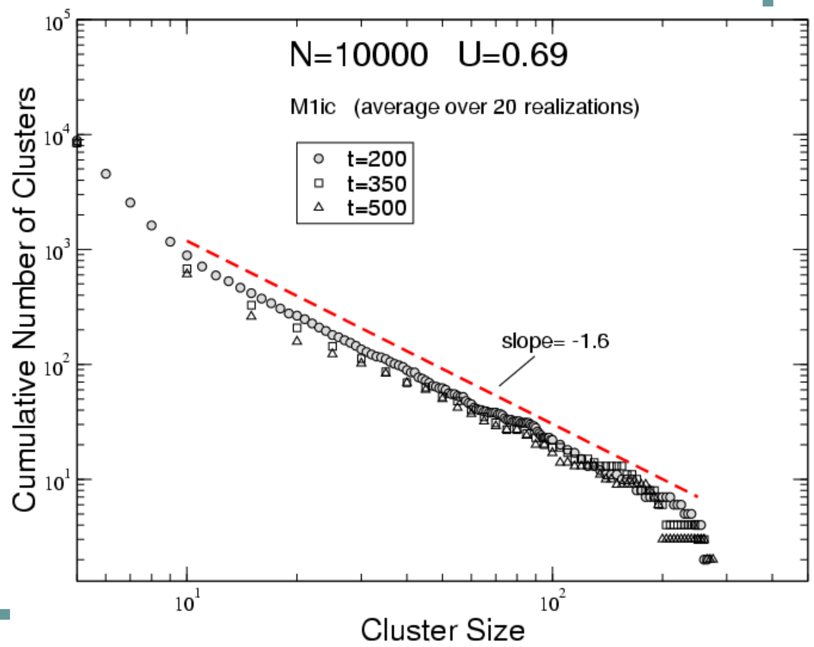
“Critical” cluster size distribution

μ-SPACE PDF

t=0 **t=50** **t=500**

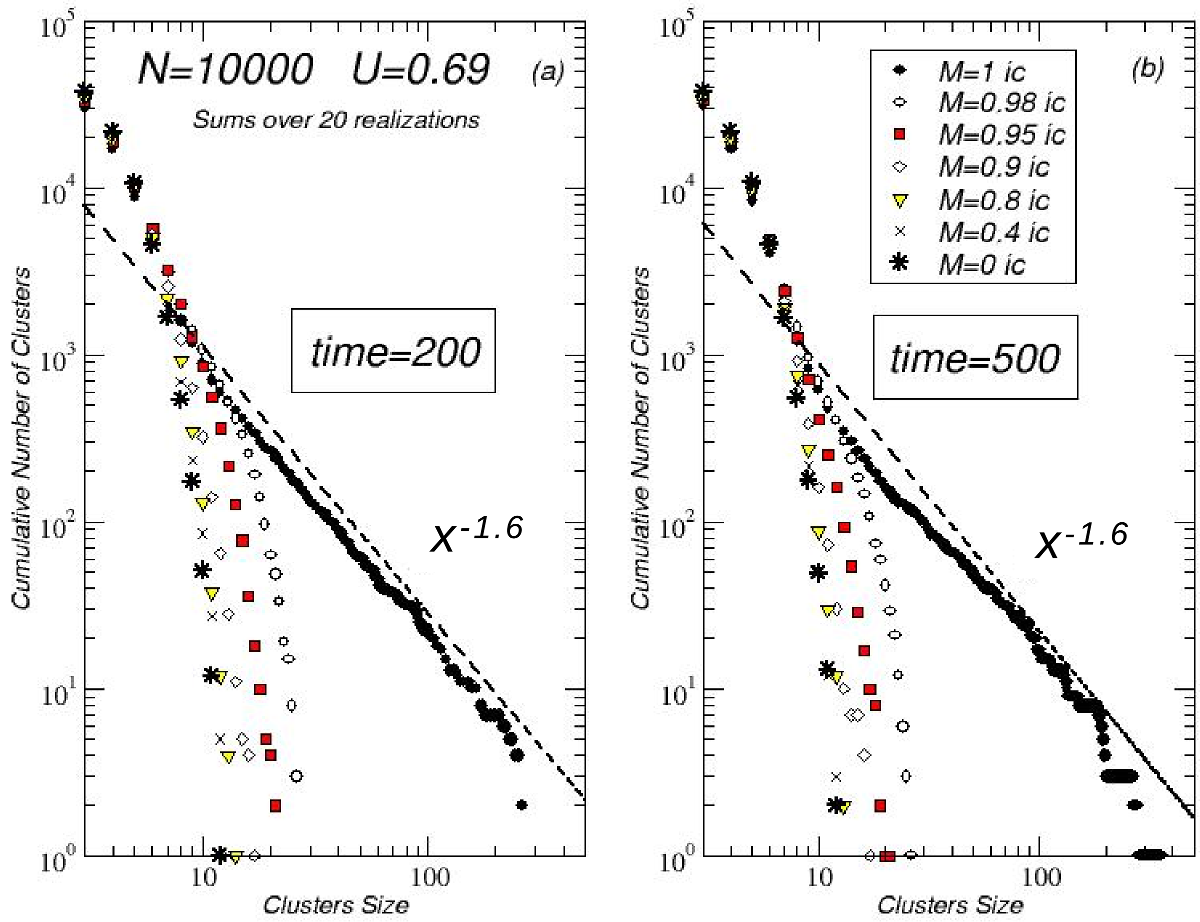


Cumulative Number of Clusters

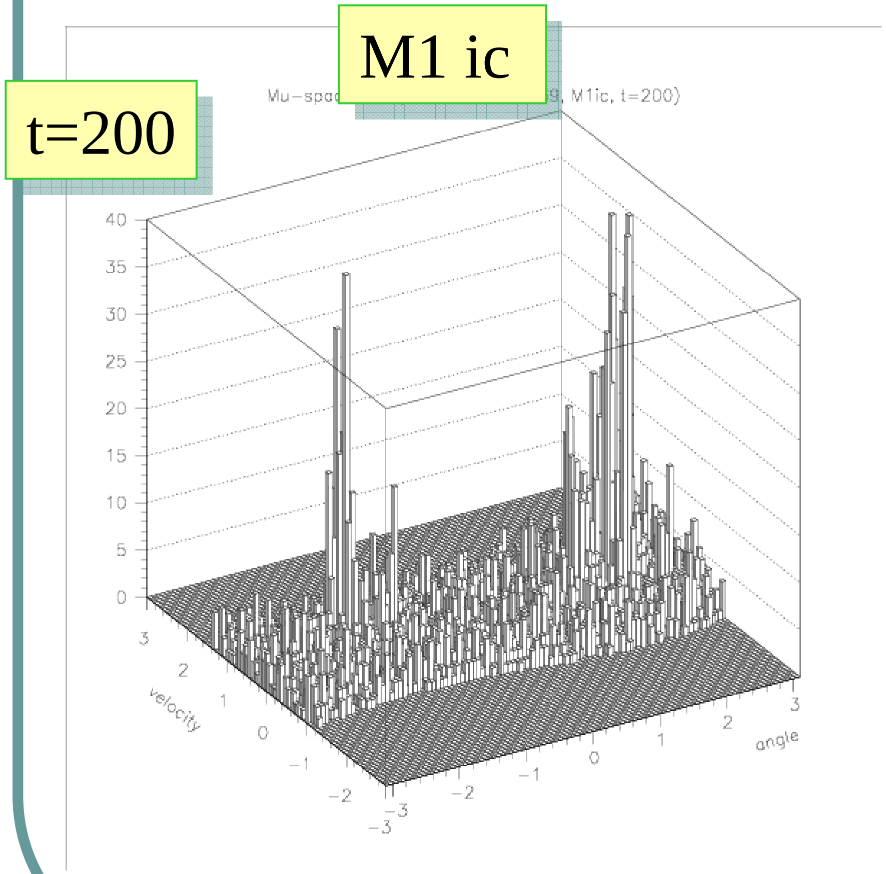


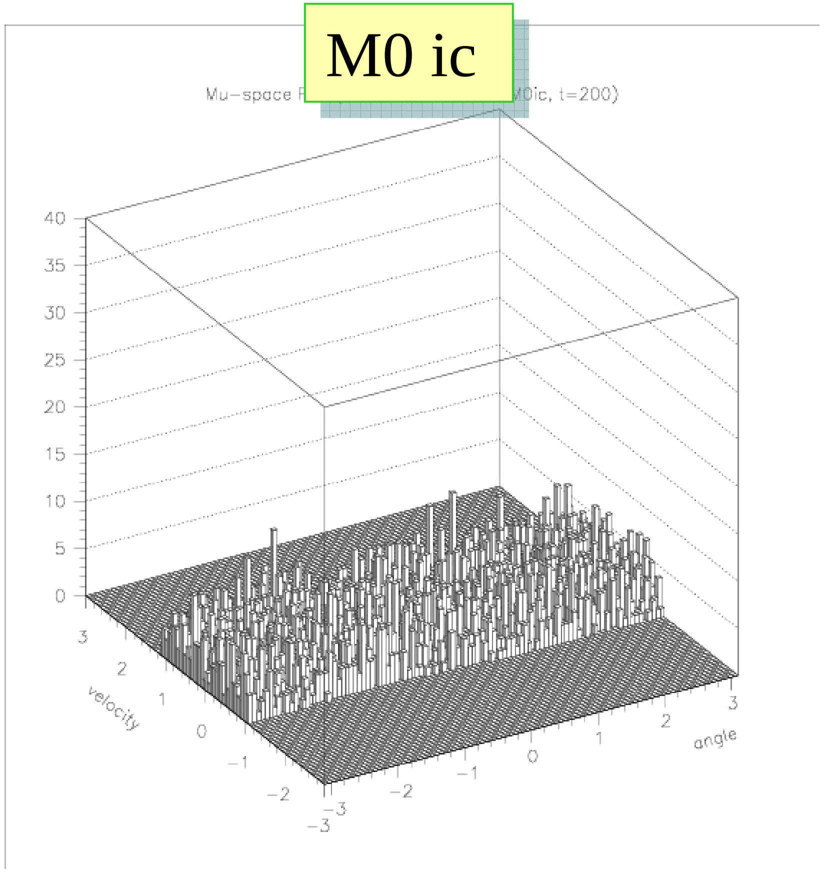
The cumulative cluster size distribution is a *power law for the QSS regime*, which is robust for the entire plateau

Cluster size distribution vs initial conditions

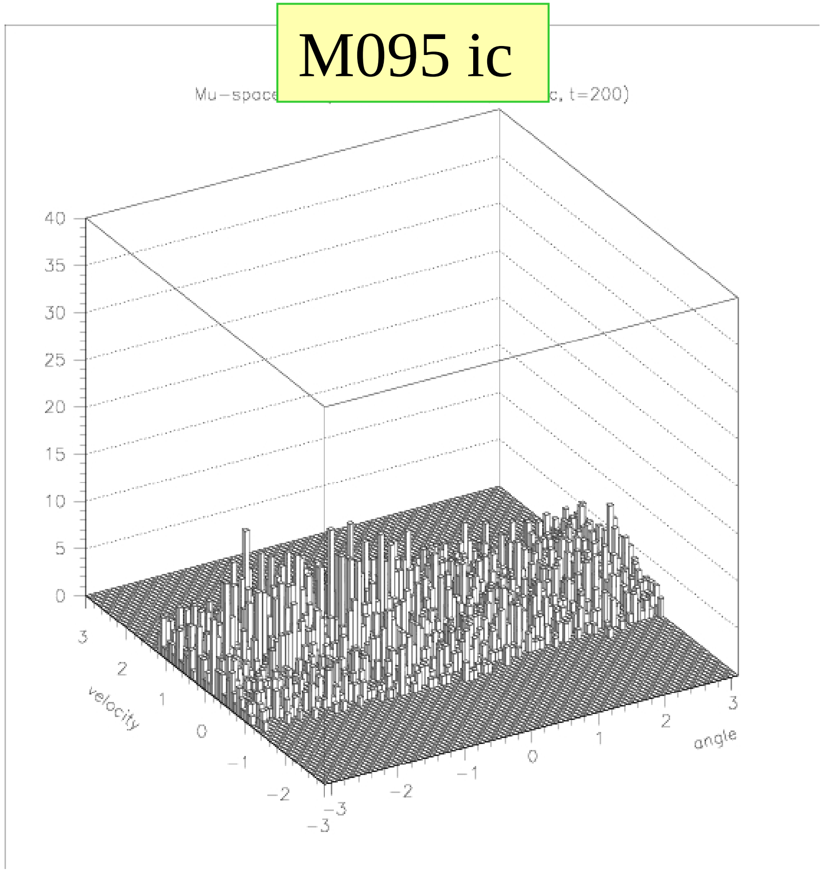


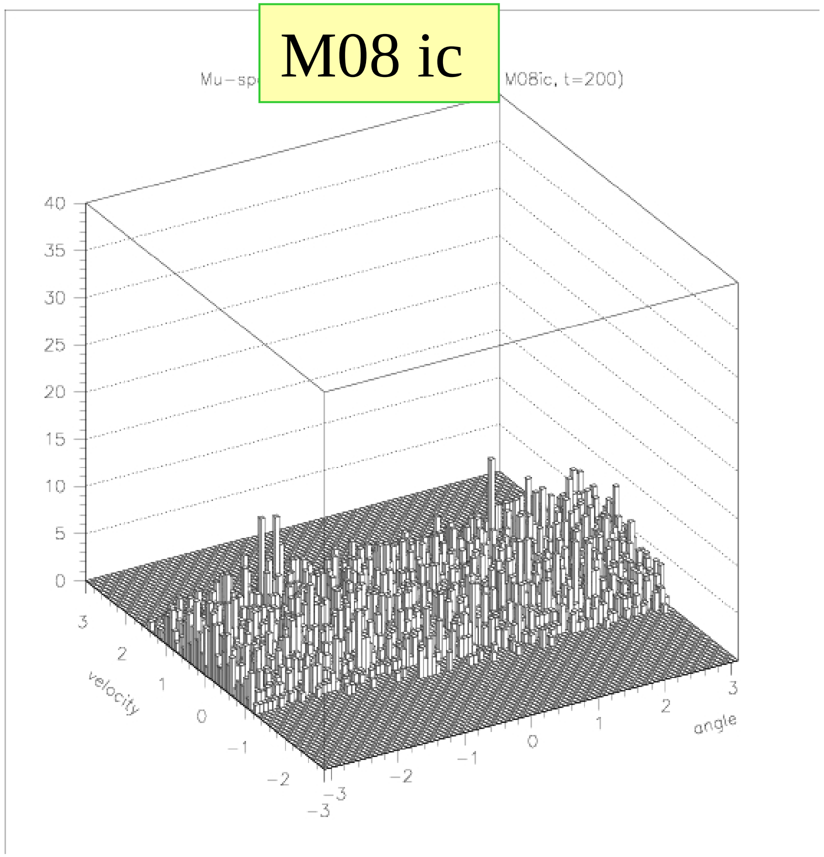
Cluster formation vs initial conditions





Cluster formation vs initial conditions





2nd lecture

The generalized HMF model

α-XY model

The HMF model has been also generalized to study the dynamic and thermodynamic behavior as a function of the range of the interaction

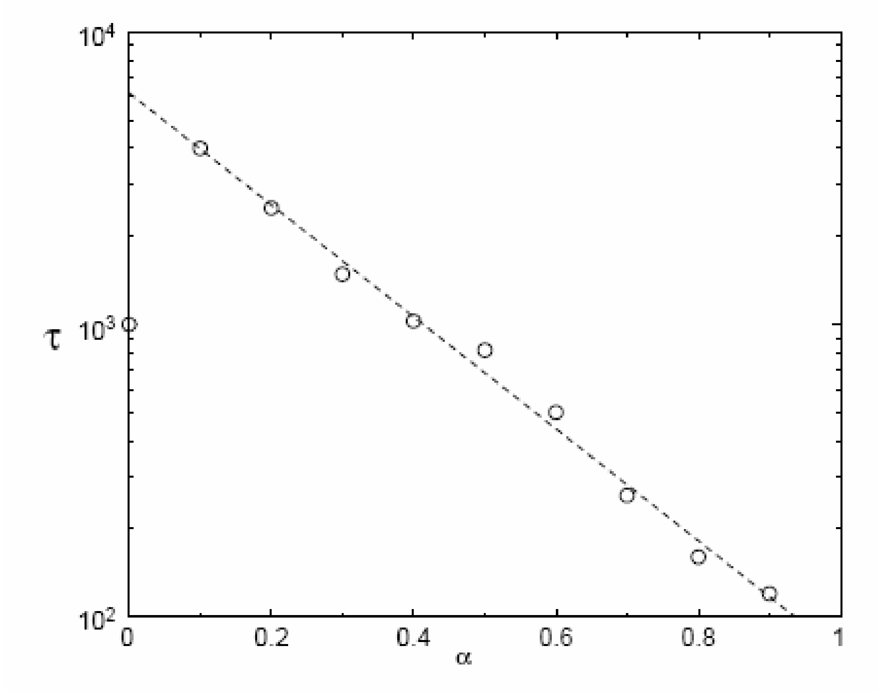
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i \neq j}^{N} \frac{\left[1 - \cos(\theta_i - \theta_j)\right]}{r_{ij}^{\alpha}}$$

- Anteneodo and Tsallis, PRL 80 (1998) 5313
- Campa, Giansanti and Moroni, PRE 62 (2000) 303
- Tamarit and Anteneodo, PRL 84 (2000) 208
- Campa, Giansanti and Moroni, J. Phys. A 36 (2003) 6897.

For $\alpha \leq d$ this generalized model reduces to HMF.

For $\alpha \to \infty$ one has interaction only among nearest neighbour spins.

α-XY model and nonextensive effects



Anomalies depend in a crucial way on the range of the interaction

The lifetime τ of the QSS does not diverge for all α…

see A. Campa et al. Physica A 305 (2002) 137

Decreasing the range of the interaction, i.e. diminishing nonextensivity $(\alpha > 0)$ this anomalous behaviour disappears:

- Relaxation is very fast $(\tau \propto e^{-\alpha})$
- No negative specific heat is observed

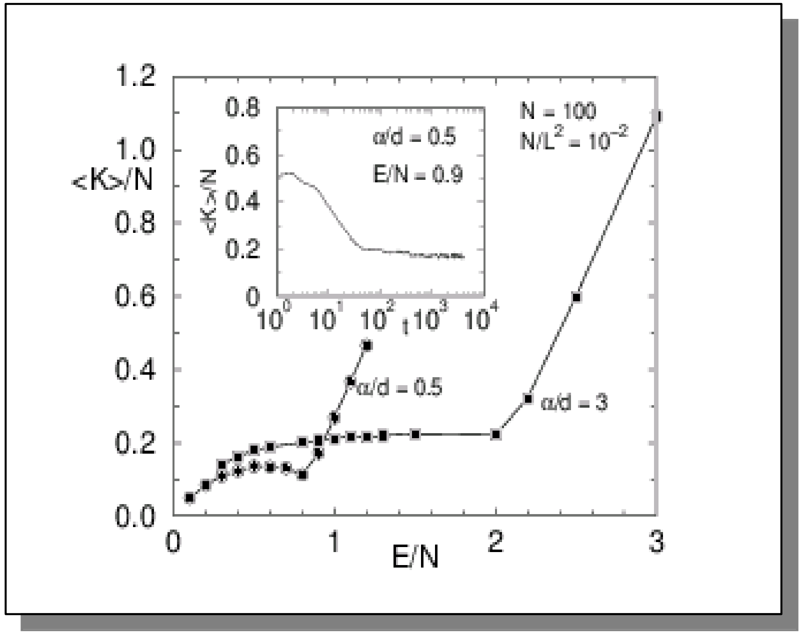
Long-range Lennard-Jones potential

Another example which support the long-range nature of the interaction as the origin of negative specific heat is the

2-d Lennard-Jones gas with attractive potential

$$V \propto r^{-\alpha}$$

Borges and Tsallis Physica A 305 (2002)148



Again decreasing the range of the interaction, i.e. *diminishing non extensivity* this anomalous specific heat disappears.
In correspondence they also find *non-Boltzmann velocity distributions*

Further similar examples have been found in self-gravitating systems, see

Sota et al PRE 64 (2001) 056133

More precisely the potential energy is

$$V = C_\alpha \sum_{i<j}^{N} \left[\left(\frac{\sigma}{r_{ij}} \right)^{12} - \left(\frac{\sigma}{r_{ij}} \right)^{\alpha} \right]$$

$$C_\alpha = \varepsilon \left(\frac{b}{\alpha^\alpha} \right)^{1/(12-\alpha)} / N(12-\alpha)$$

σ is the diameter of the particles, b is a constant and ε is the energy scale.

For α=6 one gets the usual Lennard-Jones

Interpretration of QSS regime

The anomalous QSS regime is the effect of non extensivity or, in other words, of the long-range character of the interaction.

These anomalies seems be connected to Tsallis thermostatistics

Very recently a link with a glassy phase has also been found

80 Tsallis generalized thermostatistics

In the last decade a lot of effort has been devoted to understand if thermodynamics can be generalized to nonequilibrium complex systems

In particular one of these attempts is that one started by Constantino Tsallis with his seminal paper J. Stat. Phys. 52 (1988) 479

For recent reviews see for example:

- C. Tsallis, ***Nonextensive Statistical Mechanics and Thermodynamics*** , **Lecture Notes in Physics**, eds. S. Abe and Y. Okamoto, Springer, Berlin, (2001);
- Proceedings of ***NEXT2001***, special issue of Physica A 305 (2002) eds. G. Kaniadakis, M. Lissia and A Rapisarda;
- C. Tsallis, A. Rapisarda, V. Latora and F. Baldovin in ***"Dynamics and Thermodynamics of Systems with Long-Range Interactions",*** T. Dauxois, S. Ruffo,E. Arimondo, M. Wilkens eds., **Lecture Notes in Physics** Vol. 602, Springer (2002) 140;
- ***``Nonextensive Entropy - Interdisciplinary Applications"***, C. Tsallis and M. Gell-Mann eds., **Oxford University Press** (2003).
- A. Cho, **Science** 297 (2002) 1268; S. Abe and A.K. Rajagopal, **Science** 300, (2003)249; A. Plastino, **Science** 300 (2003) 250; V. Latora, A. Rapisarda and A. Robledo, **Science** 300 (2003) 250.

For a regularly updated bibliografy: http://tsallis.cat.cbpf.br/biblio.htm

Tsallis conjecture for nonextensive systems

C.Tsallis, Braz. Jour. of Phys. 29 (1999) 1

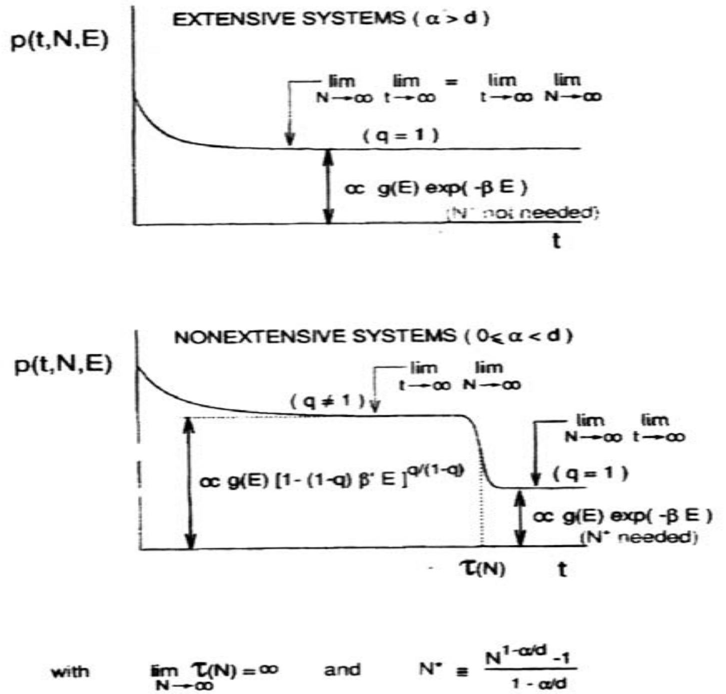
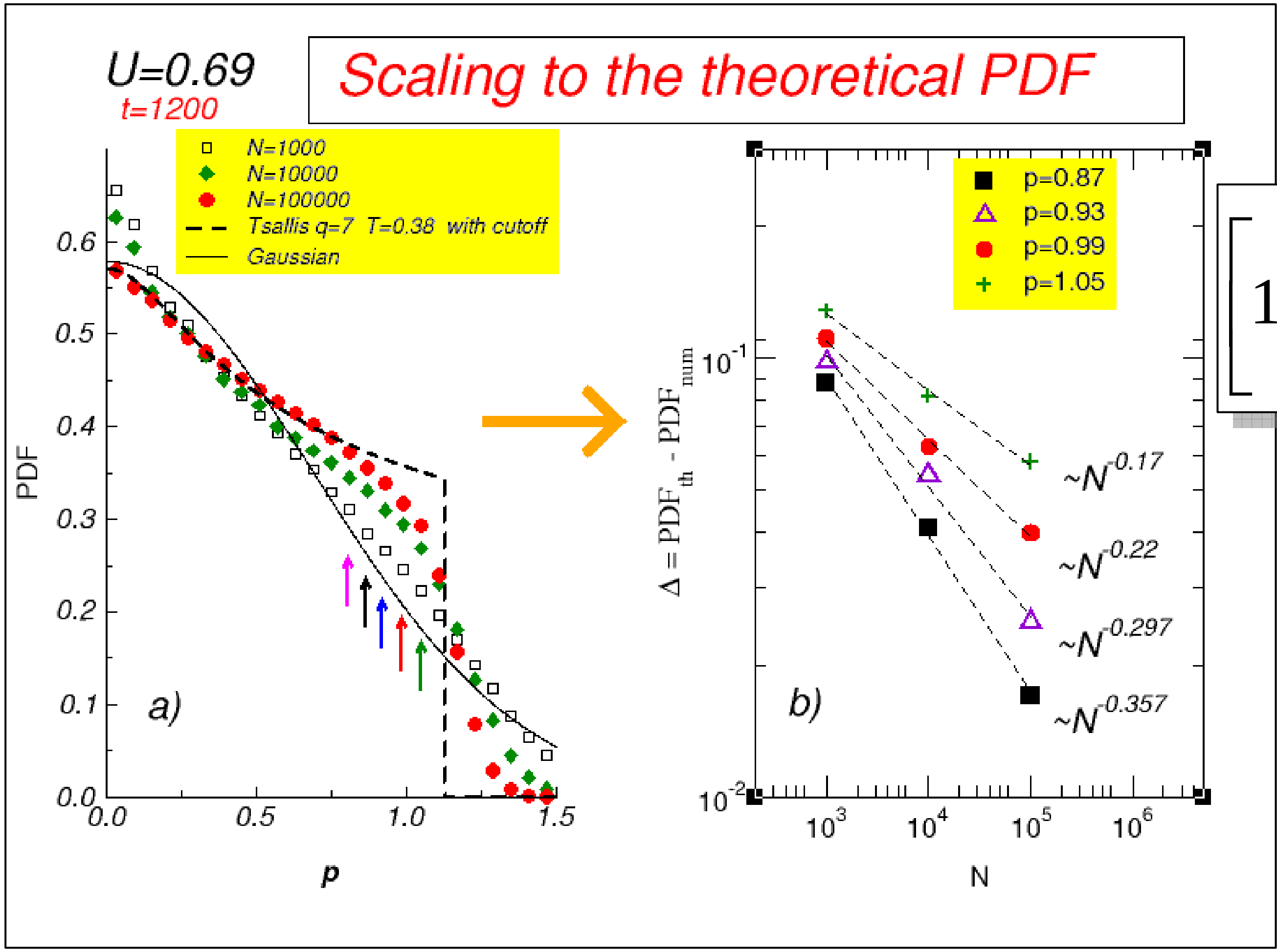


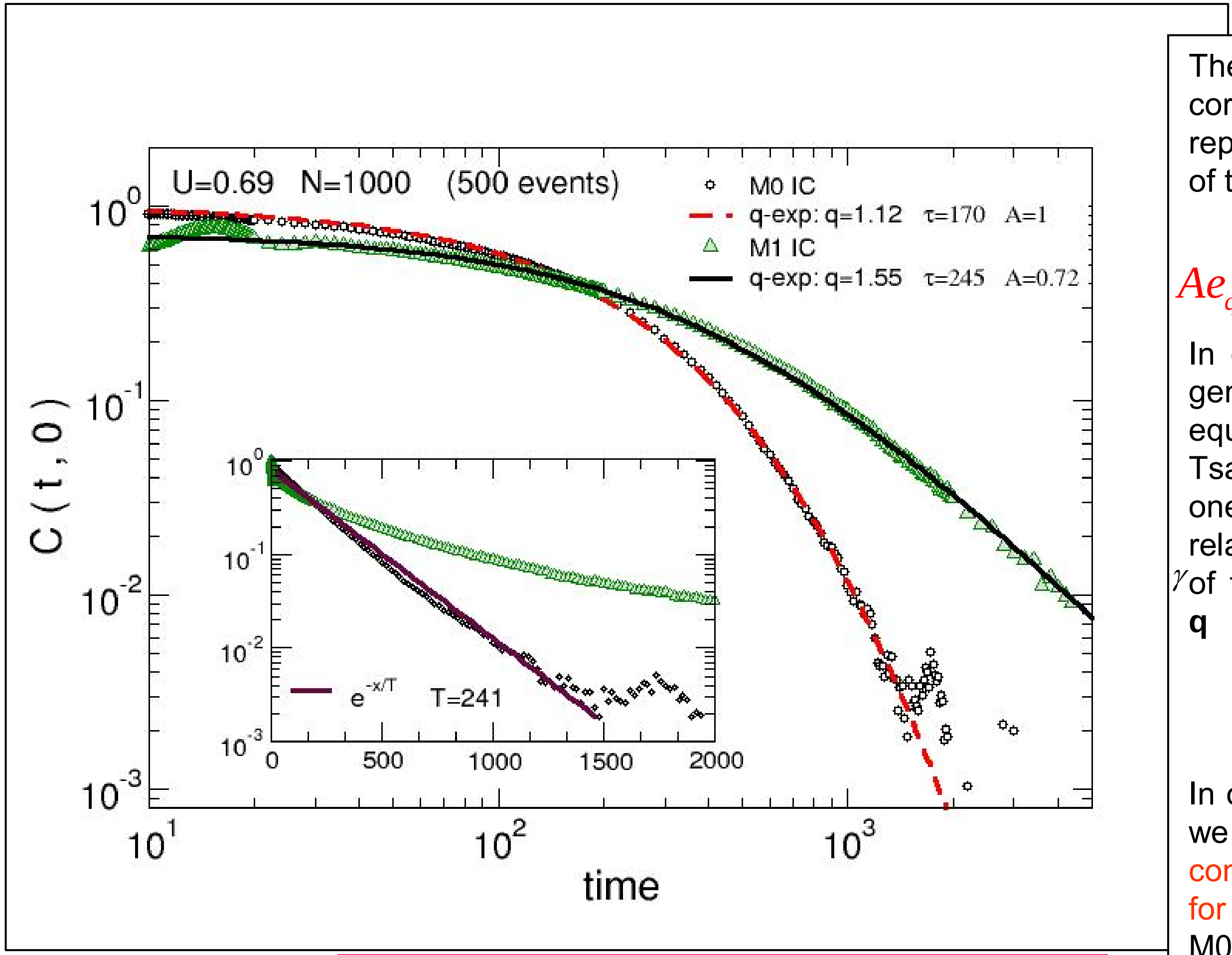
Figure 4. Central conjecture of the present work, assuming a Hamiltonian system which includes two-body (attractive) interactions which, at long distances, decay as $r^{-\alpha}$. The crossover at $t = \tau$ is expected to be slower than indicated in the figure (for space reasons).

82 Generalized velocity pdfs



$$\left[1-(1-q)\frac{p^2}{2T}\right]^{\frac{1}{1-q}}$$

83 q-exponential decay of C(t,0)



The decay of the velocity correlation function can be reproduced very well by means of the generalized q-exponential

$$Ae_q(x) = A[1+(1-q)x]^{\frac{1}{1-q}}$$

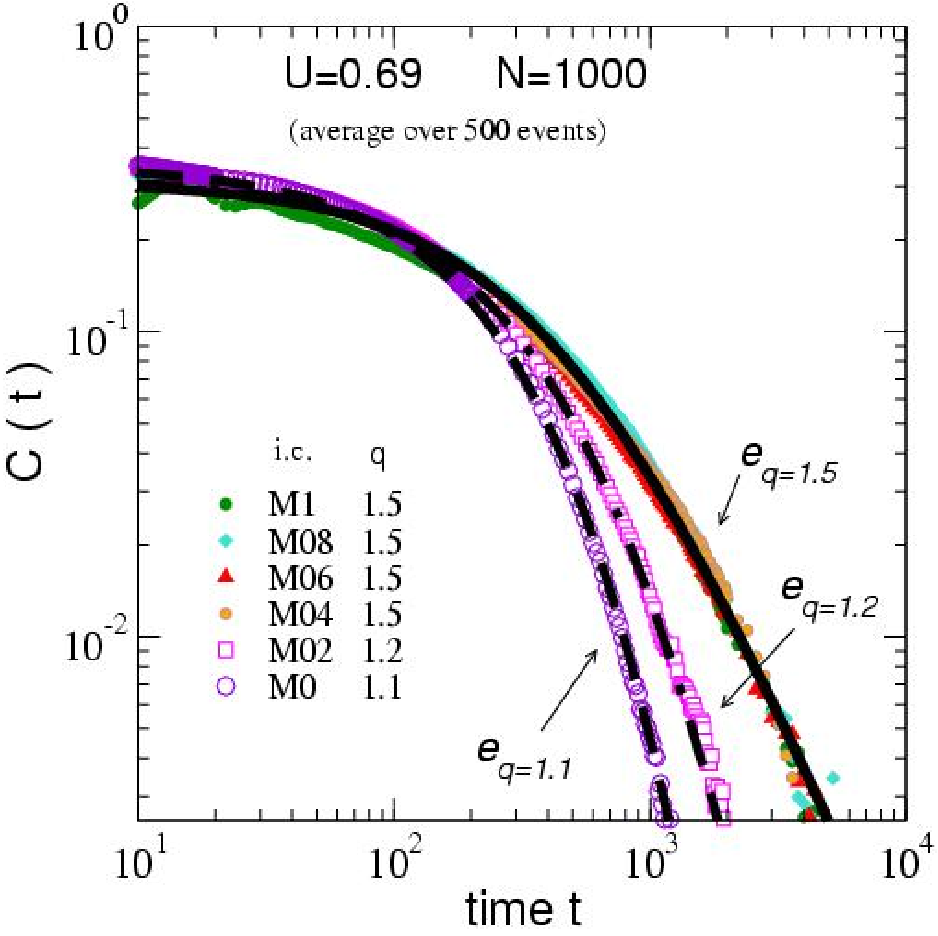
In our case $x=-t/\tau$. Within a generalized Fokker-Plank equation which generates Tsallis q-exponential pdfs [1], one can extract the following relation between the exponent γ of the anomalous diffusion and **q**

$$\gamma = \frac{2}{3-q}$$

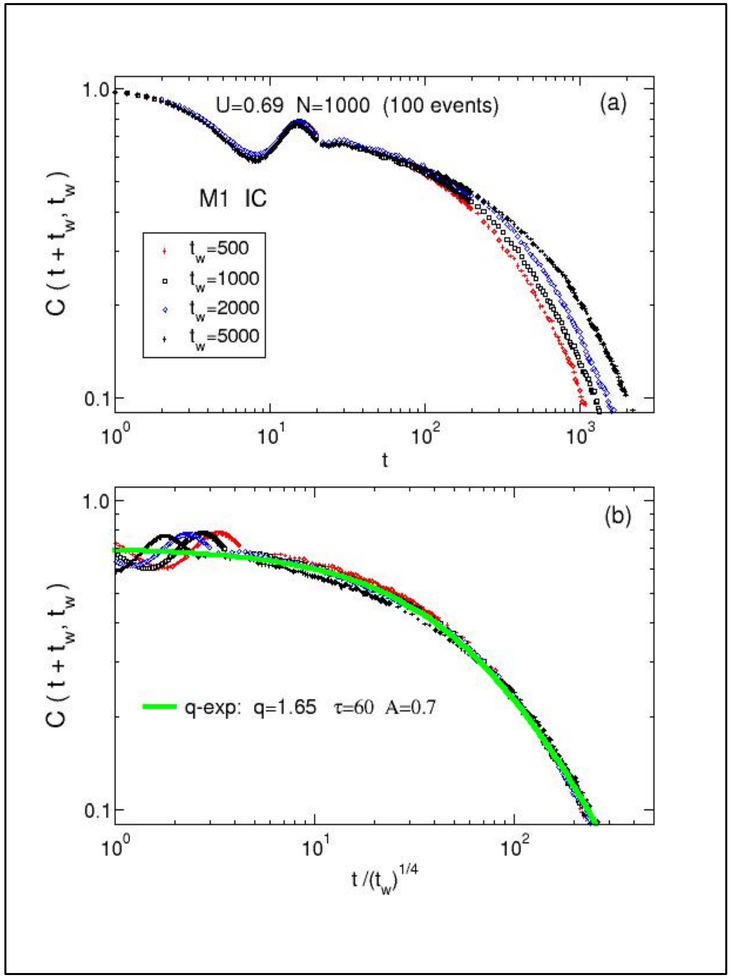
In our case γ **=1.38-1.4,** thus we expect **q=1.55-1.6,** which is confirmed by the fit in the figure for M1IC. On the other hand, for M0IC the decay is almost exponential.

[1] Tsallis and Buckman PRE 54 (1996) R2197

q-exponential decay also for different i.c.



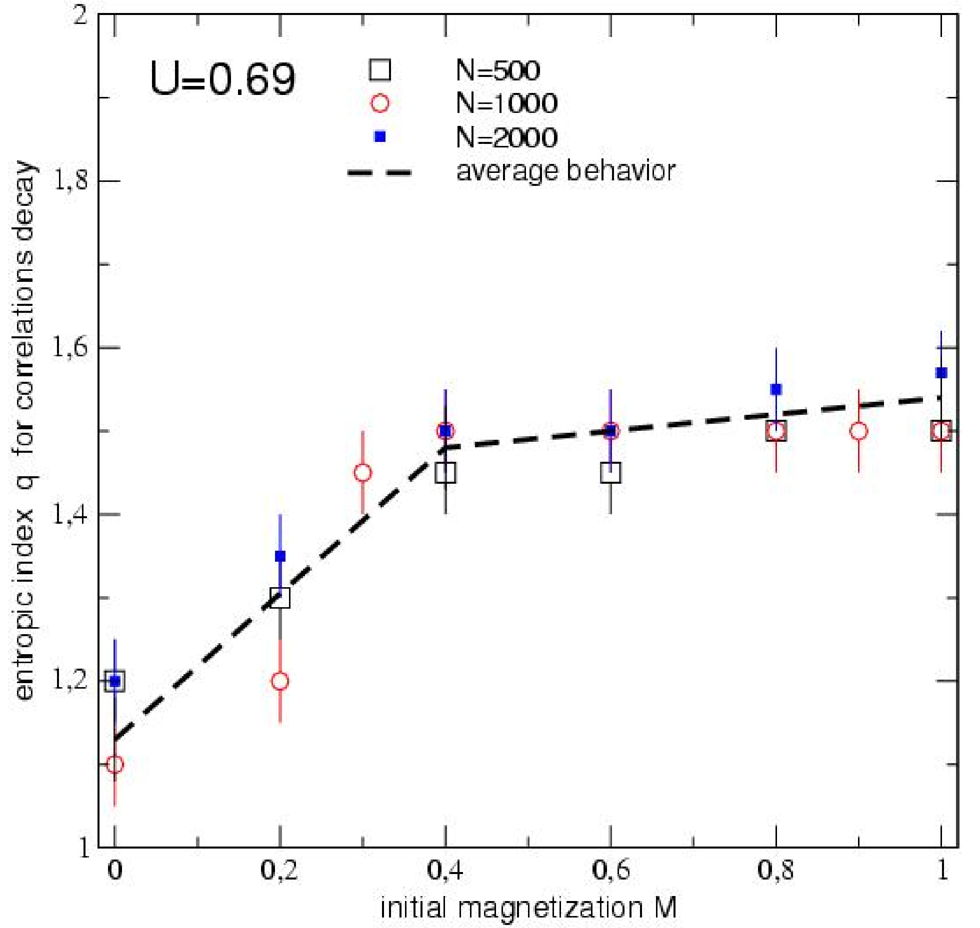
q-exponential decay for aging



Also for the aging behavior, the power law decay of the correlation functions, after a proper rescaling, can be reproduced with a q-exponential function.

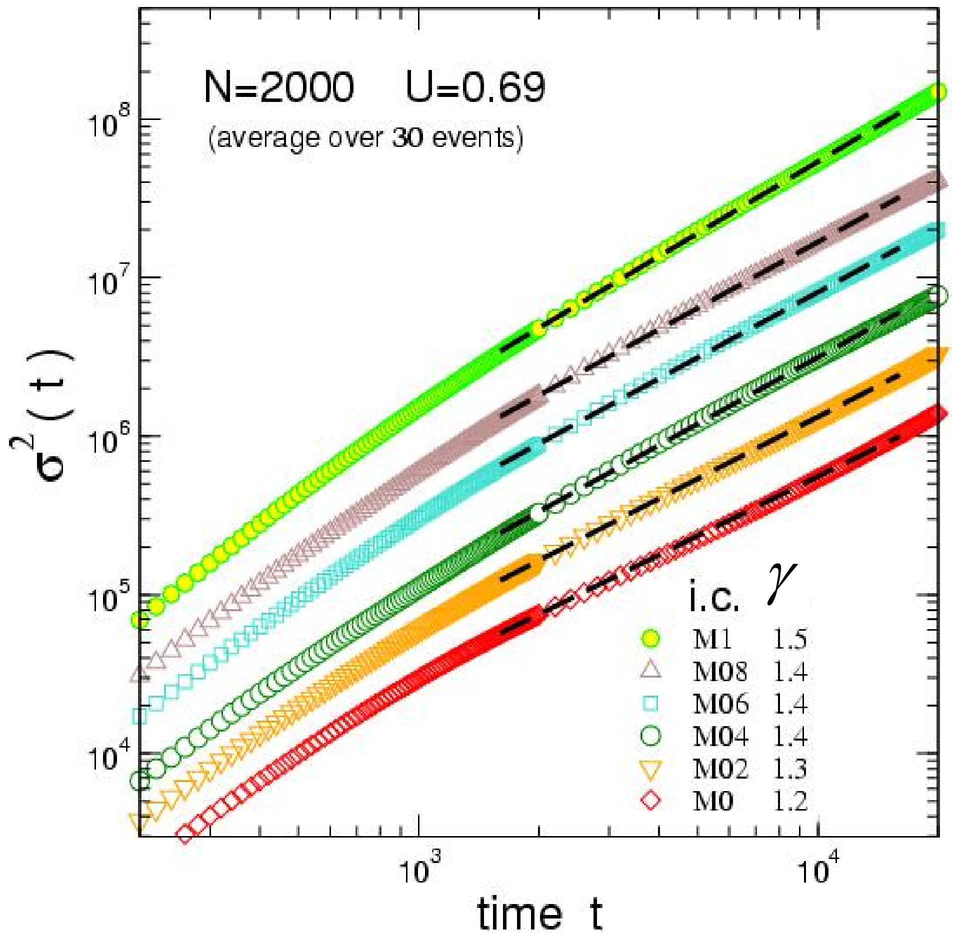
In this case we get **q=1.65**.

64 q vs initial Magnetization

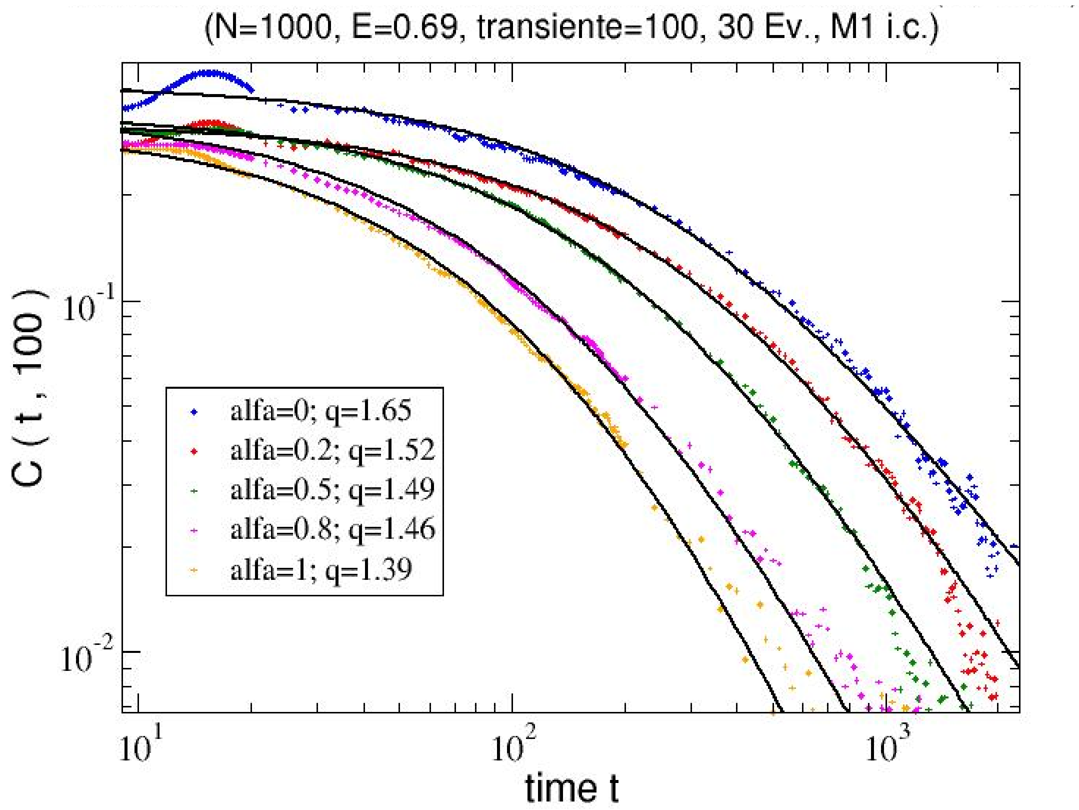


γ

63 Anomalous diffusion for different i.c.

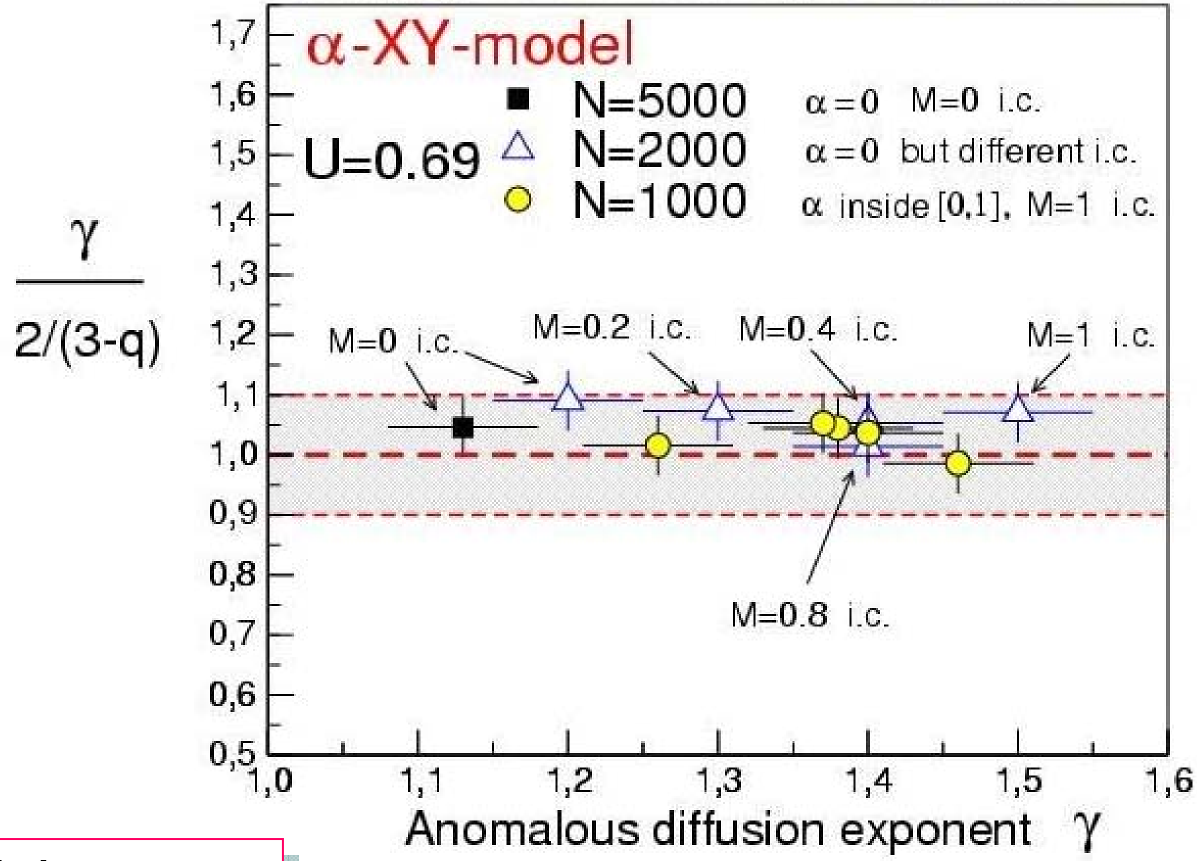


Correlation decay vs range α



Summary of most recent results regarding HMF model and q-statistics

Anomalous diffusion vs q-exponential decay



Rapisarda and Pluchino,
Europhysics News 36 (2005) 202

90 Most recent results

Some unpublished results on recent criticism

Recent criticism

A few comments on recent criticism regarding anomalous diffusion in the HMF model and q-statistics

It seems that velocity pdfs can be described using Lynden-Bell entropy within a Vlasov approach.. There are however several discrepancies…..due to dynamical effects and initial conditions

Antoniazzi et al

cond-mat/0601518

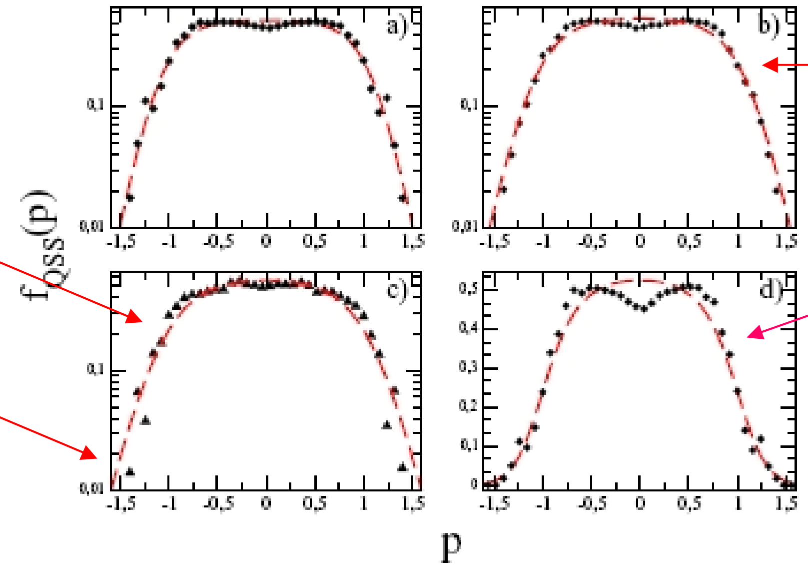
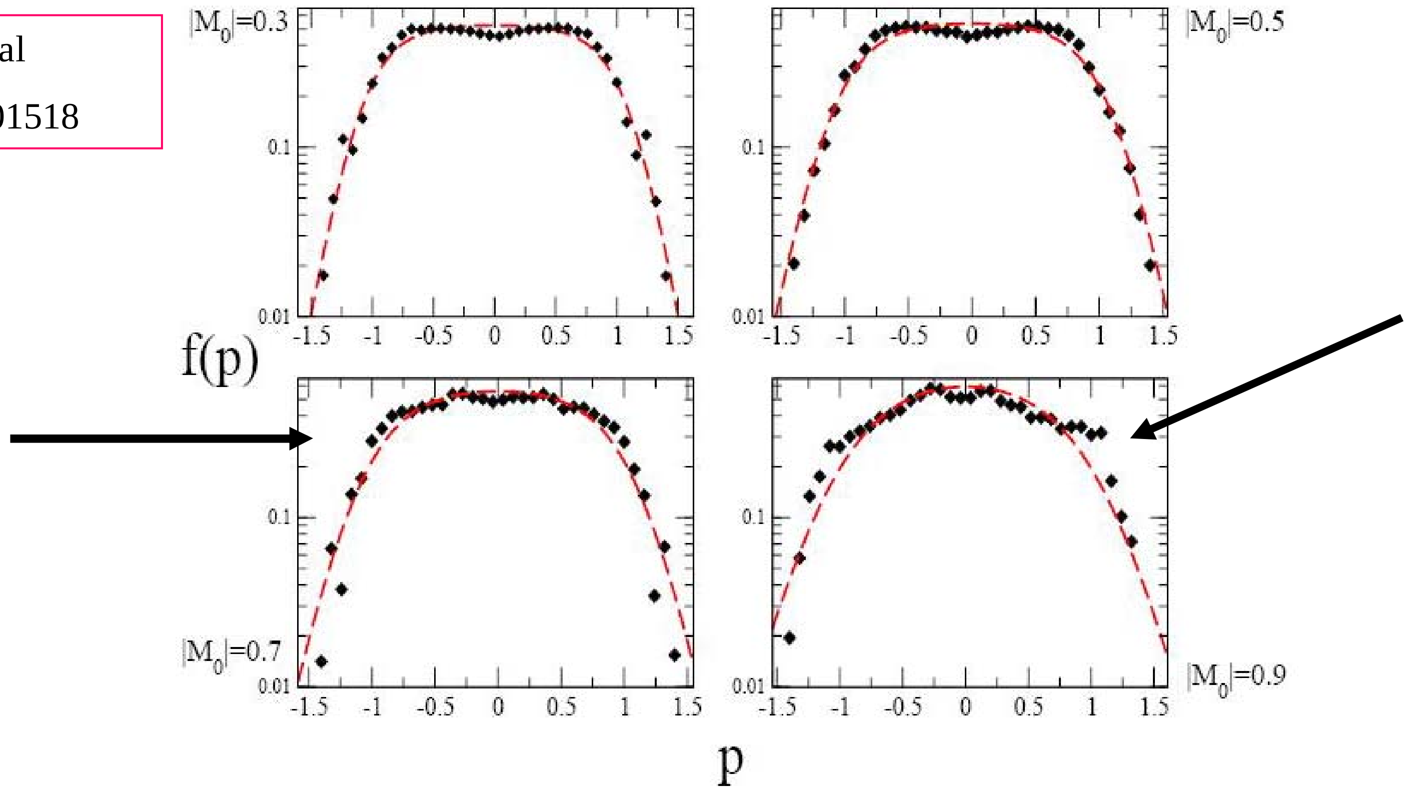


FIG. 2: Velocity distribution functions. Symbols refer to numerical simulations, while dashed solid lines stand for the theoretical profile ([9]). Panels a), b) and c) present the three cases $M_0 = 0.3$, $M_0 = 0.5$ and $M_0 = 0.7$ in lin-log scale, while panel d) shows the case $M_0 = 0.3$ in lin-lin scale. The numerical curves are computed from one single realization with $N = 10^7$ at time $t = 100$. Here $e = H/N = 0.69$.

It seems that velocity pdfs can be described using Lynden-Bell entropy within a Vlasov approach.. There are however several discrepancies…..due to dynamical effects and initial conditions

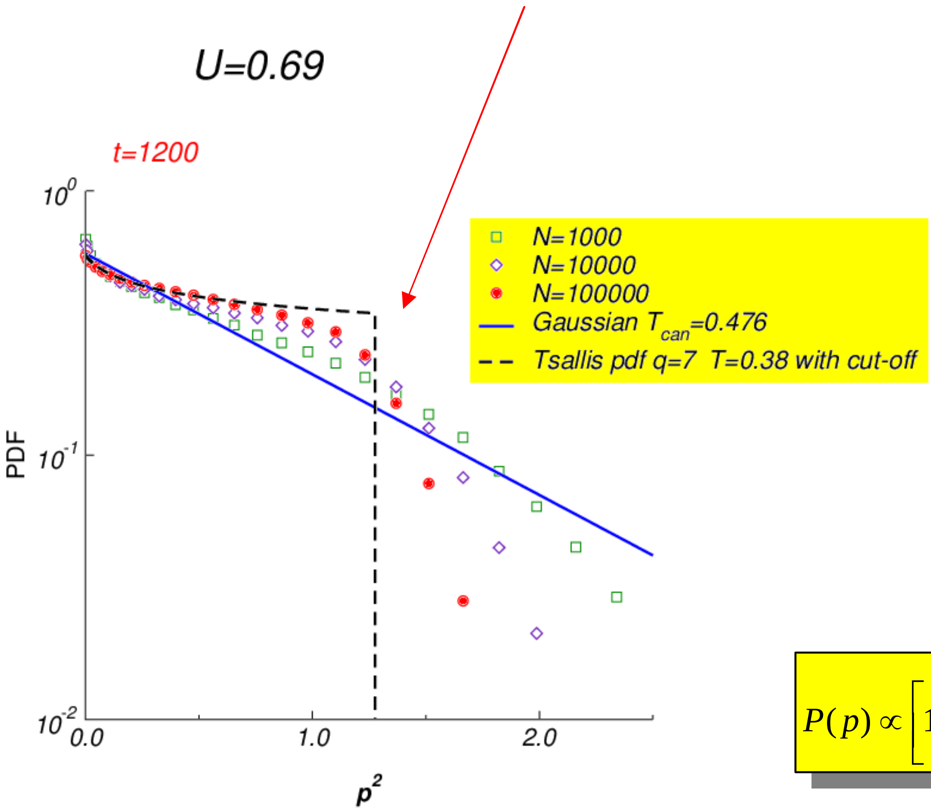
Antoniazzi et al

cond-mat/0601518



Latora, Rapisarda Tsallis

PRE 2001



$$P(p) \propto \left[1 - \frac{(1-q)}{2T}\, p^2\right]^{\frac{1}{1-q}}$$

Anomalous diffusion is a finite size effect?

Antoniazzi et al

cond-mat/0601518

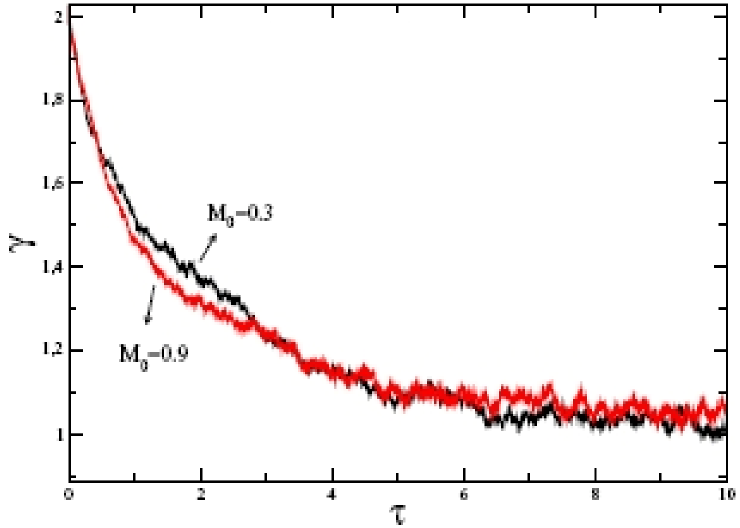


FIG. 3: The exponent $\gamma = \mathrm{d}\log(\sigma^2)/\mathrm{d}\log(t)$ is plotted as a function of the rescaled time $\tau = t/N$. Starting from the initial ballistic value 2, it converges to the normal diffusion exponent 1. Simulations refer to $M_0 = 0.3$, and $M_0 = 0.9$. Here $N = 10^5$ and $\epsilon = H/N = 0.69$.

Anomalous diffusion is a finite size effect?

Moyano and Anteneodo

cond-mat/0601518

On the diffusive anomalies in a long-range Hamiltonian system

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We scrutinize the anomalies in diffusion observed in an extended long-range system of classical rotors, the HMF model. Under suitable preparation, the system falls into long-lived quasi-stationary states presenting super-diffusion of rotor phases. We investigate the diffusive motion of phases by monitoring the evolution of their probability density function for large system sizes. These densities are shown to be of the q-Gaussian form, $P(x) \propto (1+(q-1)[x/\beta]^2)^{1/(1-q)}$, with parameter q increasing with time before reaching a steady value $q \simeq 3/2$. From this perspective, we also discuss the relaxation to equilibrium and show that diffusive motion in quasi-stationary trajectories strongly depends on system size.

PACS numbers: 05.20.-y, 05.60.Cd, 05.90.+m

I. INTRODUCTION

Systems with long-range interactions constitute a very appealing subject of research as they display a variety of dynamic and thermodynamic features very different from those of short-range systems treated in the text-books (see [1] for a review on the subject). Moreover, in recent years, the study of long-range models have raised

at energies close below ε_c[5]. In a QS state, the temperature (twice the specific mean kinetic energy) is almost constant in time and lower than the canonical value to which it eventually relaxes. However, the duration of QS states increases with the system size N, indicating that these states are indeed relevant in the $(N \to \infty)$ thermodynamical limit (TL).

Several other peculiar features have been found for

Moyano and Anteneodo

cond-mat/0601518

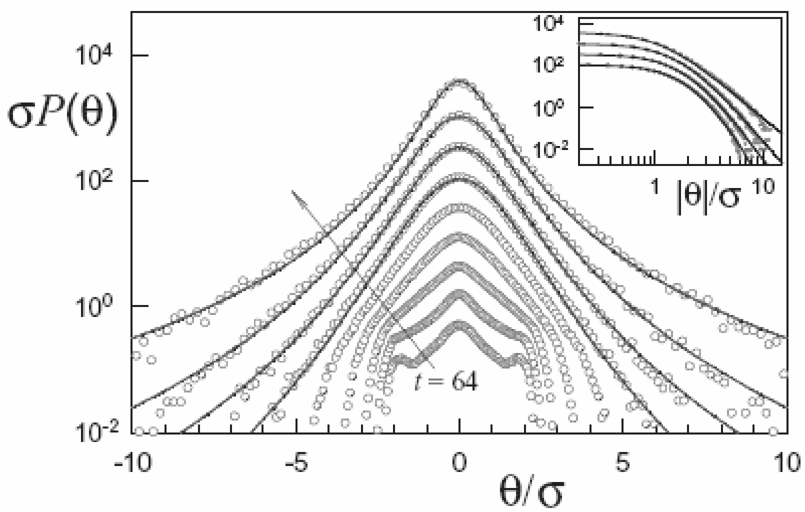

FIG. 1: Histograms of rotor phases at different instants of the dynamics (symbols). Simulations for $N = 1000$ were performed starting from regular water-bag initial conditions at $\varepsilon = 0.69$ (conditions leading to QS states). Countings were accumulated over 100 realizations, at times $t_k = 2^k$, with $k = 6, 8, ..14$, growing in the direction of the arrow up to $t = 16384$. Solid lines correspond to q-Gaussian fittings. Histograms were shifted for visualization. Inset: log-log representation of the fitted data.

Anomalous diffusion is a finite size effect?

Moyano and Anteneodo

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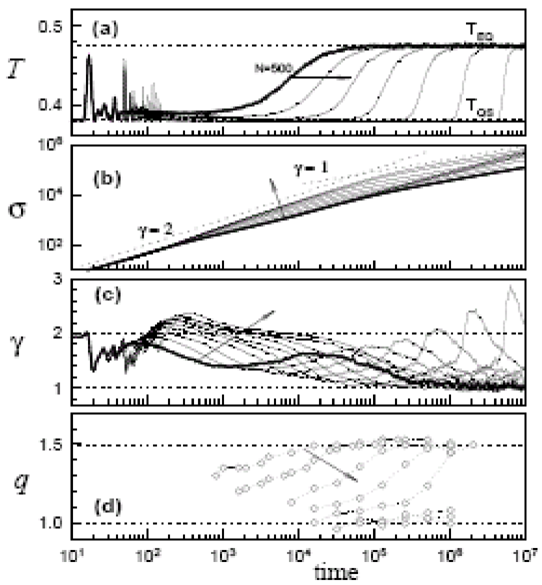


FIG. 2: Averaged time series of (a) temperature T, (b) deviation σ, (c) diffusion exponent γ and (d) parameter q, for $\varepsilon = 0.69$ and different values of N ($N = 500 \times 2^k$, with $k = 0, \ldots, 9$). Bold lines correspond to $N = 500$, as reference, and N increases in the direction of the arrows up to $N = 256000$. Averages were taken over $2.56 \times 10^5/N$ realizations, starting from a *waterbag* configuration at $t = 0$. In panel (d), the fitting error is approx. 0.03. Dotted lines are drawn as references. In (a), they correspond to temperatures at equilibrium ($T_{EQ} = 0.476$) and at QS states in the TL ($T_{QS} = 0.38$). In (b), to ballistic motion ($\gamma = 2$) and normal diffusion ($\gamma = 1$).

sponds to normal diffusion, $\gamma < 1$ to sub-diffusion and super-diffusion occurs for $\gamma > 1$. The evolution of σ is shown in Figs. 2b and 5b, for water-bag and equilibrium initial preparations, respectively. In order to detect different regimes, it is useful to obtain an instantaneous exponent γ as a function of time by taking the logarithm in both sides of Eq. (6) and differentiating with respect to $\ln t$:

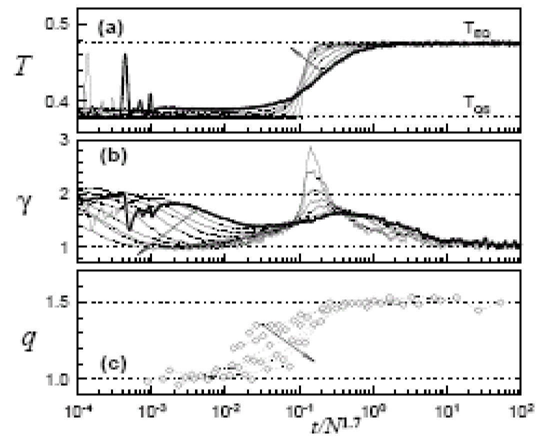


FIG. 3: Averaged time series of (a) temperature T, (b) local exponent γ and parameter q (symbols), as a function of $t/N^{1.7}$. Data are the same presented in Fig. 2.

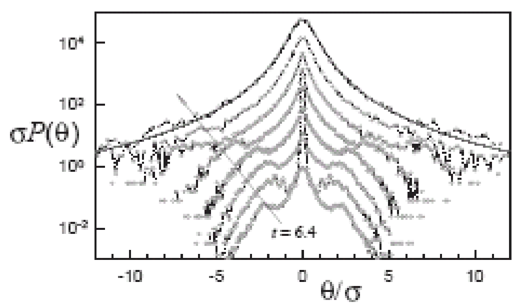
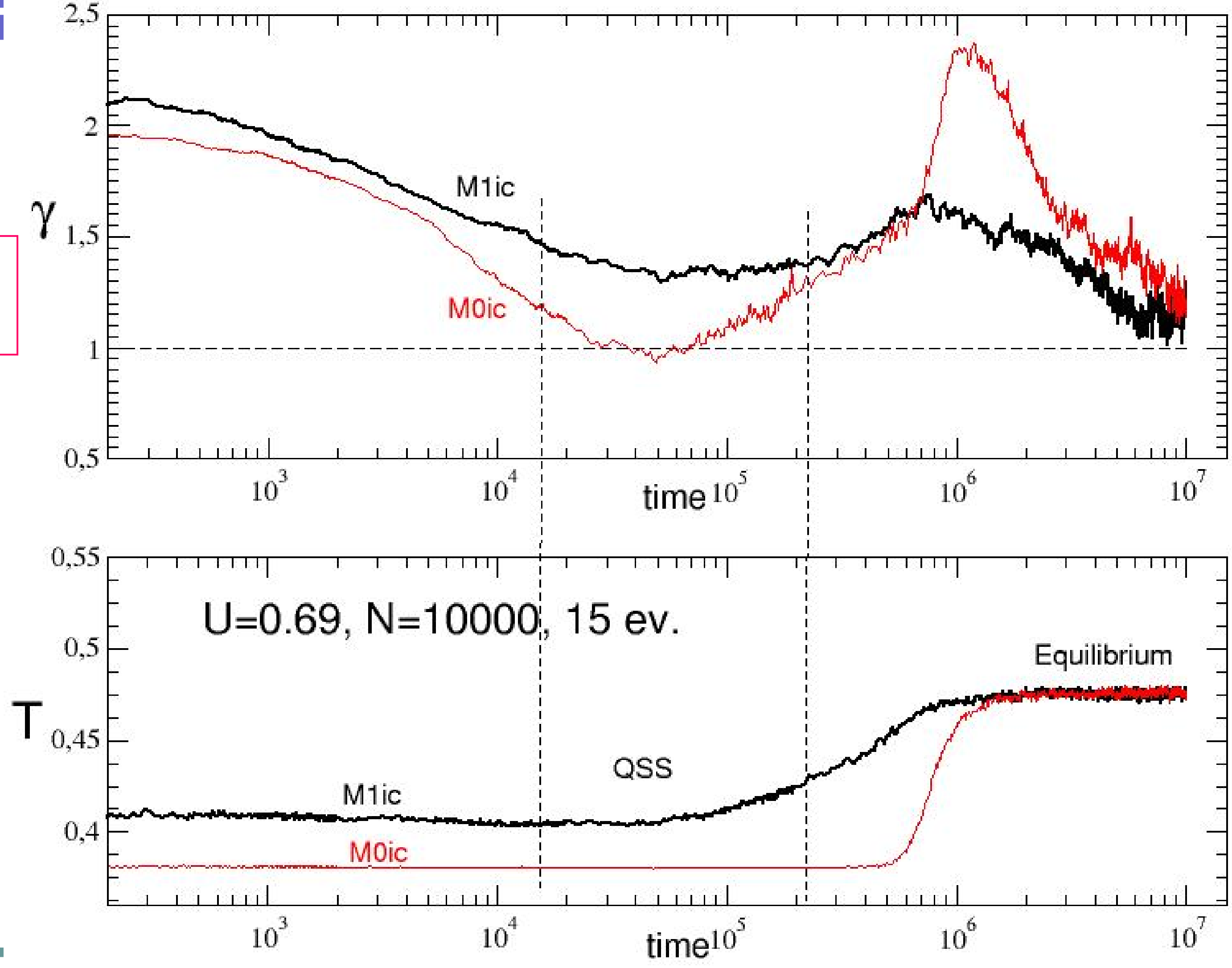


FIG. 4: Histograms of rotor phases at different instants of the dynamics (symbols). Simulations were performed for $N = 500$ and $\varepsilon = 0.69$, starting from an equilibrated initial condition. Countings were accumulated over 200 realizations, at times $t_k = 0.1 \times 4^k$, $k = 3, 4, ..10$, growing in the direction of the arrow up to $t \simeq 1.05 \times 10^5$). The q-Gaussian function with $q = 1.53$ was plotted for comparison (solid line). Histograms were shifted for visualization.

It is not completely clear if anomalous diffusion disappears at large N at the moment. However initial conditions are extremely important even for large N



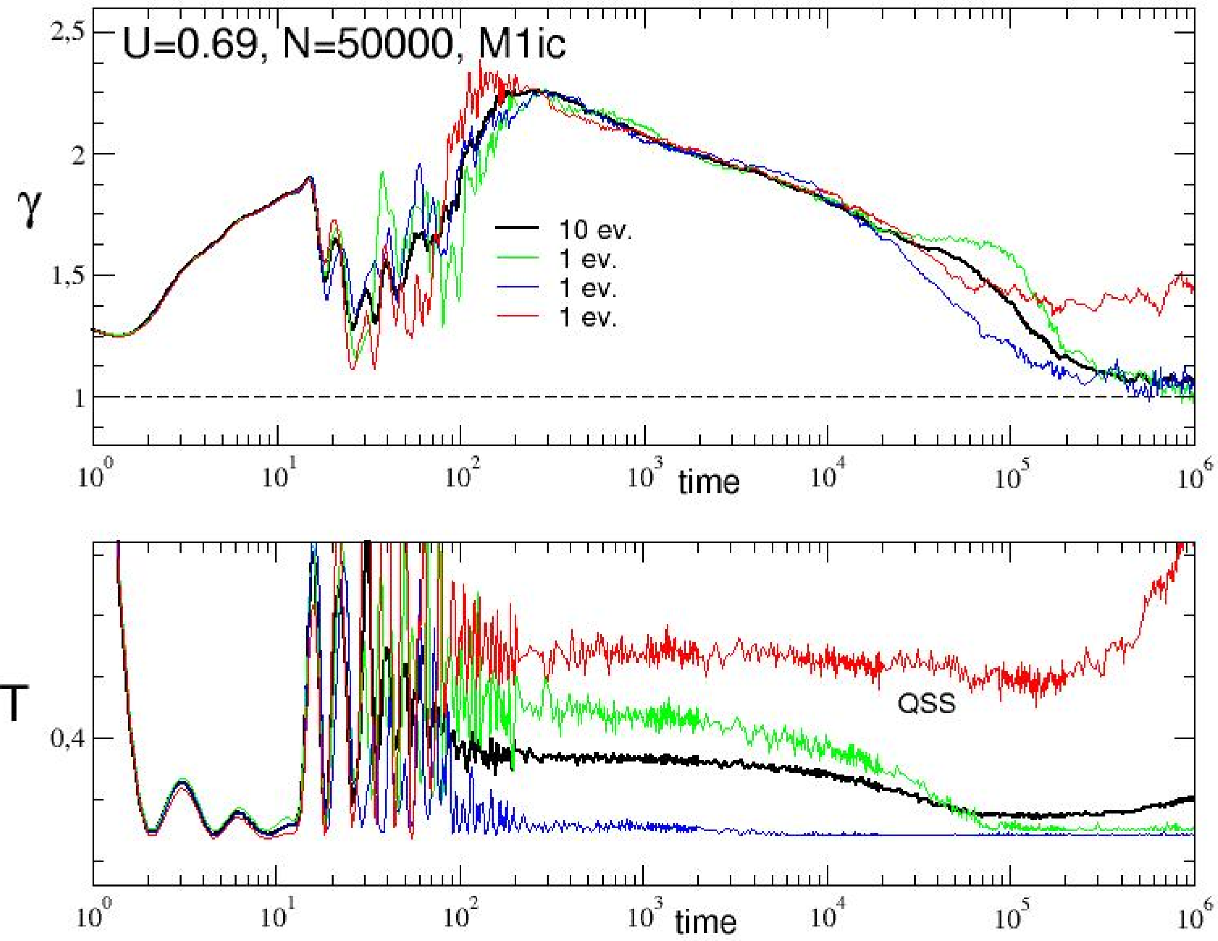
Tsallis, Rapisarda, Pluchino

in preparation

Single realizations can have a very different dynamics. So a good statistics is needed even for large N.

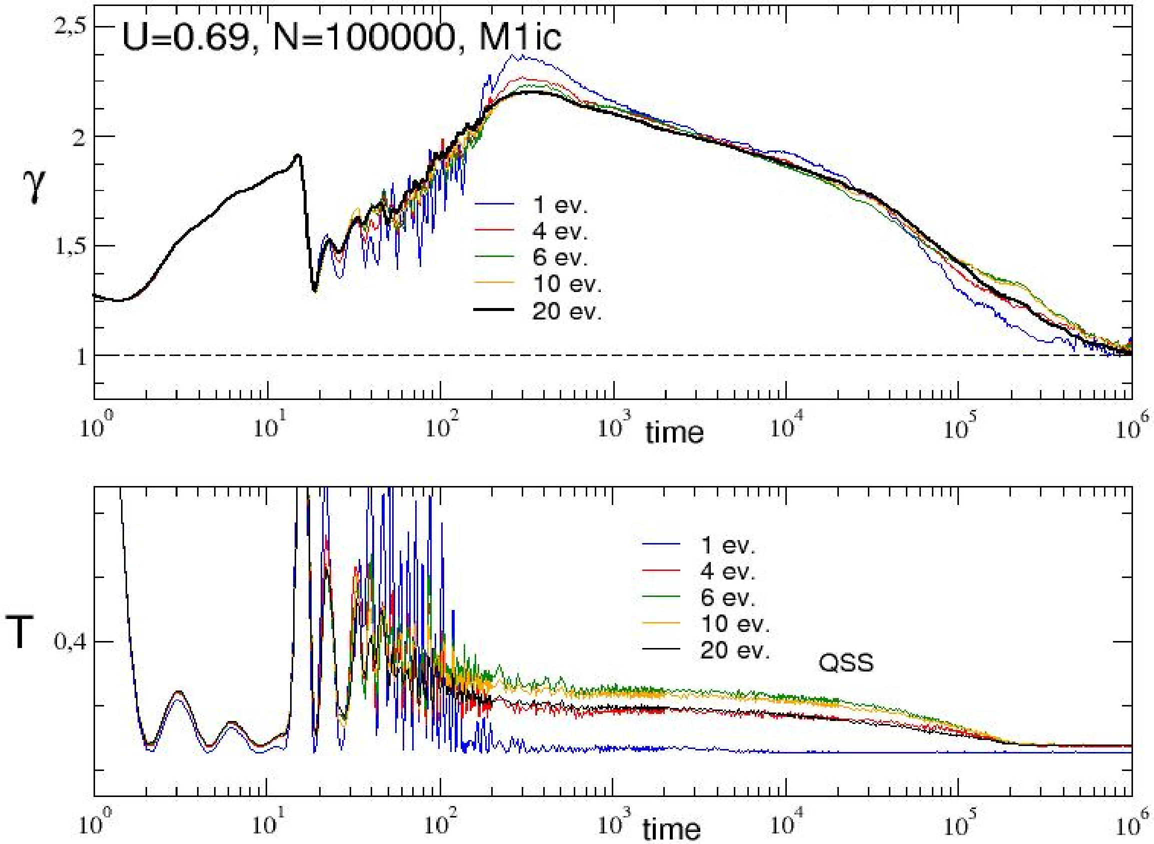
Tsallis, Rapisarda, Pluchino

in preparation



Tsallis, Rapisarda, Pluchino

in preparation



A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

Recently a dynamical transition as a function of the initial magnetization was claimed by Chavanis

Lynden-Bell and Tsallis distributions for the HMF model

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To be included later

Abstract. Systems with long-range interactions can reach a Quasi Stationary State (QSS) as a result of a violent collisionless relaxation. If the system mixes well (ergodicity), the QSS can be predicted by the statistical theory of Lynden-Bell (1967) based on the Vlasov equation. When the initial distribution takes only two values, the Lynden-Bell distribution is similar to the Fermi-Dirac statistics. Such distributions have recently been observed in direct numerical simulations of the HMF model (Antoniazzi et al. 2006). In this paper, we determine the caloric curve corresponding to the Lynden-Bell statistics in relation with the HMF model and analyze the dynamical and thermodynamical stability of spatially homogeneous solutions by using two general criteria previously introduced in the literature. We express the critical energy and the critical temperature as a function of a degeneracy parameter fixed by the initial condition. Below these critical values, the homogeneous Lynden-Bell distribution is not a maximum entropy state but an unstable saddle point. Known stability criteria corresponding to the Maxwellian distribution and the water-bag distribution are recovered as particular limits of our study. In addition, we find a critical point below which the homogeneous Lynden-Bell distribution is always stable. We apply these results to the situation considered by Antoniazzi et al. For a given energy, we find a critical initial magnetization above which the homogeneous Lynden-Bell distribution ceases to be a maximum entropy state, contrary to the claim of these authors. For an energy $U = 0.69$, this transition occurs above an initial magnetization $M_x = 0.897$. In that case, the system should reach an inhomogeneous Lynden-Bell distribution (most mixed) or an incompletely mixed state (possibly fitted by a Tsallis distribution). Thus, our theoretical study proves that the dynamics is different for small and large initial magnetizations, in agreement with numerical results of Pluchino et al. (2004). This new dynamical phase transition may reconcile the two communities.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

Recently a dynamical transition as a function of the initial magnetization was claimed by Chavanis

See Chavanis

cond-mat/0604234

For an initial magnetization M>0.897 (if U=0.69)

The homogeneous Linden-Bell distribution becomes unstable and the system can be trapped into an incomplete mixed state, where Tsallis statistics is a possible explanation

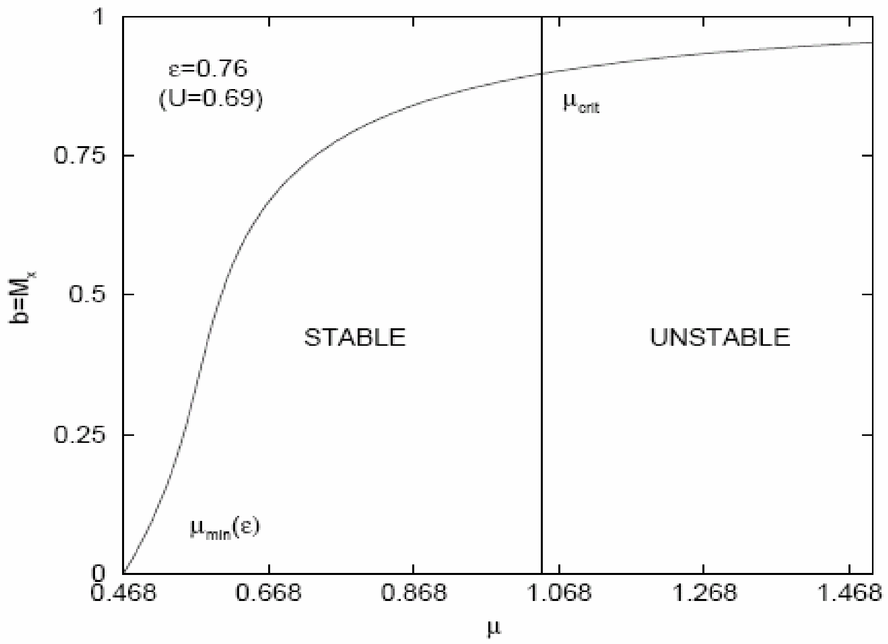

Fig. 10. Initial magnetization $b = M_x$ as a function of the degeneracy parameter μ for a given value of the energy. There exists a critical magnetization, corresponding to $\mu_{crit}(\epsilon)$, above which the homogeneous Lynden-Bell distribution is unstable.

A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

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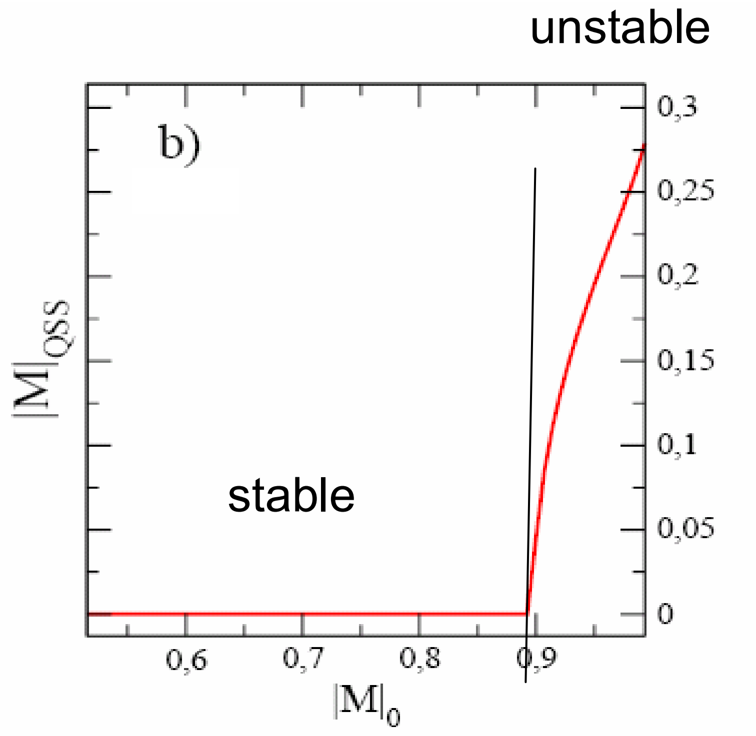
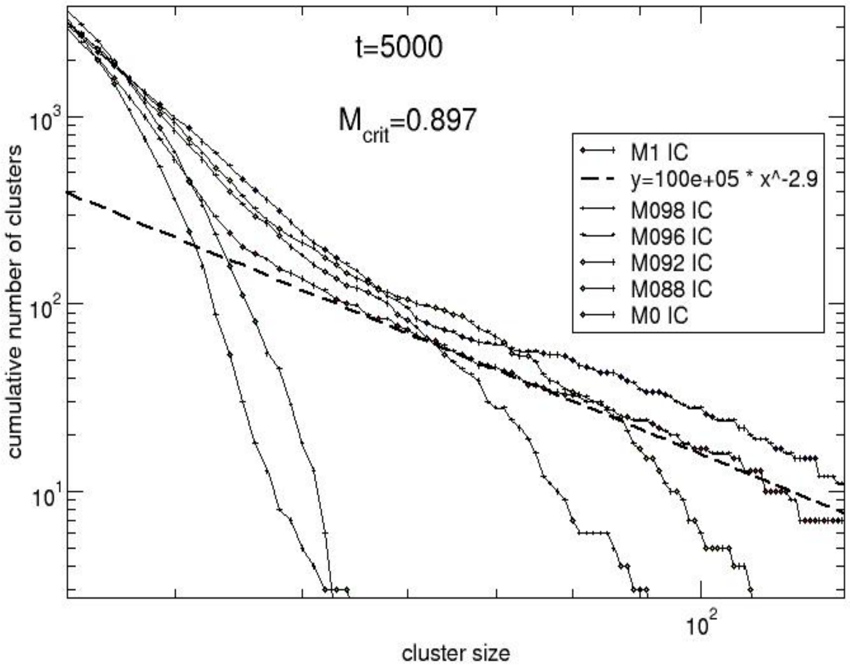


Antoniazzi et al cond-mat/0601518

A few notes on recent criticisms regarding anomalous diffusion in the HMF model and q-statistics

Several numerical simulations confirm this transition, but further investigation is needed

U=0.69 N=100000



Also Morita and Kaneko recently found anomalous collective dynamics which cannot be explained with Vlasov states

PRL 96, 050602 (2006) PHYSICAL REVIEW LETTERS week ending 10 FEBRUARY 2006

Collective Oscillation in a Hamiltonian System

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(Received 11 June 2005; published 9 February 2006)

Oscillation of macroscopic variables is discovered in a metastable state of the Hamiltonian system of the mean-field *XY* model. The duration of the oscillation is divergent with the system size. This long-lasting periodic or quasiperiodic collective motion appears through Hopf bifurcation, which is a typical route in low-dimensional dissipative dynamical systems. The origin of the oscillation is explained, with a self-consistent analysis of the distribution function, as the self-organization of a self-excited swing state through the mean field. The universality of the phenomena is discussed.

DOI: 10.1103/PhysRevLett.96.050602 PACS numbers: 05.70.Ln, 05.45.−a, 87.10.+e

Dissipative systems often show periodic, quasiperiodic, and chaotic motion at a macroscopic level, when they are far away from equilibrium. The motion is described as low-dimensional dynamics, and its discovery has marked an epoch of nonlinear dynamics studies in physics. Recalling that the microscopic degrees of freedom involved are large, such macroscopic behavior is a result of collective motion that emerges out of high-dimensional microscopic dynamics. The collective motion, indeed, has been intensively and

Letter, the essence of this discovery is briefly reported, especially in the mean-field *XY* model [8].

We adopt the Hamiltonian system of the mean-field *XY* model, or globally coupled pendula [5,9,10],

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} [1 - \cos(\theta_i - \theta_j)]. \quad (1)$$

All the N pendula interact with each other through phase

$$f^0_\theta(\theta; M_0) = \frac{1}{Z_\theta(M_0)} \exp\left[\frac{M_0}{T_{\rm eq}(M_0)} \cos\theta\right], \qquad (5)$$

where $Z_\theta(M_0)$ is the normalization. Next, the distribution

Here we note that another metastable state in this model has been intensively investigated for a decade, especially by taking a rectangular (*water bag*) initial momentum distribution [5,6]. This metastable state exists only in the region just below the critical energy of the phase transition, and there $M(t)$ and $T(t)$ take smaller values than those in equilibrium, leading to a branch of negative specific heat. This state is regarded as a reflection of a stable stationary solution of the corresponding Vlasov equation. On the other hand, the metastable state that we have discovered takes larger values of $M(t)$ and $T(t)$ than those in equilibrium and exists over a much broader region than the negative specific heat branch. Thus the present metastable state is not explained by the above stationary solution of the Vlasov equation and is a novel one.

creasing total energy, the temporal pattern of the macroscopic variable changes as stationary → periodic → quasiperiodic. This is regarded as a "bifurcation" of the collective motion. Here we note the similarity to the typical bifurcation route in low-dimensional dissipative dynamical systems, fixed point → limit cycle → torus, through Hopf bifurcations. Hence it is suggested that the present bifurcation of the collective motion is described as that of low-dimensional dynamical systems, in particular, by Hopf bifurcations.

We next investigate the bifurcation in more detail. The mean amplitude of $M(t)$ against U in the vicinity of the

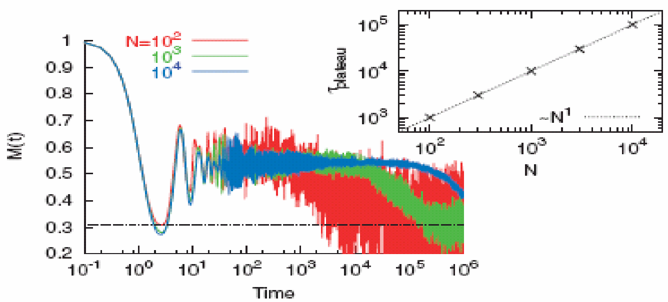


FIG. 1 (color). A time series of $M(t)$. The abscissa axis is a log scale. The dotted line is the equilibrium value. Inset: The duration of the plateau $\tau_{\rm plateau}$ against N. $U = 0.69$ [6] and $M_0 = 1$.

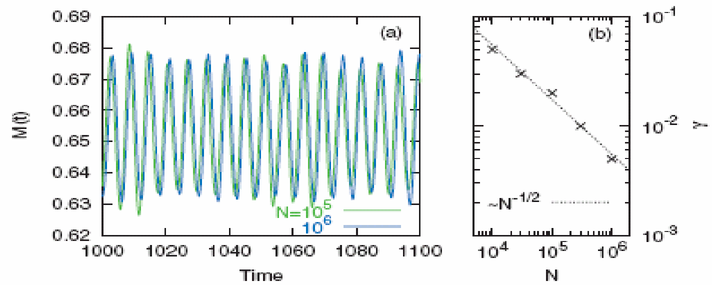


FIG. 2 (color). (a) A time series of $M(t)$ in the metastable state. (b) The decay rate γ of the amplitude of the oscillation, where $M_{\rm amp}(t) = M_{\rm amp}(t_0) - \gamma \log(t/t_0)$. $U = 0.5$ and $M_0 = 0.9$.

Microscopic dynamics can be very different according to the initial conditions (as for other fully-coupled systems, for example the Kuramoto model) and different theoretical approaches can be applied

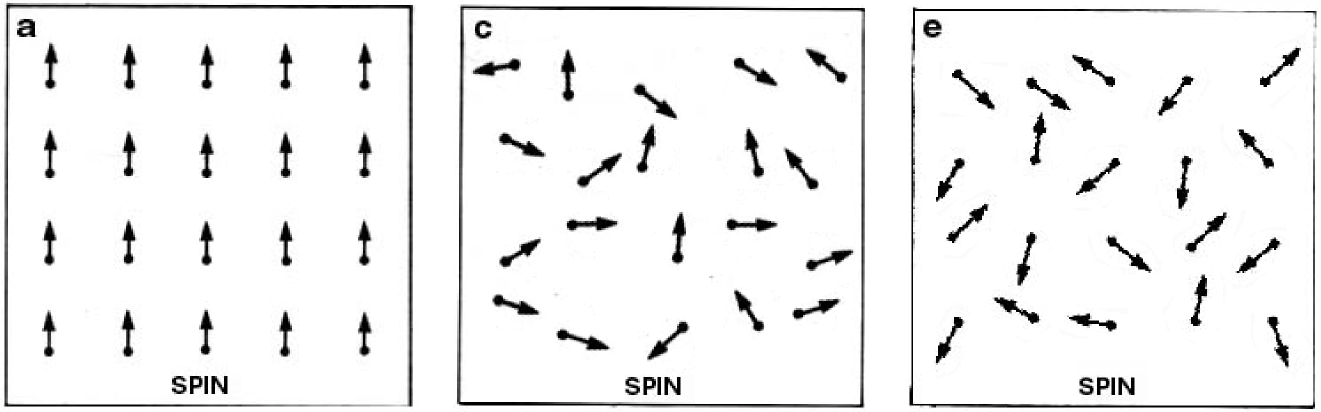
Most probably the Vlasov equation cannot explain all the anomalies found, which on the other hand, are found also in other models with long-range interactions

Anomalies have a clear dynamical origin.

q-statistics provides a coherent scenario and at the same time does not excludes other interpretations

109 Glassy phase in HMF

Ferromagnetic Phase: | **Paramagnetic Phase:** | **Glassy Phase:**



FRUSTRATION
&
QUENCHED
DISORDER

$M \neq 0$ $\qquad$ $M \cong 0$ $\qquad$ $M \cong 0$

$$M = \frac{1}{N}\left|\sum_{i=1}^{N} \vec{s}_i\right|$$

How to discriminate between Paramagnetic and Glassy phase?

110 The polarization p

One can introduce the "***elementary polarization***":

$$< \vec{s}_i > = \frac{1}{\tau}\int_1^\tau \vec{s}_i(t)\ dt$$

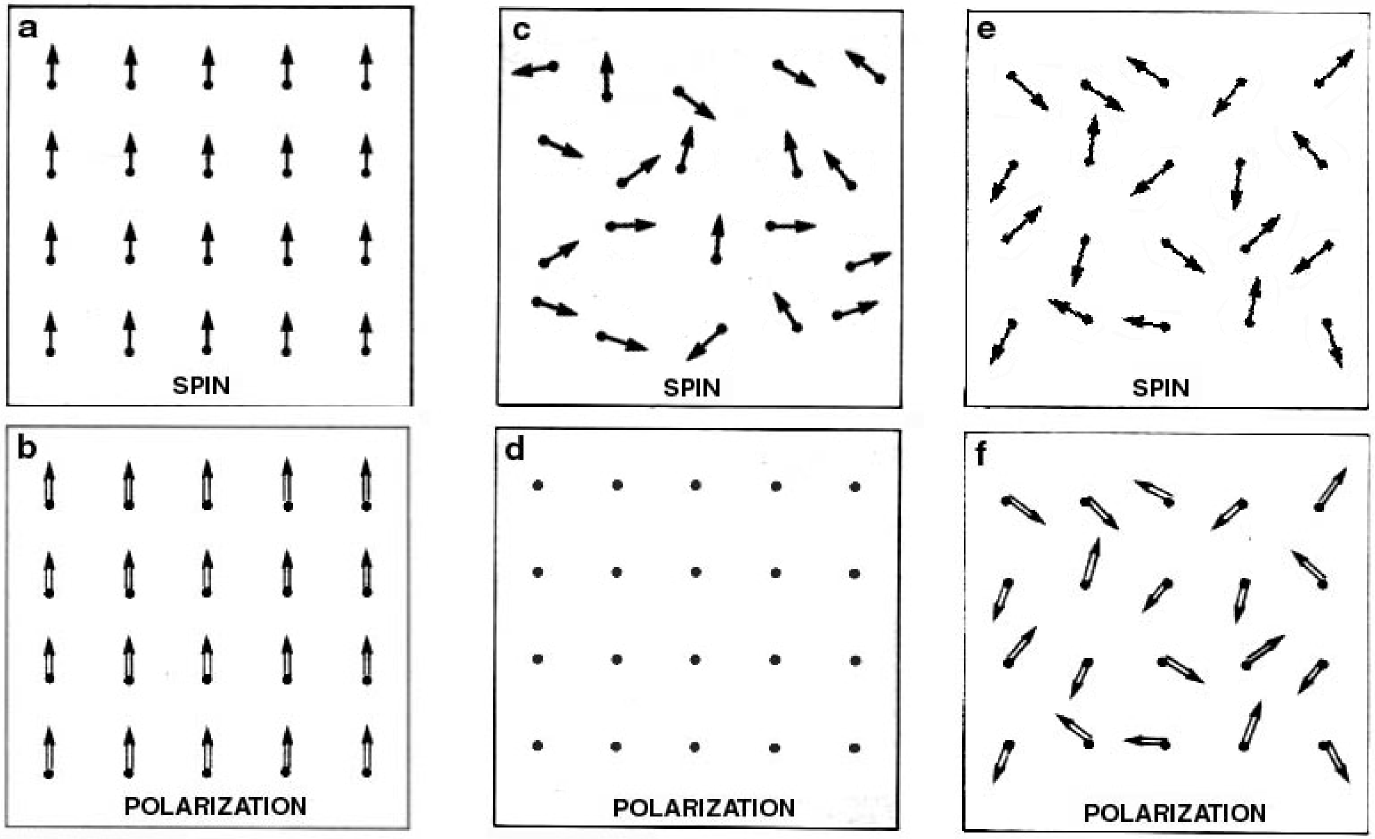
i.e. the temporal average, over a time interval **t**, of the successive positions of each elementary spin vector.

The modulus of the "*elementary polarization*" has to be furtherly averaged over the quenched disorder of the N spin configurations, to finally obtain the "***polarization*** ***" p*** :

$$p = \frac{1}{N}\sum_{i=1}^{N}\left| < \vec{s}_i > \right|$$

Polarization & Glassy phase

Ferromagnetic Phase: **Paramagnetic Phase**: **Glassy Phase**:



$M \neq 0 \quad p \neq 0$ $M \cong 0 \quad p \cong 0$ $M \cong 0 \quad p \neq 0$

One can consider the polarization p as a new order parameter in order to measure the freezing of the spins

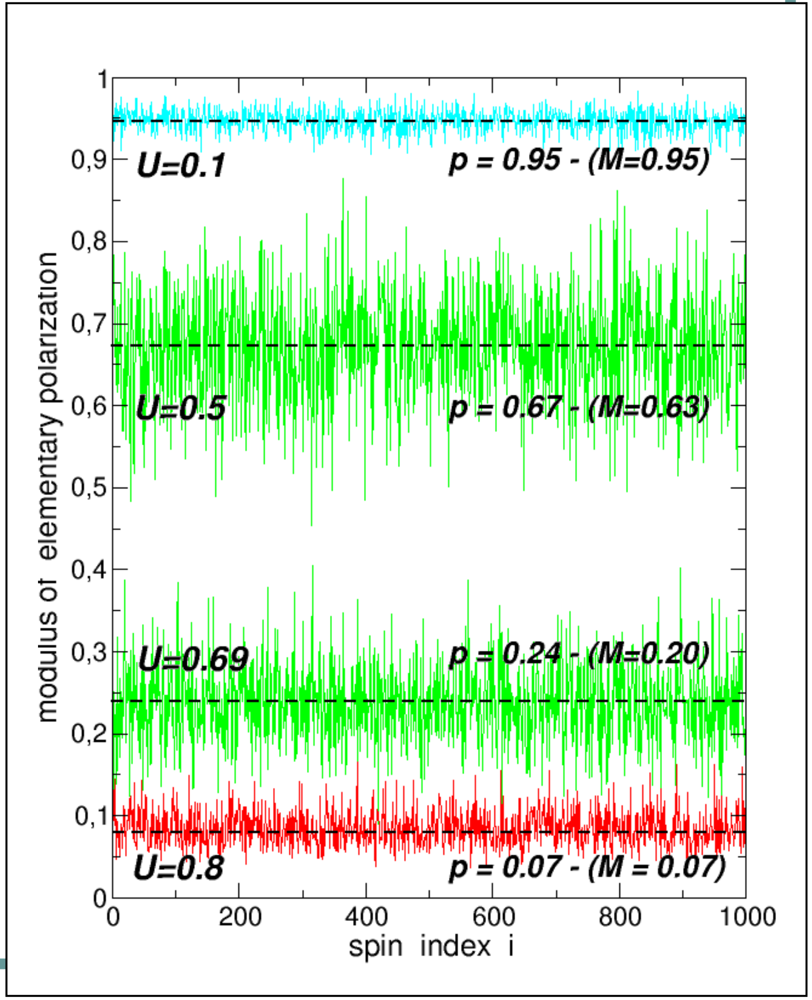
≠

112 Polarization and QSS

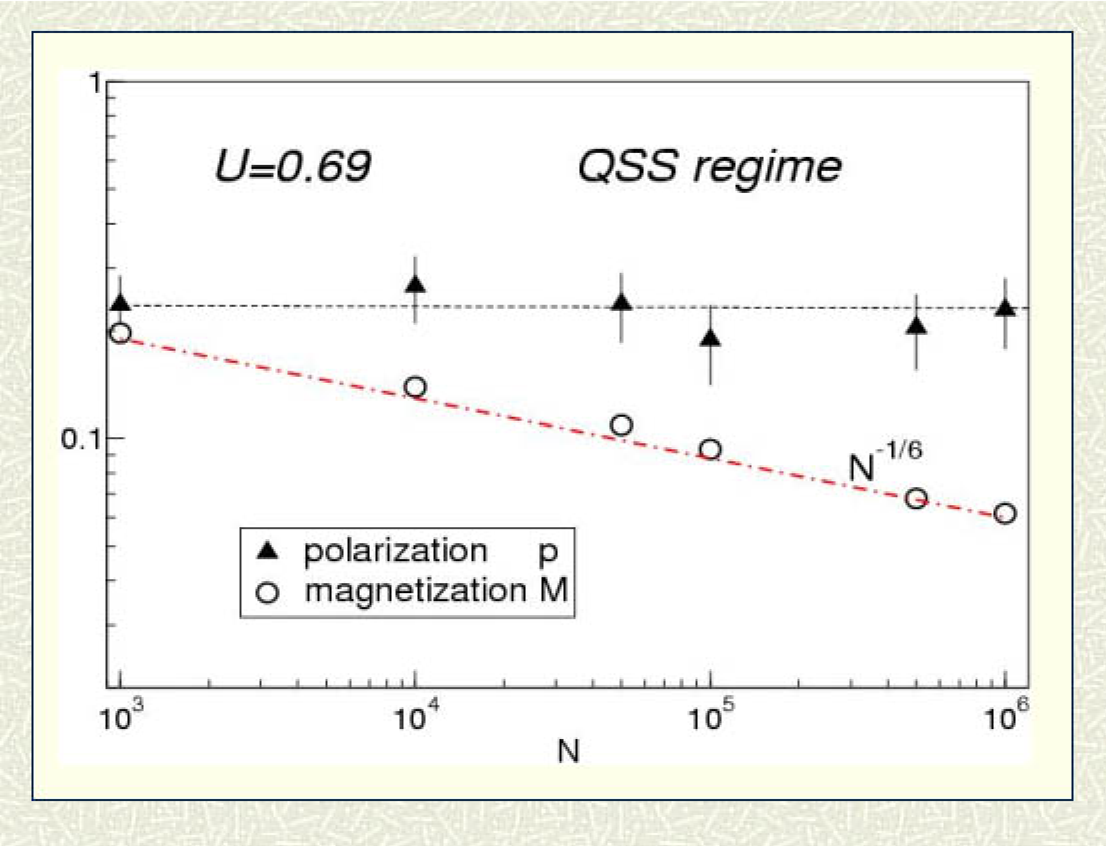
In the **HMF model** we can use the ***polarization p*** in order to characterize the ***QSS regime*** as a **Glassy Phase**, related with the dynamical frustration between clusters:

PHASE	*p*	*M*
Ferromagnetic	≠ 0	≠ 0
Paramagnetic	≅ 0	≅ 0
Glassy phase	≠ 0	≅ 0

Pluchino, Latora, Rapisarda, Phys. Rev. E 69 (2004) 056113



113 p and M in the QSS regime vs N



In the QSS regime the magnetization goes to zero with the system size as

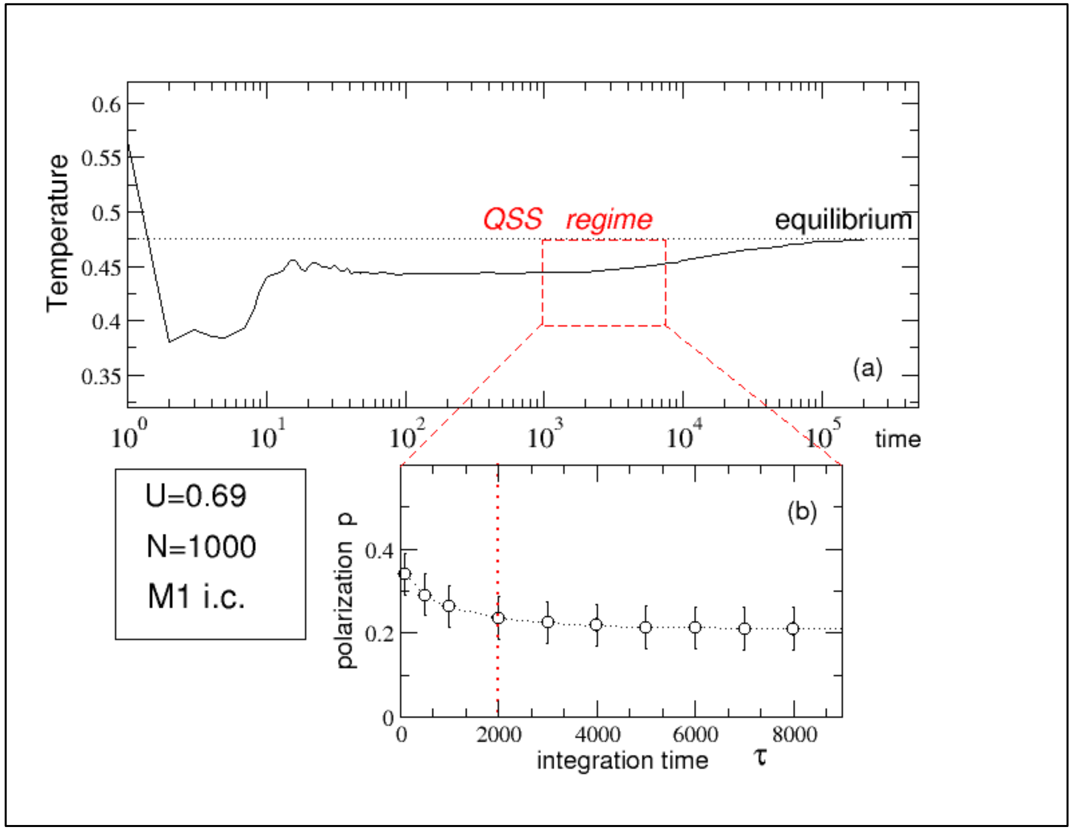
$$M \propto N^{-\frac{1}{6}}$$

while the polarization remains constant

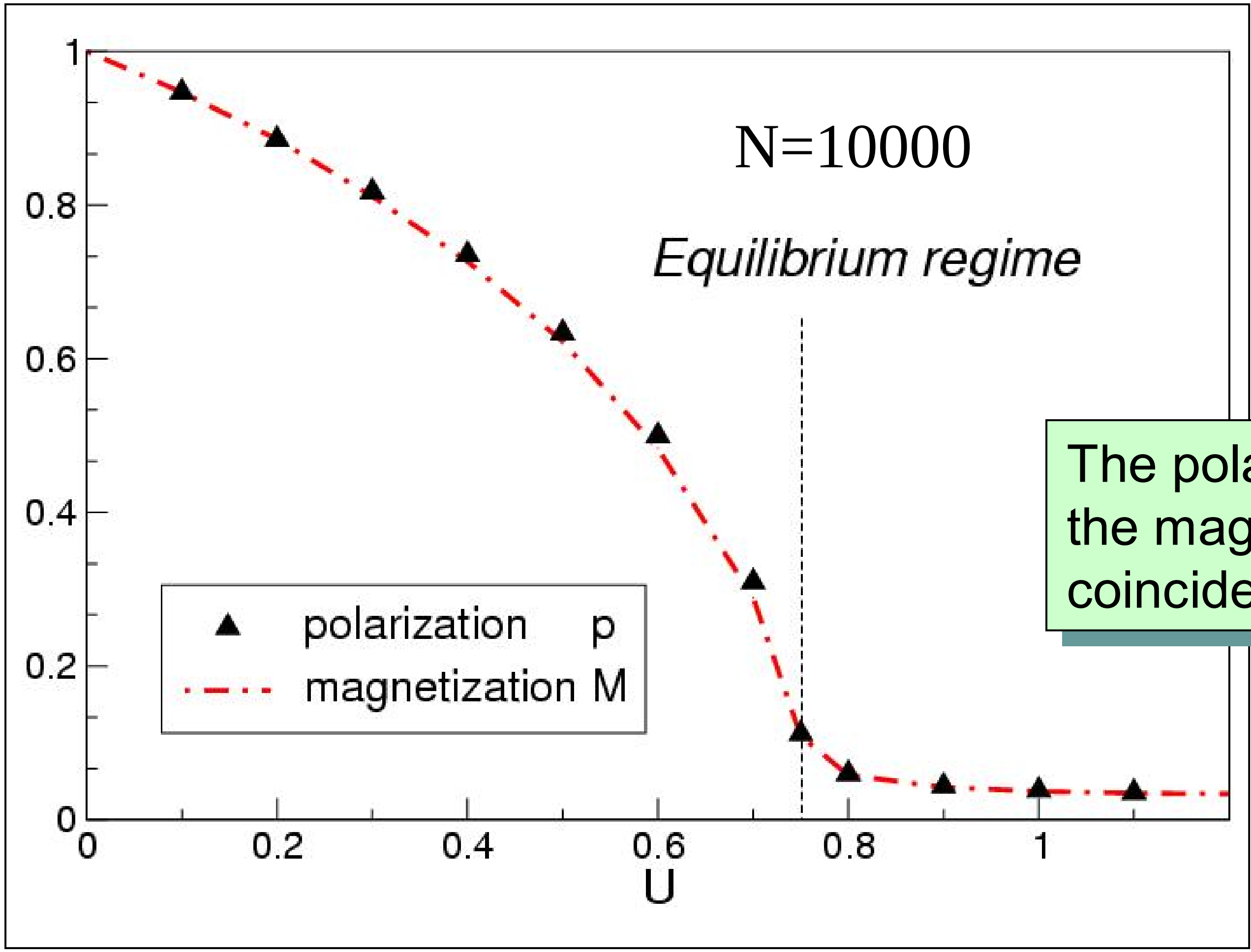
$$p \sim 0.24$$

Dependence of p on the integration time

The polarization p does not depend on the integration time interval inside the QSS regime



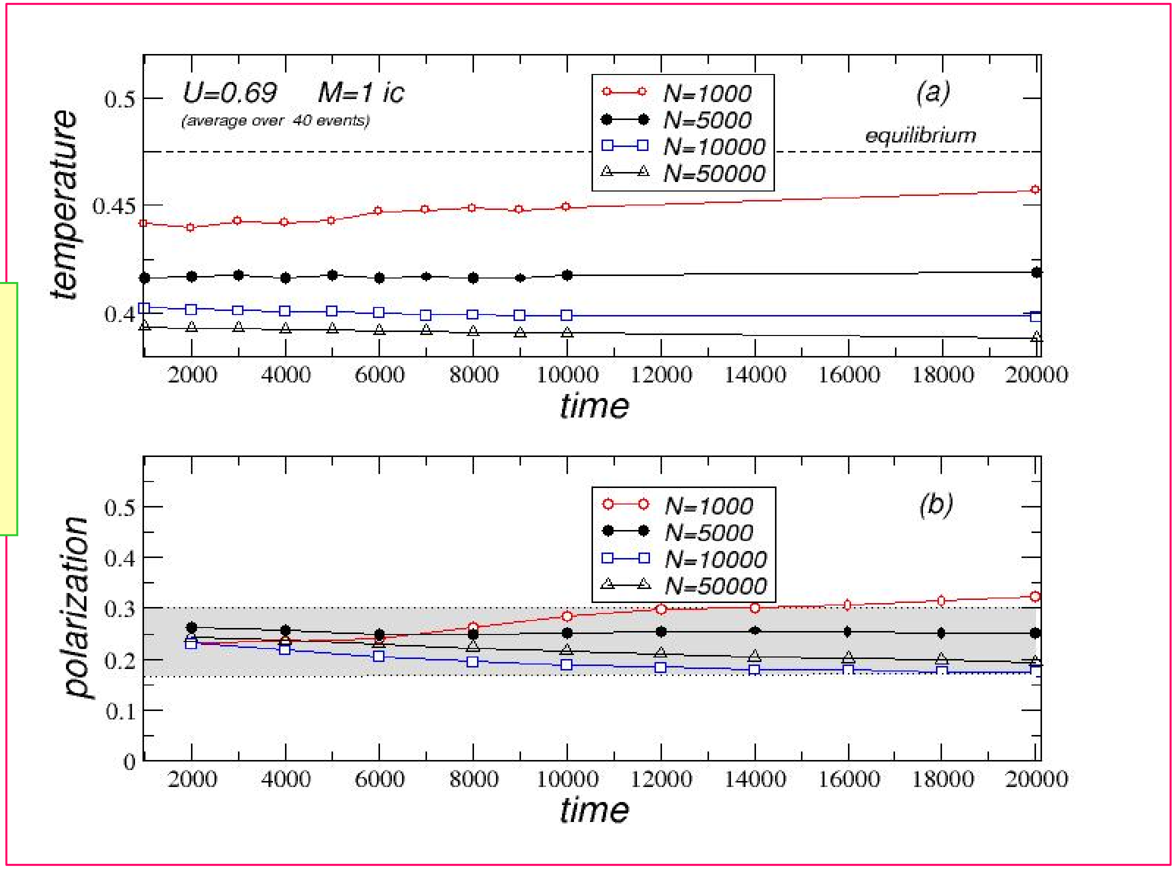
115 p and M at equilibrium



The polarization p and the magnetization M coincide at equilibrium

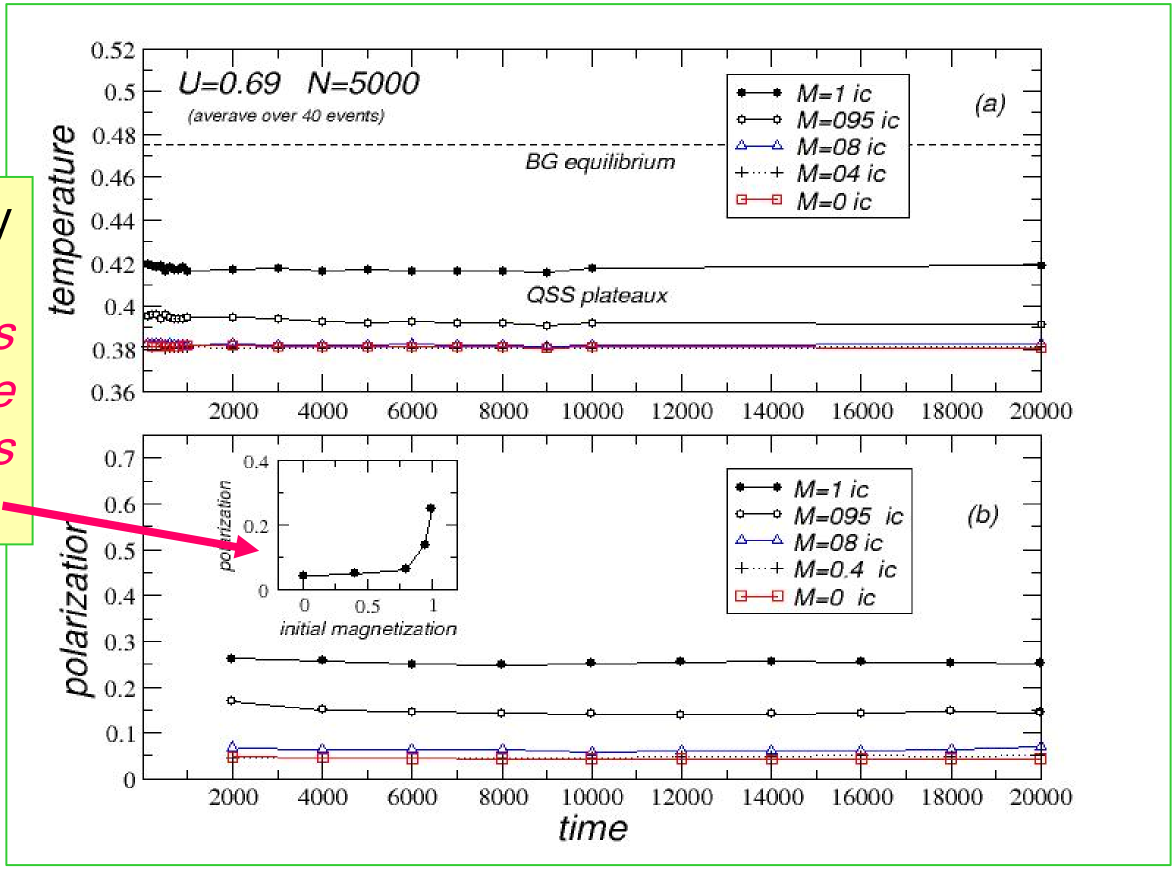
102 Polarization vs initial conditions

Polarization is very robust as a function of N and time (within the QSS regime).



103 Polarization vs initial conditions

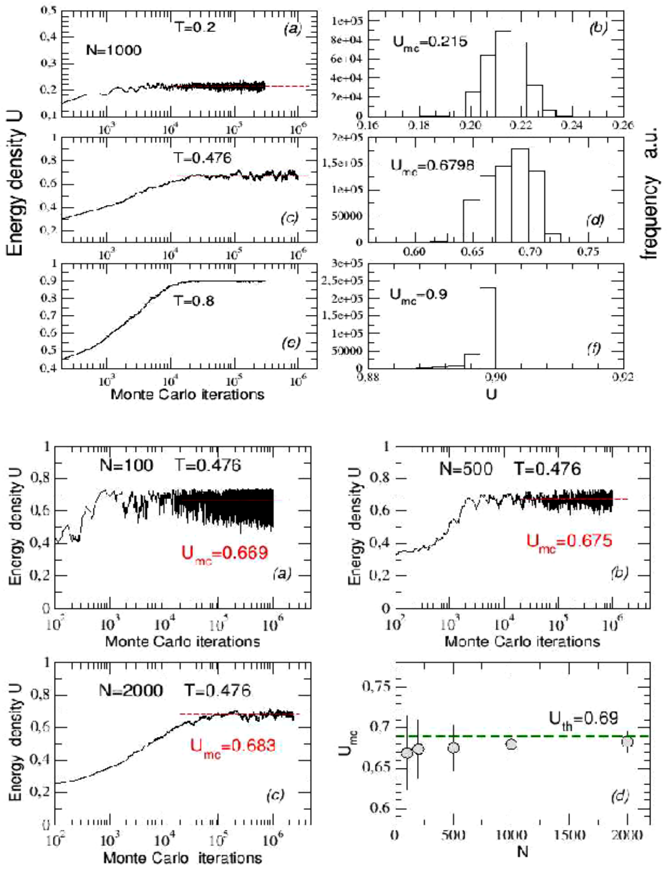
Polarization is very sensitive to the initial conditions: *it goes rapidly to zero when the initial magnetization is less than 1.*



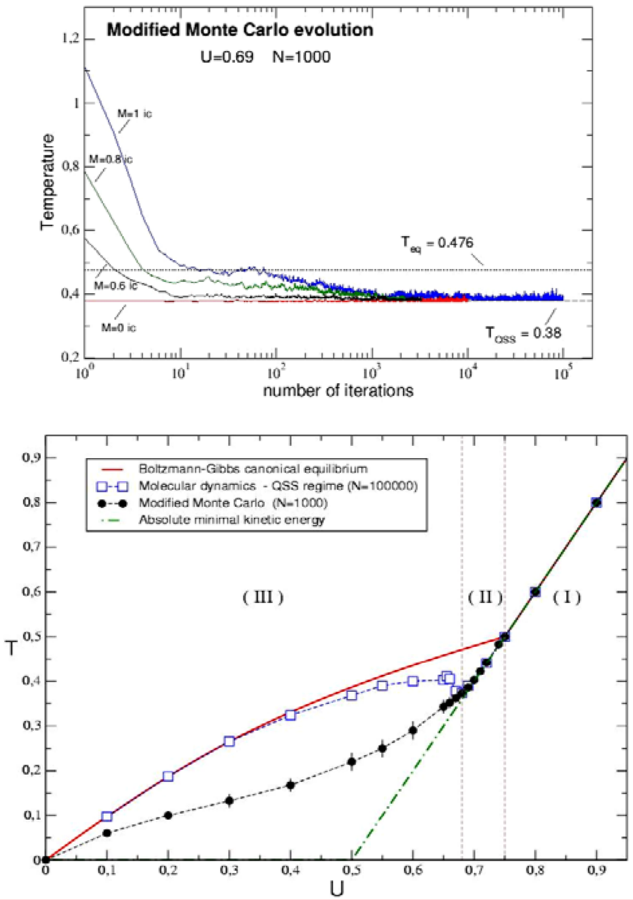
68 Monte Carlo simulations

A. Pluchino, G. Andronico, A. Rapisarda, Physica A 349 (2005) 143

Standard Metropolis



Minimization of the Temperature



Glassy thermodynamics for the QSS regime

A. Pluchino PhD thesis (2005) and cond-mat/0506665 Physica A (2006) in press

Let us start from the following effective spin-glass Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\vec{s}_i \cdot \vec{s}_j \quad (1)$$

with $\vec{s}_i = (\cos\vartheta_i, \sin\vartheta_i)$

Using the following distribution probability for the interaction J_{ij} :

$$p(J_{ij}) = (\sqrt{2\pi}\sigma_J)^{-1} \mathrm{e}^{-\frac{(J_{ij}-J_0)^2}{2\sigma_J^2}}$$

with average and variance

$$\overline{J_{ij}} = J_0 = 1/N \;, \quad \overline{J_{ij}^{\,2}} - \overline{J_{ij}}^{\,2} = \sigma_J^{\,2} = 1/N$$

This distribution implies a random coupling among spins with a probability on average $J_0=1/N$ which simulates a glassy behavior similar to what we observe in the QSS regime.

In the thermodynamic limit we have no interaction and the system remains frozen for ever.

In the thermodynamic limit the Hamiltonian (1) reduces to the potential part of the HMF Hamiltonian. One can treat the kinetic part considering a heat bath with $T=2K/N$

General HMF Hamiltonian :

$$H = K + V = \sum_{i=1}^{N}\frac{p_i^2}{2} + \frac{1}{2N}\sum_{i,j=1}^{N} J_{ij}\vec{s}_i \cdot \vec{s}_j$$

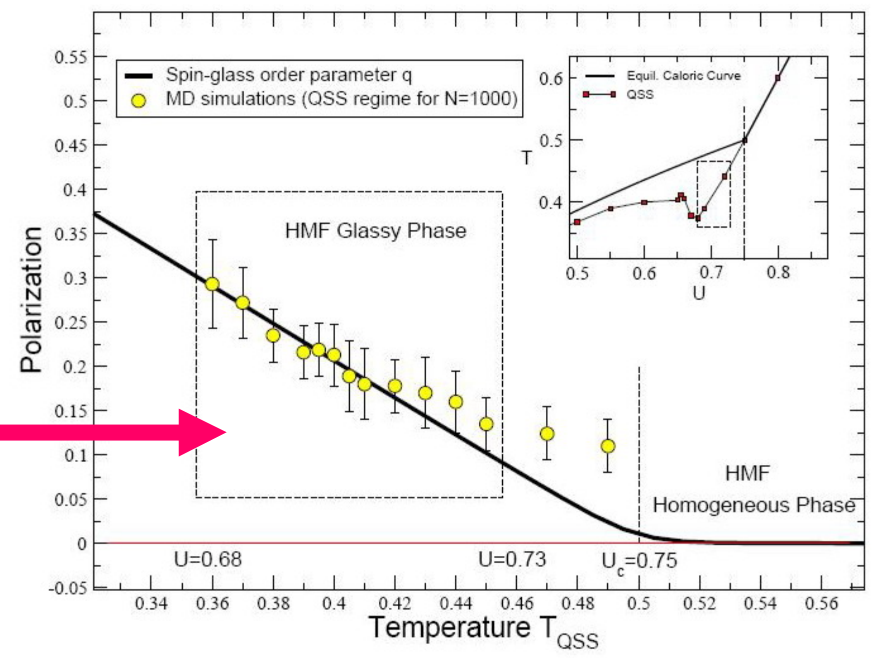
75 Comparison with numerical data

By applying the replica method, after some standard calculations one can extract from the Hamiltonian (1), the following **self-consistent equation for the spin-glass order parameter**

$$p_{SG} = 1 - \sqrt{\frac{2}{p}}\beta^{-1}\int_0^\infty r^2\, dr \exp\left(-\frac{r^2}{2}\right)\frac{I_1\left[\beta r\sqrt{p/2}\right]}{I_0\left[\beta r\sqrt{p/2}\right]}$$

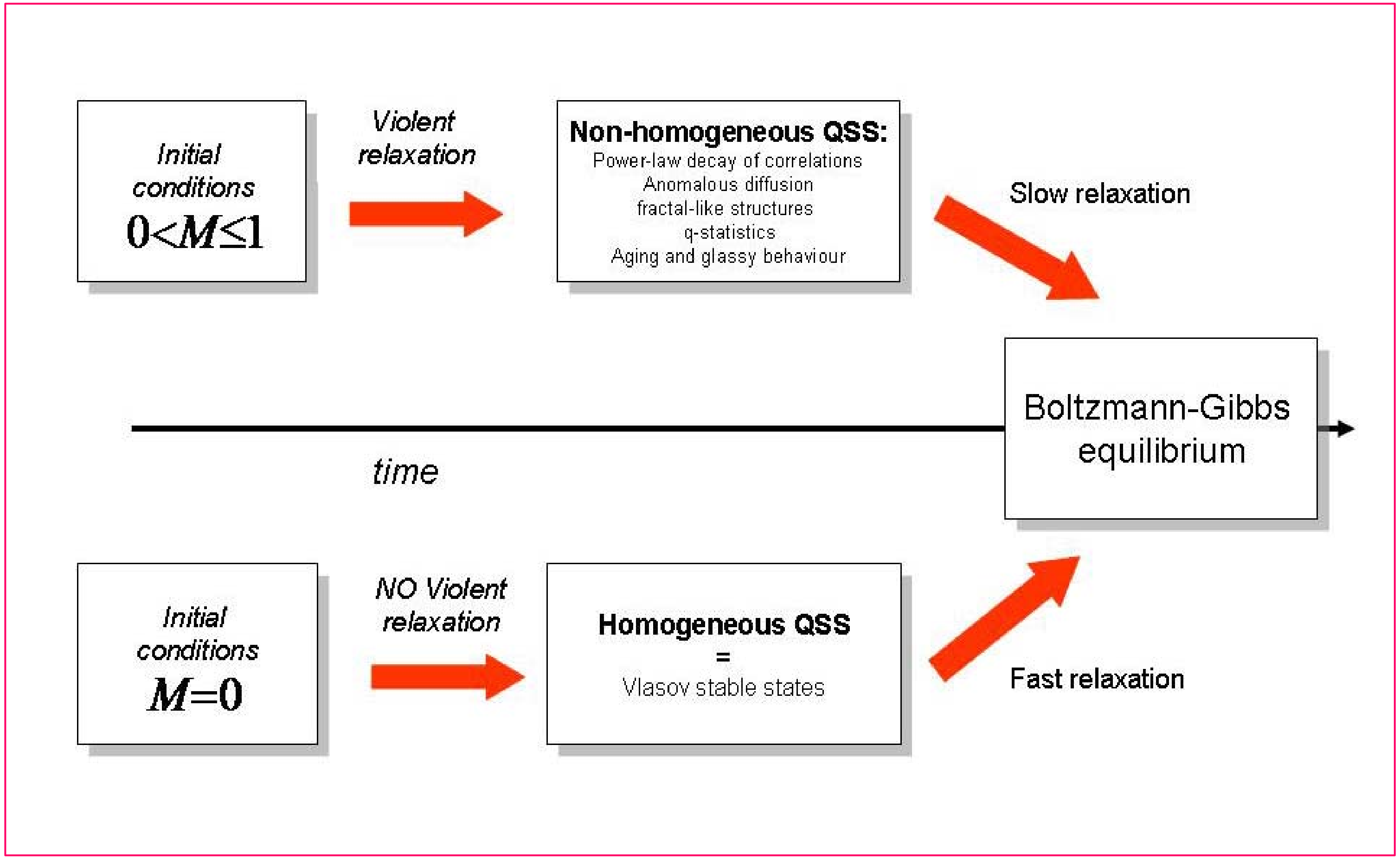
Which can be compared with numerical molecular dynamics calculation of the polarization in the **thermodynamic limit of the QSS regime** considering

$$T = T(N_\infty) = T_{QSS}$$



The comparison is good and gives further support to the connection between glassy systems and the QSS regime of the HMF model

Schematic summary



Summary for the HMF model

- Summarizing, the **Hmf model** represents a **paradigmatic model** for nonextensive (long range interacting or finite) systems, as for example self-gravitating objects, nuclear and atomic systems.

- Several **dynamical metastable anomalies** are present: negative specific heat, very slow dynamics, anomalous diffusion, power-law relaxation, fractal-like structures, vanishing of lyapunov exponents, ergodicity breaking and aging

- This **metastable anomalous behavior can become stable** if the infinite size limit is performed before the infinite time limit: ***the two limits do not commute.***

- There are links to Tsallis generalized statistics.

- It is possible to treat the QSS regime in terms of a glassy phase.

- But one can find similarities with other non Hamiltonian systems….

Some references on the HMF model

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- ✓ Latora, Rapisarda, Ruffo, Phys. Rev. Lett. 80 (1998) 698. Physica D 131 (1999) 38
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- ✓ Yamaguchi, Barrè, Bouchet, Dauxois, Ruffo, Physica A 337 (2004) 36
- ✓ Pluchino, Latora, Rapisarda, Physica D 193 (2004) 315; Phys. Rev. E 69 (2004) 056113, Physica A 338 (2004) 60, Cont. Mech. and Therm. 16 (2004) 245, cond-mat/0506665 and cond-mat/0507005, Europhysics News 36 (2005) 202
  Physica A 365 (2006) 184
- ✓ Tamarit, Maglione, Anteneodo, Stariolo, Phys. Rev. E 71 (2005) 036148

Most recent results

For the generalized version of the HMF model see:

- ✓Anteneodo and Tsallis, Phys. Rev. Lett. 80 (1998) 5313; Tamarit and Anteneodo, Phys. Rev. Lett. 84 (20
- ✓ Campa, Giansanti and Moroni, J. Phys. A 36 (2003) 6897.

124 Non Hamiltonian systems

- The Kuramoto Model
- Coupled Logistic maps with noise
- Soc models for earthquakes dynamics

The Kuramoto model

The Kuramoto Model

(Kuramoto 1975)

Eqs. for the N coupled oscillators

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N$$

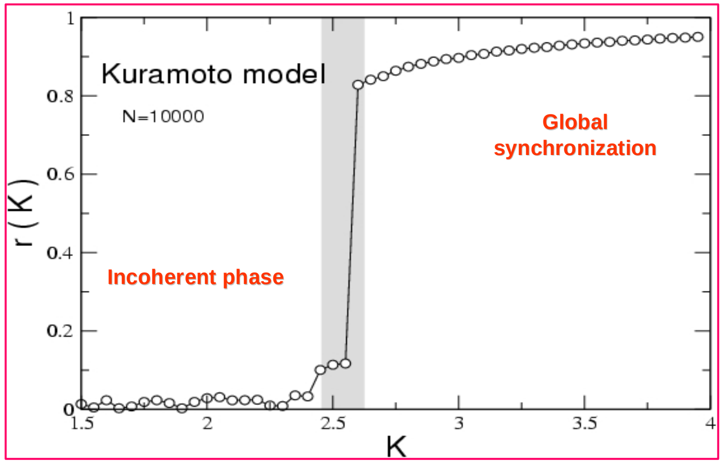
Order parameter

$$re^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

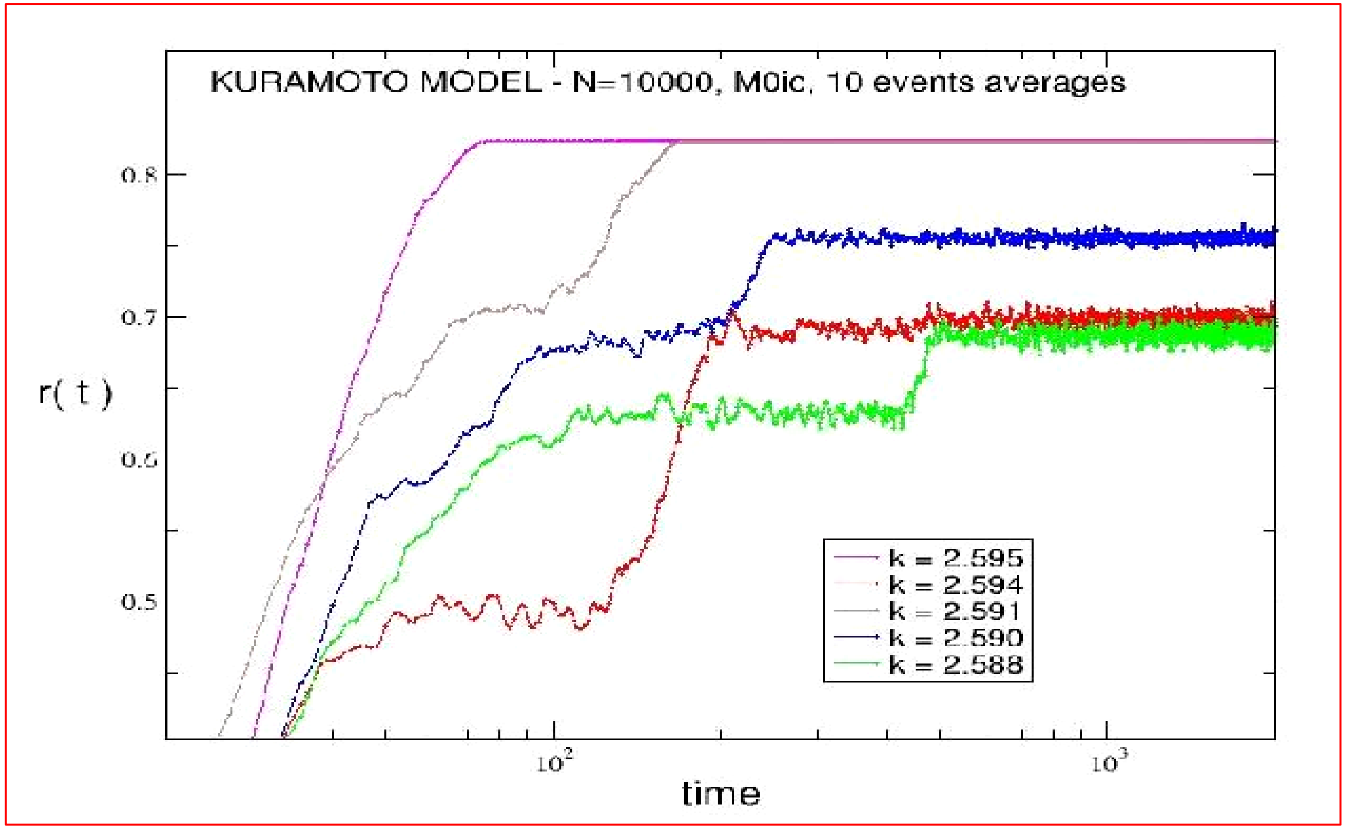
mean field equation

$$\dot{\theta}_i(t) = \omega_i + Kr\sin(\Psi - \theta_i)$$

Phase transition



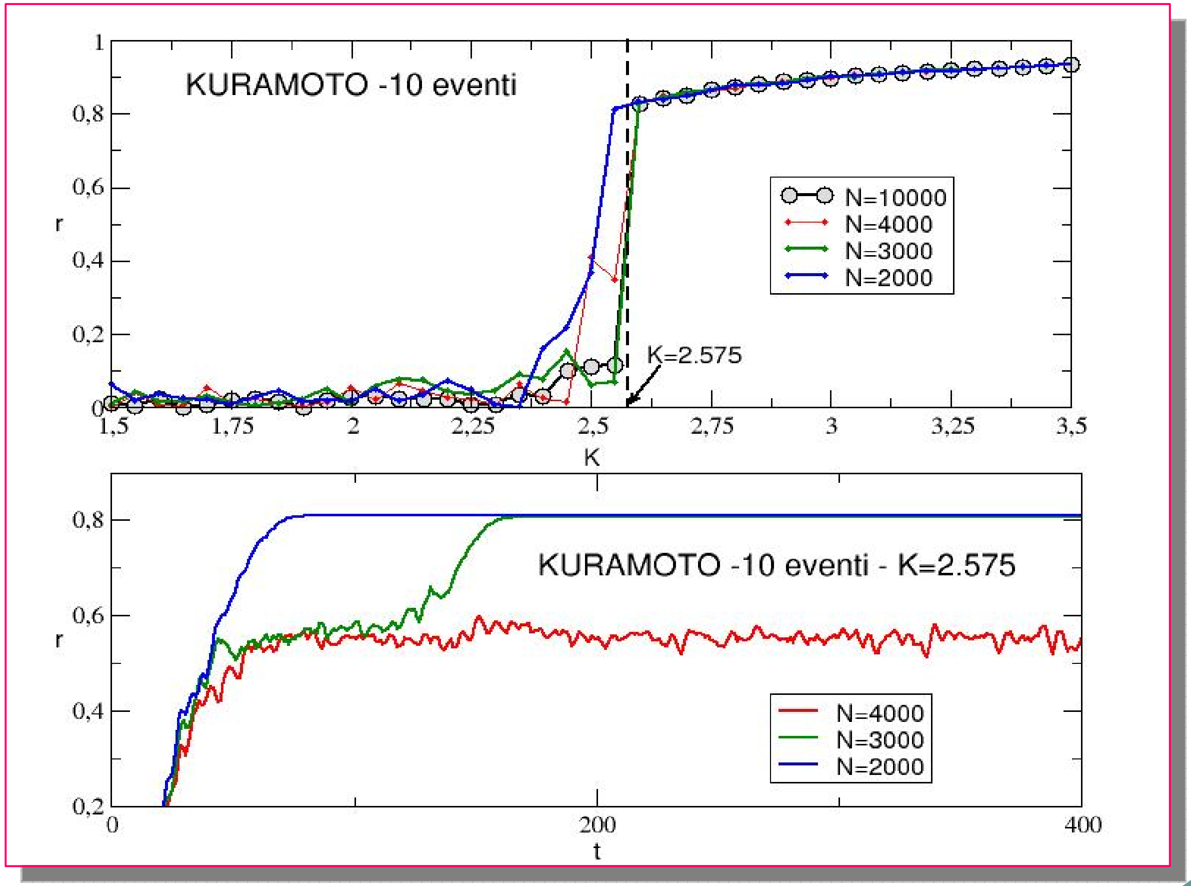
112 Metastability in the Kuramoto Model



A. Pluchino, A. Rapisarda, **Physica A** 365 (2006) 184

113 Metastability in the Kuramoto Model

Also for the Kuramoto model metastability seems to diverge with the system size



Coupled maps on a lattice

102 Coupled Maps on a lattice

We consider a model of Coupled Logistic Maps on a lattice, i.e.

$$f(x) = sign(x) \cdot \mathrm{mod}\ \{2\,[\alpha \cdot g_\mu(x) + (1 - \alpha) \cdot w]\,,\ 1\} \qquad (1)$$

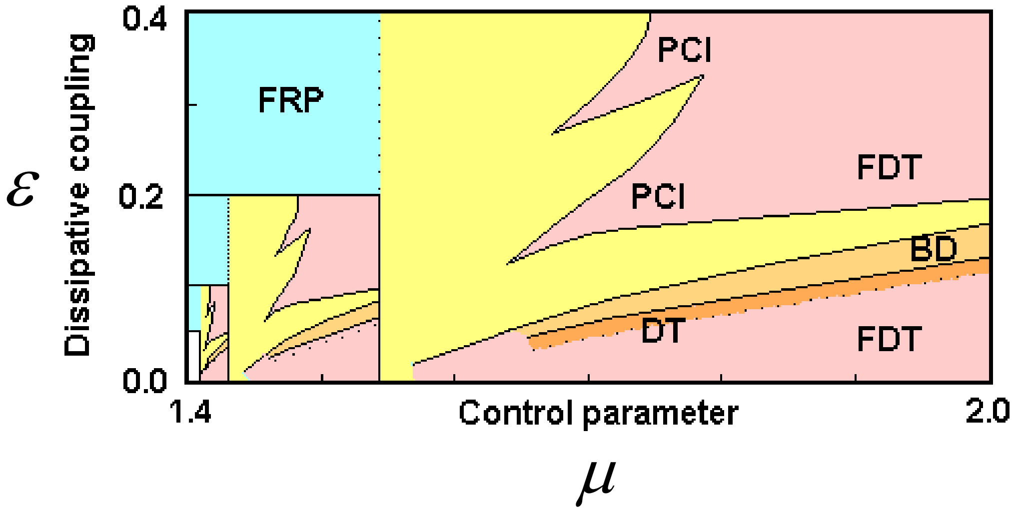
$$x_{t+1} = f(x_t(i)) + \frac{\varepsilon}{2}[f\,(x_t(i+1)) - 2f(x_t(i)) + f(x_t(i-1))] \qquad (2)$$

$$g_\mu = 1 - \mu x^2$$

Here w is a white noise term controlled by the parameter $\alpha \in [0,1]$

See the poster by S. RIZZO

Phase diagram of CML



FRP= frozen random patterns

BD = brownian motion of defect

DT = defect turbulence

PCI= pattern competition intermittency

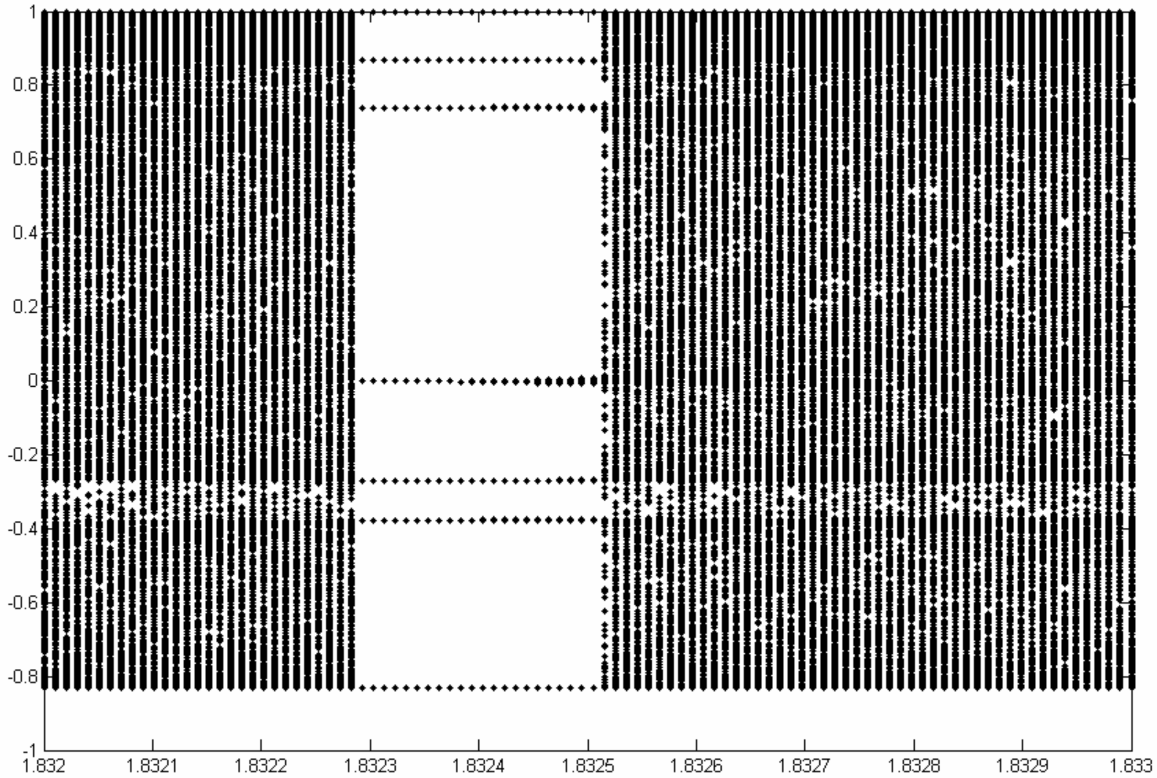
FDT= fully developed turbulence

K. Kaneko, *Simulating Physics with Coupled Map Lattices*, World Scientific (1990)

103 Coupled Maps on a lattice

We fix the control parameter at the edge of chaos

1.8322<μ <1.8323

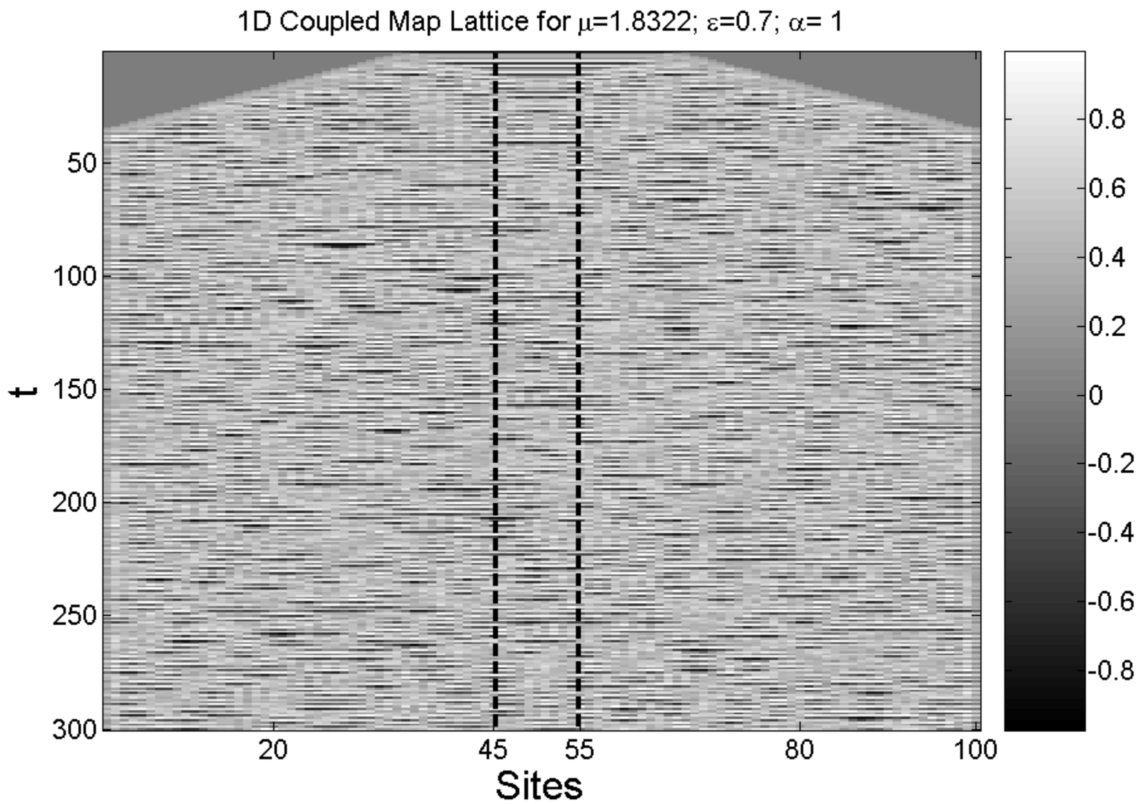


103 Dynamics of Coupled Maps on a lattice

Time evolution with no noise (α=1)

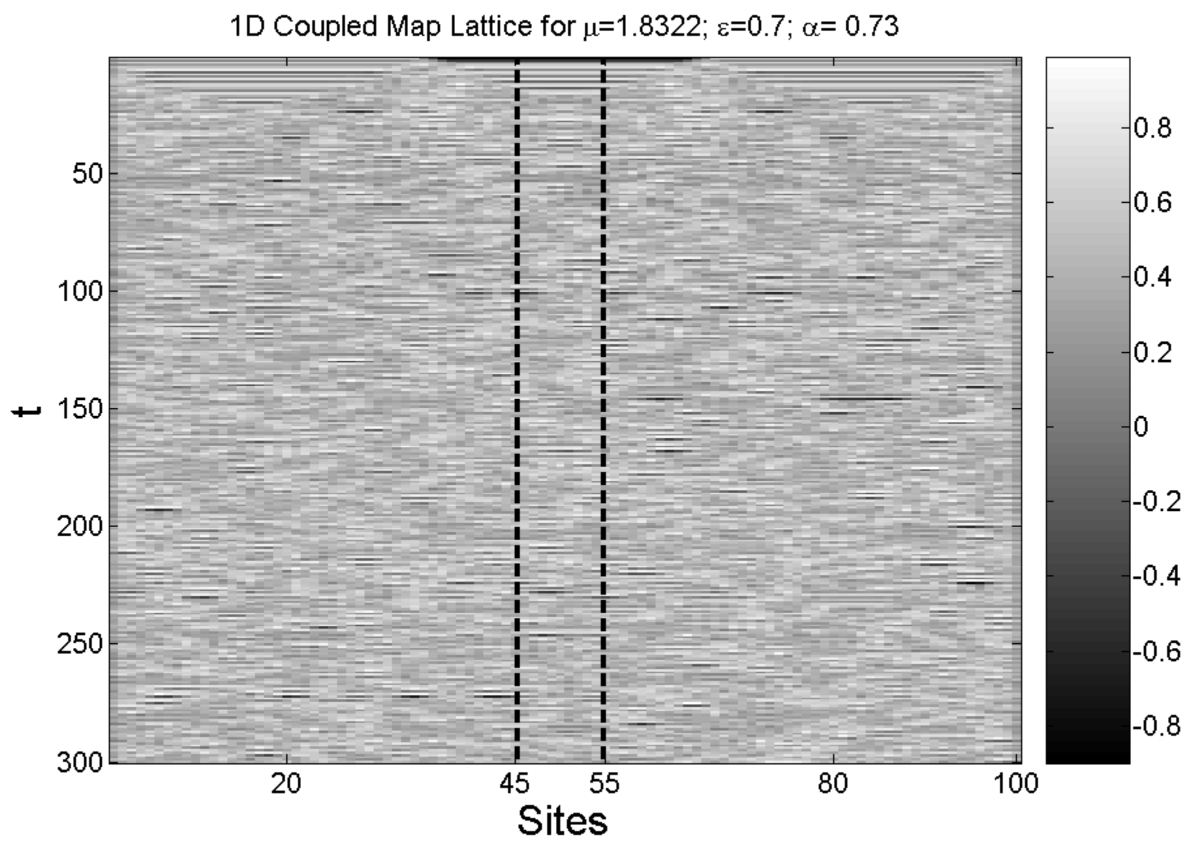
μ=1.8322

ε=0.7



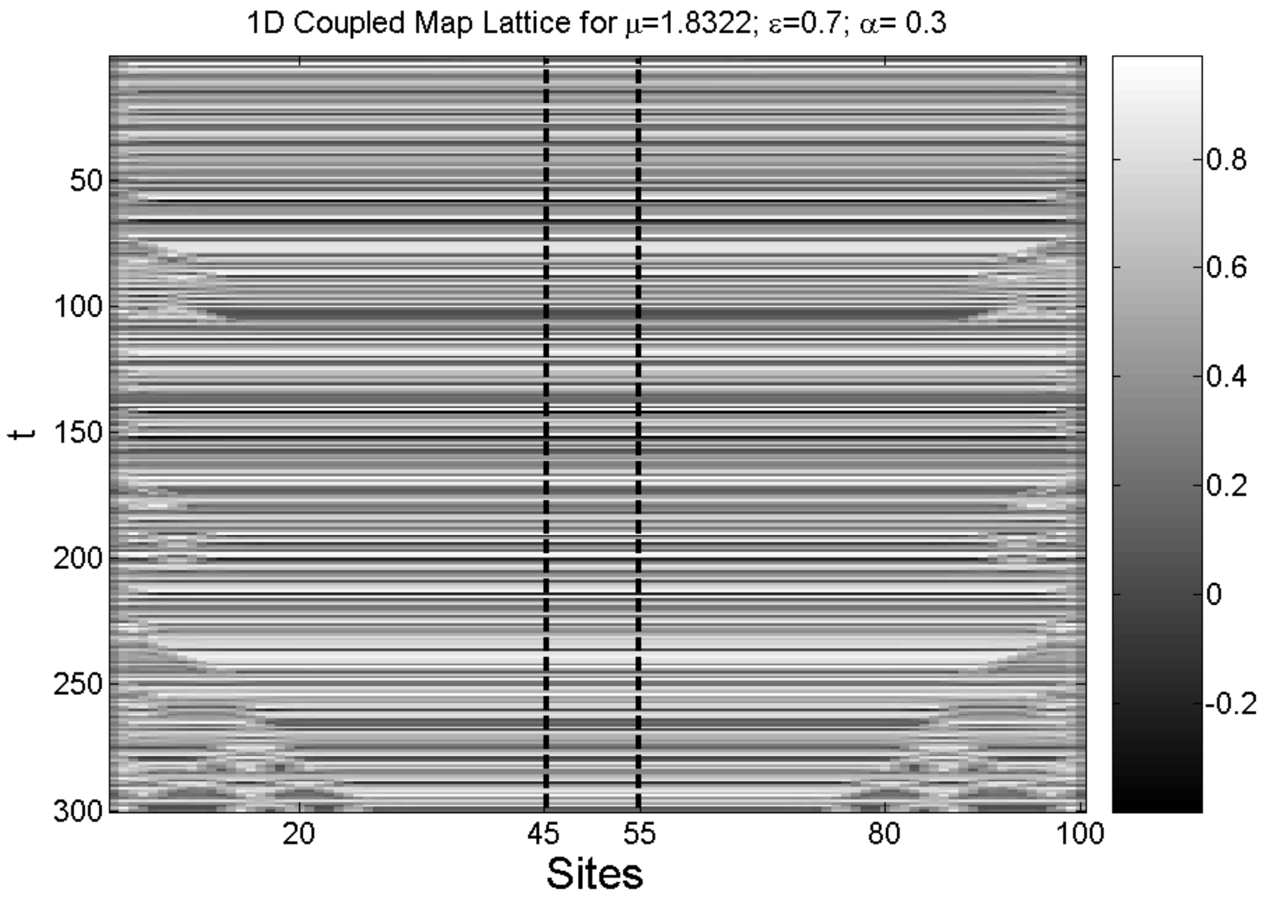
104 Dynamics of Coupled Maps on a lattice

Time evolutior with moderate noise
 (α=0.73)

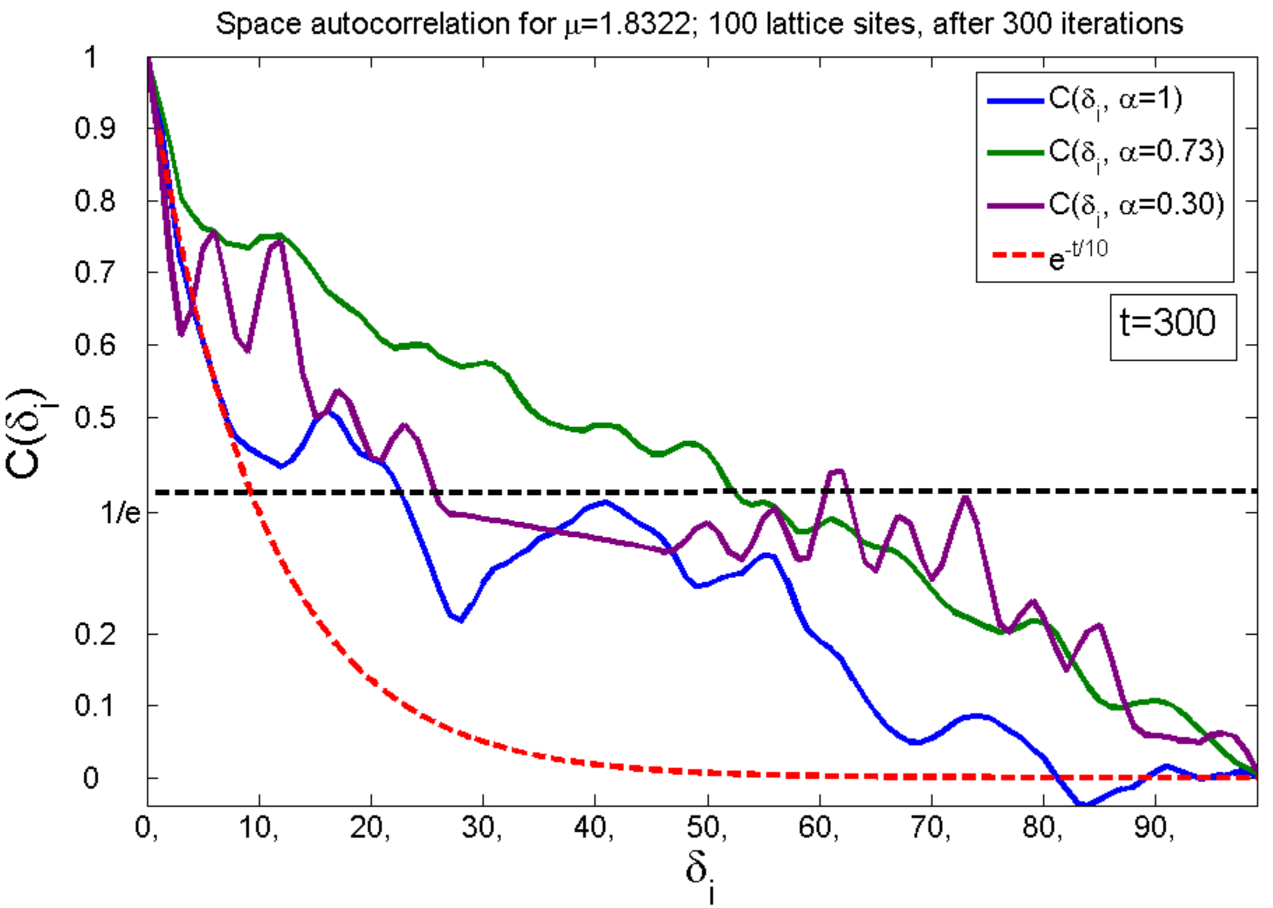


105 Dynamics of Coupled Maps on a lattice

Time evolutior
with strong
noise
(α=3)



105 Autocorrelation for Coupled Maps on a lattice

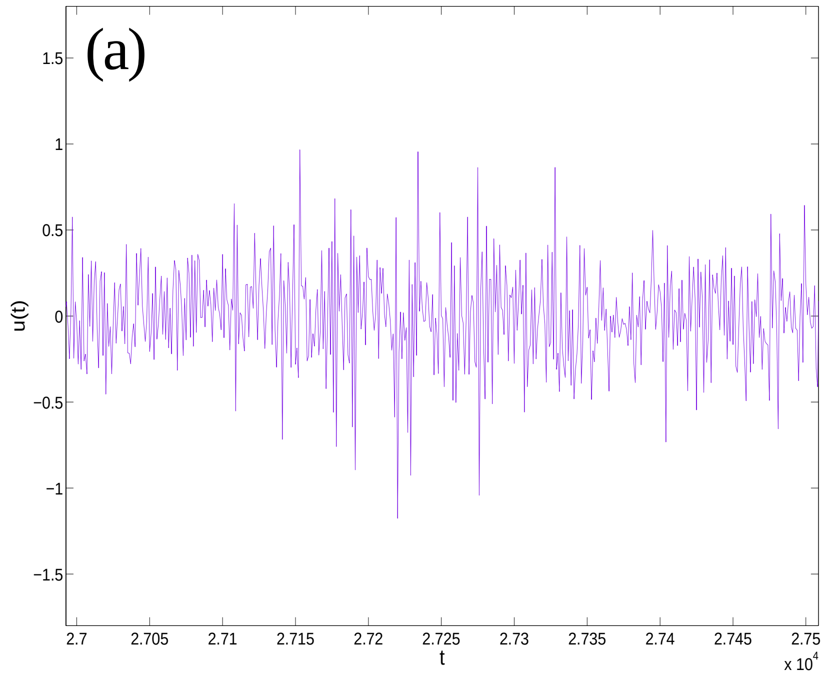


106 Fluctuations of Coupled Maps on a lattice

One can consider the difference between two maps and study the fluctuations of this new variable.

We considered for example

$$u(t) = x(45) - x(55)$$

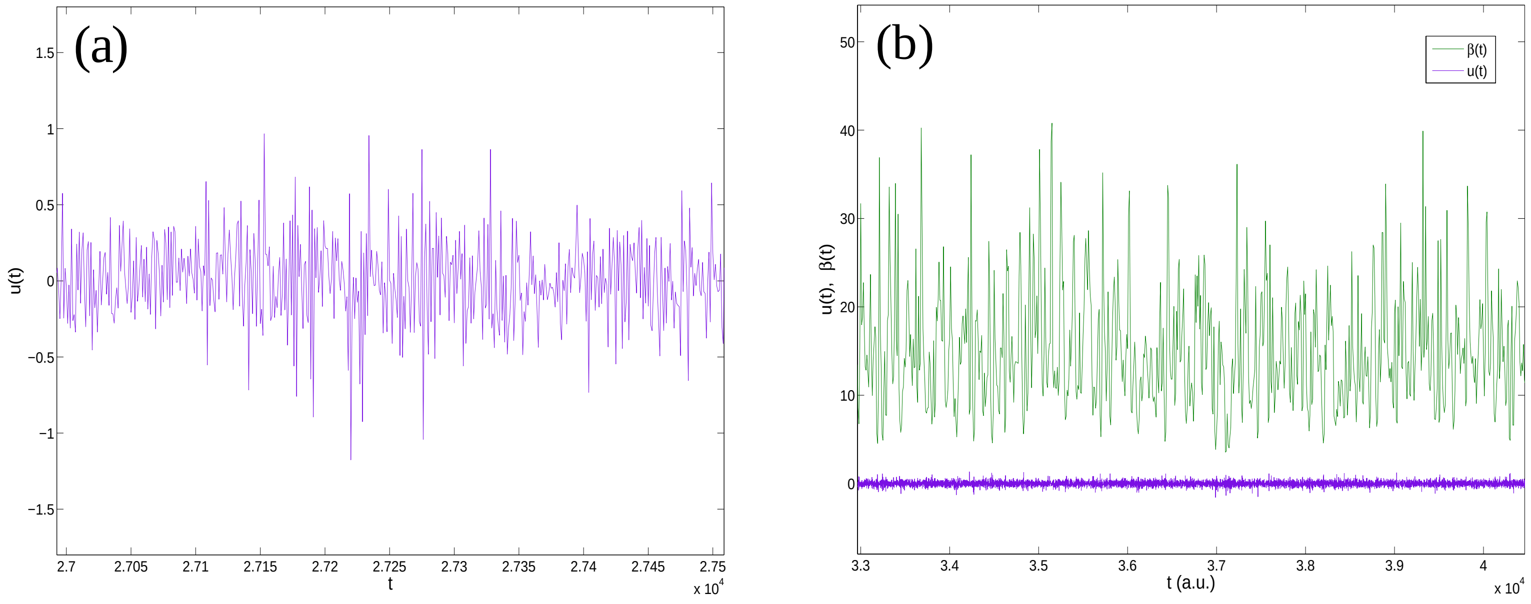


The time behavior of **u(t)** can be very intermittent and one can apply a superstatistic approach

(more details in the poster by

S. RIZZO)

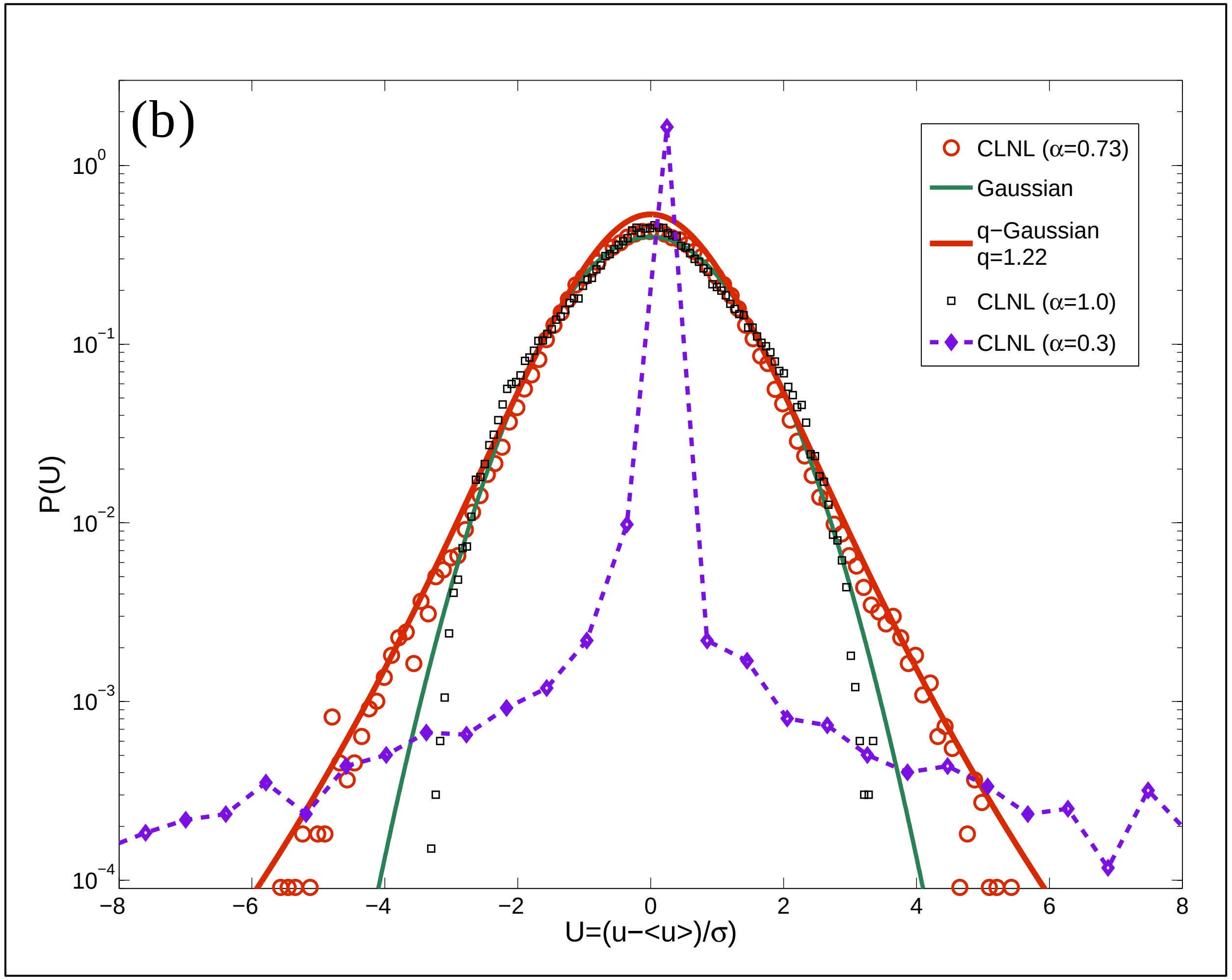
106 Coupled Maps on a lattice



107 Pdfs of Coupled Maps on a lattice

With no noise one gets pdfs very similar to the HMF case (black dots).

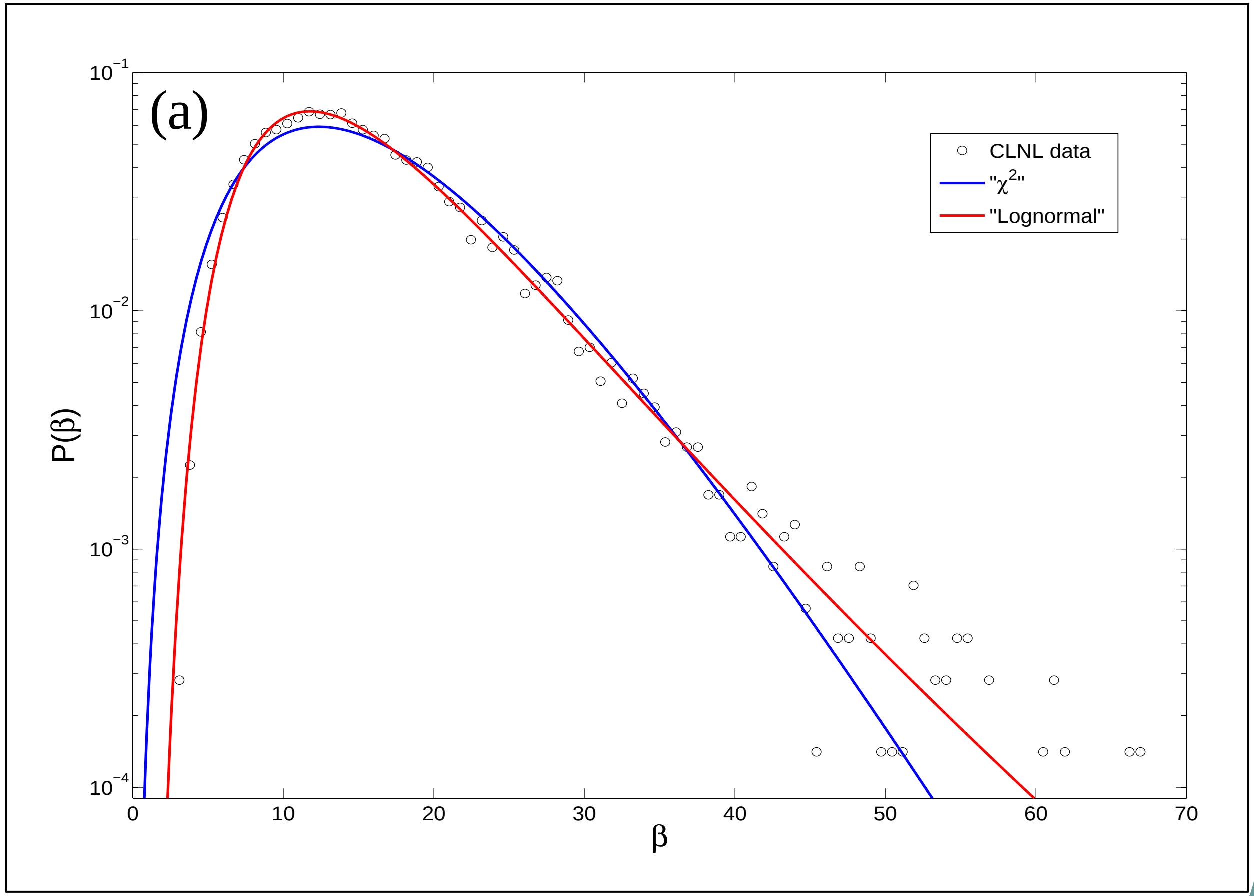
Adding some noise one gets q-gaussians which can be explained by applying the superstatistics approach proposed by Beck and Cohen



108 Coupled Maps on a lattice

The pdf of β, the inverse variance of u(t), obeys a Log-normal distribution.

But a Gamma distribution is very close to it and represents a good approximation



SOC model with long-range interactions for earthquakes dynamics

Conclusions

Summarizing, long-range interacting systems present several dynamical anomalies which pose severe problems to standard statistical mechanics and which can find a natural description in term of q-statistics

"ALL THE TRUTHS PASS THROUGH THREE STAGES:

FIRST, THEY ARE CONSIDERED RIDICULOUS,

SECOND, THEY ARE VIOLENTLY ADVERSED,

THIRD, THEY ARE ACCEPTED AND CONSIDERED SELF-EVIDENT."

A. SCHOPENHAUER